Wave-number-explicit bounds in time-harmonic scattering

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WAVE-NUMBER-EXPLICIT BOUNDS IN TIME-HARMONIC SCATTERING *

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Abstract. In this paper we consider the problem of scattering of time-harmonic acoustic waves by a bounded sound soft obstacle in two and three dimensions, studying dependence on the wave number in two classical formulations of this problem. The first is the standard variational/weak formulation in the part of the exterior domain contained in a large sphere, with an exact Dirichlet-to-Neumann map applied on the boundary. The second formulation is as a second kind boundary integral equation in which the solution is sought as a combined single- and double-layer potential. For the variational formulation we obtain, in the case when the obstacle is starlike, explicit upper and lower bounds which show that the inf-sup constant decreases like k^{-1} as the wave number k increases. We also give an example where the obstacle is not starlike and the inf-sup constant decreases at least as fast as k^{-2}. For the boundary integral equation formulation, if the boundary is also Lipschitz and piecewise smooth, we show that the norm of the inverse boundary integral operator is bounded independently of k if the coupling parameter is chosen correctly. The methods we use also lead to explicit bounds on the solution of the scattering problem in the energy norm when the obstacle is starlike in which the dependence of the norm of the solution on the wave number and on the geometry are made explicit.

Key words. Non-smooth boundary, a priori estimate, inf-sup constant, Helmholtz equation, oscillatory integral operator

AMS subject classifications. 35J05, 35J20, 35J25, 42B10, 78A45

1. Introduction. In this paper we consider the classical problem of scattering of a time-harmonic acoustic wave by a bounded, sound soft obstacle occupying a compact set $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3). The wave propagates in the exterior domain $\Omega_e = \mathbb{R}^n \setminus \Omega$ and the boundedness of the scatterer implies that there is an $R > 0$ such that $\{x \in \mathbb{R}^n : |x| > R\} \subset \Omega_e$. We suppose that the medium of propagation outside Ω_e is homogeneous, isotropic and at rest, and that a time harmonic ($\mathrm{e}^{-\mathrm{i}\omega t}$ time dependence) pressure field u^i is incident on Ω. Denoting by $c > 0$ the speed of sound, we assume that u^i is an entire solution of the Helmholtz (or reduced wave) equation with wave number $k = \omega/c > 0$. Then the problem we consider is to find the resulting time-harmonic acoustic pressure field u which satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0 \text{ in } \Omega_e \tag{1.1}$$

and the sound soft boundary condition

$$u = 0 \text{ on } \Gamma := \partial\Omega_e, \tag{1.2}$$

and is such that the scattered part of the field, $u^s := u - u^i$, satisfies the Sommerfeld radiation condition

$$\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s = o(r^{-(n-1)/2}) \tag{1.3}$$

as $r := |x| \to \infty$, uniformly in $\hat{x} := x/r$. (This latter condition expresses mathematically that the scattered field u^s is outgoing at infinity; see e.g. [14]). It is well

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known that this problem has exactly one solution under the constraint that u and ∇u be locally square integrable; see e.g. [34].

The aim of this paper is to understand the behaviour, in the important but difficult high frequency limit $k \to \infty$, of two standard reformulations of this problem. Both of these reformulations are used extensively, for theoretical analysis and for practical numerical computation. The first is a weak, variational formulation in the bounded domain $D_R := \{x \in \Omega_e : |x| < R\}$, for some $R > R_0 := \sup_{x\in\Omega} |x|$. This formulation is expressed in terms of the Dirichlet to Neumann map T_R, for the canonical domain $G_R := \{x : |x| > R\}$ with boundary $\Gamma_R := \{x : |x| = R\}$. The mapping T_R takes Dirichlet data $g \in C^\infty(\Gamma_R)$ to the corresponding Neumann data $T_R g := \frac{\partial v}{\partial r}|_{\Gamma_R}$, where v denotes the solution to the Helmholtz equation in G_R which satisfies the Sommerfeld radiation condition and the boundary condition $v = g$ on Γ_R. It is standard that the mapping T_R extends to a bounded map $T_R : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$.

Let V_R denote the closure of $\{v|_{D_R} : v \in C_0^\infty(\Omega_e)\} \subset H^1(D_R)$ in the norm of $H^1(D_R)$. It is well known (e.g. [39]), and follows easily by integration by parts, that u satisfies the scattering problem if and only if the restriction of u to D_R satisfies the variational problem: find $u \in V_R$ such that

$$b(u,v) = G(v), \quad v \in V_R. \tag{1.4}$$

Here G is an anti-linear functional that depends on the incident field (for details see §3), while $b(\cdot,\cdot)$ is the sesquilinear form on $V_R \times V_R$ defined by

$$b(u,v) := \int_{D_R} (\nabla u \cdot \nabla \bar{v} - k^2 u\bar{v})\, dx - \int_{\Gamma_R} \gamma\bar{v} T_R \gamma u\, ds, \tag{1.5}$$

where $\gamma : V_R \to H^{1/2}(\Gamma_R)$ is the usual trace operator. Equation (1.4) is our first standard reformulation of the scattering problem.

To introduce our second reformulation, let $\Phi(x,y)$ denote the standard free-space fundamental solution of the Helmholtz equation, given, in the 2D and 3D cases, by

$$\Phi(x,y) := \begin{cases} \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|), & n = 2, \\[2ex] \dfrac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|}, & n = 3, \end{cases}$$

for $x, y \in \mathbb{R}^n$, $x \neq y$. It was proposed independently by Brakhage and Werner [4], Leis [33], and Panich [40], as a means to obtain an integral equation uniquely solvable at all wave numbers, to look for a solution to the scattering problem in the form of the *combined single- and double-layer potential*

$$u^s(x) := \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi(y)\, ds(y) - \mathrm{i}\eta \int_\Gamma \Phi(x,y)\varphi(y)\, ds(y), \quad x \in \Omega_e, \tag{1.6}$$

for some non-zero value of the *coupling parameter* $\eta \in \mathbb{R}$. (In this equation $\partial/\partial\nu(y)$ is the derivative in the normal direction, the unit normal $\nu(y)$ directed into Ω_e.) It follows from standard boundary trace results for single- and double-layer potentials that u^s, given by (1.6), satisfies the scattering problem if and only if φ satisfies a second kind boundary integral equation on Γ (see §4 for details). This integral equation, in operator form, is

$$(I + K - \mathrm{i}\eta S)\varphi = 2g, \tag{1.7}$$

where I is the identity operator, S and K are single- and double-layer potential operators, defined by (4.1) and (4.2) below, and $g := -u^i|_\Gamma$ is the Dirichlet data for the scattered field on Γ.

Choosing $\eta \neq 0$ ensures that (1.6) is uniquely solvable. Precisely,

$$A := I + K - \mathrm{i}\eta S$$

is invertible as an operator on $C(\Gamma)$ when Γ is sufficiently smooth, e.g. of class C^2 (see [4] or [14]). The case of non-smooth (Lipschitz) Γ has been considered recently in [9] (and see [37]) where it is shown that A is invertible as an operator on the Sobolev space $H^s(\Gamma)$, for $0 \leq s \leq 1$.

While it is established that each of these formulations is well-posed, precisely that A^{-1} is a bounded operator on $H^s(\Gamma)$, $0 \leq s \leq 1$, in the case of (1.6), and that the sesquilinear form $b(\cdot, \cdot)$ satisfies the inf-sup condition, that

$$\alpha := \inf_{0 \neq u \in V_R} \sup_{0 \neq v \in V_R} \frac{|b(u,v)|}{\|u\|_{V_R} \|v\|_{V_R}} > 0, \tag{1.8}$$

in the case of the formulation (1.4), there is little information in the literature on how the stability constants $\|A^{-1}\|$ and α depend on k, particularly in the limit as $k \to \infty$.

This lack of theoretical understanding is unfortunate for a number of reasons. In the first place both formulations (and similar formulations for other boundary conditions on Γ) are used extensively for numerical computation. Much research has been aimed in recent years at efficient solvers in the difficult high frequency case, where the scatterer Γ, and so the region D_R, are large in diameter compared to the wavelength, so that the solution u is highly oscillatory and standard discretisation methods require very many degrees of freedom. This effort has included many important developments for the solution of (1.6) and similar integral equations, including higher order boundary element or Nyström schemes (e.g. [22]), fast multipole methods (e.g. [17]), generalised boundary element methods using oscillatory basis functions (e.g. [5, 32, 19]), and preconditioners for iterative solvers (e.g. [12]). Similarly, for the solution of (1.4) at high frequency, important recent developments have included the use of higher order *hp*-finite element methods (e.g. [2, 18]), the use of oscillatory basis functions (e.g. [31] and, for methods based on more general variational formulations, [7, 21]) and ray-based techniques (e.g. [30]).

An essential ingredient in the development of numerical analysis for these methods, in particular analysis which seeks to determine the behaviour of algorithms as the wave number increases, is an understanding of how the stability constants of numerical schemes depend on the wave number. Quantification of the dependence on k of $\|A^{-1}\|$ and α, i.e. of stability constants for the continuous formulation, is an important step in this direction.

An additional and important practical issue in connection to (1.6) is how to choose the parameter η. A natural criterion when using (1.6) for numerical computation is to choose η so as to minimise the condition number cond $A := \|A\| \, \|A^{-1}\|$ (e.g. Kress [25, 26]). To determine this optimal choice, information on the dependence of $\|A^{-1}\|$ on k and η is required and will be obtained in §4.

Given the practical importance of the questions we will address, it is not surprising that a number of relevant investigations have been carried out previously. In particular, a number of authors have studied (1.6), or related integral equations, in the canonical case when Γ is a cylinder or sphere, i.e. $\Gamma = \Gamma_R$, for some $R > 0$, especially with the aim of determining η so as to minimise the $L^2(\Gamma)$ condition number of A [25, 26, 3, 23, 6, 19]. Particularly relevant are the results of Giebermann [23] which have recently been completed and put on a rigorous footing by Dominguez et al. [19]. It is shown in [19] that, in the 2D case, if the choice $\eta = k$ is made, then

$$\|A^{-1}\|_2 \leq 1 \tag{1.9}$$

for all sufficiently large k (we are using $\|\cdot\|_2$ to denote both the norm on $L^2(\Gamma)$ and the induced operator norm on the space of bounded linear operators on $L^2(\Gamma)$). This result is obtained as a consequence of the coercivity result that

$$\Re(A\psi,\psi) \geq \|\psi\|_2^2, \quad \forall \psi \in L^2(\Gamma), \tag{1.10}$$

where $(\cdot,\cdot)$ is the usual scalar product on $L^2(\Gamma)$. The same coercivity result, but without an explicit value for the constant, is shown in the 3D case [19], so that, for the case when Γ is a sphere it also holds that $\|A^{-1}\|_2 = O(1)$ as $k \to \infty$. We note that, even for these canonical cases, establishing such bounds is not straightforward and depends on explicit calculations of the spectrum of A and careful estimates of Bessel functions uniformly in argument and order.

Research of relevance to the wave number dependence of (1.4) has also been carried out. Indeed an explicit estimate of the dependence of the inf-sup constant on the wave number has been made previously in two cases. The first is what may be thought of as a 1D analogue of (1.4), with $V_R := \{u \in H^1(0,1) : u(0) = 0\}$ and $b(\cdot,\cdot)$ defined by $b(u,v) := \int_0^1 u'\bar{v}' - k^2 u\bar{v}dx - \mathrm{i}k\bar{v}(1)u(1)$. The results for this case, due to Ihlenburg and Babuška [27, 28], summarised in [29], are obtained via explicit calculations of the Green's function for the corresponding boundary value problem, i.e. the solution of $u'' + k^2 u = \delta_y$ on $(0,1)$ with $u(0) = 0$, $u'(1) = \mathrm{i}ku(1)$, where δ_y is the delta distribution supported at $y \in (0,1)$. For this 1D problem it is shown that, for some constants $C_1 \leq C_2$, the inf-sup constant given by (1.8), with $\|u\|_{V_R}^2 = \int_0^1 |u'|^2 dx$, satisfies

$$\frac{C_1}{k} \leq \alpha \leq \frac{C_2}{k}. \tag{1.11}$$

Closer still to the results of this paper is the work of Melenk [35] (or see Cummings and Feng [16]), who consider the Helmholtz equation in a bounded domain D, which is either convex or sufficiently smooth and starlike, with the impedance boundary condition $\frac{\partial u}{\partial \nu} = \mathrm{i}k\eta u$ on ∂D, with the normal directed out of D and $\eta > 0$. The sesquilinear form in their case is $b : H^1(D) \times H^1(D) \to \mathbb{C}$ given by

$$b(u,v) := \int_D (\nabla u \cdot \nabla \bar{v} - k^2 u\bar{v})\, dx - \mathrm{i}k\eta \int_\Gamma \gamma\bar{v}\gamma u\, ds. \tag{1.12}$$

With this definition of $b(\cdot,\cdot)$ they show that their inf-sup constant α satisfies

$$\alpha \geq \frac{C}{k}, \tag{1.13}$$

for some constant $C > 0$. The technique of argument used in [35] and [16] is to derive a Rellich-type identity, this technique of wide applicability to obtain a priori estimates for solutions of boundary value problems for strongly elliptic systems of PDEs, see e.g. [38, 36, 34]. This approach, essentially a carefully chosen application of the divergence theorem, appears to depend essentially on the starlike nature of the domain to obtain the wave-number-explicit bound (1.13).

The arguments of [35] and [16] will be one ingredient of the methods we use in this paper. The general structure of the arguments, though little of the detail, will borrow heavily from two of our own recent papers [10, 8] where we show analogous results to those presented here but for the case of *rough surface scattering*, i.e. the case where Γ is unbounded, the graph of some bounded continuous function, and Ω_e is its epigraph. Assuming that the axes are oriented so that Γ is bounded in the x_n-direction, i.e. $f_- \leq x_n \leq f_+$, for $x = (x_1,\ldots,x_n) \in \Gamma$, for some constants f_- and f_+, the analogous sesquilinear form to (1.5) for this case is given by the same formula, provided one redefines D_R and Γ_R by $D_R := \{x \in \Omega_e : x_n < R\}$ and $\Gamma_R :=$

$\{x \in \Omega_e : x_n = R\}$, chooses $R \geq f_+$, and sets T_R to be the Dirichlet to Neumann map for the Helmholtz equation in the upper half-space $\{x : x_n > f_+\}$. This Dirichlet to Neumann map is given explicitly as a composition of a multiplication operator and Fourier transform operators. With this definition of the sesquilinear form $b(\cdot, \cdot)$ and with the inf-sup constant defined by (1.8) with

$$\|u\|_{V_R} := \left\{ \int_{D_R} \left(|\nabla u|^2 + k^2 |u|^2 \right) dx \right\}^{1/2}, \tag{1.14}$$

we show in [10] the explicit bound for the rough surface problem that

$$\alpha \geq (1 + \sqrt{2}\,\kappa\,(\kappa + 1)^2)^{-1}, \tag{1.15}$$

where $\kappa := k(R - f_-)$. In [8] we study an integral equation formulation for the same problem in the case when, additionally, the function that Γ is the graph of is continuously differentiable. For the integral equation formulation (1.7) for this problem (with the twist that S and K are defined with the standard fundamental solution $\Phi(x, y)$ replaced by the Dirichlet Green's function for a half-space containing Ω_e), we show the bound

$$||A^{-1}||_2 < 2 + 2L + 4L^2 + \frac{k}{\eta} \left(2 + 5L + 3L^{3/2} \right), \tag{1.16}$$

where L is the maximum surface slope.

We note that Claeys and Haddar [13] have recently adapted the arguments of [10] to study 3D acoustic scattering from an unbounded sound soft rough tubular surface, as an initial model of electromagnetic scattering by an infinite wire with a perturbed surface. They study a weak formulation which can be written in the form (1.4) with a sesquilinear form which can be written as (1.5), provided one redefines Γ_R to be the infinite cylinder $\Gamma_R := \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = R^2\}$, D_R to be that part of the region outside the tubular surface but inside Γ_R, and T_R to be the appropriate Dirichlet to Neumann map for the Helmholtz equation in the region exterior to Γ_R. Their emphasis is on showing well-posedness for this problem, including showing that the inf-sup condition (1.8) holds, rather than on obtaining explicitly the k-dependence, but their results do imply a lower bound on α, that $\alpha^{-1} = O(k^3)$ as $k \to \infty$, the same k-dependence as (1.15).

In this paper we will obtain analogous bounds to (1.11), (1.15) and (1.16) for the problem of scattering by a bounded sound soft obstacle. A major obstacle in achieving this aim is understanding the behaviour of the Dirichlet to Neumann map T_R in sufficient detail. We address this issue in §2, where our main new result is Lemma 2.1, a subtle property of radiating solutions of the Helmholtz equation, whose proof depends on a detailed understanding of monotonicity properties of Bessel functions. This lemma is essential to our results and we expect will be of value in deducing explicit bounds for a range of other wave scattering problems.

In §3 we study the formulation (1.4). Our main results are, firstly, the upper bound on the inf-sup constant (1.8), which holds with no constraint on Γ, that

$$\alpha \leq \frac{C_1}{kR} + \frac{C_2}{k^2 R^2}$$

where the constants $C_1 \geq 2\sqrt{2}$ and C_2 depend on the shape of the domain. (Our norm $\| \cdot \|_{V_R}$ in (1.8) is the wave number dependent norm given by (1.14).) In the case that the scattering obstacle Ω is starlike in the sense that $x \in \Omega$ implies $sx \in \Omega$, for $0 \leq s < 1$, we also show a lower bound, so that it holds that

$$\frac{1}{5 + 4\sqrt{2}kR} \leq \alpha \leq \frac{C_1}{kR} + \frac{C_2}{k^2 R^2}. \tag{1.17}$$

We note that this bound establishes that, when Ω is starlike, α decreases like k^{-1} as $k \to \infty$ (cf. (1.11)). Finally, we produce an example (scattering by two parallel plates) for which

$$\alpha \leq \frac{C}{k^2 R^2}$$

for some constant C and unbounded sequence of values of k, showing that the lower bound in (1.17) need not hold if Ω is not starlike. We emphasise that these appear to be the first bounds on the inf-sup constant in the literature for any problem of time-harmonic scattering by a bounded obstacle in more than one dimension which make the dependence on the wave number explicit.

We turn in §4 to the integral equation formulation (1.7). We restrict attention to the case when Ω is starlike and Γ is Lipschitz and piecewise smooth (e.g. a starlike polyhedron). Our main result is a bound on $\|A^{-1}\|_2$ (Theorem 4.3) as a function of three geometrical parameters and the ratio k/η, of the wave number to the coupling parameter. Importantly this bound shows that, if the ratio k/η is kept fixed then $\|A^{-1}\|_2$ remains bounded as $k \to \infty$. In particular, if the choice $\eta = k$ is made then, for $kR_0 \geq 1$,

$$\|A^{-1}\|_2 \leq \frac{1}{2}\left(1 + \theta(4\theta + 4n + 1)\right), \tag{1.18}$$

where $n = 2$ or 3 is the dimension, $\theta := R_0/\delta_-$ and $\delta_- > 0$ is the essential infimum of $x \cdot \nu$ over the surface Γ (for example, $\theta = 1$ for a sphere, $\theta = \sqrt{3}$ for a cube). A sharper (but more complicated) bound is given in Corollary 4.4. We note that a value of η/k in the range $1/2 \leq \eta/k \leq 1$ has been recommended based on studies of the integral operator A for circular and spherical geometries [25, 26, 3, 23, 19] as minimising, approximately, the L^2 condition number of A.

We emphasise that the only comparable results to (1.18) to date are the bounds discussed above for the case when Γ is a circle or sphere, obtained by methods specialised to circular/spherical geometry. Even for this geometry the only completely explicit bound is (1.9), shown to hold for a circle for all sufficiently large k. Our general methods give the bound (4.15) for this case, which is almost as sharp a result, implying that, for every $\beta > 5/2$, $\|A^{-1}\|_2 \leq \beta$, for all sufficiently large kR_0, where R_0 is the radius of the circle.

2. Preliminaries. It is convenient to separate off in an initial section two key lemmas which are essential ingredients in the arguments we will make to obtain wave number explicit bounds for both our formulations of the scattering problem, and to gather here other material common to both formulations.

Our arguments in this paper will depend on explicit representations for solutions of the Helmholtz equation in the exterior of a large ball. These depend in turn on explicit properties of cylindrical and spherical Bessel functions. For $\nu \geq 0$ let J_ν and Y_ν denote the usual Bessel functions of the first and second kind of order ν (see e.g. [1] for definitions) and let $H_\nu^{(1)} := J_\nu + \mathrm{i}Y_\nu$ denote the corresponding Hankel function of the first kind of order ν. Of course, where C_ν denotes any linear combination of J_ν and Y_ν, it holds that C_ν is a solution of Bessel's equation of order ν, i.e.

$$z^2 C_\nu''(z) + zC_\nu'(z) + (z^2 - \nu^2)C_\nu(z) = 0. \tag{2.1}$$

In the 3D case it is convenient to work also with the spherical Bessel functions j_m, y_m, and $h_m^{(1)} := j_m + \mathrm{i}y_m$, for $m = 0, 1, \ldots$. These can be defined directly (see e.g. Nédélec [39]) by recurrence relations which imply that $h_m^{(1)}(z) = \mathrm{e}^{\mathrm{i}z} p_m(z^{-1})$, where p_m is a polynomial of degree m with $p_m(0) = 1$. Alternatively, the spherical Bessel functions can be defined in terms of the usual Bessel functions via the relations

$$j_m(z) = \sqrt{\frac{\pi}{2z}} J_{m+1/2}(z), \quad y_m(z) = \sqrt{\frac{\pi}{2z}} Y_{m+1/2}(z). \tag{2.2}$$

It is convenient to introduce the notations

$$M_\nu(z) := |H_\nu^{(1)}(z)|, \quad N_\nu(z) := |H_\nu^{(1)\prime}(z)|.$$

The arguments we make depend on the fact that $M_\nu(z)$ is decreasing on the positive real axis for $\nu \geq 0$; indeed, for $\nu \geq \frac{1}{2}$ it holds that $zM_\nu^2(z)$ is non-increasing [42, §13.74]. This latter fact, together with the asymptotics of $M_\nu(z)$ [1, (9.2.28)] that

$$M_\nu(z) = \sqrt{\frac{2}{\pi z}} + O(z^{-5/2}) \text{ as } z \to \infty, \tag{2.3}$$

imply that

$$zM_\nu^2(z) \geq \frac{2}{\pi}, \quad \text{for } z > 0, \ \nu \geq \frac{1}{2}. \tag{2.4}$$

It follows easily from the Bessel equation (2.1) that

$$(z^2 - \nu^2)\frac{d}{dz}\left(M_\nu^2(z)\right) + \frac{d}{dz}\left(z^2 N_\nu^2(z)\right) = 0.$$

Thus, defining the function A_ν for $\nu \geq 0$ by

$$A_\nu(z) := M_\nu^2(z)(z^2 - \nu^2) + z^2 N_\nu^2(z) - \frac{4z}{\pi}, \quad z > 0, \tag{2.5}$$

it holds that

$$A_\nu'(z) = 2zM_\nu^2(z) - \frac{4}{\pi}. \tag{2.6}$$

Thus $A_\nu'(z) \geq 0$, for $\nu \geq \frac{1}{2}$ and $z > 0$ by (2.4). Further, from (2.3), and the same asymptotics for N_ν [1, (9.2.30)], that

$$N_\nu(z) = \sqrt{\frac{2}{\pi z}} + O(z^{-5/2}) \text{ as } z \to \infty,$$

it follows that $A_\nu(z) \to 0$ as $z \to \infty$, for $\nu \geq 0$. So

$$A_\nu(z) \leq 0, \quad \text{for } z > 0, \ \nu \geq \frac{1}{2}. \tag{2.7}$$

It is convenient in the following key lemma and later to use the notation $G_R := \{x : |x| > R\}$, for $R > 0$. In addition, throughout this paper $\nabla_T v$ denotes the tangential component of ∇v, i.e. $\nabla_T v := \nabla v - \nu \partial v/\partial \nu$.

LEMMA 2.1. *Suppose that, for some $R_0 > 0$, $v \in C^2(G_{R_0})$ satisfies the Helmholtz equation (1.1) in G_{R_0} and the Sommerfeld radiation condition (1.3). Then, for $R > R_0$,*

$$\Im \int_{\Gamma_R} \bar{v}\frac{\partial v}{\partial r}\, ds \geq 0, \quad \Re \int_{\Gamma_R} \bar{v}\frac{\partial v}{\partial r}\, ds \leq 0, \tag{2.8}$$

and

$$\Re \int_{\Gamma_R} \bar{v}\frac{\partial v}{\partial r}\, ds + R\int_{\Gamma_R}\left(k^2|v|^2 + \left|\frac{\partial v}{\partial r}\right|^2 - |\nabla_T v|^2\right) ds \leq 2kR\,\Im \int_{\Gamma_R} \bar{v}\frac{\partial v}{\partial r}\, ds. \tag{2.9}$$

REMARK 2.2. *The first two inequalities (2.8) are well known; see, for example, Nédélec [39]. The third inequality appears to be new, but we note that an analogous inequality ([11, Lemma 6.1], [10, Lemma 2.2]) has been used extensively in the mathematical analysis of problems of scattering by unbounded rough surfaces. This inequality (proved easily by Fourier transform methods) can be viewed as a (formal) limit of (2.9) in the limit $R \to \infty$. Closer still to (2.9) is the recent inequality of Claeys and Haddar [13, Lemma 4.4], who study the Dirchlet to Neumann map for the Helmholtz equation in the exterior of an infinite cylinder in $\mathbb{R}^3$. In fact, their inequality implies, at least formally, the following less sharp version of (2.9) in the 2D case, namely that, for every $\rho_0 > 0$ there exists a constant $C > 0$ such that, provided $kR > \rho_0$,*

$$2\Re \int_{\Gamma_R} \bar{v}\frac{\partial v}{\partial r}\, ds + R\int_{\Gamma_R} \left(k^2|v|^2 + \left|\frac{\partial v}{\partial r}\right|^2 - |\nabla_T v|^2\right) ds \leq C(1+kR)\,\Im \int_{\Gamma_R} \bar{v}\frac{\partial v}{\partial r}\, ds.$$

Proof. Note first that, by standard elliptic regularity results, it holds that $v \in C^\infty(G_{R_0})$. We now deal with the 2D and 3D cases separately.

Suppose first that $n = 2$. Choose $R_1 \in (R_0, R)$. Introducing standard cylindrical polar coordinates, we expand v on Γ_{R_1} as the Fourier series

$$v(x) = \sum_{m\in\mathbb{Z}} a_m \mathrm{e}^{\mathrm{i}m\theta},$$

where (R_1, θ) are the polar coordinates of x. Since $v \in C^\infty(\Gamma_{R_1})$ it holds that the series is rapidly converging, i.e. that $a_m = o(|m|^{-p})$ as $|m| \to \infty$, for every $p > 0$. It is standard that the corresponding Fourier series representation of v in G_{R_1} is

$$v(x) = \sum_{m\in\mathbb{Z}} a_m \mathrm{e}^{\mathrm{i}m\theta} \frac{H^{(1)}_{|m|}(kr)}{H^{(1)}_{|m|}(kR_1)}, \tag{2.10}$$

where (r, θ) are now the polar coordinates of x, and that this series, and all its partial derivatives with respect to r and θ, converge absolutely and uniformly in G_{R_1}. Hence, defining $c_m := (|a_m|^2 + |a_{-m}|^2)/|H^{(1)}_m(kR_1)|^2$ and $\rho := kR$, and using the orthogonality of $\{\mathrm{e}^{\mathrm{i}m\theta} : m \in \mathbb{Z}\}$, we see that

$$\begin{aligned}
\int_{\Gamma_R} \bar{v}\frac{\partial v}{\partial r}\, ds &= 2\pi\rho \sum_{m\in\mathbb{Z}} |a_m|^2 \frac{\overline{H^{(1)}_{|m|}(\rho)}\, {H^{(1)}_{|m|}}'(\rho)}{|H^{(1)}_{|m|}(kR_1)|^2} \\
&= 2\pi\rho \sum_{m=0}^{\infty} c_m \left(\Re\left(\overline{H^{(1)}_m(\rho)}\, {H^{(1)}_m}'(\rho)\right) + \mathrm{i}\left(J_m(\rho)Y_m'(\rho) - J_m'(\rho)Y_m(\rho)\right)\right) \\
&= \sum_{m=0}^{\infty} c_m \left(\pi\rho \frac{d}{d\rho}\left(M_m^2(\rho)\right) + 4\mathrm{i}\right),
\end{aligned} \tag{2.11}$$

where in the last step we have used the Wronskian formula [1, (9.1.16)] that

$$\pi\rho(J_\nu(\rho)Y_\nu'(\rho) - J_\nu'(\rho)Y_\nu(\rho)) = 2. \tag{2.12}$$

Since $M_m(\rho)$ is decreasing on $(0, \infty)$ we see that (2.8) holds.

Similarly, noting that $|\nabla_T v| = R^{-1}|\partial v/\partial\theta|$, we calculate that

$$R\int_{\Gamma_R} \left(k^2|v|^2 + \left|\frac{\partial v}{\partial r}\right|^2 - |\nabla_T v|^2\right) ds = 2\pi \sum_{m=0}^{\infty} c_m \left(M_m^2(\rho)(\rho^2 - m^2) + \rho^2 N_m^2(\rho)\right).$$

From this equation and (2.11), and recalling the definition (2.5), we see that we will complete the proof of (2.9) if we can show the inequality that

$$\frac{\rho}{2}\frac{d}{d\rho}\left(M_m^2(\rho)\right) + A_m(\rho) \le 0, \tag{2.13}$$

for $\rho > 0$ and $m = 0, 1, \ldots$.

By (2.7) and since M_m is decreasing on $(0, \infty)$, we see that (2.13) holds for $\rho > 0$ and $m \in \mathbb{N}$. To finish the proof of (2.9) in the case $n = 2$ we need to show (2.13) for $m = 0$, i.e. that

$$A(\rho) := \frac{\rho}{2}\frac{d}{d\rho}\left(M_0^2(\rho)\right) + A_0(\rho) \le 0, \quad \rho > 0.$$

Now $A(\rho) = \rho(J_0(\rho)J_0'(\rho) + Y_0(\rho)Y_0'(\rho)) + A_0(\rho)$ so, using (2.1) and (2.6), it follows that

$$A'(\rho) = A_0'(\rho) + \rho\left(N_0^2(\rho) - M_0^2(\rho)\right) = \rho\left(M_0^2(\rho) + N_0^2(\rho)\right) - \frac{4}{\pi} = \frac{A_0(\rho)}{\rho}.$$

Thus

$$\frac{d}{d\rho}\left(\frac{A(\rho)}{\rho}\right) = \frac{A'(\rho)}{\rho} - \frac{A(\rho)}{\rho^2} = -\frac{1}{2\rho}\frac{d}{d\rho}\left(M_0^2(\rho)\right) \ge 0.$$

Since also, from the standard large argument asymptotics of the Bessel functions, $A(\rho)/\rho \to 0$ as $\rho \to \infty$, it follows that $A(\rho) \le 0$ for $\rho > 0$. This completes the proof for $n = 2$.

We turn now to the 3D case for which we make analogous arguments, though the details are different. Again we choose $R_1 \in (R_0, R)$. Introducing standard spherical polar coordinates (r, θ, ϕ), we expand v on Γ_{R_1} as the spherical harmonic expansion

$$v(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_\ell^m Y_\ell^m(\theta, \phi), \tag{2.14}$$

where (R_1, θ, ϕ) are the spherical polar coordinates of x and the functions Y_ℓ^m, $m = -\ell, \ldots, \ell$, are the standard spherical harmonics of order ℓ (see, for example, [39, Theorem 2.4.4]). We recall (e.g. [39]) that $\{Y_\ell^m : \ell = 0, 1, \ldots,\ m = -\ell, \ldots, \ell\}$ is a complete orthonormal sequence in $L^2(S)$, where $S := \{x : |x| = 1\}$ is the unit sphere, and an orthogonal sequence in $H^1(S)$. Since $v \in C^\infty(\Gamma_{R_1}) \supset H^m(S)$, for all $m \in \mathbb{N}$, it holds that the series is rapidly converging, i.e. that $a_\ell^m = o(|\ell|^{-p})$ as $|\ell| \to \infty$, for every $p > 0$ [39].

The solution of the Dirichlet problem for the Helmholtz equation in the exterior of a sphere is discussed in detail in [39]. It follows from (2.14) and [39, (2.6.55)] that, for $x \in G_{R_1}$,

$$v(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_\ell^m Y_\ell^m(\theta, \phi)\frac{h_\ell^{(1)}(kr)}{h_\ell^{(1)}(kR_1)}, \tag{2.15}$$

where (r, θ, ϕ) are now the polar coordinates of x, and hence that [39, (2.6.70)-(2.6.74)]

$$\frac{\partial v}{\partial r}(x) = k\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_\ell^m Y_\ell^m(\theta, \phi)\frac{h_\ell^{(1)'}(kr)}{h_\ell^{(1)}(kR_1)}, \tag{2.16}$$

$$\nabla_T v(x) = \frac{1}{r}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_\ell^m \nabla_S Y_\ell^m(\theta, \phi)\frac{h_\ell^{(1)}(kr)}{h_\ell^{(1)}(kR_1)},$$

where ∇_S is the surface gradient operator on S and

$$\int_S |\nabla_S Y_\ell^m|^2 \, ds = \ell(\ell+1). \tag{2.17}$$

Hence, using the orthonormality in $L^2(S)$ of the spherical harmonics Y_ℓ^m, we see that, where $c_\ell := |h_\ell^{(1)}(kR_1)|^{-2} \sum_{m=-\ell}^{\ell} |a_\ell^m|^2$ and $\rho := kR$,

$$\begin{aligned}\int_{\Gamma_R} \bar{v} \frac{\partial v}{\partial r} \, ds = R^2 \int_S \bar{v}(R\hat{x}) \frac{\partial v}{\partial r}(R\hat{x}) \, ds(\hat{x}) &= R\rho \sum_{\ell=0}^{\infty} c_\ell \, \overline{h_\ell^{(1)}(\rho)} h_\ell^{(1)\prime}(\rho) \\ &= R \sum_{\ell=0}^{\infty} c_\ell \left(\frac{\rho}{2} \frac{d}{d\rho} \left(|h_\ell^{(1)}(\rho)|^2 \right) + \frac{\mathrm{i}}{\rho} \right), \end{aligned} \tag{2.18}$$

where in the last step we have used (2.2) and (2.12). Recalling that $|h_\ell^{(1)}(\rho)| = \sqrt{\pi/(2\rho)} M_{\ell+1/2}(\rho)$ is decreasing on $(0,\infty)$, we see that (2.8) holds.

Similarly, but using also the orthogonality of the surface gradients $\nabla_S Y_\ell^m$ in $L^2(S)$ and (2.17), we calculate that

$$\begin{aligned}&\int_{\Gamma_R} \left(k^2 |v|^2 + \left| \frac{\partial v}{\partial r} \right|^2 - |\nabla_T v|^2 \right) ds \\ &\qquad = \sum_{\ell=0}^{\infty} c_\ell \left(|h_\ell^{(1)}(\rho)|^2 (\rho^2 - \ell(\ell+1)) + \rho^2 |h_\ell^{(1)\prime}(\rho)|^2 \right).\end{aligned}$$

Thus

$$\begin{aligned}\Re \int_{\Gamma_R} \bar{v} \frac{\partial v}{\partial r} \, ds + R \int_{\Gamma_R} \left(k^2 |v|^2 + \left| \frac{\partial v}{\partial r} \right|^2 - |\nabla_T v|^2 \right) ds - 2kR \, \Im \int_{\Gamma_R} \bar{v} \frac{\partial v}{\partial r} \, ds \\ = \frac{1}{k} \sum_{\ell=0}^{\infty} c_\ell B_\ell(\rho),\end{aligned}$$

where

$$B_\ell(\rho) := \frac{\rho^2}{2} \frac{d}{d\rho} \left(|h_\ell^{(1)}(\rho)|^2 \right) + |h_\ell^{(1)}(\rho)|^2 \rho (\rho^2 - \ell(\ell+1)) + \rho^3 |h_\ell^{(1)\prime}(\rho)|^2 - 2\rho.$$

But straightforward calculations, using the definitions (2.2), yield that

$$B_\ell(\rho) = \frac{\pi}{2} A_{\ell+1/2}(\rho), \quad \rho > 0, \ \ell = 0, 1, \ldots.$$

Thus, applying (2.7), we see that $B_\ell(\rho) \le 0$ for $\ell = 0, 1, \ldots$ and $\rho > 0$, which completes the proof of (2.9). □

The following lemma is another key component in obtaining our wave-number explicit bounds. Of course, the first equation is just a special case of Green's first theorem. The second is a Rellich-Payne-Weinberger identity, essentially that used in [35] to obtain an estimate for the solution of the Helmholtz equation with impedance boundary condition in an interior domain (or see [16]). In the case $k = 0$ it is a special case of a general identity for second order strongly elliptic operators given in Lemma 4.22 of [34] (or see [38, Chapter 5]). For completeness, we include the short proof of this key step in our arguments.

LEMMA 2.3. *Suppose that $G \subset \mathbb{R}^n$ is a bounded Lipschitz domain and that $v \in H^2(G)$. Then, for every $k \ge 0$, where $g := \Delta v + k^2 v$ and the unit normal vector ν is directed out of G, it holds that*

$$\int_G \left(|\nabla v|^2 - k^2 |v|^2 + g\bar{v} \right) dx = \int_{\partial G} \bar{v} \, \frac{\partial v}{\partial \nu} \, ds \tag{2.19}$$

and

$$\int_G \left((2-n)|\nabla v|^2 + nk^2|v|^2 + 2\Re\left(gx\cdot\nabla\bar{v}\right)\right)\,dx =$$

$$\int_{\partial G}\left(x\cdot\nu\left(k^2|v|^2 + \left|\frac{\partial v}{\partial \nu}\right|^2 - |\nabla_T v|^2\right) + 2\Re\left(x\cdot\nabla_T\bar{v}\,\frac{\partial v}{\partial\nu}\right)\right)\,ds. \tag{2.20}$$

Proof. In the case $v \in C^2(\bar{G})$, these equations are a consequence of the divergence theorem, $\int_G \nabla\cdot F dx = \int_{\partial G} f\cdot\nu ds$, which holds for every vector field $F \in C^1(\bar{G})$ (see e.g. McLean [34] for the case when G is Lipschitz). Equation (2.19) follows by applying the divergence theorem to the identity $|\nabla v|^2 - k^2|v|^2 + g\bar{v} = \nabla\cdot(\bar{v}\nabla v)$ integrated over G. Equation (2.20) follows by applying the divergence theorem to the identity

$$(2-n)|\nabla v|^2 + nk^2|v|^2 + 2\Re\left(gx\cdot\nabla\bar{v}\right) = \nabla\cdot\left[x\left(k^2|v|^2 - |\nabla v|^2\right) + 2\Re\left(x\cdot\nabla\bar{v}\nabla v\right)\right]$$

integrated over G, and then noting that $x\cdot\nabla v = x\cdot\nu\dfrac{\partial v}{\partial\nu} + x\cdot\nabla_T v$ on ∂G. The extension from $C^2(\bar{G})$ to $H^2(G)$ follows by the density of $C^2(\bar{G})$ in $H^2(G)$ and by the continuity of the trace operator $\gamma : H^1(G) \to H^{1/2}(\partial G)$. □

3. The scattering problem and weak formulation. In this section we formulate the scattering problem precisely, state its weak, variational formulation, and obtain explicit lower bounds on the inf-sup constant (1.8), using the results of the previous section.

To state the scattering problem we wish to solve precisely, let $H_0^1(\Omega_e) \subset H^1(\Omega_e)$ denote the closure of $C_0^\infty(\Omega_e)$, the set of C^∞ functions on Ω_e that are compactly supported, in the norm of the Sobolev space $H^1(\Omega_e)$. Let $H_0^{1,\mathrm{loc}}(\Omega_e)$ denote the set of those functions, v, that are locally integrable on Ω_e and satisfy that $\psi\chi \in H_0^1(\Omega_e)$ for every compactly supported $\chi \in C^\infty(\bar{\Omega}_e) := \{v|_{\bar{\Omega}_e} : v \in C^\infty(\mathbb{R}^n)\}$. Then our scattering problem can be stated as follows. For simplicity of exposition we restrict attention throughout to two specific cases. The first is when the incident wave u^i is the plane wave

$$u^i(x) := \mathrm{e}^{\mathrm{i}kx_1}, \quad x \in \mathbb{R}^n. \tag{3.1}$$

The Plane Wave Scattering Problem. *Given $k > 0$, find $u \in H_0^{1,\mathrm{loc}}(\Omega_e) \cap C^2(\Omega_e)$ such that u satisfies the Helmholtz equation (1.1) in Ω_e and $u^s := u - u^i$ satisfies the Sommerfeld radiation condition (1.3), as $r = |x| \to \infty$, uniformly in $\hat{x} = x/r$.*

The above is the scattering problem that we will focus on in this paper. But it is essential to our methods of argument in this section to consider also the following scattering problem where the source of the acoustic excitation is due to a compactly supported source region in Ω_e.

The Distributed Source Scattering Problem. *Given $k > 0$ and $g \in L^2(\Omega_e)$ which is compactly supported, find $u \in H_0^{1,\mathrm{loc}}(\Omega_e)$ such that u satisfies the inhomogeneous Helmholtz equation*

$$\Delta u + k^2 u = g \text{ in } \Omega_e, \tag{3.2}$$

in a distributional sense, and u satisfies the Sommerfeld radiation condition (1.3), as $r = |x| \to \infty$, uniformly in $\hat{x} = x/r$.

Recall that, for $R > R_0 := \sup_{x\in\Omega}|x|$, we define $D_R := \{x \in \Omega_e : |x| < R\}$ and $V_R := \{v|_{D_R} : v \in H_0^1(\Omega_e)\} \subset H^1(D_R)$. We note that V_R is a closed subspace of

$H^1(D_R)$. Throughout this section, $(\cdot,\cdot)$ will denote the standard scalar product on $L^2(D_R)$ and $\|\cdot\|_2$ the corresponding norm, i.e.

$$(u,v) := \int_{D_R} u\bar{v}\, dx, \quad \|v\|_2 := (v,v)^{1/2} = \left\{ \int_{D_R} |v|^2\, dx \right\}^{1/2}.$$

It is convenient to equip V_R with a wave number dependent norm, equivalent to the usual norm on $H^1(D_R)$, and defined by (1.14).

As discussed in §1, our first reformulation of the plane wave scattering problem is as the following weak/variational formulation: find $u \in V_R$ such that

$$b(u,v) = G(v), \quad v \in V_R. \tag{3.3}$$

In this equation b is the bounded sesquilinear form on V_R given by (1.5) and $G \in V_R^*$, the dual space of V_R, is given by

$$G(v) = \int_{\Gamma_R} \bar{v} \left(\frac{\partial u^i}{\partial r} - T_R u^i \right) ds. \tag{3.4}$$

As defined in §1, the operator $T_R : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$, which occurs in both (1.5) and (3.4), is the Dirichlet to Neumann map. Explicitly, in the 2D case, if $\phi \in H^{1/2}(\Gamma_R)$ has the Fourier series expansion

$$\phi(x) = \sum_{m\in\mathbb{Z}} a_m \mathrm{e}^{\mathrm{i}m\theta},$$

where (R,θ) are the polar coordinates of x, then (see (2.10) or [29])

$$T_R\phi(x) = k \sum_{m\in\mathbb{Z}} a_m \mathrm{e}^{\mathrm{i}m\theta} \frac{H_{|m|}^{(1)\prime}(kr)}{H_{|m|}^{(1)}(kR)}, \quad x \in \Gamma_R. \tag{3.5}$$

Similarly, in the 3D case, if $\phi \in H^{1/2}(\Gamma_R)$ has the spherical harmonics expansion

$$\phi(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_\ell^m Y_\ell^m(\theta,\phi),$$

where (R,θ,ϕ) are the spherical polar coordinates of x, then (see (2.16) or [39]),

$$T_R\phi(x) = k \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_\ell^m Y_\ell^m(\theta,\phi) \frac{h_\ell^{(1)\prime}(kr)}{h_\ell^{(1)}(kR)}, \quad x \in \Gamma_R, \tag{3.6}$$

where both the series (3.5) and (3.6) are convergent in the norm of $H^{-1/2}(\Gamma_R)$. Moreover, from Lemma 2.1, we have the following key properties of T_R (see [39]).

Corollary 3.1. *For all $R > 0$ and all $\phi \in H^{1/2}(\Gamma_R)$ it holds that*

$$\Re \int_{\Gamma_R} \bar{\phi} T_R \phi\, ds \leq 0 \text{ and } \Im \int_{\Gamma_R} \bar{\phi} T_R \phi\, ds \geq 0.$$

That the plane wave scattering problem and the weak formulation (3.3) are equivalent is standard. Precisely, we have the following result (see e.g. [29] or [39]).

Theorem 3.2. *If u is a solution to the plane wave scattering problem then $u|_{D_R} \in V_R$ satisfies (3.3). Conversely, suppose $u \in V_R$ satisfies (3.3), let $F_R := \gamma u^s$ be the trace of $u^s = u - u^i$ on Γ_R, and extend the definition of $u = u^i + u^s$ to Ω_e by setting $u^s|_{G_R}$ to be the solution of the Dirichlet problem in G_R, with data F_R*

on Γ_R (this solution given explicitly by (2.10) and (2.15), in the cases $n = 2$ and $n = 3$, respectively). Then this extended function satisfies the plane wave scattering problem.

In the case that $\mathrm{supp}(g) \subset \overline{D_R}$, the distributed source scattering problem is equivalent, in the same precise sense as in the above theorem, to the following variational problem: Find $u \in V_R$ such that

$$b(u, v) = -(g, v), \quad v \in V_R. \tag{3.7}$$

It is well known that both scattering problems have exactly one solution. Indeed this follows from the above equivalence and the fact that the variational problem (3.3) has exactly one solution $u \in V_R$ for every $G \in V_R^*$ (see e.g. [29, 39]). In turn, this follows from uniqueness for the scattering problem (which follows from Rellich's lemma [14]) and from the fact that $b(\cdot, \cdot)$ satisfies a Gårding inequality [29, 39] (the first inequality in Corollary 3.1 plays a role here together with the compactness of the embedding operator from V_R to $L^2(D_R)$). Further, we have the following standard stability estimate (see [29, Remark 2.20]).

LEMMA 3.3. *The inf-sup condition (1.8) holds and, for all $u \in V_R$ and $\mathcal{G} \in V_R^*$ satisfying (3.3), it holds that*

$$\|u\|_{V_R} \leq C \|\mathcal{G}\|_{V_H^*}, \tag{3.8}$$

with $C = \alpha^{-1}$. Conversely, if there exists $C > 0$ such that, for all $u \in V_R$ and $\mathcal{G} \in V_R^*$ satisfying (3.3), the bound (3.8) holds, then the inf-sup condition (1.8) holds with $\alpha \geq C^{-1}$.

The second part of the above lemma shows that we obtain a lower bound on the inf-sup constant α if we show the bound (3.8) for all $u \in V_R$ and $\mathcal{G} \in V_R^*$ satisfying (3.3), and this will be the strategy that we will employ to obtain wave-number-explicit lower bounds on α. The following lemma reduces the problem of establishing (3.8) to that of establishing an a priori bound for solutions of the special case (3.7). The proof (very close to that of [10, Lemma 4.5]) depends on the observation that, if $u \in V_R$ satisfies (3.3), then $u = u_0 + w$, where $u_0, w \in V_R$ satisfy

$$b_0(u_0, v) = \mathcal{G}(v) \text{ and } b(w, v) = 2k^2(u_0, v), \quad \forall v \in V_R,$$

where $b_0 : V_R \times V_R \to \mathbb{C}$ is defined by

$$b_0(u, v) = (\nabla u, \nabla v) + k^2(u, v) - \int_{\Gamma_R} \gamma \bar{v}\, T_R \gamma u \, ds, \quad u, v \in V_R.$$

It follows from Corollary 3.1 that $\Re b_0(v, v) \geq \|v\|_{V_R}^2$, $v \in V_R$, so that $\|u_0\|_{V_R} \leq \|\mathcal{G}\|_{V_R^*}$ by Lax-Milgram, and (3.9) and (1.14) imply that $\|w\|_{V_R} \leq 2k\tilde{C}\|u_0\|_2 \leq 2\tilde{C}\|\mathcal{G}\|_{V_R^*}$.

LEMMA 3.4. *Suppose there exists $\tilde{C} > 0$ such that, for all $u \in V_R$ and $g \in L^2(D_R)$ satisfying (3.7) it holds that*

$$\|u\|_{V_R} \leq k^{-1}\tilde{C}\, \|g\|_2. \tag{3.9}$$

Then, for all $u \in V_R$ and $\mathcal{G} \in V_R^*$ satisfying (3.3), the bound (3.8) holds with $C \leq 1 + 2\,\tilde{C}$.

We have reduced the problem of obtaining lower bounds on the inf-sup constant to the problem of obtaining a bound on the solution to a scattering problem, namely the distributed source scattering problem stated above. We will shortly bootstrap to the case where we require no smoothness on Γ, but our first bound on the solution to this problem is restricted to the case where Γ is smooth and Ω is starlike. Specifically, we require the following assumption.

ASSUMPTION 1. *Let* $S := \{x \in \mathbb{R}^n : |x| = 1\}$. *For some* $f \in C^\infty(S, \mathbb{R})$ *with* $\min_{\hat{x} \in S} f(\hat{x}) > 0$, *it holds that* $\Gamma = \{f(\hat{x})\hat{x} : \hat{x} \in S\}$.

LEMMA 3.5. *Suppose that Assumption 1 holds, u is a solution to the distributed source scattering problem,* $R > R_0$, *and* $\mathrm{supp}(g) \subset \overline{D_R}$. *Then*

$$k\|u\|_{V_R} \le (n - 1 + 2\sqrt{2}kR)\|g\|_2. \tag{3.10}$$

Proof. Since D_R is a smooth domain, by standard elliptic regularity results [24] we have that $u \in H^{2,\mathrm{loc}}(D_R)$. Thus we can apply Lemma 2.3 to u in D_R to get, by adding $n-1$ times the real part of (2.19) to (2.20), that

$$\int_{D_R} \left(|\nabla u|^2 + k^2|u|^2 + \Re\left(g(2x \cdot \nabla \bar{u} + (n-1)\bar{u})\right)\right) dx = -\int_\Gamma x \cdot \nu \left|\frac{\partial u}{\partial \nu}\right|^2 ds$$
$$+ \int_{\Gamma_R} \left(R\left(k^2|u|^2 + \left|\frac{\partial u}{\partial r}\right|^2 - |\nabla_T u|^2\right) + \Re\left((n-1)\bar{u}\,\frac{\partial u}{\partial r}\right)\right) ds,$$

where we have also used the Dirichlet boundary condition (1.2), that $u = 0$ on Γ.

Since $x \cdot \nu > 0$ on Γ, applying Lemma 2.1 and then using (2.19), we see that

$$\int_{D_R} \left(|\nabla u|^2 + k^2|u|^2 + \Re\left(g(2x \cdot \nabla \bar{u} + (n-1)\bar{u})\right)\right) dx$$
$$\le 2kR\,\Im \int_{\Gamma_R} \bar{u}\frac{\partial u}{\partial r}\, ds = 2kR\,\Im \int_{D_R} g\bar{u}\, dx.$$

Applying the Cauchy-Schwartz inequality, and noting that

$$2ab \le \epsilon a^2 + \frac{b^2}{\epsilon}, \tag{3.11}$$

for $a, b \ge 0$, $\epsilon > 0$, we deduce that

$$\|u\|_{V_R}^2 \le (n - 1 + 2kR)\|g\|_2\|u\|_2 + 2R\|g\|_2\|\nabla u\|_2$$
$$\le \frac{1}{2}\|u\|_{V_R}^2 + \frac{\|g\|_2^2}{2k^2}(4k^2R^2 + (n-1+2kR)^2).$$

Thus

$$k^2\|u\|_{V_R}^2 \le \|g\|_2^2(4k^2R^2 + (n-1+2kR)^2),$$

from which (3.10) follows. ◻

Combining Lemmas 3.3, 3.4, and 3.5, we obtain the following corollary.

COROLLARY 3.6. *If Assumption 1 is satisfied, then the inf-sup condition (1.8) holds with* $\alpha^{-1} \le 1 + 2(n - 1 + 2\sqrt{2}kR) \le 5 + 4\sqrt{2}kR$.

We proceed now to establish that Lemma 3.5 and Corollary 3.6 hold if Ω is starlike. Precisely, we require only the following, relaxed version of Assumption 1.

ASSUMPTION 2. *It holds that* $0 \notin \Omega_e$ *and, if* $x \in \Omega_e$, *then* $sx \in \Omega_e$ *for every* $s > 1$.

To establish these generalisations we first prove the following technical lemma (cf. [10, Lemma 4.10]).

LEMMA 3.7. *If Assumption 2 holds then, for every* $\phi \in C_0^\infty(\Omega_e)$, *and* $R > R_0$, *there exists* $f \in C^\infty(S, \mathbb{R})$ *with* $\min_{\hat{x} \in S} f(\hat{x}) > 0$ *such that*

$$\mathrm{supp}\,\phi \subset \Omega_e' := \{sf(\hat{x}) \in \mathbb{R}^n : \hat{x} \in S,\, s > 1\}.$$

and $\overline{G_R} \subset \Omega_e' \subset \Omega_e$.

Proof. Clearly, it is sufficient to consider the case when $R = 1$. So suppose $R = 1$, let $U := \operatorname{supp}\phi \cup \Gamma_1$, let $B := \{sx : x \in U,\ s \geq 1\}$, and let $\delta := \operatorname{dist}(U, \Gamma)/4$, so $\operatorname{dist}(B, \Gamma) = \operatorname{dist}(U, \Gamma) = 4\delta$ and $0 < \delta \leq \frac{1}{4}$. Let $B_\delta := \{x \in \mathbb{R}^n : \operatorname{dist}(x, B) < 2\delta\}$.

Let $N \in \mathbb{N}$ and $S_j \subset S$, $j = 1, ..., N$, be such that each S_j is measurable and non-empty, $S_j \cap S_m = \emptyset$, for $j \neq m$, $S = \bigcup_{j=1}^N S_j$, and $\operatorname{diam}(S_j) \leq \delta$, $j = 1, ..., N$. For $j = 1, ..., N$ choose $\hat{x}_j \in S_j$ and let

$$f_j := \inf\{|x| : x \in B_\delta,\ x/|x| \in S_j\}.$$

Then $2\delta \leq f_j \leq 1 - 2\delta$, $j = 1, ..., N$. Define $\tilde{f} : S \to \mathbb{R}$ by

$$\tilde{f}(\hat{x}) := f_j \text{ if } \hat{x} \in S_j,\ j = 1, ..., N.$$

Then $\tilde{f} \in L^\infty(S, \mathbb{R})$; in fact $\tilde{f}$ is a simple function and $2\delta \leq \tilde{f}(\hat{x}) \leq 1 - 2\delta$, $\hat{x} \in S$. Choose ϵ with $0 < \epsilon < \delta$ and let $J \in C^\infty[0, 2]$ be such that $J \geq 0$, $J(t) = 0$ if $\epsilon^2/2 \leq t \leq 2$, and, where $e_3 := (0, 0, 1)$, such that $\int_S J(1 - e_3 \cdot \hat{y})\, ds(\hat{y}) = 1$, so that $\int_S J(1 - \hat{x} \cdot \hat{y})\, ds(\hat{y}) = 1$, $\hat{x} \in S$. Define $f \in C^\infty(S, \mathbb{R})$ by

$$f(\hat{x}) := \int_S J(1 - \hat{x} \cdot \hat{y}) \tilde{f}(\hat{y})\, ds(\hat{y}), \quad \hat{x} \in S,$$

and let Ω_e' be defined as in the statement of the lemma. Then f and Ω_e' have the properties claimed.

To see that this is true note first that, since $J(1 - \hat{x} \cdot \hat{y}) = 0$ if $|\hat{x} - \hat{y}| \geq \epsilon$,

$$\min_{|\hat{y} - \hat{x}| < \epsilon} |\tilde{f}(\hat{y})| \leq f(\hat{x}) \leq \max_{|\hat{y} - \hat{x}| < \epsilon} |\tilde{f}(\hat{y})|, \quad \hat{x} \in S, \tag{3.12}$$

so that $\overline{G_1} \subset \Omega_e'$. Now every $\hat{y} \in S$ is an element of S_j, for some $j \in \{1, \ldots, N\}$, and $\tilde{f}(\hat{y}) = f_j$ and $|\hat{y} - \hat{x}_j| \leq \delta$. Thus it follows from (3.12) that, for every $\hat{x} \in S$, $f(\hat{x}) \leq f_m$, for some m for which $|\hat{x}_m - \hat{x}| < \epsilon + \delta$. Now let $x = f_m \hat{x}$, $y = f_m \hat{x}_m$. Then $|x - y| \leq |\hat{x} - \hat{x}_m| < \epsilon + \delta$ and $\operatorname{dist}(y, B) = 2\delta$, so that

$$\operatorname{dist}(x, B) \geq \operatorname{dist}(y, B) - |x - y| \geq 2\delta - (\epsilon + \delta) > 0.$$

Thus $x \notin B$ and so $f(\hat{x})\hat{x} \notin B$. Thus $U \subset B \subset \Omega_e'$ and so $\operatorname{supp}\phi \subset U \subset \Omega_e'$.

Arguing similarly, for all $\hat{x} \in S$, $f(\hat{x}) \geq f_m$, for some m for which $|\hat{x}_m - \hat{x}| < \epsilon + \delta$. Defining $x = f_m \hat{x}$ and $y = f_m \hat{x}_m$, it holds that

$$\operatorname{dist}(x, B) \leq \operatorname{dist}(y, B) + |x - y| \leq 2\delta + \epsilon + \delta < 4\delta,$$

so that $x \in \Omega_e$ and hence $f(\hat{x})\hat{x} \in \Omega_e$. Thus, for all $\hat{x} \in S$, $s\hat{x} \in \Omega_e$ for $s > f(\hat{x})$, i.e. $\Omega_e' \subset \Omega_e$. □

With this preliminary lemma we can proceed to show that Lemma 3.5 holds whenever Assumption 2 holds. In this final lemma (cf. [10, Lemma 4.11]) we use explicitly the fact that $b(\cdot, \cdot)$ is bounded. In fact, examining the definition (1.5), clearly we have that

$$|b(u, v)| \leq c\|u\|_{V_R}\|v\|_{V_R}, \quad u, v \in V_R, \tag{3.13}$$

where $c := 1 + \|\gamma\|^2\|T_R\|$ and $\|\gamma\|$ denotes the norm of the trace operator $\gamma : V_R \to H^{1/2}(\Gamma_R)$ while $\|T_R\|$ denotes the norm of T_R as a mapping from $H^{1/2}(\Gamma_R)$ to $H^{-1/2}(\Gamma_R)$.

LEMMA 3.8. *Suppose that Assumption 2 holds, u is a solution to the distributed source scattering problem, $R > R_0$, and $\operatorname{supp}(g) \subset \overline{D_R}$. Then the bound (3.10) holds.*

Proof. Let $\tilde{V} := \{\phi|_{D_R} : \phi \in C_0^\infty(\Omega_e)\}$. Then $\tilde{V}$ is dense in V_R. Recall that the distributed source scattering problem is equivalent to (3.7). Suppose u satisfies (3.7) and choose a sequence $(u_m) \subset \tilde{V}$ such that $\|u_m - u\|_{V_R} \to 0$ as $m \to \infty$. Then $u_m = \phi_m|_{D_R}$, with $\phi_m \in C_0^\infty(\Omega_e)$, and, by Lemma 3.7, there exists $f_m \in C^\infty(S, \mathbb{R})$ with $\min f > 0$ such that $\operatorname{supp} \phi_m \subset \Omega_e^{(m)}$ and $\overline{G_R} \subset \Omega_e^{(m)} \subset \Omega_e$, where $\Omega_e^{(m)} := \{sf_m(\hat{x}) \in \mathbb{R}^n : \hat{x} \in S, s > 1\}$. Let $V_R^{(m)}$ and b_m denote the space and sesquilinear form corresponding to the domain $\Omega_e^{(m)}$. That is, where $D_R^{(m)} := \Omega_e^{(m)} \setminus \overline{G_R}$, $V_R^{(m)}$ is defined by $V_R^{(m)} := \{\phi|_{D_R^{(m)}} : \phi \in H_0^1(\Omega_e^{(m)})\}$ and b_m is given by (1.5) with D_R and V_R replaced by $D_R^{(m)}$ and $V_R^{(m)}$, respectively. Then $D_R^{(m)} \subset D_R$ and, if $v_m \in V_R^{(m)}$ and v denotes v_m extended by zero from $D_R^{(m)}$ to D_R, it holds that $v \in V_R$. Via this extension by zero, we can regard $V_R^{(m)}$ as a subspace of V_R and regard u_m as an element of $V_R^{(m)}$.

For all $v \in V_R^{(m)} \subset V_R$, we have

$$b_m(u_m, v) = b(u_m, v) = -(g, v) - b(u - u_m, v).$$

Let u_m' and $u_m'' \in V_R^{(m)}$ be the unique solutions of

$$b_m(u_m', v) = -(g, v), \quad b_m(u_m'', v) = -b(u - u_m, v), \quad \forall v \in V_R^{(m)}.$$

Clearly $u_m = u_m' + u_m''$ and, by Lemma 3.5, $\|u_m'\|_{V_R^{(m)}} \leq k^{-1}\tilde{C}\|g\|_2$, where $\tilde{C} = n - 1 + 2\sqrt{2}kR$, while, by (3.13), Lemma 3.3 and Corollary 3.6,

$$\|u_m''\|_{V_R^{(m)}} \leq c(1 + 2\tilde{C})\|u - u_m\|_{V_R}.$$

Thus $\|u\|_{V_R} = \lim_{m\to\infty} \|u_m\|_{V_R^{(m)}} \leq k^{-1}\tilde{C}\|g\|_2$. □

Combining Lemmas 3.3, 3.4, and 3.8, we obtain the following generalisation of Corollary 3.6, which is our main lower bound on the inf-sup constant and the main result of this section.

COROLLARY 3.9. *If Assumption 2 is satisfied, then the inf-sup condition (1.8) holds with* $\alpha^{-1} \leq 1 + 2(n - 1 + 2\sqrt{2}kR) \leq 5 + 4\sqrt{2}kR$.

We finish the section by obtaining two upper bounds on the inf-sup constant, which will show, among other things, that the above bound is sharp in its dependence on k in the limit $k \to \infty$.

To obtain these bounds we modify the construction of Ihlenburg [29] for a weak formulation of a 1D Helmholtz problem. We note first that, for every non-zero $w \in V_R$,

$$\alpha \leq \sup_{0 \neq v \in V_R} \frac{|b(w, v)|}{\|w\|_{V_R}\|v\|_{V_R}}.$$

Now choose $w \in V_R \cap H^2(D_R)$ such that $w = \nabla w = 0$ on Γ_R. Then, integrating by parts, for $v \in V_R$,

$$|b(w, v)| = \left| \int_{D_R} \left(\nabla w \cdot \nabla \bar{v} - k^2 w \bar{v}\right) dx \right| = \left| \int_{D_R} \left(\Delta w + k^2 w\right) \bar{v}\, dx \right|.$$

Thus, and recalling the definition (1.14),

$$\frac{|b(w, v)|}{\|w\|_{V_R}\|v\|_{V_R}} \leq \frac{\|\Delta w + k^2 w\|_2 \|v\|_2}{\|w\|_{V_R}\|v\|_{V_R}} \leq \frac{\|\Delta w + k^2 w\|_2}{k^2\|w\|_2}. \tag{3.14}$$

Now define $u(x) = \mathrm{e}^{\mathrm{i}kx_1} w(x)$. Then the above bound holds with w replaced by u, and

$$\Delta u(x) + k^2 u(x) = \left(2\mathrm{i}k \frac{\partial w(x)}{\partial x_1} + \Delta w(x) \right) \mathrm{e}^{\mathrm{i}kx_1}$$

so that

$$\frac{|b(u,v)|}{\|u\|_{V_R} \|v\|_{V_R}} \leq \frac{\|\Delta u + k^2 u\|_2}{k^2 \|u\|_2} = \frac{\|2\mathrm{i}k \frac{\partial w}{\partial x_1} + \Delta w\|_2}{k^2 \|w\|_2}.$$

We have shown most of the following lemma.

LEMMA 3.10. *Suppose $w \in V_R \cap H^2(D_R)$ is such that $\gamma w = \gamma \nabla w = 0$ and w is non-zero. Then the inf-sup constant (1.8) is bounded above by*

$$\alpha \leq \frac{C_1}{kR} + \frac{C_2}{k^2 R^2},$$

where $C_1 := 2R \left\| \frac{\partial w}{\partial x_1} \right\|_2 / \|w\|_2$, $C_2 := R^2 \|\Delta w\|_2 / \|w\|_2$ and $C_1 \geq 2\sqrt{2} \approx 2.83$.

Proof. It only remains to show the last inequality. Since $\gamma w = 0$, we can approximate w in the $H^1(D_R)$ norm arbitrarily closely by $\tilde{w} \in C_0^\infty(D_R)$. Then $C_1 \geq 2\sqrt{2}$ follows by a standard Friedrichs inequality (e.g. [10, Lemma 3.4]) which gives that $\|\tilde{w}\|_2 \leq (R/\sqrt{2}) \|\frac{\partial \tilde{w}}{\partial x_1}\|_2$. □

We note that in the case that Assumption 2 holds so that Corollary 3.9 applies, we have both upper and lower bounds on the inf-sup constant, namely (1.17), where C_1 and C_2 are as defined in Lemma 3.10.

The left hand bound in (1.17) holds for every domain Ω_e satisfying Assumption 2. To check its sharpness, let us consider the case when $\Omega = \{0\}$ and $D = \mathbb{R}^n \setminus \{0\}$. In this special case, $V_R = H^1(D_R)$ and the solution of the plane wave scattering problem is just $u = u^i$, i.e. the scattered field is zero. Taking in this case $w(x) = F(|x|/R)$, where $F(t) := (1-t^2)^2$, we calculate that

$$C_1 = 2\sqrt{\frac{\int_0^1 (F'(t))^2 t^{n-1} dt}{n \int_0^1 (F(t))^2 t^{n-1} dt}} = \begin{cases} 2\sqrt{30}/3 \approx 3.65, & n = 2, \\ 2\sqrt{33}/3 \approx 3.83, & n = 3. \end{cases}$$

Thus, defining $c_- := (4\sqrt{2})^{-1}$, for this example the bounds (1.17) bracket $kR\alpha$ fairly tightly, predicting that, in the limit $kR \to \infty$, $kR\alpha$ is in the range $[c_-, C_1]$ with $C_1/c_- \leq 8\sqrt{66}/3 \approx 21.7$.

The above results show that $k\alpha$ is bounded above in the limit $k \to \infty$ and is also bounded below if Assumption 2 holds. If Assumption 2 does not hold then α may not be bounded below by a multiple of k^{-1}. The following example shows this behaviour. It is convenient in this example to write $x \in \mathbb{R}^n$ as $x = (\tilde{x}, x_n)$ where $\tilde{x} = (x_1, \ldots, x_{n-1})$.

Choose $A > 0$ and let $\Omega := \Omega_+ \cup \Omega_-$ where $\Omega_\pm := \{x \in \mathbb{R}^n : x_n = \pm A, |\tilde{x}| \leq A\}$, so that Ω consists of two parallel lines of length $2A$ distance $2A$ apart in the 2D case, two parallel disks of radius A in the 3D case. Choose $R > R_0 = \sqrt{2}A$ and define the function w by

$$w(x) := \begin{cases} \cos(kx_n) F(|\tilde{x}|/A), & |\tilde{x}| \leq A, \ |x_n| \leq A, \\ 0, & \text{otherwise}, \end{cases}$$

and suppose that $k \in \Lambda := \{(m + 1/2)\pi/A : m \in \mathbb{N}\}$. Then $w \in V_R \cap H^2(D_R)$ and $w = \nabla w = 0$ on Γ_R, so that (3.14) holds. Further,

$$\Delta w(x) + k^2 w(x) = \begin{cases} A^{-2} \cos(kx_n) \tilde{F}(|\tilde{x}|/A), & |\tilde{x}| \leq A, \ |x_n| \leq A, \\ 0, & \text{otherwise}, \end{cases}$$

where $\tilde{F}(t) := F''(t) + (n-2)F'(t)/t$. Thus, for $k \in \Lambda$, the inf-sup constant is bounded above by

$$\alpha \leq \frac{\|\Delta w + k^2 w\|_2}{k^2\|w\|_2} = \frac{C_n^*}{k^2 A^2}, \tag{3.15}$$

where $C_n^* := \sqrt{\int_0^1 \tilde{F}^2(t)t^{n-2}dt / \int_0^1 F^2(t)t^{n-2}dt}$.

4. Integral equation formulations. In this section we will obtain estimates explicit in the wave number for integral equation formulations of scattering problems, focussing on the plane wave scattering problem introduced in §3 and on the integral equation (1.7) and its adjoint.

Throughout this section we assume, essential to the integral equation method, a degree of regularity of the domain, namely that Ω_e is Lipschitz (which implies that the interior of Ω, denoted $\Omega_i = \mathbb{R}^n \setminus \bar{\Omega}_e$ is also Lipschitz). We note that the invertibility of the integral equation (1.7) and its adjoint for the general Lipschitz case has recently been established in [9, §2], by combining known results for layer-potentials on Lipschitz domains in [41, 20, 36, 34].

Given a domain G, let $H^1(G;\Delta) := \{v \in H^1(G) : \Delta v \in L^2(G)\}$ (Δ the Laplacian in a weak sense). This is a Hilbert space with the norm $\|v\|_{H^1(G;\Delta)} := \{\int_G [|v|^2 + |\nabla v|^2 + |\Delta v|^2] dx\}^{1/2}$. If G is Lipschitz, then there is a well-defined normal derivative operator [34], the unique bounded linear operator $\partial_\nu : H^1(G;\Delta) \to H^{-1/2}(\partial G)$ which satisfies

$$\partial_\nu v = \frac{\partial v}{\partial \nu} := \nu \cdot \nabla v,$$

almost everywhere on Γ, when $v \in C^\infty(\bar{G})$.

Our integral equation formulations will be based on standard acoustic layer potentials and their normal derivatives. In the case when the domain Ω_e is Lipschitz, for $\varphi \in L^2(\Gamma)$ we define the *single-layer potential operator* by

$$S\varphi(x) \; := \; 2\int_\Gamma \Phi(x,y)\varphi(y)\, ds(y), \quad x \in \Gamma, \tag{4.1}$$

and the *double-layer potential operator* by

$$K\varphi(x) \; := \; 2\int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)\, ds(y), \quad x \in \Gamma, \tag{4.2}$$

where the normal ν is directed into Ω_e. We define also the operator K', which arises from taking the normal derivative of the single-layer potential, by

$$K'\varphi(x) = 2\int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(x)}\varphi(y)\, ds(y), \quad x \in \Gamma. \tag{4.3}$$

We note that the right hand sides of these equations are well-defined at least for almost all $x \in \Gamma$, (4.1) understood in a Lebesgue sense (that $S\varphi(x)$ is well-defined in this sense for almost all $x \in \Gamma$ follows from Young's inequality), while the double-layer potential and $K'\varphi$ must be understood as Cauchy principal values [36]. Further, all three operators are bounded operators on $L^2(\Gamma)$ [36]. In fact [34], it holds that, for $|s| \leq 1/2$,

$$\begin{aligned} S &: H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma), \\ K &: H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma), \\ K' &: H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma), \end{aligned}$$

and these mappings are bounded.

These operators can also be characterised as traces on Γ of single- and double-layer potentials defined in Ω_e and Ω_i. Introducing, temporarily, the notations $\partial_\nu^\pm$, ∂_ν^+ and ∂_ν^- denoting the exterior and interior normal derivative operators, mapping $H^1(\Omega_e;\Delta)$ and $H^1(\Omega_i;\Delta)$, respectively, to $H^{-1/2}(\Gamma)$, it holds that [34]

$$K'\varphi = \left(\partial_\nu^+\mathcal{S} + \partial_\nu^-\mathcal{S}\right)\varphi, \quad \varphi \in H^{-1/2}(\Gamma), \tag{4.4}$$

where $\mathcal{S}$ is defined by

$$\mathcal{S}\varphi(x) := \int_\Gamma \Phi(x,y)\varphi(y)\, ds(y), \quad x \in \mathbb{R}^n. \tag{4.5}$$

It is shown in e.g. McLean [34] that $\mathcal{S} : H^{-1/2}(\Gamma) \to H^{1,\mathrm{loc}}(\mathbb{R}^n)$, and clearly $(\Delta + k^2)\mathcal{S}\varphi = 0$ in $\mathbb{R}^n \setminus \Gamma$, so that the right hand side of (4.5) defines a bounded operator on $H^{-1/2}(\Gamma)$.

From [15, Theorem 3.12] and [34, Theorems 7.15, 9.6] it follows that, if u satisfies the plane wave scattering problem, then a form of Green's representation theorem holds, namely

$$u(x) = u^i(x) - \int_\Gamma \Phi(x,y)\partial_\nu u(y)\, ds(y), \quad x \in \Omega_e. \tag{4.6}$$

Two integral equations for $\partial_\nu u$ can be obtained by taking the trace and the normal derivative, respectively, of (4.6) on Γ, namely $0 = u^i - \mathcal{S}\partial_\nu u$ and $\partial_\nu u = \partial_\nu u^i - \partial_\nu\mathcal{S}\partial_\nu u$. Note that, to simplify the notation, we have not explicitly used the trace operator γ in these equations or later in this section. Its presence is assumed implicitly. Since [34] we have the jump relations that on Γ we have $2\mathcal{S}\varphi = S\varphi$ and $2\partial_\nu\mathcal{S}\varphi = -\varphi + K'\varphi$, for $\varphi \in H^{-1/2}(\Gamma)$, we can write these equations as

$$S\partial_\nu u = 2u^i, \quad \partial_\nu u + K'\partial_\nu u = 2\partial_\nu u^i.$$

It is well known (e.g. [14]) that each of these integral equations fails to be uniquely solvable if $-k^2$ is an eigenvalue of the Laplacian in Ω for, respectively, Dirichlet and Neumann boundary conditions, but that a uniquely solvable integral equation is obtained by taking an appropriate linear combination of the above equations. Clearly, for every $\eta \in \mathbb{R}$ it follows from the above equations that

$$A'\partial_\nu u = f, \tag{4.7}$$

where

$$A' := I + K' - \mathrm{i}\eta S,$$

I is the identity operator, and

$$f(x) := 2\frac{\partial u^i}{\partial \nu}(x) - 2\mathrm{i}\eta u^i(x), \quad x \in \Gamma.$$

We have shown the first part of the following theorem, which is standard (e.g. [14]) in the case when Γ is smooth; for the extension to the case of Lipschitz Γ see [9].

THEOREM 4.1. *If u satisfies the plane wave scattering problem then, for every $\eta \in \mathbb{R}$, $\partial_\nu u \in H^{-1/2}(\Gamma)$ satisfies the integral equation (4.7). Conversely, if $\phi \in H^{-1/2}(\Gamma)$ satisfies $A'\phi = f$, for some $\eta \in \mathbb{R} \setminus \{0\}$, and u is defined in Ω_e by (4.6), with $\partial_\nu u$ replaced by ϕ, then u satisfies the plane wave scattering problem and $\partial_\nu u = \phi$.*

Note that, since we know that the plane wave scattering problem is uniquely solvable, this theorem implies that the integral equation (4.7) has exactly one solution in $H^{-1/2}(\Gamma)$.

The integral equation (4.7) is an example of a so-called *direct* integral equation formulation, obtained by applying Green's theorem to the original scattering problem. A related, *indirect* integral equation formulation, dating back to Brakhage and Werner [4], Leis [33], and Panich [40], is obtained by looking for a solution to the scattering problem in the form (1.6), for some density $\varphi \in H^{1/2}(\Gamma)$ and some $\eta \in \mathbb{R} \setminus \{0\}$. This combined single- and double-layer potential is in $C^2(\Omega_e)$, satisfies the Helmholtz equation and Sommerfeld radiation condition, and [34] is in $H^{1,\mathrm{loc}}(\Omega_e)$. Thus it satisfies the plane wave scattering problem if and only if it satisfies the boundary condition that $u^s = -u^i$ on Γ. Using the standard jump relations for Lipschitz domains [34], we see that this holds if and only if the integral equation (1.7) is satisfied, i.e. if and only if

$$A\varphi = 2g$$

where $g(x) := -u^i(x)$, $x \in \Gamma$, is the required Dirichlet data on Γ, and $A := I + K - \mathrm{i}\eta S$. This is the integral equation formulation introduced in [4, 33, 40].

Note that the above mapping properties of S, K, and K' imply that, for $|s| \leq 1/2$,

$$A : H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma), \quad A' : H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma), \tag{4.8}$$

and these mappings are bounded. It is shown moreover in [9] (or see [37] for the case A' and $s = 0$), by combining the standard arguments for these integral equations when Γ is smooth (see e.g. [14]) with known properties of integral operators on Lipschitz domains [41, 20, 36, 34], that, for $\eta \in \mathbb{R} \setminus \{0\}$, these mappings are bijections, which of course implies that their inverses are bounded by the Banach theorem. Further [36], K' is the adjoint of K and S is self-adjoint, so that A' is the adjoint of A in the same sense, namely that

$$(\phi, A\psi)_\Gamma = (A'\phi, \psi)_\Gamma, \quad \text{for } \phi \in L^2(\Gamma),\ \psi \in L^2(\Gamma), \tag{4.9}$$

where $(\phi, \psi)_\Gamma := \int_\Gamma \phi\psi ds$. Since $H^1(\Gamma)$ is dense in $H^{-1}(\Gamma)$ and the mappings (4.8) are bounded, it follows by density that the duality relation (4.9) holds, more generally, for $\phi \in H^{-s-1/2}(\Gamma)$ and $\psi \in H^{s+1/2}(\Gamma)$, provided $|s| \leq 1/2$. This implies that the norms of A and A^{-1} as operators on $H^{s+1/2}(\Gamma)$ coincide with those of A' and A'^{-1}, respectively, as operators on $H^{-s-1/2}(\Gamma)$, for $|s| \leq 1/2$. In particular, we note that

$$\|A^{-1}\|_2 = \|A'^{-1}\|_2, \tag{4.10}$$

where, here and in the remainder of the paper, $\|\cdot\|_2$ denotes both the norm on $L^2(\Gamma) = H^0(\Gamma)$ and the induced norm on the space of bounded linear operators on $L^2(\Gamma)$.

Following this preparation, we show now the main result of this section, which is an explicit bound on $\|A^{-1}\|_2 = \|A'^{-1}\|_2$ in terms of the geometry of Γ and the wave number, in the case when Ω is starlike and Lipschitz. For brevity and to simplify the arguments somewhat we also assume that Γ is piecewise smooth. Precisely, we make the following assumption, which is intermediate between Assumptions 1 and 2 introduced in §3.

ASSUMPTION 3. *For some $f \in C^{0,1}(S, \mathbb{R})$ with $\min_{\hat{x} \in S} f(\hat{x}) > 0$, it holds that $\Gamma = \{f(\hat{x})\hat{x} : \hat{x} \in S\}$. Further, for some $M \in \mathbb{N}$, it holds that $S = \overline{\cup_{j=1}^M S_j}$, with each S_j open in S, $S_{sing} := S \setminus \cup_{j=1}^M S_j$ a set of zero (surface) measure, and $f|_{S_j} \in C^2(S_j, \mathbb{R})$, for $j = 1, \ldots, M$.*

REMARK 4.2. *As an important example, we note that Assumption 3 is satisfied if Γ is a polyhedron, provided the interior of Ω, $\Omega_i = \mathbb{R}^n \setminus \bar{\Omega}_e$, is starlike with respect to the origin, i.e. $x \in \Omega_i$ implies $sx \in \Omega_i$ for $0 \le s \le 1$. Explicitly the function f is then defined by $f(\hat{x}) := \max\{s > 0 : s\hat{x} \in \Omega\}$ and, if $\Gamma_1, \ldots, \Gamma_M$ denote the sides of Γ (each Γ_j open in Γ) and $\Gamma_{sing} := \Gamma \setminus \cup_{j=1}^M \Gamma_j$ the edges and corners of Γ, then Assumption 3 holds with $S_j := f^{-1}(\Gamma_j)$, $j = 1, \ldots, M$ and $\Gamma_{sing} = f(S_{sing})$.*

Note that, if Assumption 3 holds (and, more generally, whenever Γ is piecewise smooth) the integrals (4.2) and (4.3) are well-defined in the ordinary Lebesgue sense almost everywhere on Γ, in fact, provided $x \notin \Gamma_{sing} = f(S_{sing})$. Note also that if Assumption 3 holds then $0 < \delta_- \le \delta_+ \le R_0$, where

$$\delta_- := \inf_{x \in \Gamma \setminus \Gamma_{sing}} (x \cdot \nu), \quad \delta_+ := \sup_{x \in \Gamma \setminus \Gamma_{sing}} (x \cdot \nu),$$

and $R_0 = \max_{x \in \Gamma} |x|$. Let us also define

$$\delta^* := \sup_{x \in \Gamma \setminus \Gamma_{sing}} |x - (x \cdot \nu)\nu| \le R_0.$$

The main result of this section is the following theorem. We postpone the proof until the end of the section.

THEOREM 4.3. *Suppose that Assumption 3 holds and $\eta \in \mathbb{R} \setminus \{0\}$. Then*

$$\|A^{-1}\|_2 = \|A'^{-1}\|_2 \le B \tag{4.11}$$

where

$$B := \frac{1}{2} + \left[\left(\frac{\delta_+}{\delta_-} + \frac{4\delta^{*2}}{\delta_-^2} \right) \left[\frac{\delta_+}{\delta_-} \left(\frac{k^2}{\eta^2} + 1 \right) + \frac{n-2}{\delta_- |\eta|} + \frac{\delta^{*2}}{\delta_-^2} \right] + \frac{(1 + 2kR_0)^2}{2\delta_-^2 \eta^2} \right]^{1/2}.$$

To help make the expression for B more comprehensible, let us consider some examples. Suppose first that Γ is a circle or sphere, i.e. $\Gamma = \{x : |x| = R_0\}$. Then $\delta_- = \delta_+ = R_0$ and $\delta^* = 0$ so

$$B = \frac{1}{2} + \left[1 + \frac{k^2}{\eta^2} + \frac{n-2}{R_0|\eta|} + \frac{(1 + 2kR_0)^2}{2R_0^2\eta^2} \right]^{1/2}. \tag{4.12}$$

In the 2D case that Γ is a regular polygon (centred on the origin) with M sides, $\delta_- = \delta_+ = R_0 \cos(\pi/M)$ and $\delta^* = R_0 \sin(\pi/M)$, so

$$B = \frac{1}{2} + \left[\left(1 + 4\tan^2 \frac{\pi}{M} \right) \left[1 + \frac{k^2}{\eta^2} + \tan^2 \frac{\pi}{M} \right] + \frac{(1 + 2kR_0)^2}{2R_0^2 \eta^2 \cos^2(\pi/M)} \right]^{1/2}. \tag{4.13}$$

In the limit $M \to \infty$ this recovers (4.12), and for a square ($M = 4$) this simplifies to

$$B = \frac{1}{2} + \left[10 + 5\frac{k^2}{\eta^2} + \frac{(1 + 2kR_0)^2}{R_0^2 \eta^2} \right]^{1/2}. \tag{4.14}$$

Similarly, for the cube $\Omega = \{x : |x_j| \le a,\ j = 1, 2, 3\}$ of side-length $2a$ we have $\delta_- = \delta_+ = a$, $\delta^* = \sqrt{2}a$, $R_0 = \sqrt{3}a$, so that

$$B = \frac{1}{2} + 3\left[3 + \frac{k^2}{\eta^2} + \frac{1}{a|\eta|} + \frac{(1 + 2\sqrt{3}ka)^2}{18a^2\eta^2} \right]^{1/2}.$$

We note that (4.12) can be compared with the results of Dominguez et al. [19] who have shown, when Γ is a circle, the bound (1.9) that $\|A^{-1}\|_2 \le 1$ for all sufficiently large k, if the choice $\eta = k$ is made. Our results Theorem 4.3 and (4.12) predict for the circle, if we choose $\eta = k$, that

$$\|A^{-1}\|_2 \leq \frac{1}{2} + \left[2 + \frac{(1+2kR_0)^2}{2k^2R_0^2}\right]^{1/2}. \tag{4.15}$$

The right hand side of this equation is a decreasing function of kR_0 on $(0,\infty)$ which approaches the limit 2.5 as $kR_0 \to \infty$. Thus our results show for a circle that, for every $\theta > 2.5$, $\|A^{-1}\|_2 \leq \theta$ for all sufficiently large kR_0. This bound is close to the result of [19] although we use much more general methods than the explicit calculation of eigenfunctions and eigenvectors used in [19], which are only available for a circular geometry. On the other hand the authors also show, importantly, the coercivity (1.10), which our methods do not seem to be well adapted to obtain.

If we follow Dominguez et al. [19] and choose $\eta = k$ we obtain the following simplification of the bound in Theorem 4.3 for the case $kR_0 \geq 1$. To obtain the second inequality we use that $\delta_+/\delta_- \leq \theta$ and $\delta^*/\delta_- \leq \theta$.

Corollary 4.4. *If Assumption 3 holds, $\eta = k$, and $kR_0 \geq 1$, then*

$$\begin{aligned}\|A^{-1}\|_2 = \|A'^{-1}\|_2 &\leq \frac{1}{2} + \left[\left(\frac{\delta_+}{\delta_-} + \frac{4\delta^{*2}}{\delta_-^2}\right)\left[2\frac{\delta_+}{\delta_-} + \frac{(n-2)R_0}{\delta_-} + \frac{\delta^{*2}}{\delta_-^2}\right] + \frac{9R_0^2}{2\delta_-^2}\right]^{1/2}\\ &\leq \frac{1}{2}\left(1 + \theta(4\theta + 4n + 1)\right),\end{aligned}$$

where $\theta := R_0/\delta_-$.

We finish the section by providing a proof of Theorem 4.3. Clearly, given that we already know that A and A' are invertible as operators on $L^2(\Gamma)$ and we have (4.10), this theorem is implied as a corollary of the following lemma (cf. [8, Lemma 3.3]).

Lemma 4.5. *Suppose that Assumption 3 holds and $\eta \in \mathbb{R} \setminus \{0\}$. Then, for all $\varphi \in L^2(\Gamma)$,*

$$||A'\varphi||_2 \geq B^{-1}||\varphi||_2. \tag{4.16}$$

Proof. Let $Y \subset L^2(\Gamma)$ denote the set of those functions φ that are Hölder continuous and are supported in $\Gamma \setminus \Gamma_{sing}$. Since Y is dense in $L^2(\Gamma)$ and A' is bounded on $L^2(\Gamma)$ it is sufficient to show that (4.16) holds for all $\varphi \in Y$.

So suppose $\varphi \in Y$, and define the single-layer potential u by

$$u(x) := \int_\Gamma \Phi(x,y)\varphi(y)ds(y) = \int_{\tilde{\Gamma}} \Phi(x,y)\varphi(y)ds(y), \quad x \in \mathbb{R}^n,$$

where $\tilde{\Gamma} \subset \Gamma\setminus\Gamma_{sing}$ is the support of φ. From standard properties of the single-layer potential (e.g. [14]) we have that $u \in C(\mathbb{R}^3) \cap C^2(\mathbb{R}^n \setminus \tilde{\Gamma})$. Further, it follows from [14, Theorem 2.17] that ∇u can be continuously extended from Ω_e to $\bar{\Omega}_e$ and from Ω_i to Ω, with limiting values on Γ given by

$$\nabla u_\pm(x) = \int_{\tilde{\Gamma}} \nabla_x \Phi(x,y)\varphi(y)ds(y) \mp \frac{1}{2}\varphi(x)\nu(x), \quad x \in \Gamma, \tag{4.17}$$

where, as before, $\nu(x)$ is the unit normal vector at x, directed into Ω_e, and

$$\nabla u_\pm(x) := \lim_{\epsilon\to 0+} \nabla u(x \pm \epsilon\nu(x)), \quad x \in \Gamma.$$

We note from (4.17) that the tangential part of ∇u, $\nabla_T u$, is continuous across Γ. On the other hand, the normal derivative jumps across Γ, with

$$\frac{\partial u_\pm}{\partial \nu}(x) = \frac{1}{2}\left[K'\varphi(x) \mp \varphi(x)\right], \quad x \in \Gamma \setminus \Gamma_{sing}. \tag{4.18}$$

Since also $u(x) = \frac{1}{2}S\varphi(x)$, $x \in \Gamma$, defining

$$g := \frac{1}{2}A'\varphi = \frac{1}{2}(I + K' - \mathrm{i}\eta S)\varphi,$$

we see that

$$\frac{\partial u_-}{\partial \nu}(x) - \mathrm{i}\eta u(x) = g(x), \quad x \in \Gamma \setminus \Gamma_{sing}. \tag{4.19}$$

Further, from (4.18) we see that

$$\frac{\partial u_-}{\partial \nu}(x) - \frac{\partial u_+}{\partial \nu}(x) = \varphi(x), \quad x \in \Gamma \setminus \Gamma_{sing}. \tag{4.20}$$

Note that to complete the proof we have to show that

$$||\varphi||_2 \leq 2B||g||_2. \tag{4.21}$$

We will achieve this by bounding the normal derivatives of u on Γ via applications of Lemma 2.3 in Ω_i and in D_R, for some $R > R_0$.

Before proceeding we note first that equations (2.19) and (2.20) do hold with v replaced by u and $G = \Omega_i$ or $G = D_R$, although we have not shown that $u \in H^2(G)$ so that we cannot apply Lemma 2.3 directly. To derive these equations when $G = \Omega_i$, we can first apply Lemma 2.3 with $v = u$ and $G = s\Omega_i$, for $s \in (0,1)$, and then take the limit $s \to 1^-$, noting that $\Delta u + k^2 u = 0$ in Ω_i and $u \in C^1(\Omega)$. Arguing similarly, these equations also hold with v replaced by u and $G = D_R$. Thus, recalling that our normal vector ν on Γ points out of Ω_e, we have, taking the imaginary part of (2.19) with $v = u$ and $G = \Omega_i$ and $G = D_R$, the identities

$$\Im \int_\Gamma \bar{u}\, \frac{\partial u_-}{\partial \nu}\, ds = 0, \tag{4.22}$$

$$\Im \int_\Gamma \bar{u}\, \frac{\partial u_+}{\partial \nu}\, ds = \Im \int_{\Gamma_R} \bar{u}\, \frac{\partial u}{\partial r}\, ds. \tag{4.23}$$

Taking $v = u$ and $G = \Omega_i$ and adding (2.20) to $(n-2)$ times the real part of (2.19) gives

$$\begin{aligned} 2k^2 \int_\Omega |u|^2\, dx = \int_\Gamma \Bigg(x \cdot \nu \left(k^2|u|^2 + \left|\frac{\partial u_-}{\partial \nu}\right|^2 - |\nabla_T u|^2 \right) \\ + \Re \left([(n-2)\bar{u} + 2x \cdot \nabla_T \bar{u}]\, \frac{\partial u_-}{\partial \nu} \right) \Bigg)\, ds. \end{aligned} \tag{4.24}$$

Finally, taking $v = u$ and $G = D_R$, for some $R > R_0$, and adding (2.20) to the real part of (2.19), we have

$$\begin{aligned} \int_{D_R} \left((3-n)|\nabla u|^2 + (n-1)k^2|u|^2 \right)\, dx = \\ -\int_\Gamma \left(x \cdot \nu \left(k^2|u|^2 + \left|\frac{\partial u_+}{\partial \nu}\right|^2 - |\nabla_T u|^2 \right) + \Re \left([\bar{u} + 2x \cdot \nabla_T \bar{u}]\, \frac{\partial u_+}{\partial \nu} \right) \right) ds \\ + \int_{\Gamma_R} \left(\left(k^2|u|^2 + \left|\frac{\partial u}{\partial r}\right|^2 - |\nabla_T u|^2 \right) + \Re \left(\bar{u}\, \frac{\partial u}{\partial r} \right) \right) ds. \end{aligned} \tag{4.25}$$

Using these four identities and Lemma 2.1 we will complete the proof.

We start by using (4.19) to replace $\partial u_-/\partial\nu$ in (4.22). Applying Cauchy-Schwarz, we see that

$$|\eta|\,\|u\|_2^2 = \left|\Im\int_\Gamma \bar{u}g ds\right| \leq \|u\|_2\|g\|_2,$$

so that

$$\|u\|_2 \leq |\eta|^{-1}\|g\|_2. \tag{4.26}$$

Alternatively, from (4.22) we have that

$$\Re\int_\Gamma \mathrm{i}\eta\bar{u}\frac{\partial u_-}{\partial\nu}ds = 0,$$

and, using (4.19) and Cauchy-Schwarz, we see that

$$\left\|\frac{\partial u_-}{\partial\nu}\right\|_2 \leq \|g\|_2. \tag{4.27}$$

It remains to bound the L^2 norm of $\partial u_+/\partial\nu$ in terms of $\|g\|_2$. To achieve this goal we first bound $\|\nabla_T\|_2$ using (4.24). From this equation we have that

$$\delta_-\|\nabla_T u\|_2^2 \leq \int_\Gamma x\cdot\nu|\nabla_T u|^2\,ds \leq$$

$$\delta_+k^2\|u\|_2^2 + \delta_+\left\|\frac{\partial u_-}{\partial\nu}\right\|_2^2 + [(n-2)\|u\|_2 + 2\delta^*\|\nabla_T u\|_2]\left\|\frac{\partial u_-}{\partial\nu}\right\|_2,$$

where we have used that $|x\cdot\nabla_T u| = |(x-(x\cdot\nu)\nu)\cdot\nabla_T u|$. From (4.26) and (4.27) it follows that

$$\delta_-\|\nabla_T u\|_2^2 \leq \left[\delta_+\left(\frac{k^2}{\eta^2}+1\right) + \frac{n-2}{|\eta|}\right]\|g\|_2^2 + 2\delta^*\|\nabla_T u\|_2\|g\|_2.$$

Finally, applying (3.11) to the last term on the right hand side, we deduce that

$$\frac{\delta_-}{2}\|\nabla_T u\|_2^2 \leq \left[\delta_+\left(\frac{k^2}{\eta^2}+1\right) + \frac{n-2}{|\eta|} + 2\frac{\delta^{*2}}{\delta_-}\right]\|g\|_2^2$$

so that

$$\|\nabla_T u\|_2 \leq \left[2\frac{\delta_+}{\delta_-}\left(\frac{k^2}{\eta^2}+1\right) + \frac{2(n-2)}{\delta_-|\eta|} + 2\frac{\delta^{*2}}{\delta_-^2}\right]^{1/2}\|g\|_2. \tag{4.28}$$

To finish the proof, we start from (4.25), apply Lemma 2.1, valid since u is a radiating solution of the Helmholtz equation, and then use equation (4.23), to see that

$$\delta_-\left\|\frac{\partial u_+}{\partial\nu}\right\|_2^2 \leq \int_\Gamma x\cdot\nu\left|\frac{\partial u_+}{\partial\nu}\right|^2 ds \leq$$

$$\int_\Gamma\left(x\cdot\nu|\nabla_T u|^2 + \Re\left([\bar{u}+2x\cdot\nabla_T\bar{u}]\frac{\partial u_+}{\partial\nu}\right) + 2kR\Im\left(\bar{u}\frac{\partial u_+}{\partial\nu}\right)\right)ds.$$

Applying Cauchy-Schwarz and (3.11), we see that

$$\begin{aligned}\delta_-\left\|\frac{\partial u_+}{\partial\nu}\right\|_2^2 &\leq \delta_+\|\nabla_T u\|_2^2 + (1+2kR)\|u\|_2\left\|\frac{\partial u_+}{\partial\nu}\right\|_2 + 2\delta^*\|\nabla_T u\|_2\left\|\frac{\partial u_+}{\partial\nu}\right\|_2\\ &\leq \left(\delta_+ + \frac{4\delta^{*2}}{\delta_-}\right)\|\nabla_T u\|_2^2 + \frac{\delta_-}{2}\left\|\frac{\partial u_+}{\partial\nu}\right\|_2^2 + \frac{(1+2kR)^2}{\delta_-}\|u\|_2^2.\end{aligned}$$

Hence, and using (4.26) and (4.28),

$$\left\|\frac{\partial u_+}{\partial \nu}\right\|_2^2 \leq 2\left(\frac{\delta_+}{\delta_-}+\frac{4\delta^{*2}}{\delta_-^2}\right)\|\nabla_T u\|_2^2+\frac{2(1+2kR)^2}{\delta_-^2}\|u\|_2^2$$
$$\leq 4\left[\left(\frac{\delta_+}{\delta_-}+\frac{4\delta^{*2}}{\delta_-^2}\right)\left[\frac{\delta_+}{\delta_-}\left(\frac{k^2}{\eta^2}+1\right)+\frac{(n-2)}{\delta_-|\eta|}+\frac{\delta^{*2}}{\delta_-^2}\right]+\frac{(1+2kR)^2}{2\delta_-^2\eta^2}\right]\|g\|_2^2.$$

This bound holds for all $R > R_0$ and hence also for $R = R_0$. Combining this bound with (4.27) we see that we have shown (4.21) and so finished the proof of the lemma. $\square$

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965.

[2] M. Ainsworth. Discrete dispersion relation for *hp*-version finite element approximation at high wave number. *SIAM J. Numer. Anal.*, 42 (2004), pp. 553–575.

[3] S. Amini, *On the choice of the coupling parameter in boundary integral equation formulations of the exterior acoustic problem*, Appl. Anal., 35 (1990), pp. 75–92.

[4] H. Brakhage and P. Werner, *Über das Dirichletsche Außenraumproblem für die Helmholtzsche Schwingungsgleichung*, Arch. Math., 16 (1965), pp. 325–329.

[5] O. P. Bruno, C. A. Geuzaine, J. A. Monro Jr, and F. Reitich, *Prescribed error tolerances within fixed computational times for scattering problems of arbitrarily high frequency: the convex case*, Phil. Trans. R. Soc. Lond A, 362 (2004), pp. 629–645.

[6] A. Buffa and S. Sauter, *On the acoustic single layer potential: Stabilisation and Fourier analysis*, SIAM J. Sci. Comput. 28 (2006), pp. 1974–1999.

[7] O. Cessenat and B. Després. Using plane waves as base functions for solving time harmonic equations with the Ultra Weak Variational Formulation. *Journal of Computational Acoustics*, 11 (2003), pp. 227–38.

[8] S. N. Chandler-Wilde, E. Heinemeyer, and R. Potthast, *A well-posed integral equation formulation for 3D rough surface scattering*, to appear in Proc. R. Soc. Lon. A.

[9] S. N. Chandler-Wilde and S. Langdon, *A Galerkin boundary element method for high frequency scattering by convex polygons*, to appear in SIAM J. Numer. Anal.

[10] S. N. Chandler-Wilde and P. Monk, *Existence, uniqueness and variational methods for scattering by unbounded rough surfaces*, SIAM J. Math. Anal. 37 (2005), pp. 598–618.

[11] S. N. Chandler-Wilde and B. Zhang, *Electromagnetic scattering by an inhomogenous conducting or dielectric layer on a perfectly conducting plate*, Proc. R. Soc. Lon. A, 454 (1998), pp. 519–542.

[12] S. H. Christiansen and J. C. Nédélec, *Preconditioners for the numerical solution of boundary integral equations from electromagnetism*, C.R. Acad. Sci. I-Math, 331 (2000), pp. 733–738.

[13] X. Claeys and H. Haddar, *Scattering from infinite rough tubular surfaces*, Math. Meth. Appl. Sci., 30 (2006), pp. 389–414.

[14] D. L. Colton and R. Kress, *Integral equation methods in scattering theory*, John Wiley, New York, 1983.

[15] ———, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer Verlag, 2nd Ed., 1992.

[16] P. Cummings and X. Feng, *Sharp regularity coefficient estimates for complex-valued acoustic and elastic Helmholtz equations,* Math. Models and Methods in the Applied Sciences, 16 (2006), pp. 139–160.

[17] E. Darve and P. Havé, *A fast multipole method for Maxwell equations stable at all frequencies*, Phil. Trans. R. Soc. Lond A, 362 (2004), pp. 603–628.

[18] L.F. Demkowicz and K. Gerdes. Solution of 3D Laplace and Helmholtz equations in exterior domains using *hp* infinite element methods. *Comput. Meth. Appl. Mech. Eng.*, 137 (1996), pp. 239–74.

[19] V. Dominguez, I. G. Graham, and V. P. Smyshlyaev, *A hybrid numerical-asymptotic boundary integral method for high-frequency acoustic scattering*, to appear in Numer. Math.

[20] J. Elschner, *The double-layer potential operator over polyhedral domains I: solvability in weighted Sobolev spaces*, Appl. Anal., 45 (1992), pp. 117–134.

[21] C. Farhat, I. Harari, and U. Hetmaniuk. A discontinuous Galerkin method with Lagrange multipliers for the solution of Helmholtz problems in the mid-frequency regime. *Comput. Meth. Appl. Mech. Eng.*, 192 (2003), pp. 1389–1419.

[22] M. Ganesh and I. G. Graham, *A high-order algorithm for obstacle scattering in three dimensions*, J. Comput. Phys., 198 (2004), pp. 211–242.

[23] K. Giebermann, *Schnelle Summationsverfahren zur numerischen Lösung von Integralgleichungen für Streuprobleme im* $\mathbb{R}^3$, PhD Thesis, Universität Karlsruhe, Germany, 1997.
[24] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 2nd Ed., 1983.
[25] R. Kress and W. T. Spassov, *On the condition number of boundary integral operators for the exterior Dirichlet problem for the Helmholtz equation*, Numer. Math., 42 (1983), 77–95.
[26] R. Kress, *Minimizing the condition number of boundary integral-operators in acoustic and electromagnetic scattering*, Q. J. Mech. Appl. Math., 38 (1985), 323–341.
[27] F. Ihlenburg and I. Babuška, *Finite element solution of the Helmholtz equation with high wave number. 1. the h version of the FEM*, Comput. Math. Appl., 30 (1995), pp. 9–37.
[28] ———, *Finite element solution of the Helmholtz equation with high wave number. 2. the* $h-p$ *version of the FEM*, SIAM J. Numer. Anal., 34 (1997), pp. 315–358.
[29] F. Ihlenburg, *Finite Element Analysis of Acoustic Scattering*, Springer, Berlin, 1998.
[30] S.J. Kim and J.B. Keller. High-frequency asymptotics for the numerical solution of the Helmholtz equation. *Applied Math. Letters*, 18 (2005), pp. 797–804.
[31] O. Laghrouche, P. Bettess, and R.J. Astley. Modelling of short wave diffraction problems using approximating systems of plane waves. *Int. J. Numer. Meth. Eng.*, 54 (2002), pp. 1501–33.
[32] S. Langdon and S. N. Chandler-Wilde, *A wavenumber independent boundary element method for an acoustic scattering problem*, SIAM J. Numer. Anal., 43 (2006), pp. 2450–2477.
[33] R. Leis, *Zur Dirichtletschen Randwertaufgabe des Aussenraums der Schwingungsgleichung*, Math. Z., 90 (1965), pp. 205–211.
[34] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, 2000.
[35] J. M. Melenk, *On Generalized Finite Element Methods*, PhD thesis, University of Maryland, College Park, MD, USA. 1995.
[36] Y. Meyer and R. Coifman, *Wavelets: Calderón-Zygmund and Multilinear Operators*, CUP, 2000.
[37] M. Mitrea, *Boundary value problems and Hardy spaces associated to the Helmholtz equation in Lipschitz domains*, J. Math. Anal. Appl., 202 (1996), pp. 819–842.
[38] J. Necas, *Les Methodes Directes en Theorie des Equations Elliptiques*, Academia, Praha, and Masson et Cie, Editeurs, Paris, 1967.
[39] J.-C. Nédélec, *Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems*, Springer, New York, 2001.
[40] O. I. Panich, *On the question of the solvability of the exterior boundary-value problems for the wave eaution and Maxwell's equations*, Usp. Mat. Nauk, 20A (1965), pp. 221–226.
[41] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal., 59 (1984), 572-611.
[42] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 2nd Ed., 1944.