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Generalised Dirichlet-to-Neumann map in time-dependent domains

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Abstract

We study the heat, linear Schrödinger and linear KdV equations in the domain $l(t) < x < \infty$, $0 < t < T$, with prescribed initial and boundary conditions and with $l(t)$ a given differentiable function. For the first two equations, we show that the unknown Neumann or Dirichlet boundary value can be computed as the solution of a linear Volterra integral equation with an explicit weakly singular kernel. This integral equation can be derived from the formal Fourier integral representation of the solution. For the linear KdV equation we show that the two unknown boundary values can be computed as the solution of a system of linear Volterra integral equations with explicit weakly singular kernels. The derivation in this case makes crucial use of analyticity and certain invariance properties in the complex spectral plane.

The above Volterra equations are shown to admit a unique solution.

1 Introduction

We study linear evolution problems posed on a time-dependent domain, assuming that the dependence of the boundary on time is known and described by a function $l(t)$. Namely, the domain we consider is of the form

$$D(t) = \{(x,s) : l(s) < x < \infty,\ 0 < s < t\}, \tag{1.1}$$

where $l(t)$ is a given, real, differentiable function.
The three illustrative examples we consider are the linear Schrödinger (LS), the heat and and linear KdV (LKdV) equations:

$$(LS) \qquad iq_t + q_{xx} = 0, \tag{1.2}$$

$$(heat) \qquad q_t - q_{xx} = 0, \tag{1.3}$$

$$(LKdV) \qquad q_t + q_{xxx} = 0. \tag{1.4}$$

We consider these PDEs in the domain $D(T)$ for some fixed positive constant $T > 0$, and assume that initial and appropriate boundary conditions are prescribed.

Such boundary value problems generally can only be well posed when a subset of all possible boundary values is prescribed. It is therefore necessary, as part of the solution of the problem, to characterise the unknown boundary values. The contribution of this paper is a proof that the unknown boundary values can be obtained as the unique solution of a linear Volterra integral equation for either a scalar or a vector-valued function. The kernels defining these integral equations are computed explicitly; these kernels are weakly singular and their properties guarantee, under suitable boundedness and regularity assumption on the given initial and boundary data, that a solution indeed exists and is unique.

We also discuss the applicability of our general methodology in higher dimensions, as well as to more general PDEs.

The problem considered in this paper has been considered in earlier papers [5, 10, 14] for the heat, linear Schrödinger and linear KdV equations, but only for the case that the function $l(t)$ describing the boundary is a convex function. In these earlier papers, the unknown boundary values were characterised through Volterra linear integral equations, with rather complicated kernels defined as contour integrals. No rigorous analysis of these kernels was presented.

In the present work the derivation of the Volterra integral equation is substantially simplified and generalised. Furthermore, we do not impose the convexity restriction required in our earlier work, and we require only that $l(t)$ be a differentiable function, with bounded derivative.

There exists an important difference between the second order cases and the third order one. This difference originates in the fact that in the second order cases one unknown boundary function must be determined, while in the third order case two such functions have to be determined as part of the solution process. Consequently the Volterra integral equation for the single unknown boundary function can be derived directly from the formal Fourier integral representation. On the otehr hand, in order to obtain the solution in the case of two unknown boundary functions, we need to exploit certain invariance properties of the so-called *global relation* in the complex spectral plane.

We present the details of the derivation for the three examples considered in our previous work: we either derive the Volterra integral equation either directly from the Fourier representation of the solution, or we take the more generally applicable route of the analysis of the global relation in the complex plane. For the second order cases, this is done for pedagogical reasons in order to prepare the reader for the third order example.

In all three cases, we give an explicit representation of the integral equation characterising the Dirichlet to Neumann (or Neumann to Dirichlet) map, and we prove that the the kernels involved are weakly singular, for both Dirichlet and Neumann type of boundary value problems. In addition, we include the rigorous analysis of the existence and uniqueness of the solution of the associated linear integral equations.

We also comment briefly on how our method can be extended to treat higher dimensional situations, and boundary value problems in the presence of forcing terms. In particular, the solution of forced problems can be used to derive quantitative estimates for perturbations of our basic equations (1.2-1.4), for example when the coefficients are variable but close, in an appropriate sense, to a constant.

In order to illustrate the wide applicability of the method presented here, we also present *explicit* formulas for the case that the boundary is a small perturbation of a linearly moving boundary, $l(t) = t + \varepsilon L(t)$, where $L(t)$ is a bounded function and ε a small parameter. As a further example, we discuss the case that the given boundary conditions on the moving boundary are periodic functions of time.

The main result for each of the three examples we consider is stated below:

Theorem 1.1 *Let $q(x,t)$ denote the solution of the partial differential equation (1.2) satisfying the initial and boundary conditions*

$$q(x,0) = q_0(x),\ 0 < x < \infty, \qquad q(l(t),t) = g_0(t),\ 0 < t < T. \tag{1.5}$$

Assume that the given functions $q_0(x)$ and $g_0(t)$ satisfy the following conditions:

(a) $q_0(x) \in \mathbf{C}^1([0,\infty))$ *and* $q_0'(x) \in \mathbf{L}^1([0,\infty))$;

(b) $g_0(t) \in \mathbf{C}^1([0,T])$.

Let $f(t)$ denote the unknown boundary value of $q(x, l(t)$ at $x = l(t)$:

$$f(t) = q_x(l(t),t), \quad 0 < t < T. \tag{1.6}$$

The function $f(t)$ defined by (1.6) is the unique solution of the Volterra integral equation

$$\pi f(t) = N(t) - \int_0^t K(s,t) f(s) ds, \quad 0 < t < T, \tag{1.7}$$

where $K(s,t)$ is defined by

$$K(s,t) = \frac{(1-i)\sqrt{\pi}}{2\sqrt{2}} \frac{l(t)-l(s)}{(t-s)} \frac{\mathrm{e}^{i\frac{(l(t)-l(s))^2}{4(t-s)}}}{(t-s)^{1/2}}, \qquad 0 < s < t < T, \tag{1.8}$$

with the known function $N(t)$ given by

$$N(t) = \frac{(1-i)\sqrt{\pi}}{\sqrt{2}} \left[\frac{1}{\sqrt{t}} \int_0^\infty \mathrm{e}^{i\frac{(l(t)-x)^2}{4t}} q_0'(x) dx - \int_0^t \frac{\mathrm{e}^{i\frac{(l(t)-l(s))^2}{4(t-s)}}}{\sqrt{t-s}} g_0'(s) ds \right]. \tag{1.9}$$

The Volterra integral equation (1.7) admits a unique solution in $\mathbf{C}([0,T])$.

Theorem 1.2 *Let $q(x,t)$ denote the solution of the partial differential equation (1.3) satisfying the initial and boundary conditions (1.5), subject to assumptions (a) and (b) of theorem 1.1.*
The function $f(t)$ given by (1.6) is characterised as the solution of the following Volterra integral equation:

$$\pi f(t) = N(t) + \int_0^t K(s,t) f(s) ds, \qquad 0 < t < T, \tag{1.10}$$

where the integral kernel $K(s,t)$ is defined by

$$K(s,t) = \frac{\sqrt{\pi}}{2} \frac{l(t)-l(s)}{t-s} \frac{\mathrm{e}^{-\frac{(l(s)-l(t))^2}{4(t-s)}}}{(t-s)^{1/2}}, \qquad 0 < s < t < T \tag{1.11}$$

with the known function $N(t)$ given by

$$N(t) = \sqrt{\pi} \left[\frac{1}{\sqrt{t}} \int_0^\infty \mathrm{e}^{-\frac{(l(t)-x)^2}{4t}} q_0'(x)dx - \int_0^t \frac{\mathrm{e}^{-\frac{(l(t)-l(s))^2}{4(t-s)}}}{\sqrt{t-s}} g_0'(s)ds \right], \qquad 0 < t < T. \tag{1.12}$$

The Volterra linear integral equation (1.10) admits a unique solution in $\mathbf{C}([0,T])$.

Similar results hold for the case of Neumann rather than Dirichlet boundary conditions, see Proposition 5.3.

Theorem 1.3 *Let $q(x,t)$ denote the solution of the partial differential equation (1.4) satisfying the initial and boundary conditions (1.5).*
Assume that the given functions $q_0(x)$ and $g_0(t)$ satisfy the following conditions:

(a) $q_0(x) \in \mathbf{C}^2([0,\infty))$ *and* $q_0'(x),\ q_0''(x) \in \mathbf{L}^1([0,\infty))$*;*

(b) $g_0(t) \in \mathbf{C}^1([0,T])$*.*

Let $f_1(t)$, $f_2(t)$ denote the unknown boundary values of $q(x,t)$:

$$f_1(t) = q_x(l(t),t), \qquad f_2(t) = q_{xx}(l(t),t), \quad 0 < t < T. \tag{1.13}$$

These functions are characterised as the solution of the following system of Volterra linear integral equations:

$$\begin{cases} \pi f_1(t) = N_1(t) - \int_0^t K_2(s,t) f_1(s)ds - \int_0^t K_1(s,t) f_2(s)ds, \\ \pi f_2(t) = N_2(t) - \int_0^t K_2(s,t) f_2(s)ds - \int_0^t K_{lr}(s,t) f_1(s)ds, \end{cases} \quad 0 < t < T, \tag{1.14}$$

where $N_1(t)$, $N_2(t)$ are the two known functions

$$N_1(t) = 2\pi \left[\frac{1}{(3t)^{1/3}} \int_{l(t)}^\infty Ai\left(\frac{l(t)-x}{(3t)^{1/3}}\right) q_0'(x)dx - \int_0^t Ai\left(\frac{l(t)-l(s)}{(3(t-s))^{1/3}}\right) \frac{g_0'(s)}{(3(t-s))^{1/3}} ds \right], \tag{1.15}$$

$$N_2(t) = -2\pi \left[\frac{1}{(3t)^{2/3}} \int_{l(t)}^\infty Ai\left(\frac{l(t)-x}{(3t)^{1/3}}\right) q_0''(x)dx - \int_0^t Ai'\left(\frac{l(t)-l(s)}{(3(t-s))^{1/3}}\right) \frac{g_0'(s)}{(3(t-s))^{2/3}} ds \right], \quad 0 < t < T, \tag{1.16}$$

with the integral kernels $K_1(s,t)$, $K_2(s,t)$ and $K_{lr}(s,t)$, $0 < s < t < T$ defined by

$$K_1(s,t) = \frac{2\pi i}{(3(t-s))^{2/3}} Ai'\left(\frac{l(t)-l(s)}{(3(t-s))^{1/3}}\right), \tag{1.17}$$

$$K_2(s,t) = -\frac{2\pi}{(3(t-s))^{1/3}} \frac{l(t)-l(s)}{3(t-s)} Ai\left(\frac{l(t)-l(s)}{(3(t-s))^{1/3}}\right), \tag{1.18}$$

$$K_{lr}(s,t) = \frac{2\alpha^2}{1-\alpha} Re \int_{\mathbb{R}} \mathrm{e}^{i\lambda^3}\left[\mathrm{e}^{i\alpha\lambda\frac{(l(t)-l(s))}{(t-s)^{1/3}}} - \mathrm{e}^{i\alpha^2\lambda\frac{(l(t)-l(s))}{(t-s)^{1/3}}}\right] \frac{i\lambda^3}{t-s} d\lambda, \tag{1.19}$$

where $\alpha = \mathrm{e}^{2\pi i/3}$ and $Ai(\cdot)$ denotes the Airy function.
The kernel K_{lr} admits also the following explicit representation:

$$K_{lr}(s,t) = \frac{2}{\sqrt{3}(t-s)^{1/3}} \frac{l(t)-l(s)}{(t-s)} \left[\sum_{m=0}^{\infty} \frac{\left(\frac{l(t)-l(s)}{t-s}\right)^{3m} (t-s)^{2m}}{(3m+1)!} \Gamma\left(\frac{3m+5}{3}\right)\right],$$

where $\Gamma(\cdot)$ denotes the Gamma function.

The paper is organised as follows.
In section 2, we derive formally a representation of the solution of a boundary value problem for a general linear evolution PDE posed in a domain of the form $D(t)$, as well as a relation that the boundary values must satisfy. This relation, known as the *global relation* (see equation (2.10)), is the starting point for the analysis that follows.

In section 3, after introducing some notation, we summarise known results for the solution of Volterra linear integral equations with singular kernels.
In section 4 we solve the Dirichlet problem for the linear Schrödinger equation by obtaining a linear Volterra integral equation for the unknown Neumann boundary value $q_x(l(t),t)$ directly from the Fourier integral representation of the solution. This is possible because only one boundary function needs to be characterised, but this method does not generalise to the case of the third-order problem $q_t + q_{xxx} = 0$, where two boundary functions must be determined.
In section 5 we consider the Dirichlet and Neumann problems for the heat equation. The analysis we present of this problem is based on the global relation, and encompasses the choice of a suitable integral contour, the inversion of the global relation on this contour in order to obtain a well-defined Volterra integral equation for the unknown boundary value, and finally the explicit computation and analysis of the integral kernel. This analysis is presented as an illustration of the general theory, in preparation of the solution of the third order problem.
In section 6, we consider the linear KdV equation. For this equation the analysis depends crucially on manipulating various integrals in the complex plane; however, even in this case we can express the final answer in terms of real integrals, see theorem 1.3.
Finally in section 7 we present concrete examples and explicit formulas for the case that the time-dependence of the boundary is approximately linear.

2 Formal solution representation via Green's formula

We consider the PDE

$$q_t + \omega(-i\partial_x)q = 0, \qquad (x,t) \in D(T), \tag{2.1}$$

where:

- T denotes a fixed positive constant;
- the given function $l : [0,T] \to \mathbb{R}$ is such that $l(0) = 0$, $l(t) \in \mathbf{C^1}([0,T])$;
- $D(T) \subset \mathbb{R}^2$ is the domain defined in (1.1);
- $\omega(\lambda)$ is a polynomial in λ of degree n,

$$\omega(\lambda) = \alpha_n\lambda^n + \alpha_{n-1}\lambda^{n-1} + ... + \alpha_1\lambda + \alpha_0, \quad \alpha_n \neq 0, \tag{2.2}$$

  such that $Re\ \omega(\lambda) \geq 0$ for $\lambda \in \mathbb{R}$ (this ensures that the pure initial value problem for this equation, posed on $\mathbb{R}$, is well posed).
- We are interested in constructing solutions of (2.1) such that $\{\partial_x^j q(x,t)\}_{j=0}^{n-1}$ decay as $x \to \infty$, for all t. This is a tacit assumption throughout all that follows.

Let

$$A(x,t,\lambda) = \mathrm{e}^{-i\lambda x + \omega(\lambda)t} q(x,t), \qquad B(x,t,\lambda) = \mathrm{e}^{-i\lambda x+\omega(\lambda)t}\sum_{k=0}^{n-1} c_k(\lambda)\partial_x^k q(x,t), \tag{2.3}$$

where $c_k(\lambda)$ are defined by the identity

$$\sum_{k=0}^{n-1} c_k(\lambda)\partial_x^k = i\frac{\omega(\lambda)-\omega(l)}{\lambda - l}|_{l=-i\partial_x}.$$

The PDE (2.1) can be written in the following divergence form:

$$A_t - B_x = 0. \tag{2.4}$$

Using the two-dimensional Green's theorem in the domain $D(t)$, for any fixed $t > 0$, we obtain

$$\int_{\partial D(t)} [Adx + Bds] = 0, \qquad t > 0, \quad Im(\lambda) \leq 0,$$

where $\partial D(t)$ denotes the oriented boundary of the domain $D(t)$. This equation yields

$$-\int_0^\infty A(x,0,\lambda)dx + \int_0^t [A(l(s),s,\lambda)l'(s) + B(l(s),s,\lambda)]ds + \int_{l(t)}^\infty A(x,t,\lambda)dx = 0. \tag{2.5}$$

Using (2.3), we find

$$\int_0^\infty \mathrm{e}^{-i\lambda x} q(x,0)dx - \mathrm{e}^{\omega(\lambda)t}\int_{l(t)}^\infty \mathrm{e}^{-i\lambda x}q(x,t)dx = \tag{2.6}$$

$$\int_0^t \mathrm{e}^{-i\lambda l(s)+\omega(\lambda)s}[q(l(s),s)l'(s) + \sum_{k=0}^{n-1} c_k(\lambda)\partial_x^k q(l(s),s)]ds, \qquad t > 0, \quad Im(\lambda) \leq 0.$$

This equation can be viewed either as the formal representation of the solution, or as the starting point for determining the unknown boundary values. Indeed, we let

$$\hat{q}_0(\lambda)=\int_0^\infty \mathrm{e}^{-i\lambda x}q(x,0)dx, \qquad \hat{q}(t,\lambda)=\int_{l(t)}^\infty \mathrm{e}^{-i\lambda x}q(x,t)dx, \qquad 0<t<T, \quad Im(\lambda)\le 0. \tag{2.7}$$

We assume that $q(x,0)$ and the boundary values $f_k(t)=\partial_x^k q(l(t),t)$ are sufficiently regular functions (to avoid technical issues, we also assume the initial and boundary conditions to be compatible at x=0=l(0)).
Inverting the Fourier transform in (2.6) for $q(x,t)$, we obtain the following formal representation of the solution:

$$q(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{i\lambda x-\omega(\lambda)t}\left[\hat{q}_0(\lambda)-\int_0^t \mathrm{e}^{-i\lambda l(s)+\omega(\lambda)s}[q(l(s),s)l'(s)+\sum_{k=0}^{n-1}c_k(\lambda)\partial_x^k q(l(s),s)]ds\right]d\lambda,$$

$$(x,t)\in D(T). \tag{2.8}$$

Assuming $q(x,t)=0$ for $x<l(t)$, equation (2.8) is also formally valid at $x=l(t)$, where it yields

$$q(l(t),t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{e}^{i\lambda l(t)-\omega(\lambda)t}\left[\hat{q}_0(\lambda)-\int_0^t \mathrm{e}^{-i\lambda l(s)+\omega(\lambda)s}[q(l(s),s)l'(s)+\sum_{k=0}^{n-1}c_k(\lambda)\partial_x^k q(l(s),s)]ds\right]d\lambda. \tag{2.9}$$

Remark 2.1 To avoid technicalities arising from the lack of continuity at $x=0$, we assume that $q_0(x)$ and the corresponding boundary conditions vanish at $x=0$ with all derivatives up to order $n-1$.

The global relation

Equation (2.6) can also be written in the form of the following *global relation*:

$$\hat{q}_0(\lambda)-\mathrm{e}^{\omega(\lambda)t}\hat{q}(t,\lambda)=\int_0^t \mathrm{e}^{-i\lambda l(s)+\omega(\lambda)s}[q(l(s),s)l'(s)+\sum_{k=0}^{n-1}c_k(\lambda)\partial_x^k q(l(s),s)]ds, \tag{2.10}$$

$$0<t<T, \qquad Im(\lambda)\le 0,$$

with $\hat{q}_0(\lambda)$, $\hat{q}(t,\lambda)$ given by (2.7).

Remark 2.2 A formula similar to (2.8) and a relation analogous to (2.10) are also valid for the solution of higher-dimensional PDEs of the form

$$q_t+\omega_1(-i\partial_{x_1})q+\omega_2(-i\partial_{x_2})q=0, \qquad 0<t<T, \quad l_1(t)<x_1<\infty, \; l_2(t)<x_2<\infty. \tag{2.11}$$

Using these relaitons, the solution of a given Dirichlet problem for equation (2.11) can be written in the form of a weakly singular Volterra integral equation with explicit kernels as in the one-dimensional case.

Indeed, the PDE (2.11) can be written in the form (2.4) with

$$A(x_1, x_2, t) = \mathrm{e}^{-i\lambda_1 x_1 - i\lambda_2 x_2 + \omega(\lambda)t} q(x,t), \quad B(x_1, x_2, t) = \mathrm{e}^{-i\lambda_1 x_1 - i\lambda_2 x_2 + \omega(\lambda)t}(B_1 + B_2), \tag{2.12}$$

where

$$B_i(x_1, x_2, t) = \sum_{k=0}^{n-1} c_k^{(i)}(\lambda)\partial_x^k q(x_1, x_2, t), \qquad i = 1, 2, \tag{2.13}$$

where $c_k^{(i)}(\lambda_i)$ are defined by the identity

$$\sum_{k=0}^{n-1} c_k^{(i)}(\lambda_i)\partial_x^k = i\frac{\omega_i(\lambda_i) - \omega_i(\delta)}{\lambda_i - \delta}|_{\delta=-i\partial_{x_i}}, \quad i = 1, 2.$$

See [8] for details.

Remark 2.3 In the case of forced problems, of the form

$$q_t + \omega(-i\partial_x)q = F(x,t), \qquad (x,t) \in D(T), \tag{2.14}$$

equation (2.8) is still valid, with $\hat{q}_0(\lambda)$ replaced by

$$\hat{q}_0(\lambda) + \int_0^t \int_{l(t)}^\infty \mathrm{e}^{-i\lambda\xi + \omega(\lambda)s} F(\xi, s) d\xi ds. \tag{2.15}$$

All formulas derived for the homogeneous case are still valid, provided $\hat{q}_0(\lambda)$ is replaced by the expression in (2.15). In particular, denoting by $f(t)$ the solution of D-to-N map for the forced problem and by f_{hom} the solution for the corresponding homogeneous problem, we find

$$f(t) - f_{hom}(t) = \int_{\mathbb{R}} \lambda \mathrm{e}^{i\lambda l(t) - \omega(\lambda)t} \left[\int_0^t \int_{l(t)}^\infty \mathrm{e}^{\omega(\lambda)s - i\lambda\xi} F(\xi, s) d\xi ds\right] d\lambda. \tag{2.16}$$

Nonlinear problems, or problems involving variable coefficients, can be considered as forced linear constant coefficient PDEs of the form (2.14). By using the general formula (2.16), it is then possible to obtain estimates for the more general problems, for sufficiently small times or when an appropriate norm of the forcing term is small.

3 Notation and background results

Notation

In what follows, we use the following convention:

- the letter λ denotes a complex variable,

$$\lambda = \lambda_R + i\lambda_I, \qquad \lambda_R,\ \lambda_I \in \mathbb{R}.$$

- $\Gamma(\lambda)$ denotes the Gamma function, and $Ai(\lambda)$ denotes the Airy function, see chapters 5 and 9 respectively of [1].

3.1 The D-to-N map - general remarks

A boundary value problem for equation (2.1) in the domain D is well posed if $n/2$ (n even) or $(n\pm 1)/2$ (n odd, sign depending on the sign on c_n) boundary conditions are prescribed [11]. Hence for $n \geq 2$, at least one boundary value must be determined as part of the solution of the problem. The determination of the unknown boundary values, or more precisely their expression in terms of the known data of the problem, is known as the (generalised) *Dirichlet-to-Neumann* map.

To determine this map, we solve the global relation for the unknown boundary value. In order to do this effectively, it is necessary to eliminate the term involving $q(x,t)$. This can be achieved by multiplying the global relation by $\mathrm{e}^{i\lambda l(t)-\omega(\lambda)t}$ and integrating along an appropriate contour.

The choice of integration contour

We multiply the left hand side of (2.10) by $\mathrm{e}^{i\lambda l(t)-\omega(\lambda)t}$:

$$\mathrm{e}^{i\lambda l(t)-\omega(\lambda)t}\left[\mathrm{e}^{\omega(\lambda)t}\hat{q}(t,\lambda)\right] = \int_{l(t)}^{\infty} \mathrm{e}^{-i\lambda(x-l(t))}q(x,t)dx. \tag{3.1}$$

Note that

- The identity (3.1) is valid as $\lambda \to \infty$ only along curves with the property that $|\mathrm{e}^{\pm\omega(\lambda)t}| \neq 0$ (asymptotically in λ).
- The right hand side of (3.1) defines as analytic function of λ for $Im\ \lambda < 0$, which is of order
$$\frac{q(l(t),t)}{i\lambda} + O\left(\frac{1}{\lambda^2}\right) \quad as\ \ \lambda \to \infty.$$

The above properties constrain our choice to those contours L_0 which are contained in the closed lower half plane $Im(\lambda) \leq 0$, and which are asymptotic to the contour

$$L = \{\lambda \in \mathbb{C}^- : Re(\omega(\lambda)) = 0\}. \tag{3.2}$$

Let $L_0 \subset \mathbb{C}^- \setminus \{0\}$ be such a contour. It is shown in [11] that L_0 can be chosen to be the oriented boundary of a union of triangular sectors of the form $\theta_1 \leq arg(\lambda) \leq \theta_2$ for some θ_1, θ_2 satisfying $\pi \leq \theta_1 < \theta_2 \leq 2\pi$ (indented to avoid zero). Integrating along the boundary of any such sector, we find

$$\int_{L_0}\left[\int_{l(t)}^{\infty} \mathrm{e}^{-i\lambda(x-l(t))}q(x,t)dx\right]d\lambda = \int_{L_0}\frac{q(l(t),t)}{i\lambda}d\lambda = (\theta_1-\theta_2)q(l(t),t). \tag{3.3}$$

Similarly, by subtracting appropriate boundary terms and integrating by parts, we find representations for $q_x(l(t),t)$, $q_{xx}(l(t),t)$:

$$\int_{L_0}\left[\int_{l(t)}^{\infty} \lambda\mathrm{e}^{-i\lambda(x-l(t))}q(x,t)dx + iq(l(t),t)\right]d\lambda = (\theta_1-\theta_2)iq_x(l(t),t), \tag{3.4}$$

$$\int_{L_0}\left[\int_{l(t)}^{\infty}\lambda^2 \mathrm{e}^{-i\lambda(x-l(t))}q(x,t)dx + i\lambda q(l(t),t) + q_x(l(t),t)\right]d\lambda = (\theta_1-\theta_2)q_{xx}(l(t),t). \quad (3.5)$$

Similar expressions are valid for $\partial_x^m q(l(t),t)$ for $m \geq 3$.

Remark 3.1 The terms appearing in the global relation, considered as functions of the complex variable λ, are exponential, and hence entire, functions. Hence it is possible to deform the contours in the complex λ plane along which the global relation is integrated within any bounded ball, or more generally, to any contour with the same asymptotic behaviour. In particular, the integration contours can always be deformed to avoid $\lambda = 0$. In the sequel, we will not mention explicitly this technical point in our derivations.

Volterra integral equations with a weakly singular kernel

We summarise the main result for linear Volterra integral equations of the second kind, see [2, 13].

Definition 3.1 *The kernel $K(s,t)$ is weakly singular of order γ, $0<\gamma<1$ if there exists a function $\tilde{K}(s,t):[0,T]\times[0,T]\to\mathbb{R}$, such that $\tilde{K}(s,t)$ is well defined at $s=t$, $|\tilde{K}(t,t)|<\infty$ and*

$$K(s,t) = \frac{\tilde{K}(s,t)}{(t-s)^{\gamma}}. \quad (3.6)$$

The following results are proved in [13], and slightly improved in [2]; see also [3, 4].

Proposition 3.1 *Consider the linear Volterra integral equation*

$$\pi f(t) = N(t) + \int_0^t K(s,t)f(s)ds, \qquad 0 \leq t < T, \quad (3.7)$$

and assume that the associated kernel $K(s,t)$ is of the form (3.6), for some $0<\gamma<1$.
If $N(t)\in \mathbf{L^1}(0,T)$ and $\tilde{K}(s,t)\in \mathbf{L}^{\infty}[0\leq s\leq t<T]$, the solution $f(t)$ of (3.7) is unique and continuous in $[0,T]$. This solution in general is not smooth at $t=0$.

For the three examples we consider in this paper, a boundary value problem is well posed when one initial and one boundary condition are prescribed. Assuming that the given conditions are $q(x,0)=q_0(x)$ and $g_0(t)=q(l(t),t)$ and the unknown boundary datum in the second order cases is $f(t)=q_x(l(t),t)$, the function $N(t)$ of (3.7) is given by an expression of the form

$$N(t) = \int_{\mathbb{R}} \mathrm{e}^{i\lambda l(t)-\omega(\lambda)t}\left[\int_0^{\infty}\mathrm{e}^{-i\lambda x}q_0'(x)dx - \int_0^t \mathrm{e}^{-i\lambda l(s)+\omega(\lambda)s}g_0'(s)ds\right]d\lambda, \quad (3.8)$$

where $\omega(\lambda)$ depends on the particular PDE. A similar expression holds for the Neumann problem, when the prescribed boundary condition is $q_x(l(t),t)$ instead of $q(l(t),t)$. Hence the regularity and decay of the function $N(t)$ depends in an explicit way on the regularity and decay properties of the given data.
Similar considerations are also valid for the third order case.

4 The linear Schrödinger equation $iq_t + q_{xx} = 0$

We consider the linear Schrödinger equation (1.2), in the time-dependent domain D, with the given initial and boundary conditions (1.5).

The formal solution representation at $x = l(t)$

Equation (2.6) in this case becomes

$$\int_0^{\infty} \mathrm{e}^{-i\lambda x} q(x,0)dx - \mathrm{e}^{i\lambda^2 t} \int_{l(t)}^{\infty} \mathrm{e}^{-i\lambda x} q(x,t)dx = \tag{4.1}$$
$$\int_0^t \mathrm{e}^{-i\lambda l(s)+i\lambda^2 s}[q(l(s),s)(l'(s)-\lambda) + iq_x(l(s),s)]ds, \qquad t>0, \quad Im(\lambda) \le 0.$$

Hence the formal solution representation at $x = l(t)$ is

$$\begin{aligned} q(l(t),t) &= \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{e}^{i\lambda l(t) - i\lambda^2 t}\left[\int_0^{\infty} \mathrm{e}^{-i\lambda x} q(x,0)dx + \qquad 0<t<T, \quad \lambda_I \le 0 \right. \\ & \left. \int_0^t \mathrm{e}^{i\lambda^2 s - i\lambda l(s)}[(l'(s)-\lambda)q(l(s),s) + iq_x(l(s),s)]ds\right] d\lambda. \end{aligned} \tag{4.2}$$

This representation can be used directly to solve the Neumann problem, that is, to characterise $q(l(s),s)$ in terms of $q_x(l(s),s)$. However, to solve the Dirichlet problem, we need a representation of $q_x(l(t),t)$. A direct differentiation of (4.2) yields an expression for $q_x(x,t)$ only for $x > l(t)$. In addition, the integrand in this representation is not guaranteed to be integrable on $\mathbb{R}$, and differentiation under the integral sign is not justified.
To obtain a well-defined representation of $q_x(l(t),t)$ we first multiply (4.1) by $i\lambda$. Then, integrating by parts, setting $q_x(l(s),s) = g_0(s)$ and assuming $g_0(0) = 0$, we obtain that the term involving the known function $g_0(s)$ is given by

$$\int_0^t \mathrm{e}^{-i\lambda l(s)+i\lambda^2 s}(i\lambda l'(s) - i\lambda^2)g_0(s)ds = -\mathrm{e}^{-i\lambda l(t)+i\lambda^2 t}g_0(t) + \int_0^t \mathrm{e}^{-i\lambda l(s))+i\lambda^2 t} g_0'(s)ds.$$

Hence from (4.1) we obtain

$$i\lambda\hat{q}_0(\lambda) - i\lambda \mathrm{e}^{i\lambda^2 t}\int_{l(t)}^{\infty} \mathrm{e}^{-i\lambda x}q(x,t)dx = \int_0^t \mathrm{e}^{-i\lambda l(s)+i\lambda^2 s}[g_0'(s)ds - \lambda q_x(l(s),s)]ds - \mathrm{e}^{-i\lambda l(t)+i\lambda^2 t}g_0(t),$$

where $\hat{q}_0(\lambda)$ is given by (2.7). Thus,

$$\int_{l(t)}^{\infty} \mathrm{e}^{-i\lambda x} i\lambda q(x,t)dx - \mathrm{e}^{-i\lambda l(t)}g_0(t) = i\lambda \mathrm{e}^{-i\lambda^2 t}\hat{q}_0(\lambda) - \int_0^t \mathrm{e}^{-i\lambda l(s) - i\lambda^2(t-s)}[g_0'(s)ds - \lambda q_x(l(s),s)]ds.$$

The term on the left hand side of this expression is equal to the Fourier transform of $q_x(x,t)$. Inverting this transform and evaluating it at $x = l(t)$, we obtain the desired representation:

$$q_x(l(t),t) = \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{e}^{i\lambda l(t)}\left[i\lambda \mathrm{e}^{-i\lambda^2 t}\hat{q}_0(\lambda) - \int_0^t \mathrm{e}^{-i\lambda l(s) - i\lambda^2(t-s)}[g_0'(s)ds - \lambda q_x(l(s),s)]ds\right] d\lambda. \tag{4.3}$$

The Volterra integral equation

Using the definition (1.6), we write (4.3) as

$$\pi f(t) = N(t) - \int_{\mathbb{R}} \int_0^t \mathrm{e}^{-i\lambda^2(t-s)+i\lambda(l(t)-l(s))} f(s) ds \, \lambda d\lambda,$$

where

$$N(t) = \int_{\mathbb{R}} \mathrm{e}^{i\lambda l(t)-i\lambda^2 t} \left[i\lambda \hat{q}_0(\lambda) - \int_0^t \mathrm{e}^{i\lambda^2 s - i\lambda l(s)} g_0'(s) ds \right] d\lambda, \tag{4.4}$$

and $\hat{q}_0(\lambda)$ is given by (2.7) .

• *Claim 1*

$$\int_{\mathbb{R}} \int_0^t \mathrm{e}^{-i\lambda^2(t-s)+i\lambda(l(t)-l(s))} f(s) ds \, \lambda d\lambda = \int_0^t K(t,s) f(s) ds, \tag{4.5}$$

where $K(t,s)$ is given by (1.8).

Setting

$$\mathcal{E}(t,s) = \mathrm{e}^{-i\lambda^2(t-s)+i\lambda(l(t)-l(s))}, \tag{4.6}$$

and interchanging the order of integration in (4.5), we find

$$\int_{\mathbb{R}} \left[\int_0^t \lambda \mathcal{E}(t,s,\lambda) f(s) ds \right] d\lambda = \int_0^t K(t,s) f(s) ds, \qquad K(t,s) = \int_{\mathbb{R}} \lambda \mathcal{E}(t,s,\lambda) d\lambda. \tag{4.7}$$

After changing variable to $\mu = \lambda/\sqrt{t-s}$ and completing the square in the integral defining $K(t,s)$, a principal value calculation (along the dotted contour indicated in figure **??**) yields expression (1.8).

• *Claim 2: The known term $N(t)$ is given by (1.9).*

Using the decay properties of $q_0(x)$ to show that $i\lambda \hat{q}_0(\lambda) = \widehat{q_0'}(\lambda)$, and changing the order of integration, we can rewrite expression (4.4) for $N(t)$ as

$$N(t) = \int_0^\infty \left[\int_{\mathbb{R}} \mathrm{e}^{-i\lambda^2 t + i\lambda(l(t)-x)} d\lambda \right] q_0'(x) dx - \int_0^t \left[\int_{\mathbb{R}} \mathcal{E}(s,t,\lambda) d\lambda \right] g_0'(s) ds.$$

where $\mathcal{E}$ is given by (4.6). Evaluating explicitly the integrals along $\mathbb{R}$, we find (1.9).

To finish the proof of Theorem 1.1, we need to show that the Volterra integral equation (1.7) admits a unique solution. This is the content of the following proposition.

Proposition 4.1 *Assume that the functions $q_0(x)$ and $g_0(t)$ of (1.5) satisfy the following conditions:*

(a) $q_0(x) \in \mathbf{C}^1([0,\infty))$ and $q_0'(x) \in \mathbf{L}^1([0,\infty))$;

(b) $g_0(t) \in \mathbf{C}^1([0,T])$.

Then the Volterra linear integral equation (1.7), which expresses $q_x(l(t),t)$ in terms of $q_0(x)$ and $g_0(t)$, admits a unique solution in $\mathbf{C}([0,T])$.

Proof: The kernel $K(s,t)$ is weakly singular, of order 1/2. Indeed, we can write

$$K(s,t) = \frac{\tilde{K}(s,t)}{(t-s)^{1/2}}, \qquad \tilde{K}(s,t) = \frac{(1-i)\sqrt{\pi}}{\sqrt{2}} \frac{l(t)-l(s)}{t-s} \mathrm{e}^{i\frac{(l(s)-l(t))^2}{4(t-s)}}. \tag{4.8}$$

Noting that

$$|\frac{l(t)-l(s)}{t-s}\mathrm{e}^{i\frac{(l(t)-l(s))^2}{t-s}}| \leq |l'(\tau)|, \qquad \textit{some}\ \tau:\ s \leq \tau \leq t < T,$$

and using that $l'(t)$ is bounded in $[0,T]$, it follows that the function $\tilde{K}(s,t)$ is in $\mathbf{L}^\infty[0,T]$ as a function of both s and t . This function is also continuously differentiable in both variables.
If the given data $q_0(x)$, $g_0(t)$ of the boundary value problem are such that the function $N(t)$ given by (1.12) is in $\mathbf{L}^1[0,T]$, then the general result given in theorem 3.1 guarantees that the problem admits a unique solution, and that this solution is continuous in $[0,T]$. This is indeed the case provided that

$$\int_0^T \left| \int_0^\infty \frac{\mathrm{e}^{-\frac{(l(t)-x)^2}{4t}}}{\sqrt{t}} q_0'(x)dx - \int_0^t \frac{\mathrm{e}^{-\frac{(l(t)-l(s))^2}{4(t-s)}}}{\sqrt{t-s}} g_0'(s)ds \right| dt < \infty.$$

Since q_0 or g_0 could independently be equal to zero, both double integrals must be finite. Hence the required condition splits into two conditions:

$$\int_0^T \frac{1}{\sqrt{t}} \left| \int_0^\infty \mathrm{e}^{-\frac{(l(t)-x)^2}{4t}} q_0'(x)dx \right| dt < \infty, \tag{4.9}$$

$$\int_0^T \left| \int_0^t \frac{\mathrm{e}^{-\frac{(l(t)-l(s))^2}{4(t-s)}}}{\sqrt{t-s}} g_0'(s)ds \right| dt < \infty. \tag{4.10}$$

For condition (4.9) we have

$$\int_0^T \frac{1}{\sqrt{t}} \left| \int_0^\infty \mathrm{e}^{-\frac{(l(t)-x)^2}{4t}} q_0'(x)dx \right| dt \leq 2M \int_0^\infty |q_0'(x)|dx, \qquad M = \int_0^T \frac{1}{\sqrt{t}}dt = \sqrt{T}.$$

Similarly , for condition (4.10) we write

$$\int_0^T \left| \int_0^t \frac{\mathrm{e}^{-\frac{(l(t)-l(s))^2}{4(t-s)}}}{\sqrt{t-s}} g_0'(s)ds \right| dt \leq \int_0^T \left[\int_s^T \frac{1}{\sqrt{t-s}}dt \right] |g_0'(s)|ds \leq M \int_0^T |g_0'(s)|ds.$$

Hence our assumptions guarantee that $N(t)$ is in $\mathbf{L}^1[0,T]$.
QED

The Neumann problem is solved in an analogous manner, and a theorem analogous to theorem (1.1) holds provided the assumptions (a), (b) are suitably replaced, see proposition 5.3.

5 The heat equation

We now consider the heat equation (1.3) in the domain $D(T)$, with given the initial and boundary conditions (1.5). In this section we will give a proof of Theorem 1.2 employing the analysis of the global relation in the complex λ plane. Although in this case this analysis is equivalent to the proof given in the previous section, this complex variable approach can be generalised to the third order case, as we will show in the last section.

The global relation

In this case, the global relation (2.6) becomes

$$\int_0^\infty e^{-i\lambda x} q(x,0)dx - e^{\lambda^2 t}\int_{l(t)}^\infty e^{-i\lambda x} q(x,t)dx = \int_0^t e^{\lambda^2 s - i\lambda l(s)}[q(l(s),s)(i\lambda + l'(s)) + q_x(l(s),s)]ds,$$
$$0 < t < T, \quad \lambda_I \leq 0. \tag{5.1}$$

The integration contour

We first identify an integration contour suitable to derive a Volterra integral equation, see section 3.1.

Definition 5.1 *For any $a \in \mathbb{R}$, we define the simply connected domain $\Omega_a \subset \mathbb{C}^-$ by*

$$\Omega_a = \{\lambda \in \mathbb{C}^- : \ \lambda_R^2 - \lambda_I^2 + \lambda_I a < 0\}. \tag{5.2}$$

We denote the boundary of Ω_a, oriented clockwise around the domain, by L_a:

$$L_a = \{\lambda \in \mathbb{C}^- : \ \lambda_R^2 = \lambda_I^2 - \lambda_I a\}. \tag{5.3}$$

For $a = 0$, we have the distinguished domain Ω_0, a triangular sector of opening $\pi/2$:

$$\Omega_0 = \{\lambda \in \mathbb{C}^- : \ \lambda_R^2 - \lambda_I^2 < 0\} = \{\lambda : \ \frac{5\pi}{4} < arg(\lambda) < \frac{7\pi}{4}\}. \tag{5.4}$$

The boundary of Ω_0 is the curve L_0 given by

$$L_0 = \{\lambda \in \mathbb{C}^- : \ \lambda_R^2 = \lambda_I^2\}. \tag{5.5}$$

Proposition 5.1 *For all $a \in \mathbb{R}$, the domain Ω_a is asymptotic to the sector Ω_0.*

Proof: It suffices to compare the behaviour, as $|\lambda| \to \infty$, of the boundaries of Ω_a and Ω_0, i.e. the contours L_a given in (5.3) and L_0.
For $\lambda \in L_0$, by definition we have $\frac{\lambda_I^2}{\lambda_R^2} = 1$ while for $\lambda \in L_a$,

$$\frac{\lambda_I^2}{\lambda_R^2} = \frac{\lambda_I^2}{\lambda_I(\lambda_I - a)} \to_{|\lambda_I| \to \infty} 1.$$

Hence in the large λ limit, both curves are described by the same two rays $\rho e^{i\phi}$, $\phi = 5\pi/4$ or $\phi = 7\pi/4$.
QED

The Dirichlet problem

Denote by $f(t)$ the unknown Neumann boundary value, see equation (1.6). We let $\hat{q}_0(\lambda)$ be given by (2.7), and $G_0(t,\lambda)$ denote the following known function:

$$G_0(t,\lambda) = \int_0^t \mathrm{e}^{-i\lambda l(s)+\lambda^2 s} g_0(s)[i\lambda + l'(s)]ds, \quad \lambda \in \mathbb{C}. \tag{5.6}$$

We also set

$$\mathcal{E}(t,s,\lambda) = \mathrm{e}^{\lambda^2(s-t)-i\lambda(l(s)-l(t))}, \qquad \lambda \in \mathbb{C}, \quad 0 < s < t < T. \tag{5.7}$$

Using (5.6), equation (5.1) can be written as

$$\int_0^t \mathrm{e}^{\lambda^2 s - i\lambda l(s)} f(s)ds - \hat{q}_0(\lambda) + G_0(t,\lambda) = -\mathrm{e}^{\lambda^2 t} \int_{l(t)}^{\infty} \mathrm{e}^{-i\lambda x} q(x,t)dx, \;\; 0 < t < T, \tag{5.8}$$

where $\hat{q}_0(\lambda)$ is defined by (2.7), $G_0(t,\lambda)$ by (5.6) and $f(t)$ is defined by (1.6). Our aim is to invert the global relation (5.8), and obtain an equation for $f(t)$ in terms of the known data. In order to make use of equation (3.4), we first perform the following steps:

(1) multiply both sides of the equation by $\lambda \mathrm{e}^{-\lambda^2 t + i\lambda l(t)}$;

(2) subtract the term $ig_0(t)$ from both sides of the resulting equation;

(3) integrate the resulting equation with respect to λ along the contour L_0 given in (5.5).

These steps yield the following equation, where all terms are well defined:

$$\int_{L_0} \lambda \mathrm{e}^{-\lambda^2 t + i\lambda l(t)} \left[\int_0^t \mathrm{e}^{\lambda^2 s - i\lambda l(s)} f(s)ds - \hat{q}_0(\lambda) + G_0(t,\lambda) - \frac{i}{\lambda} g_0(t) \mathrm{e}^{\lambda^2 t - i\lambda l(t)} \right] d\lambda =$$

$$= - \int_{L_0} \left[\int_{l(t)}^{\infty} \lambda \mathrm{e}^{-i\lambda(x-l(t))} q(x,t)dx + ig_0(t) \right] d\lambda. \tag{5.9}$$

We note that:

• *The right hand side of (5.9) equals* $\frac{i\pi}{2} f(t)$.

This follows from equation (3.4), with $\theta_1 = 5\pi/4$ and $\theta_2 = 7\pi/4$.

• *The first term of the left hand side of (5.9) satisfies the identity*

$$\int_{L_0} \left[\int_0^t \mathcal{E}(t,s,\lambda) f(s)ds \right] \lambda d\lambda = \int_0^t K(s,t) f(s)ds - \frac{i\pi}{2} f(t), \qquad t > 0, \tag{5.10}$$

where $\mathcal{E}(t,s,\lambda)$ *is defined by (5.7) and* $K(s,t)$ *is given by (1.11).*

This is a consequence of the fact that, for $\lambda \in \mathbb{C}^- \setminus \Omega_0$, $\mathcal{E}(t,s,\lambda)$ is a bounded, analytic function of λ. To show this property, we only need to verify that as $|\lambda| \to \infty$ in $\mathbb{C}^- \setminus \Omega_0$, the real part of the exponent of $\mathcal{E}$ is nonpositive, hence the exponential is bounded. Invoking the mean value theorem for the differentiable function $l(t)$, we can write

$$l(s) - l(t) = (s-t)l'(\tau), \quad \textit{for some } \tau \in [s,t] \tag{5.11}$$

and the real part of the exponent of $\mathcal{E}$ is given by

$$Re[\lambda^2(s-t) - i\lambda(l(s)-l(t))] = (s-t)[\lambda_R^2 - \lambda_I^2 + \lambda_I l'(\tau)], \quad s \le \tau \le t. \tag{5.12}$$

Since $l'(t)$ is assumed to be bounded, we can consider the domain Ω_a for $a = l'(\tau)$. Outside this domain,

$$\lambda_R^2 - \lambda_I^2 + \lambda_I l'(\tau) > 0 \quad \Longrightarrow (s-t)[\lambda_R^2 - \lambda_I^2 + \lambda_I l'(\tau)] < 0, \quad \lambda \notin \Omega_{l'(\tau)}. \tag{5.13}$$

By proposition 5.1, as $|\lambda| \to \infty$ each domain $\Omega_{l'(\tau)}$ is asymptotic to Ω_0. Hence the boundedness claim follows.
The boundedness of $\mathcal{E}(t,s,\lambda)$ implies that in $\mathbb{C}^- \setminus \Omega_0$ the integrand in the left hand side of (5.10) is of order $O\left(\frac{1}{\lambda}\right)$. Indeed, integrating by parts, we find

$$\int_0^t \mathcal{E}(t,s,\lambda) f(s) ds = \frac{f(t)}{\lambda} + O\left(\frac{1}{\lambda^2}\right), \quad \lambda \in \mathbb{C}^- \setminus \Omega_a.$$

Hence the contour L_0 can be deformed to $\mathbb{R}$ to yield

$$\begin{aligned}\int_{L_0}\left[\int_0^t \mathcal{E}(t,s,\lambda) f(s) ds\right]\lambda d\lambda &= \int_{\mathbb{R}}\left[\int_0^t \mathcal{E}(t,s,\lambda) f(s) ds\right]\lambda d\lambda + f(t)\left[\int_{\frac{5\pi}{4}}^{\pi} + \int_{2\pi}^{\frac{7\pi}{4}}\right] i d\theta \\ &= \int_{\mathbb{R}}\left[\int_0^t \mathcal{E}(t,s,\lambda) f(s) ds\right]\lambda d\lambda - \frac{i\pi}{2} f(t). \end{aligned} \tag{5.14}$$

We now interchange the order of integration of the integral on the right hand side of (5.14) and define $K(t,s)$ by

$$K(s,t) = \int_{\mathbb{R}} \lambda \mathcal{E}(t,s,\lambda) d\lambda. \tag{5.15}$$

The integral (5.15) can be computed explicitly by changing variable to $\mu = \lambda/\sqrt{t-s}$ and completing the square of the relevant exponent. This computation yields

$$\int_{\mathbb{R}} \lambda \mathcal{E}(t,s,\lambda) d\lambda = \frac{i\sqrt{\pi}}{2} \frac{l(t)-l(s)}{t-s} \frac{\mathrm{e}^{-\frac{(l(t)-l(s))^2}{4(t-s)}}}{\sqrt{t-s}}. \tag{5.16}$$

Hence we obtain (1.11).

- The remaining terms are given by a function $N(t)$ defined by

$$N(t) = \int_{L_0}\left[\mathrm{e}^{-\lambda^2 t + i\lambda l(t)}\left(G_0(t,\lambda) - \hat{q}_0(\lambda)\right) - \frac{i}{\lambda} g_0(t)\right]\lambda d\lambda. \tag{5.17}$$

Using the analyticity properties of the exponential $\mathcal{E}(t,s,\lambda)$, we can deform the integration contour in (5.17) to the real axis. Then, integrating by parts, we find that we can write $N(t)$ as

$$N(t) = \int_{\mathbb{R}} \mathrm{e}^{i\lambda l(t)} \mathrm{e}^{-\lambda^2 t}\left[-i\int_0^t \mathrm{e}^{-i\lambda l(s) + \lambda^2 s} g_0'(s) ds - \lambda \hat{q}_0(\lambda)\right] d\lambda.$$

Observing that $i\lambda\hat{q}_0(\lambda) = \widehat{q_0'}(\lambda)$, and inverting the order of integration we find

$$N(t) = i\int_0^\infty \left[\int_{\mathbb{R}} \mathrm{e}^{i\lambda(l(t)-x)-\lambda^2 t} d\lambda\right] q_0'(x)dx - \int_0^t \left[\int_{\mathbb{R}} \mathcal{E}(t,s,\lambda)d\lambda\right] g_0'(s)ds.$$

Computing the integrals along $\mathbb{R}$ explicitly we obtain (1.12).

In summary, the integral relation (5.9) yields (1.10).

QED

We now analyse the properties of the solution of the Volterra integral equation (1.10).

Proposition 5.2 *Consider the heat equation (1.3), with the given initial and boundary conditions (1.5).*
Assume that the given functions $q_0(x)$ and $g_0(t)$ satisfy the following conditions:

(a) $q_0(x) \in \mathbf{C}^1([0,\infty))$ *and* $q_0'(x) \in \mathbf{L}^1([0,\infty))$;

(b) $g_0(t) \in \mathbf{C}^1([0,T])$ *and* $\int_0^T \left[\sqrt{T-s}\,|g_0'(s)|\right] ds < \infty$.

Then the Volterra linear integral equation (1.10), which expresses $q_x(l(t,t)$ in terms of $q_0(x)$ and $g_0(t)$, admits a unique solution in $\mathbf{C}([0,T])$.

The proof is analogous to the proof of Proposition 4.1.

The Neumann problem

For completeness, we sketch the analogous analysis for the case of the Neumann problem. In this case, we seek the solution of the heat equation (1.3) which satisfies the following, sufficiently smooth, initial and boundary conditions:

$$q(x,0) = q_0(x),\ l(t) < x < \infty, \qquad q_x(l(t),t) = g_1(t),\ 0 < t < T. \tag{5.18}$$

We denote by $f_0(t)$ the unknown Dirichlet boundary value, i.e.

$$f_0(t) = q(l(t),t), \qquad 0 < t < T. \tag{5.19}$$

Proposition 5.3 *Let $q(x,t)$ denote the solution of the initial boundary value problem for the heat equation (1.3) defined by (5.18). Assume the the prescribed initial and boundary conditions satisfy:*

(a') $q_0(x) \in \mathbf{C}^1([0,\infty)) \cap \mathbf{L}^1([0,\infty))$;

(b') $g_1(t) \in \mathbf{C}([0,T])$ *and* $\int_0^T \sqrt{T-s}\,|g_1(s)|ds < \infty$.

The function $f_0(t)$ defined by (5.19) satisfies the Volterra integral equation

$$\pi f_0(t) = \int_0^t K_N(s,t)f_0(s)ds + N(t), \qquad 0 < t < T, \tag{5.20}$$

where the known function $N(t)$ is given by

$$N(t)=\sqrt{\pi}\left[\frac{1}{\sqrt{t}}\int_0^\infty \mathrm{e}^{-\frac{(l(t)-x)^2}{4t}}q_0(x)dx-\int_0^t \frac{\mathrm{e}^{-\frac{(l(t)-l(s))^2}{4(t-s)}}}{\sqrt{t-s}}g_1(s)ds\right],\quad 0<t<T,\quad (5.21)$$

and the integral kernel is given by

$$K_N(t,s)=\sqrt{\pi}\left(\frac{1}{2}\frac{l(t)-l(s)}{t-s}-l'(s)\right)\frac{\mathrm{e}^{-\frac{(l(s)-l(t))^2}{4(t-s)}}}{\sqrt{t-s}},\qquad 0<s<t<T.\qquad (5.22)$$

This Volterra integral equation admits a unique solution $f_0(t)\in\mathbf{C}[0,T]$.

Proof: The inversion of the global relation can now be obtained using (3.3). Setting

$$G_1(t,\lambda)=\int_0^t \mathrm{e}^{\lambda^2 s-i\lambda l(s)}g_1(s)ds,\qquad (5.23)$$

we write the global relation as

$$\int_0^t \mathrm{e}^{\lambda^2 s-i\lambda l(s)}(i\lambda+l'(s))f_0(s)ds-\hat{q}_0(\lambda)+G_1(t,\lambda)=-\mathrm{e}^{\lambda^2 t}\int_{l(t)}^\infty \mathrm{e}^{-i\lambda x}q(x,t)dx,\ \ 0<t<T,$$

we multiply this relation by $\mathrm{e}^{-\lambda^2 t+i\lambda l(t)}$, and integrate along L_0. The right hand side of the resulting equation equals $-\frac{\pi}{2}f_0(t)$.
For the first term on the left hand side of the same equation, which contains the unknown function $f_0(t)$, we proceed as for the Dirichlet problem: this term is given by

$$\int_{L_0}\int_0^t \mathcal{E}(t,s,\lambda)[i\lambda+l'(s)]f_0(s)dsd\lambda.$$

We compute the integral along L_0 to obtain

$$\int_{L_0}\int_0^t \mathcal{E}(t,s,\lambda)[i\lambda+l'(s)]f_0(s)dsd\lambda=\int_{\mathbb{R}}\int_0^t \mathcal{E}(t,s,\lambda)[i\lambda+l'(s)]f_0(s)dsd\lambda+\frac{\pi}{2}f_0(t).$$

Exchanging the order of integration on the right hand side, we find

$$\int_{\mathbb{R}}\left[\int_0^t \mathcal{E}(t,s,\lambda)[i\lambda+l'(s)]f_0(s)ds\right]d\lambda=-\int_0^t K_N(t,s)f_0(s)ds,\quad K_N(t,s)=\int_{\mathbb{R}}(i\lambda+l'(s))\mathcal{E}(t,s,\lambda)d\lambda,$$

where $K_N(t,s)$ can be evaluated explicitly and is given by (5.22).

It remains to show that the Volterra integral equation (5.20) admits a unique continuous solution. Indeed, as in the Dirichlet case, the kernel $K_N(s,t)$ is a well defined, in general weakly singular, integral kernel of order $\frac{1}{(t-s)^{1/2}}$. In addition, if $l(t)$ is Hölder continuous of order $\beta>\frac{1}{2}$, the kernel $K_N(s,t)$ is regular.
QED

6 The linear KdV equation

In this section, we give the proof of Theorem 1.3. This proof relies crucially on the analyticity properties of the functions involved in the global relation, as functions of the spectral complex parameter λ.

The global relation

For this equation, the global relation (2.10), well defined for $\lambda \in \mathbb{C}^-$, is given by

$$\hat{q}_0(\lambda) - \mathrm{e}^{-i\lambda^3 t}\hat{q}(t,\lambda) = \int_0^t \mathrm{e}^{-i\lambda l(s) - i\lambda^3 s}[q(l(s),s)(l'(s)+\lambda^2) - i\lambda q_x(l(s),s) - q_{xx}(l(s),s)]ds, \tag{6.1}$$

where $\hat{q}_0(\lambda)$ and $\hat{q}(t,\lambda)$ are given by (2.7).

The integration contour and domain decomposition

Since the global relation (2.6) is well defined as $|\lambda| \to \infty$ for $\lambda_I \leq 0$, we consider domains contained in in the lower half space of the λ complex plane.

Definition 6.1 *For $a \in \mathbb{R}$ we define*

$$\Omega_a = \{\lambda \in \mathbb{C}^- : \ \lambda_I^2 - 3\lambda_R^2 - a < 0\}. \tag{6.2}$$

The boundary of the domain Ω_a is given by

$$\partial\Omega_a = \begin{cases} L_a \cup \mathbb{R}, \qquad L_a = \left\{\lambda \in \mathbb{C}^- : \ \lambda_I = -\sqrt{3\lambda_R^2 + a}\right\} & a \geq 0 \\ \left\{\lambda_I = 0, \ \ |\lambda_R| \geq \sqrt{\frac{-a}{3}}\right\} \cup \left\{\lambda_I = -\sqrt{3\lambda_R^2 + a}, \ \ |\lambda_R| > \sqrt{\frac{-a}{3}}\right\} & a < 0. \end{cases} \tag{6.3}$$

For $a = 0$, we have the distinguished domain

$$\Omega_0 = \{\lambda \in \mathbb{C}^- : \ \lambda_I^2 - 3\lambda_R^2 < 0\}. \tag{6.4}$$

The boundary of Ω_0 is given by

$$\partial\Omega_0 = L_0 \cup \mathbb{R}, \qquad L_0 = \left\{\lambda \in \mathbb{C}^- : \ \lambda_I = -\sqrt{3\lambda_R^2}\right\}, \tag{6.5}$$

see figure **??**.

Lemma 6.1 *As $|\lambda| \to \infty$ the domain Ω_a given by (6.2) is asymptotic to the domain Ω_0 given by (6.4).*

Proof: It suffices to compare the behaviour, as $|\lambda| \to \infty$, of the boundaries of Ω_a and Ω_0, i.e. the contours L_a and L_0.
For $\lambda \in L_0$, and $Im(\lambda) \neq 0$, by definition we have $\frac{\lambda_R^2}{\lambda_I^2} = \frac{1}{3}$, while for $\lambda \in L_a$, and $Im(\lambda) \neq 0$,

$$\frac{\lambda_R^2}{\lambda_I^2} = \frac{1}{3} + \frac{a}{3\lambda_I^2} \to_{|\lambda_I| \to \infty} \frac{1}{3};$$

hence in the large λ limit, both curves are described by the same semilines $\rho e^{i\phi}$, $\phi = 4\pi/3$, $\phi = 5\pi/3$, or by $\mathbb{R}$.

QED

Note that the domain Ω_0 can be described as

$$\Omega_0 = \{\lambda : \ \pi < arg(\lambda) < \frac{4\pi}{3}\} \cup \{\lambda : \ \frac{5\pi}{3} < arg(\lambda) < \frac{2}{\pi}\}. \tag{6.6}$$

We also consider the decomposition of Ω_0 in its simply connected components:

$$\Omega_0 = \Omega_0^l \cup \Omega_0^r, \quad \begin{cases} \Omega_0^l = \{\lambda : \ \pi < \arg(\lambda) < \frac{4\pi}{3}\}, \\ \Omega_0^r = \{\lambda : \ \frac{5\pi}{3} < \arg(\lambda) < 2\pi\}. \end{cases} \tag{6.7}$$

The exponential involved in the computations below is

$$\mathcal{E}(t,s,\lambda) = e^{i\lambda^3(t-s)+i\lambda(l(t)-l(s))}, \qquad \lambda \in \mathbb{C}, \quad 0 < t < T. \tag{6.8}$$

Using the decomposition (6.7), any integral along $\partial\Omega_0^l$ that involves the exponential $\mathcal{E}$ multiplied by a function F, can be transformed to an integral along $\partial\Omega_0^r$ in two ways:
(1) by using a rotation of $4\pi/3$, namely

$$\int_{\partial\Omega_0^l} \mathcal{E}(t,s,\lambda)F(\lambda)d\lambda = \alpha^2 \int_{\partial\Omega_0^r} \mathcal{E}(t,s,\alpha^2\lambda)F(\alpha^2\lambda)d\lambda, \qquad \alpha = e^{2\pi i/3}. \tag{6.9}$$

(2) By using the change of variable $\lambda \to -\bar{\lambda}$, and the observation that

$$\mathcal{E}(s,t,-\bar{\lambda}) = \overline{\mathcal{E}(t,s,\lambda)},$$

so that

$$\int_{\partial\Omega_0^l} \overline{\mathcal{E}(s,t,-\bar{\lambda})F(-\bar{\lambda})d(-\bar{\lambda})} = \int_{\partial\Omega_0^r} \mathcal{E}(t,s,\lambda)\overline{F(-\bar{\lambda})}d\lambda. \tag{6.10}$$

In particular, the above observations imply the following preliminary lemma.

Lemma 6.2 *Let $\mathcal{E}(t,s,\lambda)$ be the exponential defined by (6.8), and Ω_0, Ω_0^r, Ω_0^l be the domains defined by (6.2), (6.7) respectively. Then*

$$\int_{\partial\Omega_0} \mathcal{E}(t,s,\lambda)i\lambda^3 d\lambda = 2Re \int_{\partial\Omega_0^r} \mathcal{E}(t,s,\lambda)i\lambda^3 d\lambda = 2Re \int_{\partial\Omega_0^l} \mathcal{E}(t,s,\lambda)i\lambda^3 d\lambda. \tag{6.11}$$

Proof: Let $F(\lambda) = i\lambda^3$. Then $\overline{F(-\bar{\lambda})} = \overline{-i\bar{\lambda}^3} = i\lambda^3$, so that

$$\overline{\int_{\partial\Omega_0^l} \mathcal{E}(t,s,\lambda)i\lambda^3 d\lambda} = \int_{\partial\Omega_0^r} \mathcal{E}(t,s,\lambda)i\lambda^3 d\lambda.$$

Hence

$$\int_{\partial\Omega_0^l} \mathcal{E}(t,s,\lambda)i\lambda^3 d\lambda + \int_{\partial\Omega_0^r} \mathcal{E}(t,s,\lambda)i\lambda^3 d\lambda = \int_{\partial\Omega_0^l} \mathcal{E}(t,s,\lambda)i\lambda^3 d\lambda + \overline{\int_{\partial\Omega_0^l} \mathcal{E}(t,s,\lambda)i\lambda^3 d\lambda}.$$

QED

The inversion of the global relation

We denote by $f_1(t)$ and $f_2(t)$ the two unknown boundary values given by (1.13), and we let $\hat{q}_0(\lambda)$ be given by (2.7) and $G_0(t,\lambda)$ be given by

$$G_0(t,\lambda)=\int_0^t \mathrm{e}^{-i\lambda l(s)-i\lambda^3 s} g_0(s)[l'(s)+\lambda^2]ds, \quad \lambda\in\mathbb{C}, \quad 0<t<T, \tag{6.12}$$

where $g_0(t)$ is the known boundary datum given by (1.5).
Using the definition (6.12), equation (2.10) can be written as

$$G_0(t,\lambda)-\hat{q}_0(\lambda)-\int_0^t \mathrm{e}^{-i\lambda^3 s-i\lambda l(s)} f_2(s)ds-\int_0^t \mathrm{e}^{-i\lambda^3 s-i\lambda l(s)} i\lambda f_1(s)ds$$
$$=-\mathrm{e}^{-i\lambda^3 t}\int_{l(t)}^\infty \mathrm{e}^{-i\lambda x} q(x,t)dx, \qquad 0<t<T. \tag{6.13}$$

Our aim is to obtain equations for $f_1(t)$, $f_2(t)$ in terms of the known data. To this end, since we have to characterise the *two* unknown boundary values $q_x(l(t),t)$ and $q_{xx}(l(t),t)$, we shall make use of both relations (3.4) and (3.5). Hence we multiply the global relation by either $\lambda\mathrm{e}^{i\lambda l(t)+i\lambda^3 t}$ or $\lambda^2\mathrm{e}^{i\lambda l(t)+i\lambda^3 t}$, subtract from both sides the terms $i\ g_0(t)$ or $i\lambda g_0(t)-f_1(t)$ respectively, and integrate the resulting two equations around $\partial\Omega_0$. This results in the following equation, where all integrals involved are well defined:

$$\int_{\partial\Omega_0}\left\{\lambda\mathrm{e}^{i\lambda^3 t+i\lambda l(t)}\left[G_0(t,\lambda)-\int_0^t \mathrm{e}^{-i\lambda^3 s-i\lambda l(s)}[f_2(s)+i\lambda f_1(s)]ds-\hat{q}_0(\lambda)\right]-ig_0(t)\right\}d\lambda=$$
$$=-\int_{\partial\Omega_0}\left[\int_{l(t)}^\infty \lambda\mathrm{e}^{-i\lambda(x-l(t))}q(x,t)dx+ig_0(t)\right]d\lambda, \tag{6.14}$$

$$\int_{\partial\Omega_0}\left\{\lambda^2\mathrm{e}^{i\lambda^3 t+i\lambda l(t)}\left[G_0(t,\lambda)-\int_0^t \mathrm{e}^{-i\lambda^3 s-i\lambda l(s)}[f_2(s)+i\lambda f_1(s)]ds-\hat{q}_0(\lambda)\right]-i\lambda g_0(t)+f_1(t)\right\}d\lambda=$$
$$=-\int_{\partial\Omega_0}\left[\int_{l(t)}^\infty \lambda^2\mathrm{e}^{-i\lambda(x-l(t))}q(x,t)dx+i\lambda g_0(t)+f_1(t)\right]d\lambda. \tag{6.15}$$

Note that

- The left hand side of (6.14) is the sum of three terms, $I_3-I_1-I_2$, where

$$I_1(t) = \int_{\partial\Omega_0}\int_0^t \lambda^2\mathrm{e}^{i\lambda^3(t-s)+i\lambda(l(t)-l(s))} if_1(s)dsd\lambda, \tag{6.16}$$

$$I_2(t) = \int_{\partial\Omega_0}\int_0^t \lambda\mathrm{e}^{i\lambda^3(t-s)+i\lambda(l(t)-l(s))} f_2(s)dsd\lambda, \tag{6.17}$$

$$I_3(t) = \int_{\partial\Omega_0}\lambda\mathrm{e}^{i\lambda^3 t+i\lambda l(t)}\left[G_0(t,\lambda)-\frac{i}{\lambda}g_0(t)\mathrm{e}^{-i\lambda^3 t-i\lambda l(t)}-\hat{q}_0(\lambda)\right]d\lambda. \tag{6.18}$$

- The left hand side of (6.15) is the sum of three terms, $J_3 - J_1 - J_2$, where

$$J_1(t) = \int_{\partial\Omega_0} \left[\int_0^t i\lambda^3 e^{i\lambda^3(t-s)+i\lambda(l(t)-l(s))} f_1(s)ds + f_1(t) \right] d\lambda, \tag{6.19}$$

$$J_2(t) = \int_{\partial\Omega_0} \int_0^t \lambda^2 e^{i\lambda^3(t-s)+i\lambda(l(t)-l(s))} f_2(s)dsd\lambda, \tag{6.20}$$

$$J_3(t) = \int_{\partial\Omega_0} \lambda^2 e^{i\lambda^3 t+i\lambda l(t)} \left[G_0(t,\lambda) - \frac{i}{\lambda} g_0(t) e^{-i\lambda^3 t - i\lambda l(t)} - \hat{q}_0(\lambda) \right] d\lambda. \tag{6.21}$$

We now find:
• *The contribution of the right hand side of (6.14) is equal to* $\frac{2\pi}{3} i f_1(t)$; *the contribution of the right hand side of (6.15) is equal to* $\frac{2\pi}{3} f_2(t)$.
This follows from equations (3.4) and (3.5).
• *The following identities hold:*

$$I_1(t) = \int_0^t K_2(s,t) i f_1(s)ds + \frac{\pi}{3} i f_1(t), \quad J_2(t) = \int_0^t K_2(s,t) f_2(s)ds + \frac{\pi}{3} f_2(t); \tag{6.22}$$

$$I_2(t) = \int_0^t K_1(s,t) f_2(s)ds, \tag{6.23}$$

where $K_1(s,t)$, $K_2(s,t)$ *are given by (1.17), (1.18) respectively.*
Note that $\mathcal{E}$ is bounded and analytic in $\mathbb{C}^- \setminus \Omega_0$. Indeed, the real part of the exponent of $\mathcal{E}$ is given by

$$Re[i\lambda^3(t-s) + i\lambda(l(t)-l(s))] = -(t-s)\lambda_I[3\lambda_R^2 - \lambda_I^2 + l'(\tau)], \quad s \le \tau \le t, \tag{6.24}$$

hence by definition, $\mathcal{E}$ is bounded and analytic in $\mathbb{C}^- \setminus \Omega_a$, with $a = l'(\tau)$. Since each domain Ω_a is asymptotic to Ω_0 by lemma 6.1, the claim follows.
Integration by parts and the analyticity properties of $\mathcal{E}$ imply

$$\int_0^t \lambda^2 e^{i\lambda^3(t-s)+i\lambda(l(t)-l(s))} i f_1(s)ds = -\frac{f_1(t)}{\lambda} + O\left(\frac{1}{\lambda^2}\right), \quad \lambda \in \mathbb{C}^- \setminus \Omega_0.$$

Therefore

$$\int_{L_0} \left[\int_0^t \lambda^2 e^{i\lambda^3(t-s)+i\lambda(l(t)-l(s))} i f_1(s)ds \right] d\lambda = \frac{\pi}{3} i f_1(t),$$

where L_0 is defined in (6.5), and it follows that

$$\int_{\partial\Omega_0} \left[\int_0^t \lambda^2 e^{i\lambda^3(t-s)+i\lambda(l(t)-l(s))} i f_1(s)ds \right] d\lambda = \int_{\mathbb{R}} \left[\int_0^t \lambda^2 e^{i\lambda^3(t-s)+i\lambda(l(t)-l(s))} i f_1(s)ds \right] d\lambda + \frac{\pi}{3} i f_1(t).$$

We now interchange of order of integration, and obtain

$$I_1(t) = \int_0^t K_2(t,s) i f_1(s)]ds + \frac{\pi}{3} i f_1(t), \qquad K_2(t,s) = \int_R \lambda^2 \mathcal{E}(t,s,\lambda)d\lambda. \tag{6.25}$$

The integral defining K_2 in (6.25) can be evaluated explicitly. After changing variable to $\mu = \lambda(t-s)^{1/3}$, the resulting integral yields the Airy function [1] (as can also be verified by using *Mathematica* [16]):

$$K_2(t,s) = \frac{1}{(t-s)}\int_{\mathbb{R}} \mu^2 \mathrm{e}^{i\mu^3 + i\mu\frac{l(t)-l(s)}{(t-s)^{1/3}}} d\mu = \frac{1}{(3(t-s))^{1/3}}\left[-\frac{2\pi}{3}\frac{l(t)-l(s)}{(t-s)} Ai\left(\frac{l(t)-l(s))}{(3(t-s))^{1/3}}\right)\right].$$

Hence the first of identities equation (6.22) follows.
The second of identities (6.22) is derived in an analogous manner.
Finally, to show (6.23) we note that by definition

$$I_2(t) = \int_{L_0}\int_0^t \lambda\mathcal{E}(t,s,\lambda) f_2(s) ds d\lambda + \int_{\mathbb{R}}\int_0^t \lambda\mathcal{E}(t,s,\lambda) f_2(s) ds d\lambda.$$

By the analyticity of $\mathcal{E}$ in the domain $\mathbb{C}^- \setminus \Omega_0$, whose boundary is L_0, the first integral on the right hand side vanishes, and the order of integration in the second integral can be exchanged. Then the explicit evaluation of $\int_{\mathbb{R}} \lambda\mathcal{E}(t,s,\lambda)d\lambda$ yields (6.23).

- *The term $I_3(t)$ equals $N_1(t)$ given by (1.15); the term $J_3(t)$ equals $N_2(t)$ given by (1.16)*

Using the analyticity properties of $\mathcal{E}$, the integral defining I_3 can be transformed to an integral along $\mathbb{R}$. A direct evaluation shows that I_3 is well defined. Indeed, integrating by parts and using that $g_0(0) = q_0(0)$, I_3 can be written as

$$I_3(t) = i\int_{\mathbb{R}}\left[\mathrm{e}^{i\lambda^3 t + i\lambda l(t)}\widehat{q_0'}(\lambda) - \int_0^t \mathcal{E}(t,s,\lambda) g_0'(s) ds\right] d\lambda. \tag{6.26}$$

Similarly, using the analyticity properties of $\mathcal{E}(t,s,\lambda)$ and integration by parts, the term J_3 can be written as

$$J_3(t) = i\int_{\mathbb{R}} \mathrm{e}^{i\lambda^3 + i\lambda l(t)}\left[\widehat{q_0''}(\lambda) - \int_0^t \mathrm{e}^{-i\lambda^3 s - i\lambda l(s)}\lambda g_0'(s) ds\right] d\lambda. \tag{6.27}$$

To show that (6.26) and (6.27) can be written in the form (1.15) and (1.16) respectively, we interchange the order of integration in (6.26) and (6.27), and then use explicit computations analogous to the computations of the kernels K_1 and K_2.

- *The following identity holds:*

$$J_1(t) = \int_0^t K_{lr}(s,t) f_1(s) ds, \tag{6.28}$$

where $K_{lr}(t,s,\lambda)$ is given by (1.19) (or (1.20)).

In the definition (6.19) of the term J_1 there appears a term $f_1(t)$. This term is essential for the integrability of the integrand. Indeed, integrating by parts the inner integral we find

$$\int_0^t i\lambda^3 \mathrm{e}^{i\lambda^3(t-s) + i\lambda(l(t)-l(s))} f_1(s) ds + f_1(t) = \mathrm{e}^{i\lambda^3 t + i\lambda l(t)} f_1(0) + O\left(\frac{1}{\lambda^3}\right). \tag{6.29}$$

Hence, for $\lambda \in \partial\Omega_0$ the λ-integrand of J_1 is $O\left(\frac{1}{\lambda^3}\right)$.

However, since the term $f_1(t)$ cannot be separated from the t-integral, its presence prevents the interchanging of the order of integration between the t and λ integrals. We can overcome this difficulty, by rewriting J_1 in a different form. Recalling the notation (6.7), we write $J_1 = J_1^r + J_1^l$ where J_1^r denotes that Ω_0 is replaced by Ω_0^r in the definition (6.19) of J_1, and similarly for J_1^l. Now, using (6.9), we find that we can cancel the term $f_1(t)$ and write

$$J_1^r - \alpha J_1^l = \int_{\partial\Omega_0^r}\left[\int_0^t i\lambda^3 \mathrm{e}^{i\lambda^3(t-s)}[\mathrm{e}^{i\lambda(l(t)-l(s))} - \mathrm{e}^{i\alpha^2\lambda(l(t)-l(s))}]f_1(s)ds\right]d\lambda. \tag{6.30}$$

However, by (6.11), we have $J_1 = 2ReJ_1^r = 2ReJ_1^l$, hence

$$Re(J_1^r - \alpha J_1^l) = (1-\alpha)ReJ_1^r = \frac{(1-\alpha)}{2}J_1. \tag{6.31}$$

Using (6.31) in (6.30) we arrive at an expression for J_1 that does not involve the term $f_1(t)$:

$$J_1(t) = \frac{2}{1-\alpha}Re\left[\int_{\partial\Omega_0^r}\left[\int_0^t i\lambda^3[\mathcal{E}(t,s,\lambda) - \mathcal{E}(s,t,\alpha^2\lambda)]f_1(s)ds\right]d\lambda\right]. \tag{6.32}$$

Note that both exponentials $\mathcal{E}(t,s,\lambda)$ and $\mathcal{E}(s,t,\alpha^2\lambda)$ are bounded along $\partial\Omega_0^r$. Furthermore, integration by parts shows that their difference is $O\left(\frac{1}{\lambda^3}\right)$. Indeed, we have

$$\int_0^t i\lambda^3\mathrm{e}^{i\lambda^3(t-s)}[\mathrm{e}^{i\lambda(l(t)-l(s))} - \mathrm{e}^{i\alpha^2\lambda(l(t)-l(s))}]f_1(s)ds = \mathrm{e}^{i\lambda^3 t}[\mathrm{e}^{i\lambda l(t)} - \mathrm{e}^{i\alpha^2\lambda l(t)}]f_1(0) + O\left(\frac{1}{\lambda^3}\right),$$

where λ lies on the boundary of the region where the exponential involved on the right hand side are decaying for $\lambda \to \infty$.
We can now interchange the order of integration and write

$$J_1 = \int_0^t K_{lr}(s,t)f_1(s)ds,$$

$$K_{lr}(s,t) = \frac{2}{1-\alpha}Re\left[\int_{\partial\Omega_0^r} i\lambda^3\mathrm{e}^{i\lambda^3(t-s)}[\mathrm{e}^{i\lambda(l(t)-l(s))} - \mathrm{e}^{i\alpha^2\lambda(l(t)-l(s))}]d\lambda\right]. \tag{6.33}$$

It is shown in the appendix how the kernel K_{lr} given by (6.33) can be written in the form (1.19) or, computing explicitly the relevant integrals, in the form (1.20). Regarding the latter expression, note that

$$\Gamma\left(\frac{3m+5}{3}\right) \sim m! \Longrightarrow \frac{\Gamma\left(\frac{3m+5}{3}\right)}{(3m+1)!} \leq \frac{1}{m^3}.$$

Moreover the boundedness of $l'(t)$ guarantees that

$$\left(\frac{l(t)-l(s)}{t-s}\right)^3 (t-s)^2 < l'(t)^3 T^2 \leq MT^2, \quad M \text{ constant}.$$

Hence the series in (1.20) converges.

In summary, we obtain

$$\frac{2\pi}{3} i f_1(t) = i\int_{\mathbb{R}} \left[\mathrm{e}^{i\lambda^3 t + i\lambda l(t)} \widehat{q_0'}(\lambda) - \int_0^t \mathcal{E}(t,s,\lambda) g_0'(s) ds \right] d\lambda$$
$$- \int_0^t K_2(t,s) f_1(s) ds - \frac{\pi}{3} i f_1(t) - \int_0^t K_1(t,s) f_2(s) ds, \tag{6.34}$$
$$\frac{2\pi}{3} f_2(t) = i\int_{\mathbb{R}} \mathrm{e}^{i\lambda^3 + i\lambda l(t)} \left[\widehat{q_0'}(\lambda) - \int_0^t \mathrm{e}^{-i\lambda^3 s - i\lambda l(s)} g_0'(s) ds \right] \lambda d\lambda$$
$$- \int_0^t K_{lr}(t,s) f_1(s) ds - \int_0^t K_2(t,s) f_2(s) ds - \frac{\pi}{3} f_2(t). \tag{6.35}$$

These equations are precisely equations (1.14).

The solvability of equations (1.14)

The explicit form of the kernels implies that $K_1(s,t)$, $K_2(s,t)$ and $K_{lr}(s,t)$ are well defined, weakly singular kernels of order 2/3, 1/3 and 1/3 respectively.
Moreover, using the properties of the Airy function and the form of the series appearing in the representation (1.20), it can be shown that these kernels are all of the form $\frac{\tilde{K}(t,s)}{(t-s)^\gamma}$, with $\tilde{K} \in \mathbf{L}^\infty[0,T]$ as a function of both s and t.
The functional dependence of $N_1(t)$, $N_2(t)$ on the given data can be computed explicitly. Indeed, the function $N_1(t) \in \mathbf{L}^1[0,T]$ provided that

$$\int_0^T \left| \int_{l(t)}^\infty \frac{1}{t^{1/3}} Ai\left(\frac{l(t)-x}{(3t)^{1/3}} \right) q_0'(x) dx \right| dt < \infty,$$

and

$$\int_0^T \left| \int_0^t \frac{1}{\sqrt[3]{3}(t-s)^{1/3}} Ai\left(\frac{l(t)-l(s)}{\sqrt[3]{3}(t-s)^{1/3}} \right) g_0'(s) ds \right| dt < \infty.$$

These conditions hold provided that

$$q_0' \in \mathbf{L}^1[0,\infty) \quad \text{and} \quad \int_0^T (T-s)^{2/3} |g_0'(s)| ds < \infty. \tag{6.36}$$

Similarly, $N_2(t) \in \mathbf{L}^1[0,T]$ if

$$q_0'' \in \mathbf{L}^1[0,\infty) \quad \text{and} \quad \int_0^T (T-s)^{1/3} |g_0'(s)| ds < \infty. \tag{6.37}$$

Hence the conditions (a), (b) in the statement of the theorem 1.3 guarantee that $N_1(t)$, $N_2(t)$ are in $\mathbf{L}^1[0,T]$.
The above conditions on the kernels and on the known functions $N_i(t)$ guarantee the existence and uniqueness of a solution for a scalar Volterra integral equation. In our case, we have two scalar coupled integral equations, which can be written as a single vector equation:

$$\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \frac{1}{\pi} \begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix} - \frac{1}{\pi} \int_0^t \mathbf{K}(t,s) \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} ds, \tag{6.38}$$

$$\text{with} \qquad \mathbf{K}(t,s) = \begin{pmatrix} K_2(s,t) & K_1(s,t) \\ K_{lr}(s,t) & K_2(s,t) \end{pmatrix}. \tag{6.39}$$

The contraction principle arguments and estimates used to prove the scalar case (see [13]) can be generalised in a straightforward way to the vector case, using the $\mathbf{L}^\infty$ matrix norm:

$$\|\mathbf{K}\|_\infty = max(\|K_2\|_\infty + \|K_1\|_\infty, \|K_2\|_\infty + \|K_{lr}\|_\infty).$$

Hence the existence of a unique solution of class $\mathbf{C^1}[0,T]$ follows from the conditions (6.36)-(6.37), that guarantee that the vector function $(N_1(t), N_2(t))^\tau$ is in $\mathbf{L^1}(0,T)$.

This completes the proof of the theorem.

7 Explicit examples

We have already remarked in the introduction (see also Remarks 2.2 and 2.3) that it is possible to use the formulas obtained in this paper to derive explicit expressions in several interesting cases, and it is also possible to generalise our approach to higher dimensional linear evolution equations and to forced problems.

In particular, the formal inclusion of forcing allows us to derive qualitative estimates of the behaviour of more general problems, obtained as small perturbations of basic linear problems of the form (2.1).

In what follows we show that, by using a perturbation expansion, it is possible to obtain explicit formulas. These formulas allow us to obtain qualitative information about the solution.

7.1 Perturbation of a linearly moving boundary

Consider the boundary value problem for the heat equation (1.3) in the domain $D(T)$, with the given initial and boundary conditions (1.5), where we assume that the equation of the moving boundary has the form

$$l(t) = t + \varepsilon L(t), \qquad t > 0, \tag{7.1}$$

where $L(t)$ is a differentiable function of t such that $0 \le L'(t) \le M$ for some constant $M > 0$.

Then, retaining terms only to order ε, we find

$$\frac{l(t)-l(s)}{t-s} = 1 + \varepsilon\frac{L(t)-L(s)}{t-s}, \qquad \frac{(l(t)-l(s))^2}{4(t-s)} = \frac{t-s}{4} + \frac{\varepsilon(L(t)-L(s))}{2}.$$

Using the first-order approximation

$$\mathrm{e}^{-\frac{\varepsilon}{2}(L(t)-L(s))} \sim 1 - \frac{\varepsilon}{2}(L(t)-L(s)),$$

the kernel (1.11) of the Volterra integral equation becomes

$$K(s,t) = \frac{\sqrt{\pi}}{2}\frac{\mathrm{e}^{-\frac{t-s}{4}}}{\sqrt{t-s}}\left(1 + \varepsilon\frac{L(t)-L(s)}{t-s} - \frac{\varepsilon}{2}(L(t)-L(s))\right), \qquad 0 < s < t < \infty \tag{7.2}$$

and the known function $N(t)$ given by (1.12) becomes

$$N(t) = -\sqrt{\pi} \int_0^t \frac{\mathrm{e}^{-\frac{t-s}{4}}}{\sqrt{t-s}} \Big(1 - \frac{\varepsilon}{2}(L(t) - L(s))\Big) g_0'(s) ds, \qquad 0 < t < \infty. \tag{7.3}$$

We now consider the equation (1.10) in the limit as $\varepsilon \to 0$. We denote by $f_0(t)$,$N_0(t)$ and $K_0(s,t)$ the limits of the functions $f(t)$ and $N(t)$ and of the kernel $K(s,t)$ as $\varepsilon \to 0$. Explicitly, we have

$$N_0(t) = -\sqrt{\pi} \int_0^t \mathrm{e}^{-\frac{t-s}{4}} \frac{\dot{g}_0(s)}{\sqrt{t-s}} ds, \qquad K_0(s,t) = \frac{\sqrt{\pi}}{2} \frac{\mathrm{e}^{-\frac{t-s}{4}}}{\sqrt{t-s}}, \qquad 0 < s < t < \infty. \tag{7.4}$$

In this limit, after rearranging the equation (1.10), we find for the unknown Neumann datum the following Abel equation of the second kind:

$$f_0(t)\mathrm{e}^{\frac{t}{4}} = \frac{1}{\pi} N_0(t)\mathrm{e}^{\frac{t}{4}} + \frac{1}{2\sqrt{\pi}} \int_0^t \frac{f_0(s)\mathrm{e}^{\frac{s}{4}}}{\sqrt{t-s}} ds. \tag{7.5}$$

This equation can be solved explicitly (see e.g. [15]) and its solution is given by

$$f_0(t) = G_0(t) + \frac{1}{4}\int_0^t G_0(s) ds, \qquad G_0(t) = \frac{1}{\pi} N_0(t) + \frac{1}{2\pi^{3/2}} \int_0^t \frac{N_0(s)\mathrm{e}^{-\frac{t-s}{4}}}{\sqrt{t-s}} ds. \tag{7.6}$$

Then, denoting

$$f_p = f - f_0, \quad K_p = K - K_0, \quad N_p = N - N_0, \tag{7.7}$$

we can write the Volterra integral equation (1.10) in the form

$$\pi(f_0 + f_p)(t) = N_0(t) + N_p(t) + \int_0^t (K_0 + K_p)(s,t)(f_0 + f_p)(s) ds.$$

In summary, neglecting the term $K_p f_p$ which is of higher order in ε, we obtain for $f_p(t)$ the following equation:

$$\pi f_p(t) \sim \left[N_p(t) + \int_0^t K_p(s,t) f_0(s) ds \right] + \int_0^t K_0(s,t) f_p(s) ds. \tag{7.8}$$

The expression in square bracket is known, thus this equation has the same form as (7.5) and can be solved in the same way, namely

$$f_p(t) = G_p(t) + \frac{1}{4}\int_0^t G_p(s) ds, \tag{7.9}$$

where

$$G_p(t) = \frac{1}{\pi}\tilde{N}_p(t) + \frac{1}{2\pi^{3/2}} \int_0^t \frac{\tilde{N}_p(s)\mathrm{e}^{-\frac{t-s}{4}}}{\sqrt{t-s}} ds, \qquad \tilde{N}_p = N_p(t) + \int_0^t K_p(s,t) f_0(s) ds. \tag{7.10}$$

Expression (7.6) and (7.9) yield an *explicit* approximation of order ε to the solution $f = f_0 + f_p$.

7.2 Periodic boundary conditions on a linearly moving boundary

We investigate the case that the given Dirichlet boundary condition is a periodic function, with zero initial condition, when the boundary of the domain has linear dependence on time, $l(t)=t,\ t>0$.
Consider the heat equation (1.3), with zero initial condition and t^*-periodic, smooth Dirichlet boundary conditions:

$$q(x,0)=0, 0<x<\infty; \qquad q(l(t),t)=g_0(t): \begin{cases} g_0(t+t^*)=g_0(t) & (\text{fixed } t^*>0), \\ \dot{g}_0(t) \text{ bounded} \end{cases} \qquad t>0. \tag{7.11}$$

We prove that the solution of the D-to-N is periodic with the same periodicity, at least in the limit as $t\to\infty$. We denote by $N_0(t)$ and $K_0(s,t)$ the values of $N(t)$ and $K(s,t)$ appearing in (1.10) corresponding to the case $l(t)=t$, and denote by $f_0(t)$ the corresponding solution of the integral equation, given in this case by (7.6). Note that since $\dot{g}_0(t)$ is bounded, so is $N_0(t)$. Indeed, using

$$\int_0^t \frac{\mathrm{e}^{-\frac{t-s}{4}}}{\sqrt{t-s}}ds = 2\sqrt{\pi}\mathrm{Erf}\left(\frac{\sqrt{t}}{2}\right), \tag{7.12}$$

we find

$$|N_0(t)| \le \|\dot{g}_0\|_\infty \frac{\sqrt{\pi}}{2}\int_0^t \frac{\mathrm{e}^{-\frac{t-s}{4}}}{\sqrt{t-s}}ds = \|\dot{g}_0\|_\infty \pi \mathrm{Erf}\left(\frac{\sqrt{t}}{2}\right) \le \pi\|\dot{g}_0\|_\infty \quad \forall\, t>0.$$

Similarly, this implies that $G_0(t)$ given by (7.6) is bounded.

Proposition 7.1 *Consider the boundary value problem for the heat equation posed on the domain $\{t>0, t<x<\infty\}$, with the initial and boundary conditions (7.11).*
Then the solution $f_0(t)$ of the Dirichlet-to-Neumann map, given by (7.6), is asymptotically t^*-periodic:

$$\lim_{t\to\infty}[f_0(t+t^*)-f_0(t)]=0.$$

Proof: Using the expression (7.6) for $f_0(t)$, we can write

$$f_0(t+t^*)-f_0(t) = \frac{1}{\pi}[N_0(t+t^*)-N_0(t)] + \frac{1}{4}\int_t^{t+t^*} G_0(s)ds$$
$$+\frac{1}{2\pi^{3/2}}\left\{\int_0^{t+t^*}\frac{N_0(s)\mathrm{e}^{-\frac{t+t^*-s}{4}}}{\sqrt{t+t^*-s}}ds - \int_0^t \frac{N_0(s)\mathrm{e}^{-\frac{t-s}{4}}}{\sqrt{t-s}}ds\right\}$$

and, after splitting and changing variables in the first integral, we find

$$f_0(t+t^*)-f_0(t) = \frac{1}{\pi}[N_0(t+t^*)-N_0(t)] + \frac{1}{4}\int_t^{t+t^*} G_0(s)ds$$
$$+\frac{1}{2\pi^{3/2}}\left\{\int_0^{t^*}\frac{N_0(s)\mathrm{e}^{-\frac{t+t^*-s}{4}}}{\sqrt{t+t^*-s}}ds - \int_0^t \frac{[N_0(s+t^*)-N_0(s)]\mathrm{e}^{-\frac{t-s}{4}}}{\sqrt{t-s}}ds\right\}.$$

Hence

$$\lim_{t\to\infty}[f_0(t+t^*)-f_0(t)] = \frac{1}{\pi}\lim_{t\to\infty}[N_0(t+t^*)-N_0(t)] + \frac{1}{4}\lim_{t\to\infty}\int_t^{t+t^*} G_0(s)ds$$

$$+\frac{1}{2\pi^{3/2}}\lim_{t\to\infty}\left\{\int_0^{t^*}\frac{N_0(s)\mathrm{e}^{-\frac{t+t^*-s}{4}}}{\sqrt{t+t^*-s}}ds - \int_0^t \frac{[N_0(s+t^*)-N_0(s)]\mathrm{e}^{-\frac{t-s}{4}}}{\sqrt{t-s}}ds\right\}. \tag{7.13}$$

The proof of the proposition now follows from the four claims below:

Claim 1:

$$\lim_{t\to\infty}\int_t^{t+t^*} G_0(s)ds = 0.$$

A direct computation, using the definitions of G_0 and N_0, yields

$$\int_t^{t+t^*} G_0(w)dw = \frac{1}{\pi}\int_t^{t+t^*} N_0(w)dw + \frac{1}{2\pi^{3/2}}\int_t^{t+t^*}\int_0^w \frac{N_0(s)\mathrm{e}^{-\frac{w-s}{4}}}{\sqrt{w-s}}dsdw$$

$$= -\frac{1}{\sqrt{\pi}}\int_t^{t+t^*}\int_0^w \frac{\dot{g}_0(s)\mathrm{e}^{-\frac{w-s}{4}}}{\sqrt{w-s}}dsdw + \frac{1}{2\pi^{3/2}}\int_t^{t+t^*}\int_0^w \frac{N_0(s)\mathrm{e}^{-\frac{w-s}{4}}}{\sqrt{w-s}}dsdw.$$

Exchanging order of integration, we find that the inner integral in both terms is given by

$$\int_t^{t+t^*}\frac{\mathrm{e}^{-\frac{w-s}{4}}}{\sqrt{w-s}}dw = 2\sqrt{\pi}\left[Erf\frac{\sqrt{t^*+t-s}}{2} - Erf\frac{\sqrt{t-s}}{2}\right].$$

In summary

$$\lim_{t\to\infty}\int_t^{t+t^*} G_0(s)ds = \lim_{t\to\infty}\int_0^w\left[\frac{1}{2\pi^{3/2}}N_0(s) - \frac{1}{\sqrt{\pi}}\dot{g}_0(s)\right]2\sqrt{\pi}\left[Erf\frac{\sqrt{t^*+t-s}}{2} - Erf\frac{\sqrt{t-s}}{2}\right]ds.$$

Computing the limit under the integral term, using

$$\lim_{t\to\infty}\left[Erf\frac{\sqrt{t^*+t-s}}{2} - Erf\frac{\sqrt{t-s}}{2}\right] = 0,$$

we obtain our claim.

Claim 2: $N_0(t)$ *is asymptotically* t^**-periodic, with exponential rate:*

$$\mathrm{e}^t\left[N_0(t+t^*)-N_0(t)\right] \to_{t\to\infty} 0.$$

Indeed, we have

$$\mathrm{e}^{\frac{t}{4}}N_0(t) = -\sqrt{\pi}\int_0^t \mathrm{e}^{\frac{s}{4}}\frac{\dot{g}_0(s)}{\sqrt{t-s}}ds.$$

Hence, after rearranging the integral on the right hand side and changing variables, we can write

$$\mathrm{e}^{\frac{t}{4}}N_0(t+t^*)\mathrm{e}^{\frac{t^*}{4}} = -\sqrt{\pi}\int_0^{t^*}\mathrm{e}^{\frac{s}{4}}\frac{\dot{g}_0(s)}{\sqrt{t+t^*-s}}ds - \sqrt{\pi}\int_0^t \mathrm{e}^{\frac{s}{4}}\frac{\dot{g}_0(s+t^*)\mathrm{e}^{\frac{t^*}{4}}}{\sqrt{t-s}}ds$$

$$\Longrightarrow \mathrm{e}^{\frac{t}{4}}\left[N_0(t+t^*)-N_0(t)\right] = -\sqrt{\pi}\left\{\int_0^{t^*} \mathrm{e}^{-\frac{t^*}{4}}\mathrm{e}^{\frac{s}{4}}\frac{\dot{g}_0(s)}{\sqrt{t+t^*-s}}ds + \int_0^t \mathrm{e}^{\frac{s}{4}}\frac{\dot{g}_0(s+t^*)-\dot{g}_0(s)}{\sqrt{t-s}}ds\right\}.$$

The integrand of the second term on the right hand side is zero, while the first integral on the right hand side has fixed integration limits and is of order $t^{-1/2}$. Hence the claim.

Claim 3:

$$\lim_{t\to\infty}\int_0^{t^*}\frac{N_0(s)\mathrm{e}^{-\frac{t+t^*-s}{4}}}{\sqrt{t+t^*-s}}ds = 0.$$

This follows immediately from the decay of the integrand and the fixed limits of integration.

Claim 4:

$$\lim_{t\to\infty}\int_0^{t}\frac{[N_0(s+t^*)-N_0(s)]\mathrm{e}^{-\frac{t-s}{4}}}{\sqrt{t-s}}ds = 0.$$

Since $N_0(t+t^*)-N_0(t)$ tends ot zero exponentially, there exists $T>0$ such that for $s>T$, $|N_0(s+t^*)-N_0(s) < s^{-\alpha}$, for some $\alpha>1$. Hence

$$\int_0^{t}\frac{[N_0(s+t^*)-N_0(s)]\mathrm{e}^{-\frac{t-s}{4}}}{\sqrt{t-s}}ds \le \int_0^{T}\frac{[N_0(s+t^*)-N_0(s)]\mathrm{e}^{-\frac{t-s}{4}}}{\sqrt{t-s}}ds + \int_T^t \frac{s^{-\alpha}}{\sqrt{t-s}}ds$$

Both these integrals can be shown explicitly to have a zero limit as $t\to\infty$, see also [12].

QED

8 Conclusions

We have derived explicit integral formulas for evaluating the Dirichlet to Neumann map for boundary value problems for linear evolution PDEs in time-dependent domain; the shape of the boundary is assumed known and of class $\mathbf{C^1}$, but not necessarily convex. To illustrate our methodology, which is applicable to any linear, constant coefficient evolution PDE, we have solved three specific examples of relevance for mathematical physics and applications. Our approach is based on the global relation, whose importance in the analysis of boundary value problems has been elucidated by one of the authors [7], and used in several contexts, fro both linear and nonlinear problems. Although the use of the analysis of this relation in the complex spectral plane is not necessary to solve the second order problems presented here, such technique is crucial in order to solve the third order case, when two boundary functions need to be determined.

The present work is a substantial extension and simplification of previous results by the same authors.

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Appendix

We give details of how the kernel K_{lr} given by (6.33) can be written in the form (1.20), namely we derive the following identity:

$$K_{lr}(s,t) = \frac{2}{1-\alpha} Re\left[\int_{\partial\Omega_a^r} i\lambda^3 e^{i\lambda^3(t-s)}[e^{i\lambda(l(t)-l(s))} - e^{i\alpha^2\lambda(l(t)-l(s))}]d\lambda\right] =$$

$$= \frac{2}{\sqrt{3}(t-s)^{1/3}} \frac{l(t)-l(s)}{(t-s)} \left[\sum_{m=0}^{\infty} \frac{\left(\frac{l(t)-l(s)}{t-s}\right)^{3m}(t-s)^{2m}}{(3m+1)!}\Gamma\left(\frac{3m+5}{3}\right)\right].$$

First, we exploit the boundedness properties of the exponentials involved in the integrand of the left hand side, and the known asymptotics of the integration contour $\partial\Omega_a^r$, to deform the latter contour to the line $\{\lambda : arg(\lambda) = \frac{\pi}{3}\} \cup \{\lambda : arg(\lambda) = \frac{4\pi}{3}\}$. Then using a rotation of α^2, we find that this integral can be written as an integral along $\mathbb{R}$:

$$K_{lr}(s,t) = \frac{2\alpha^2}{1-\alpha} Re\left[\int_{\mathbb{R}} i\lambda^3 e^{i\lambda^3(t-s)}[e^{i\alpha\lambda(l(t)-l(s))} - e^{i\alpha^2\lambda(l(t)-l(s))}]d\lambda\right].$$

Changing variable in the above integral to $\mu = \lambda(t-s)^{1/3}$, we find (1.19). We now write

$$\begin{aligned} e^{i\alpha\lambda\frac{(l(t)-l(s))}{(t-s)^{1/3}}} - e^{i\alpha^2\lambda\frac{(l(t)-l(s))}{(t-s)^{1/3}}} &= i\lambda\frac{(l(t)-l(s))}{(t-s)^{1/3}}(\alpha-\alpha^2) + \frac{1}{2}(i\lambda\frac{(l(t)-l(s))}{(t-s)^{1/3}})^2(\alpha^2-\alpha) \\ &+ \frac{1}{3!}(i\lambda\frac{(l(t)-l(s))}{(t-s)^{1/3}})^3(\alpha^3-\alpha^6) + ... \end{aligned}$$

Note that all powers that are multiple of 3 vanish, because $\alpha^3 = \alpha^6 = 1$. Hence we can write this as the infinite series

$$\frac{2\alpha^2}{1-\alpha}\frac{e^{i\alpha\lambda\frac{(l(t)-l(s))}{(t-s)^{1/3}}} - e^{i\alpha^2\lambda\frac{(l(t)-l(s))}{(t-s)^{1/3}}}}{t-s} = \frac{2i\lambda}{(t-s)^{1/3}}\frac{l(t)-l(s)}{(t-s)}\sum_{k=0}^{\infty}\frac{\left(i\lambda\frac{l(t)-l(s)}{t-s}(t-s)^{2/3}\right)^k}{(k+1)!}\eta_k, \tag{8.14}$$

where

$$\eta_k = \begin{cases} 1 & k = 0 \mod (3) \\ -1 & k = 1 \mod (3) \\ 0 & k = 2 \mod (3). \end{cases}$$

Thus

$$K_{lr} = \frac{2i}{(t-s)^{1/3}}\frac{l(t)-l(s)}{(t-s)}\sum_{k=0}^{\infty}\frac{\left(\frac{l(t)-l(s)}{t-s}(t-s)^{2/3}\right)^k i^{k+1}}{(k+1)!}\eta_k\int_{\mathbb{R}}\lambda^{k+4}e^{i\lambda^3}d\lambda, \tag{8.15}$$

where the integral is interpreted as a principal value. Since

$$\begin{aligned} \text{pv}\int_{\mathbb{R}}\lambda^{k+4}e^{i\lambda^3}d\lambda &= \frac{e^{i\pi/6}}{3}\Gamma\left(\frac{k+5}{3}\right)\left[e^{i(k+4)\pi/6} - e^{i5(4+k)\pi/6+2i\pi/3}\right] \\ &= \begin{cases} \frac{i^\kappa}{\sqrt{3}}\Gamma\left(\frac{k+5}{3}\right) & k = 0 \mod (3) \\ 0 & k = 1 \mod (3) \end{cases} \end{aligned}$$

with $\kappa+2=m(mod4)$ when $k=3m$. We thus arrive at the expression

$$K_{lr}=\frac{2}{\sqrt{3}(t-s)^{1/3}}\frac{l(t)-l(s)}{(t-s)}\sum_{m=0}^{\infty}\frac{\left(\frac{l(t)-l(s)}{t-s}\right)^{3m}(t-s)^{2m}}{(3m+1)!}\Gamma\left(\frac{3m+5}{3}\right). \tag{8.16}$$

(in the sum, the power of i is $3m+\kappa+2=4m$ (mod 4), so that the contribution of this term is always equal to 1). Hence we find (1.20).

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