Ω results for Beurling's zeta function and lower bounds for the generalised Dirichlet divisor problem


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To link to this article DOI: http://dx.doi.org/10.1016/j.jnt.2009.09.008

Publisher: Elsevier

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Ω-results for Beurling’s zeta function and lower bounds for the generalised Dirichlet divisor problem

Titus W. Hilberdink

Abstract

In this paper we study generalised prime systems for which the integer counting function $N_P(x)$ is asymptotically well-behaved, in the sense that $N_P(x) = \rho x + O(x^\beta)$, where $\rho$ is a positive constant and $\beta < \frac{1}{2}$. For such systems, the associated zeta function $\zeta_P(s)$ is holomorphic for $\Re s > \beta$. We prove that for $\beta < \sigma < \frac{1}{2}$, \[ \int_0^T |\zeta_P(\sigma + it)|^2 dt = \Omega(T^{2-2\sigma-\varepsilon}) \] for any $\varepsilon > 0$, and also for $\varepsilon = 0$ for all such $\sigma$ except possibly one value.

The Dirichlet divisor problem for generalised integers concerns the size of the error term in $N_k P(x) - \text{Res}_{s=1} (\zeta_P(s)^k x^s / s)$, which is $O(x^\theta)$ for some $\theta < 1$. Letting $\alpha_k$ denote the infimum of such $\theta$, we show that $\alpha_k \geq \frac{1}{2} - \frac{1}{2k}$.

Keywords: Beurling’s generalised primes, Dirichlet divisor problem.

1. Introduction
A generalised prime system (or $g$-prime system) $P$ is a sequence of positive reals $p_1, p_2, p_3, \ldots$ satisfying

$$1 < p_1 \leq p_2 \leq \cdots \leq p_n \leq \cdots$$

and for which $p_n \to \infty$ as $n \to \infty$. From these can be formed the system $\mathcal{N}$ of generalised integers or Beurling integers; that is, the numbers of the form

$$p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

where $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \mathbb{N}_0$. Such systems were first introduced by Beurling [2] and have been studied by many authors since then (see in particular [1]). Define the g-integer counting function $N_P(x)$ and the associated Beurling zeta function, respectively, by

$$N_P(x) = \sum_{n \in \mathcal{N}, n \leq x} 1, \quad \zeta_P(s) = \sum_{n \in \mathcal{N}} \frac{1}{n^s}.$$

(Here, $\sum_{n \in \mathcal{N}}$ means a sum over all the g-integers, counting multiplicities.) In this paper, we shall be concerned with g-prime systems for which

$$N_P(x) = \rho x + O(x^\beta), \quad \text{(1.1)}$$

for some $\beta < \frac{1}{2}$ and $\rho > 0$. Then $\zeta_P(s)$ is defined and holomorphic for $\Re s > 1$, and has an analytic continuation to the half-plane $\Re s > \beta$ except for a simple pole at $s = 1$ with residue $\rho$. Furthermore, $\zeta_P(s)$ has finite order for $\Re s > \beta$: i.e. $\zeta_P(\sigma + it) = O(|t|^\lambda)$ for some $\lambda$ for $\sigma > \beta$. Let $\mu_P(\sigma)$ denote the infimum of all such $\lambda$. It is well-known that $\mu_P(\sigma)$ is non-negative,
decreasing, and convex (and hence continuous) (see, for example, [5]). For \( P = \mathbb{P} \) (so that \( \mathcal{N} = \mathbb{N} \)), the Lindelöf Hypothesis is the conjecture that \( \mu_P(\sigma) = \mu_0(\sigma) \) for all \( \sigma \), where

\[
\mu_0(\sigma) = \begin{cases} 
\frac{1}{2} - \sigma & \text{if } \sigma < \frac{1}{2} \\
0 & \text{if } \sigma \geq \frac{1}{2}.
\end{cases}
\]

In [4], it was proven that for all g-prime systems satisfying (1.1), \( \mu_P(\sigma) \) must be at least as large as \( \mu_0(\sigma) \): i.e. \( \mu_P(\sigma) \geq \frac{1}{2} - \sigma \) for \( \sigma \in (\beta, \frac{1}{2}) \). In this paper we prove a stronger result by considering the mean square behaviour of \( \zeta_P(\sigma + it) \). For \( \sigma > \beta \), define \( \nu_P(\sigma) \) to be the infimum of numbers \( \lambda \) such that

\[
\int_1^T |\zeta_P(\sigma + it)|^2 dt = O(T^{1+2\lambda}).
\]

As in the case of \( \mu_P(\sigma) \), \( \nu_P(\sigma) \) is non-negative and convex decreasing (cf. [6], §7.8). Trivially, \( \nu_P(\sigma) \leq \mu_P(\sigma) \). We show here that \( \nu_P(\sigma) \geq \mu_0(\sigma) \). In fact we prove slightly more.

**Theorem 1**

Let \( P \) be a g-prime system for which (1.1) holds for some \( \beta < \frac{1}{2} \) and \( \rho > 0 \). Then \( \nu_P(\sigma) \geq \mu_0(\sigma) \) for \( \sigma \in (\beta, \frac{1}{2}) \). Furthermore,

\[
\int_0^T |\zeta_P(\sigma + it)|^2 dt = O(T^{2-2\sigma})
\]

(1.2)

can hold for at most one value of \( \sigma \) in this range. In this case \( T^{2\sigma - 2} \int_0^T |\zeta_P(\sigma + it)|^2 dt \) is unbounded for all other values of \( \sigma \).

**Remark.** For \( P = \mathbb{P} \), we have \( \nu_P(\sigma) = \mu_0(\sigma) \), which shows the first part of Theorem 1 is best possible. However, in this case we have the asymptotic formula

\[
\int_1^T |\zeta(\sigma + it)|^2 dt \sim \frac{\zeta(2-2\sigma)}{(2\pi)^{1-2\sigma}(2-2\sigma)} T^{2-2\sigma}
\]

for \( 0 < \sigma < \frac{1}{2} \), showing that the exceptional value need not exist. In fact it seems unlikely an exceptional value exists and hence that \( \int_0^T |\zeta_P(\sigma + it)|^2 dt = \Omega(T^{2-2\sigma}) \) for all \( \sigma \in (\beta, \frac{1}{2}) \), but we cannot quite show this. Furthermore it seems plausible that we should have \( \int_0^T |\zeta_P(\sigma + it)|^2 dt \geq C_\sigma T^{2-2\sigma} \) for some \( C_\sigma > 0 \).

**2. Dirichlet divisor problems for g-primes**

For a g-prime system satisfying (1.1) (with \( \beta < 1 \)), we can study the equivalent of the Dirichlet divisor problem concerning the error term in the asymptotic formula for the average of the ‘generalised divisor’ function. For \( k \in \mathbb{N} \), let \( kP \) denote the g-prime system obtained from \( P \) by letting every g-prime from \( P \) be counted \( k \) times. (If an original g-prime has multiplicity \( m \), then in the new system it will have multiplicity \( km \).) The Beurling zeta function of \( kP \) is

\[
\zeta_{kP}(s) = \zeta_P(s)^k.
\]

By standard methods using Perron’s formula,

\[
N_{kP}(x) = \text{Res}_{s=1} \left\{ \frac{\zeta_P(s)^k}{s} x^s \right\} + \Delta_{P,k}(x) = xP_{k-1}(\log x) + \Delta_{P,k}(x),
\]

2
where \( P_{k-1}(\cdot) \) is a polynomial of degree \( k - 1 \) and \( \Delta_{P,k}(x) = O(x^\theta) \) for some \( \theta < 1 \), depending on \( k \). Let \( \alpha_k \) denote the infimum of such \( \theta \). The \textit{generalised Dirichlet divisor problem} is the problem of determining \( \alpha_k \). Also let \( \beta_k \) denote the infimum of \( \phi \) for which

\[
\int_0^x \Delta_{P,k}(y)^2 \, dy = O(x^{1+2\phi}).
\]

Trivially, \( \beta_k \leq \alpha_k \).

For \( P \), it is known that

\[
\alpha_k \geq \beta_k \geq \frac{1}{2} - \frac{1}{2k}
\]

and it is conjectured that there is equality throughout (actually \( \beta_k = \frac{1}{2} - \frac{1}{2k} \) for all \( k \) is equivalent to the Lindelöf Hypothesis — see [6], Theorem 13.4). We use Theorem 1 to show that (2.1) remains true for \( P \) satisfying (1.1). In fact we have the following two corollaries:

**Corollary 2**

Let \( P \) satisfy (1.1) for some \( \beta < \frac{1}{2} \). Then for \( \sigma \in (\beta, \frac{1}{2} - \frac{1}{2k}) \),

\[
\int_{-\infty}^{\infty} \left| \zeta_P(\sigma + it) \right|^{2k} \left| \sigma + it \right| \, dt
\]

diverges. Further, if \( \frac{1}{2} - \frac{1}{2k} \) is not the exceptional value in (1.2), then the integral also diverges for \( \sigma = \frac{1}{2} - \frac{1}{2k} \).

**Corollary 3**

Let \( P \) satisfy (1.1) for some \( \beta < \frac{1}{2} \). With \( \alpha_k \) and \( \beta_k \) as above, \( \alpha_k \geq \beta_k \geq \max\{\beta, \frac{1}{2} - \frac{1}{2k}\} \).

**3. Proofs**

**Proof of Theorem 1.** If \( \nu_P(\sigma') < \frac{1}{2} - \sigma' \) for some \( \sigma' \in (\beta, \frac{1}{2}) \) then, by continuity of \( \nu_P(\cdot) \), \( \nu_P(\sigma) < \frac{1}{2} - \sigma \) throughout some interval around \( \sigma' \) and (1.2) holds for all such \( \sigma \); in particular for two such values. We shall show that this is impossible.

Suppose, for a contradiction, that (1.2) holds for \( \sigma = \sigma_0, \sigma_1 \) where \( \beta < \sigma_0 < \sigma_1 < \frac{1}{2} \).

For \( N \geq 1 \) let \( \zeta_{N,P}(s) = \sum_{n \leq N} n^{-s} \), where the sum ranges over \( n \in \mathbb{N} \). As was stated in [4] (and shown in [3]), for \( \sigma < \frac{1}{2} \) there exist constants \( c_1, c_2 > 0 \) such that for \( R \geq c_1 N \),

\[
\sum_{r=1}^{R} \int_0^{2r-1} |\zeta_{N,P}(\sigma + it)|^2 \, dt \geq c_2 R^2 N^{1-2\sigma}.
\]

(3.1)

Also, writing \( s = \sigma + it \), and following the arguments in [3], we have

\[
\zeta_{N,P}(s) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta_P(s + w)N^w}{w} \, dw + O\left(\frac{N^c}{T(c + \sigma - 1)}\right) + O\left(\frac{N^1-\sigma}{T}\right) \sum_{\frac{N}{2} < n < 2N} \sum_{\alpha \in \mathbb{N}} \frac{1}{|n - N|},
\]

for \( |t| < T, c > 1 - \sigma \) and \( N \notin \mathbb{N} \). We shall put \( c = 1 - \sigma + \frac{1}{\log N} \) and choose \( N \) in such a way that \( (N - \alpha, N + \alpha) \cap \mathbb{N} = \emptyset \). (As was shown in [4], this is possible for arbitrarily large \( N \) if

\[
3
\]
0 < \alpha < \frac{1}{2\pi}. With this choice of \( N \), the final sum in (3.2) was shown to be \( O(\sqrt{N}) \). As such (3.2) becomes

\[
\zeta_N,\rho(s) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta_p(s + w)N^w}{w} \, dw + O\left(\frac{N^{\frac{1}{2} - \sigma}}{T}\right). \tag{3.3}
\]

Now put \( \sigma = \sigma_1 \) and push the contour in the integral to the left as far as \( \Re w = \sigma_0 - \sigma_1 < 0 \), picking up the residues at \( w = 0 \) and \( w = 1 - \sigma \) (since \( |t| < T \)).

The contribution along the horizontal line \([\sigma_0 - \sigma_1 + iT, c + iT]\) is, in modulus, less than

\[
\frac{1}{2\pi i} \int_{\sigma_0 - \sigma_1}^{c} N^y |\zeta_p(\sigma_1 + y + i(t + T))| \, dy.
\]

Using the uniform bound \( |\zeta_p(\sigma + it)| = O(t^{\frac{1}{2} - \beta + \varepsilon}) \), this is at most a constant times

\[
\frac{1}{T} \int_{\sigma_0 - \sigma_1}^{\sigma_0 - \sigma_1 + \frac{1}{2\pi}N} T^y N^y \, dy + \frac{1}{T} \int_{1 - \sigma}^{\sigma_0 - \sigma_1 + \frac{1}{2\pi}N} T^y N^y \, dy = O\left(T^{\frac{1}{2} - \sigma + \varepsilon} N^{\sigma_0 - \sigma_1}\right) + O\left(T^{\varepsilon - 1} N^{1 - \sigma_1}\right).
\tag{3.4}
\]

Similarly on \([\sigma_0 - \sigma_1 - iT, c - iT]\).

The integral along \( \Re w = \sigma_0 - \sigma_1 \) is at most

\[
\frac{N^{\sigma_0 - \sigma_1}}{2\pi} \int_{-T}^{T} \frac{|\zeta_p(\sigma_0 + i(t + y))|}{\sqrt{(\sigma_0 - \sigma_1)^2 + y^2}} \, dy = O\left(N^{\sigma_0 - \sigma_1} \int_{1}^{2T} \frac{|\zeta_p(\sigma_0 + iy)|}{y} \, dy\right)
\]

\[
= o\left(N^{\sigma_0 - \sigma_1} T^{\frac{1}{2} - \sigma_0}\right), \tag{3.5}
\]

using\(^3\) the hypothetical bound \(\int_{0}^{T} |\zeta_p(\sigma_0 + it)|^2 \, dt = o(T^{2 - 2\sigma_0})\).

The residues at \( w = 0 \) and \( w = 1 - \sigma \) are, respectively, \( \zeta_p(s) \) and \( \rho N^{1-s}/(1-s) = O\left(N^{1-\sigma_1}\right) \).

Putting (3.3), (3.4), and (3.5) together gives

\[
\zeta_N,\rho(\sigma_1 + it) = \zeta_p(\sigma_1 + it) + O\left(\frac{N^{1-\sigma_1}}{|t| + 1}\right) + O\left(N^{\frac{1}{2} - \sigma_1} T^{\varepsilon - 1}\right) + o\left(N^{\sigma_0 - \sigma_1} T^{\frac{1}{2} - \sigma_0}\right) + O\left(\frac{N^{\frac{1}{2} - \sigma_1}}{T}\right),
\]

for \( |t| < T \). (Note that the first \( O \)-term in (3.4) is superfluous since \( \frac{\sigma_0 - \sigma_1}{1-\sigma} < \frac{1}{2} - \sigma_0 \).) Hence, using \((a + b + c + d + e)^2 \leq 5(a^2 + b^2 + c^2 + d^2 + e^2)\), we have

\[
|\zeta_N,\rho(\sigma_1 + it)|^2 \leq 5|\zeta_p(\sigma_1 + it)|^2 + O\left(\frac{N^{2-2\sigma_1}}{t^2 + 1}\right) + O\left(N^{2-2\sigma_1} T^{2\varepsilon - 2}\right) + o\left(N^{2\sigma_0 - 2\sigma_1} T^{1-2\sigma_0}\right) + O\left(\frac{N^{3-2\sigma_1}}{T^2}\right).
\]

Now apply \( \sum_{r=1}^{R} \int_{0}^{2r-1} \ldots dt \) to both sides to give (for \( 2R - 1 < T \))

\[
\sum_{r=1}^{R} \int_{0}^{2r-1} |\zeta_N,\rho(s_1 + it)|^2 \, dt = O\left(\sum_{r=1}^{R} \int_{0}^{2r-1} |\zeta_p(s_1 + it)|^2 \, dt\right) + o\left(\sum_{r=1}^{R} \int_{0}^{2r-1} N^{2-2\sigma_1} (t + 1)^2 \, dt\right)
\]

\[
+ O\left(R^2 N^{2-2\sigma_1} T^{2\varepsilon - 2}\right) + O\left(\frac{R^2 N^{3-2\sigma_1}}{T^2}\right) + o\left(R^2 N^{2(\sigma_0 - \sigma_1)} T^{1-2\sigma_0}\right)
\]

\[
= o\left(R^{3-2\sigma_1}\right) + O\left(R N^{2-2\sigma_1}\right) + O\left(R^2 N^{2-2\sigma_1} T^{2\varepsilon - 2}\right) + O\left(\frac{R^2 N^{3-2\sigma_1}}{T^2}\right) + o\left(R^2 N^{2(\sigma_0 - \sigma_1)} T^{1-2\sigma_0}\right)
\]

\(^3\)If \( f \geq 0 \) and \( \int_{0}^{T} f^2 \, dt = o(T^{\lambda}) \) (some \( \lambda > 1 \)), then \( \int_{T/2}^{T} \frac{f(y)}{y} \, dy \leq \frac{T}{2} \int_{0}^{T} f \leq \frac{3}{2} \sqrt{T} \int_{0}^{T} f^2 = o(T^{\lambda - 1}) \), and \( \int_{T/2}^{T} \frac{|f(y)|}{y} \, dy = o(T^{\lambda - 2}) \) follows.
using (1.2) for $\sigma_1$. Let $T = 2R$. The left-hand side above is at least $c_2 R^2 N^{1-2\sigma_1}$ by (3.1) if $R \geq c_1 N$. Dividing both sides through by $R^2 N^{1-2\sigma_1}$ gives

$$c_2 \leq o\left(\left(\frac{R}{N}\right)^{1-2\sigma_1}\right) + O\left(\frac{N}{R}\right) + O\left(NR^{2\sigma-2}\right) + O\left(\frac{N^2}{R^2}\right) + o\left(\frac{R}{N}\right)^{1-2\sigma_0}. \tag{3.6}$$

Put $R = KN$ where $K \geq c_1$ is a fixed, but arbitrary, constant. Letting $N \to \infty$, the $o$-terms both tend to zero as does the middle $O$-term. Hence

$$c_2 \leq \frac{A}{K} + \frac{B}{K^2}$$

for some absolute constants $A, B$. But $K$ can be made arbitrarily large, so this gives a contradiction.

For the final part, suppose (1.2) holds for $\sigma = \sigma_0$ say. If $\int_0^T |\zeta_p(\sigma^\prime + it)|^2 \, dt = O(T^{2-2\sigma^\prime})$ for some $\sigma^\prime \in (\beta, \frac{1}{2})$ with $\sigma^\prime \neq \sigma_0$, then (1.2) actually holds for all $\sigma$ between $\sigma_0$ and $\sigma^\prime$. (This follows from the Phragmen-Lindelöf Theorem for a strip (see [6], §7.8, with $\varepsilon$ in the place of $C$)). This was shown to be impossible, and hence $T^{2\sigma-2} \int_0^T |\zeta_p(\sigma^\prime + it)|^2 \, dt$ must be unbounded for all $\sigma \neq \sigma_0$.

Now we apply Theorem 1 to find lower bounds in the Dirichlet divisor problem. Note that Theorem 1 actually shows that given $\varepsilon > 0$,

$$\int_{T/2}^T |\zeta_p(\sigma + it)|^2 \, dt = \Omega(T^{2-2\sigma-\varepsilon}),$$

for if it was $o(T^{2-2\sigma-\varepsilon})$, then by telescoping it would follow that $\int_0^T |\zeta_p(\sigma + it)|^2 \, dt = o(T^{2-2\sigma-\varepsilon})$ which is false.

**Proofs of Corollaries 2 and 3.** By Hölder’s inequality,

$$\int_{T/2}^T |\zeta_p(\sigma + it)|^{2k} \, dt \geq \frac{2^{k-1}}{T^{k-1}} \left(\int_{T/2}^T |\zeta_p(\sigma + it)|^2 \, dt\right)^k,$$

for every $k \in \mathbb{N}$. By Theorem 1, given $\varepsilon > 0$, $\int_{T/2}^T |\zeta_p(\sigma + it)|^2 \, dt \geq aT^{2-2\sigma-\varepsilon}$ for some $a > 0$ and some arbitrarily large $T$. Hence for such $T$,

$$\int_{T/2}^T |\zeta_p(\sigma + it)|^{2k} \, dt \geq a^k T^{k(1-2\sigma)+1-\varepsilon k}.$$

It follows that

$$\int_{T/2}^T \frac{|\zeta_p(\sigma^\prime + it)|^{2k}}{|\sigma + it|^2} \, dt \geq a^\prime T^{k(1-2\sigma)-1-\varepsilon k}$$

for some $\sigma^\prime > 0$. But for $\sigma < \frac{1}{2} - \frac{1}{2K}$, we have $k(1-2\sigma) - 1 > 0$. Hence for $\varepsilon$ sufficiently small, $k(1-2\sigma) - 1 - \varepsilon k > 0$ also, and so $\int_{T/2}^T \frac{|\zeta_p(\sigma + it)|^{2k}}{|\sigma + it|^2} \, dt \neq 0$ as $T \to \infty$, and Corollary 2 follows. Of course, if $\frac{1}{2} - \frac{1}{2K}$ is not the exceptional value in Theorem 1, then we can take $\varepsilon = 0$ in the above and the result also holds for $\sigma = \frac{1}{2} - \frac{1}{2K}$.
Let $\gamma_k$ be the infimum of $\sigma$ (with $\sigma > \beta$) for which $\int_{-\infty}^{\infty} |\zeta_P(\sigma + it)|^{2k} dt$ converges. By Corollary 2, $\gamma_k \geq \frac{1}{2} - \frac{1}{2K}$. An identical argument as in the $P = \mathbb{P}$ case (see [6], Theorem 12.5) shows that $\gamma_k = \beta_k$. (The argument is simply based upon Parseval’s formula for Mellin transforms, which in this case is the identity

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\zeta_P(\sigma + it)|^{2k} dt = \int_{0}^{\infty} \frac{\Delta_{P,k}(x)^2}{x^{1+2\sigma}} dx$$

for $\sigma$ in some interval $(\theta, 1)$ with $\theta < 1$.) Hence $\beta_k \geq \frac{1}{2} - \frac{1}{2K}$. \hfill \Box

4. On the line $\sigma = \frac{1}{2}$
In this article, we have considered the mean-value along vertical lines $Re s = \sigma$ with $\sigma < \frac{1}{2}$. This raises the question of what happens on the line $\sigma = \frac{1}{2}$. For $P = \mathbb{P}$, we have $\int_{0}^{T} |\zeta_P(\frac{1}{2} + it)|^{2} dt \sim T \log T$, so do we have $\int_{0}^{T} |\zeta_P(\frac{1}{2} + it)|^{2} dt = \Omega(T \log T)$ in general? As in the $\sigma < \frac{1}{2}$ case, we relate the behaviour of the mean-square value at $\sigma = \frac{1}{2}$ to the behaviour of the mean-square for some $\sigma = \sigma_0 < \frac{1}{2}$.

Theorem 4
Let $P$ be a g-prime system for which (1.1) holds. If $\int_{1}^{T} |\zeta_P(\sigma + it)| dt = o((T \log T)^{1-\tau})$ for some $\sigma \in (\beta, \frac{1}{2})$, then $\int_{0}^{T} |\zeta_P(\frac{1}{2} + it)|^{2} dt = \Omega(T \log T)$.

Note that the assumption is implied by $\int_{1}^{T} |\zeta_P(\sigma + it)|^{2} dt = o(T^{1-2\sigma}(\log T)^{1-2\sigma})$.

Sketch of Proof. We follow the proof of Theorem 1 as much as possible, this time taking $\sigma_1 = \frac{1}{2}$.

Using the argument in [3] for $\sigma = \frac{1}{2}$, (3.1) becomes: there exist constants $c_1, c_2 > 0$ such that for $R \geq c_1 N / \log N$,

$$\sum_{r=1}^{R} \int_{0}^{2r-1} |\zeta_{N,P}(\frac{1}{2} + it)|^{2} dt \geq c_2 R^2 \log N. \tag{4.1}$$

To see this, note that we have

$$\int_{0}^{T} |\zeta_{N,P}(\frac{1}{2} + it)|^{2} dt \geq T \sum_{n \leq N} \frac{1}{n} + 2 \sum_{n \leq N} \frac{1}{\sqrt{n}} \sum_{m \leq n} S_{m,n}(T) / \sqrt{m},$$

where $S_{m,n}(T) = \frac{\sin(T \log(n/m))}{\log(n/m)}$. (Here $m, n \in \mathbb{N}$ and the $*$ indicates that any multiplicities must be squared.) In any case, we have $\sum_{n \leq N} \frac{1}{n} \geq \sum_{n \leq N} \frac{1}{n} \geq k_1 \log N$ for some $k_1 > 0$.\footnote{This follows readily from $N_{\mathbb{P}}(x) \sim \rho x$.} For $m \leq \frac{n}{2}$, $|S_{m,n}(T)| \leq 1 / \log 2$, so this part of the double sum is $O(\sum_{n \leq N} \frac{1}{\sqrt{n}} \sum_{m \leq n/2} \frac{1}{\sqrt{m}}) = O(N)$. Thus, for some positive constants $k_1, k_2$, independent of $T$ and $N$,

$$\int_{0}^{T} |\zeta_{N,P}(\frac{1}{2} + it)|^{2} dt \geq k_1 T \log N + 2 \sum_{n \leq N} \frac{1}{\sqrt{n}} \sum_{\frac{n}{2} < m < n} S_{m,n}(T) / \sqrt{m} - k_2 N.$$

Putting $T = 2r - 1$ for $r = 1, 2, \ldots, R$, and summing both sides gives, on noticing that

$$\sum_{r=1}^{R} \int_{0}^{2r-1} |\zeta_{N,P}(\frac{1}{2} + it)|^{2} dt \geq k_1 R^2 \log N - k_2 R N,$$
and (4.1) follows.

In (3.2), we need a better estimate for the final sum. Let $M \in \mathbb{N}$. Then, with $N$ such that $(N - \alpha, N + \alpha) \cap \mathcal{N} = \emptyset$,

$$\sum_{\substack{\mathcal{N} \ni n \leq 2N \setminus \mathcal{N} \ni n \leq 2N}} \frac{1}{|n-N|} = \sum_{m=1}^{M} \sum_{\alpha N^{m-1} \leq |n-N| < \alpha N^{m}} \frac{1}{|n-N|} + O(1)$$

$$\leq \frac{1}{\alpha} \sum_{m=1}^{M} \frac{1}{N^{m-1}} \left( N(N + \alpha N^{m/M}) - N(N - \alpha N^{m/M}) \right) + O(1)$$

$$= O(N^{1/M}) + O(N^{\beta}),$$

using (1.1). Since $M$ is arbitrary, this is $O(N^{\beta+\epsilon})$ for every $\epsilon > 0$ in any case. Thus (3.3) becomes

$$\zeta_{N,P}(s) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta_P(s+w)N^w}{w} dw + O\left(\frac{N^{1+\beta+\epsilon}}{T}\right).$$

The analysis up to (3.5) remains the same (with $\sigma_0 = \sigma$ and $\sigma_1 = \frac{1}{2}$) but in (3.5) we use the bound assumed in the statement to give $o(N^{\sigma-\frac{1}{2}}(T \log T)^{\frac{1}{2}-\sigma})$. The arguments following (3.5) remain valid and we put $T = 2R$ again, but this time we divide through by $R^2 \log N$. On assuming $\int_0^T |\zeta_P(\frac{1}{2} + it)|^2 dt = o(T \log T)$, (3.6) now becomes

$$c_2 \leq o\left(\frac{\log R}{\log N}\right) + O\left(\frac{N}{R \log N}\right) + O\left(\frac{NR^{2\epsilon-2}}{\log N}\right) + O\left(\frac{N^{1+2\beta+2\epsilon}}{R^2}\right) + o\left(\frac{(R \log R)^{1-2\sigma}}{N \log N}\right).$$

Put $R = KN/\log N$ where $K \geq c_1$ is a fixed, but arbitrary, constant. Letting $N \to \infty$, all the terms tend to zero except the first $O$-term. Hence

$$c_2 \leq \frac{A}{K}$$

for some absolute constant $A$. As $K$ can be made arbitrarily large, this gives a contradiction. Hence $\int_0^T |\zeta_P(\frac{1}{2} + it)|^2 dt = \Omega(T \log T)$.

□

References


