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# Generalised prime systems with periodic integer counting function<sup>1</sup>

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## Abstract

We study generalised prime systems (both discrete and continuous) for which the ‘integer counting function’  $N(x)$  has the property that  $N(x) - cx$  is periodic for some  $c > 0$ . We show that this is extremely rare. In particular, we show that the only such system for which  $N$  is continuous is the trivial system with  $N(x) - cx$  constant, while if  $N$  has finitely many discontinuities per bounded interval, then  $N$  must be the counting function of the g-prime system containing the usual primes except for finitely many.

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## Introduction

In a recent paper [7], we discussed Mellin transforms  $\hat{N}(s)$  of integrators  $N$  for which  $N(x) - x$  is periodic in order to study flows of holomorphic functions converging to  $\zeta(s)$ . Here we consider the question when such an  $N$  determines a g-prime system; i.e. that  $N(x)$  is the ‘integer counting function’ of a generalised prime system — see section 1.3 for the definition.

An example of such a flow  $\hat{N}_\lambda(s)$  was given (in [7]) but it was unclear whether or not they determined g-prime systems. As a consequence of our results, we show that none of them does.

In fact, we investigate more generally when an increasing function  $N$  for which  $N(x) - cx$  is periodic determines a g-prime system for a constant  $c > 0$ . (At the outset we assume that  $N$  is right-continuous,  $N(1) = 1$ , and  $N(x) = 0$  for  $x < 1$ .) For example,  $N(x) = cx + 1 - c$  for  $x \geq 1$  determines a continuous g-prime system for  $0 < c \leq 2$  at least.

As for discontinuous examples, we have the prototype  $N(x) = [x]$  for the usual primes and integers. For other examples, consider the g-prime system containing the usual primes except given primes  $p_1, \dots, p_k$ . This has integer counting function

$$N(x) = \sum_{\substack{n \leq x \\ (n, P) = 1}} \left[ \frac{x-n}{P} + 1 \right],$$

where  $P = p_1 p_2 \dots p_k$ . In this case  $N(x+P) = N(x) + \varphi(P)$  where  $\varphi$  is Euler’s function, and  $N(x) - \frac{\varphi(P)}{P}x$  has period  $P$ .

Our results split quite naturally into continuous and discontinuous cases. In section 2, where we consider the continuous case, the main result is that for  $N$  sufficiently ‘nice’ (eg. continuously differentiable),  $N$  determines a g-prime system only for the trivial case where  $N(x) - cx$  is constant; i.e.  $N(x) = cx + 1 - c$ .

For discontinuous  $N$  the picture is less straightforward. A useful tool is to consider its ‘jump’ function  $N_J$ , which must necessarily also have  $N_J(x) - c'x$  periodic (for some  $c' > 0$ ) and which also determines a g-prime system if  $N$  does (Theorem 1.1). We show that if such an  $N$  has only finitely many discontinuities in any interval but is otherwise ‘smooth’, then  $N$  must be a step function, the discontinuities must occur at *integer* points and the period, say  $P$ , must be a natural number. Then, denoting the jump at  $n$  by  $a_n$ , we show that  $a_n$  is even<sup>2</sup> (mod  $P$ ) and multiplicative. This allows us to deduce our main result.

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<sup>1</sup>To appear in *Acta Arithmetica*.

<sup>2</sup>That is;  $a_n = a_{(n, P)}$ .

**Theorem A**

Let  $N \in T$  be such that  $N(x) - cx$  has period  $P$ , and suppose that  $N$  determines a  $g$ -prime system. Then  $P \in \mathbb{N}$  and

$$N(x) = \sum_{\substack{n \leq P \\ (n, P) = 1}} \left[ \frac{x-n}{P} + 1 \right].$$

i.e.  $N$  is the integer-counting function of the  $g$ -prime system  $\mathbb{P} \setminus \{p_1, \dots, p_k\}$  where  $p_1, \dots, p_k$  are the prime divisors of  $P$ .

(For the definition of  $T$ , see section 1.2.) This actually shows that the smallest period must be squarefree and that  $c = \frac{\varphi(P)}{P}$ . Our set up includes all the usual ‘discrete’  $g$ -prime systems.

In proving Theorem A, we prove the following result on Dirichlet series with periodic coefficients, which may be of independent interest.

**Theorem B**

Let  $\{a_n\}_{n \in \mathbb{N}}$  be periodic,  $a_1 = 1$ , and suppose  $a_n = \exp_* b_n$  for some  $b_n \geq 0$ . Then  $a_n$  is multiplicative.

Here  $*$  refers to Dirichlet convolution. Thus  $a_n$  and  $b_n$  are related by  $\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \exp\{\sum_{n=1}^{\infty} \frac{b_n}{n^s}\}$ .

**§1. Preliminaries**

**1.1 Riemann-Stieltjes convolution**

Let  $S$  denote the space of functions  $f : \mathbb{R} \rightarrow \mathbb{C}$  which are zero on  $(-\infty, 1)$ , right-continuous, and of local bounded variation. (See e.g. [3], pp.50-70.) This is a vector space over addition. Let  $S^+$  denote the subspace of  $S$  consisting of increasing functions. Also, for  $\alpha \in \mathbb{R}$ , let  $S_\alpha = \{f \in S : f(1) = \alpha\}$ , while  $S_\alpha^+ = S^+ \cap S_\alpha$ .

For functions  $f, g \in S$ , define the *convolution* (or *Mellin-Stieltjes convolution*) by<sup>3</sup>

$$(f * g)(x) = \int_{1-}^x f\left(\frac{x}{t}\right) dg(t).$$

We note that  $S$  is closed under  $*$  and that  $*$  is commutative and associative. The identity (w.r.t.  $*$ ) is  $i(x) = 1$  for  $x \geq 1$  and zero otherwise.

- (a) If  $f$  or  $g$  is continuous (on  $\mathbb{R}$ ), then  $f * g$  is continuous.
- (b) *Exponentials.* For  $f \in S_1$ , there exists  $g \in S_0$  such that  $f = \exp_* g$ ; i.e.

$$f = \sum_{n=0}^{\infty} \frac{g^{*n}}{n!},$$

where  $g^{*n} = g * g^{*(n-1)}$  and  $g^{*0} = i$ . Also  $f = \exp_* g$  if and only if  $f * g_L = f_L$  (see [5]), where  $f_L \in S$  is the function defined for  $x \geq 1$  by  $f_L(x) = \int_1^x \log t df(t)$ .

- (c) For  $f \in S$ , define the *Mellin transform* of  $f$  by  $\hat{f}(s) = \int_{1-}^{\infty} x^{-s} df(x)$ . This exists if  $f(x) = O(x^A)$  for some  $A$ . Note that  $\widehat{f * g} = \hat{f} \hat{g}$  and  $\widehat{\exp_* f} = \exp \hat{f}$ .
- (d) Let  $f, g \in S$  be continuously differentiable on  $(1, \infty)$ . Let  $g_1(x) = \int_{1-}^x \frac{1}{t} dg(t)$ . Then  $f * g$  is also continuously differentiable on  $(1, \infty)$  with

$$(f * g)' = f' * g_1 + f(1)g'.$$

*Proof.* Let  $x > 1$  and consider  $(f * g)(x+h) - (f * g)(x)$  for  $h$  small. Consider  $h > 0$  first. We have

$$\frac{(f * g)(x+h) - (f * g)(x)}{h} = \int_{1-}^x \frac{f\left(\frac{x+h}{t}\right) - f\left(\frac{x}{t}\right)}{h} dg(t) + \frac{1}{h} \int_x^{x+h} f\left(\frac{x+h}{t}\right) dg(t). \quad (1.1)$$

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<sup>3</sup>All limits of integration are understood to be + (i.e. from the right) except where they are explicitly stated to be -.

The integrand in the first integral tends pointwise to  $\frac{1}{t}f'(\frac{x}{t})$ , so by the continuity of  $f'$  this integral tends to (see [1], p.218)

$$\int_{1-}^x \frac{f'(\frac{x}{t})}{t} dg(t) = (f' * g_1)(x) \quad \text{as } h \rightarrow 0.$$

The second term equals

$$f(1) \frac{g(x+h) - g(x)}{h} + \frac{1}{h} \int_x^{x+h} \left( f\left(\frac{x+h}{t}\right) - f(1) \right) dg(t).$$

The first term tends to  $f(1)g'(x)$  while the integrand tends to 0 by right-continuity of  $f$  at 1. Hence so does the integral.

If  $h < 0$ , write  $h = -k$  and split up as  $\frac{1}{k} \int_1^{x-k}$  and  $\frac{1}{k} \int_{x-k}^x$  and argue as before. □

For the proofs of (a)-(c) see [3] and [5].

## 1.2 The 'jump' function

**Definition 1.1:** (i) For  $f \in S$  and each  $x \in \mathbb{R}$ , we denote by  $\Delta f(x)$  the left-hand jump of  $f$  at  $x$ ; i.e.

$$\Delta f(x) = f(x) - f(x-) = \lim_{h \rightarrow 0^+} (f(x) - f(x-h)).$$

This is well-defined for monotone  $f$  and hence for  $f \in S$ . Note also that  $\Delta f$  is non-zero on a countable set only ([1], p.162).

(ii) For  $f \in S^+$ , let  $f_J$  denote the *jump function* of  $f$ ; i.e.

$$f_J(x) = \sum_{x_r \leq x} \Delta f(x_r),$$

where the  $x_r$  denote the discontinuities of  $f$ .

The function  $f_J$  is increasing and  $f = f_J + f_C$ , where  $f_C$  is continuous and increasing ([1], p.186).

Let  $\delta_a$  denote the function which is 1 on  $[a, \infty)$  and zero otherwise. Note that  $\delta_a * \delta_b = \delta_{ab}$ . Letting  $D_f$  denote the (countable) set of discontinuities of  $f$ , we may write

$$f_J = \sum_{\alpha \in D_f} \Delta f(\alpha) \delta_\alpha. \tag{1.2}$$

The series has only non-negative terms and converges absolutely.

**Properties.** Let  $f, g \in S^+$ .

(a)  $(f * g)_J = f_J * g_J$ .

Write  $f = f_J + f_C$  and similarly for  $g$ . Then

$$f * g = (f_J + f_C) * (g_J + g_C) = f_J * g_J + f_J * g_C + f_C * g_J + f_C * g_C. \tag{1.3}$$

The last three terms are all continuous, and so their jump functions are identically zero. Therefore we need to show  $(f_J * g_J)_J = f_J * g_J$ .

To see this, use (1.2) for  $f_J$  and  $g_J$ . Hence

$$f_J * g_J = \sum_{\alpha \in D_f} \sum_{\beta \in D_g} \Delta f(\alpha) \Delta g(\beta) \delta_\alpha * \delta_\beta = \sum_{\alpha \in D_f} \sum_{\beta \in D_g} \Delta f(\alpha) \Delta g(\beta) \delta_{\alpha\beta},$$

which is a sum of the form  $\sum_\gamma c_\gamma \delta_\gamma$ ; i.e. a jump function. Thus  $(f_J * g_J)_J = f_J * g_J$  as required.

(b) For  $x \geq 1$ , we have

$$\Delta(f * g)(x) = \sum_{\substack{\alpha\beta = x \\ \alpha \in D_f, \beta \in D_g}} \Delta f(\alpha)\Delta g(\beta). \quad (1.4)$$

Take  $\Delta$  of both sides of (1.3). As the last three terms are all continuous,  $\Delta = 0$  for these functions. For the remaining term

$$\Delta(f_J * g_J)(x) = \sum_{\alpha \in D_f, \beta \in D_g} \Delta f(\alpha)\Delta g(\beta)\Delta\delta_{\alpha\beta}(x) = \sum_{\substack{\alpha\beta = x \\ \alpha \in D_f, \beta \in D_g}} \Delta f(\alpha)\Delta g(\beta),$$

since  $\Delta\delta_a(x) = 1$  for  $x = a$  and zero otherwise.

(c)  $D_{f*g} = D_f D_g = \{\alpha\beta : \alpha \in D_f, \beta \in D_g\}$ .

If  $x \notin D_f D_g$  (i.e.  $x \neq \alpha\beta$  for any  $\alpha \in D_f$  and  $\beta \in D_g$ ), then there is no contribution to the sum in (1.4). Hence  $\Delta(f * g)(x) = 0$  and  $x \notin D_{f*g}$ . Thus  $D_{f*g} \subset D_f D_g$ .

For the converse, if  $x \in D_f D_g$  then  $x = \alpha\beta$  for some  $\alpha \in D_f$  and  $\beta \in D_g$ , so that

$$\Delta(f * g)(x) = \Delta(f * g)(\alpha\beta) \geq \Delta f(\alpha)\Delta g(\beta) > 0,$$

as all the other terms in (1.4) are non-negative. Hence  $x \in D_{f*g}$  and  $D_{f*g} = D_f D_g$  follows.

(d) For  $f \in S$ , let  $f_L$  denote the function  $f_L(x) = \int_1^x \log t df(t)$ . Then  $\Delta f_L(x) = \Delta f(x) \log x$  (see [3], p.341) and hence  $(f_J)_L = (f_L)_J$ . (Both sides equal  $\sum_{\alpha \in D_f} \Delta f(\alpha) \log \alpha \delta_\alpha$ .)

### The subspace $T$

Consider those functions in  $S$  whose right-hand derivative exists and is continuous in  $(1, \infty)$ ; i.e.

$$f'_+(x) = \lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h}$$

exists for each  $x > 1$  and  $f'_+$  is continuous here. Let  $T$  denote the subspace of such functions which have a finite number of discontinuities per bounded interval. For example, all step functions in  $S$  lie in  $T$  with  $f'_+ \equiv 0$ . Further for  $f \in T$ ,  $f'_+ \equiv 0$  if and only if  $f$  is a step function. This follows from the fact that if  $f$  is continuous on an interval, and  $f$  has a continuous one sided derivative, then in fact  $f'$  exists (and of course equals the one-sided derivative) – see [9], p.355. Thus on each interval where  $f$  is continuous and  $f'_+ \equiv 0$ , we must have  $f' \equiv 0$  so that  $f$  is constant here.

Part (d) of 1.1 generalises to functions in  $T$ : if  $f, g \in T$  then  $f * g \in T$  and

$$(f * g)'_+ = f'_+ * g_1 + f_{J,1} * g'_+,$$

where  $g_1$  is as before and  $f_{J,1} = (f_J)_1$ .

*Proof.* By 1.2(c),  $D_{f*g} \subset D_f D_g$ , so  $f * g$  has at most finitely many discontinuities per bounded interval.

We have, on  $(1, \infty)$ ,

$$(f * g)'_+ = (f_J * g_J)'_+ + (f_J * g_C)'_+ + (f_C * g_J)'_+ + (f_C * g_C)'_+.$$

Now  $f_J * g_J$  is again a step function, so  $(f_J * g_J)'_+ = 0$ . Also,  $f'_+ = (f_C)'_+$  so  $f_C$  is continuously differentiable and similarly for  $g_C$ . By 1.1(d),  $(f_C * g_C)'_+ = f'_C * g_{C,1}$ . For the remaining terms

$$(f_J * g_C)'_+(x) = \left( \sum_{\alpha \in D_f} \Delta f(\alpha) g_C\left(\frac{x}{\alpha}\right) \right)'_+ = \sum_{\alpha \in D_f} \frac{\Delta f(\alpha)}{\alpha} g'_C\left(\frac{x}{\alpha}\right).$$

This is clear for  $x \notin D_f$  (since then  $\alpha \neq x$ ), but also true if  $x \in D_f$  since  $g_C(\frac{x}{\alpha}) = 0$  for  $x \leq \alpha$ . Thus  $(f_J * g_C)'_+ = f_{J,1} * g'_C$  and similarly  $(f_C * g_J)'_+ = f'_C * g_{J,1}$ . Putting these together gives

$$(f * g)'_+ = f_{J,1} * g'_C + f'_C * g_{J,1} + f'_C * g_{C,1} = f_{J,1} * g'_+ + f'_+ * g_1.$$

Thus  $(f * g)'_+$  is continuous and  $f * g \in T$ . □

### 1.3 Generalized prime systems

We distinguish between two different types of g-prime system.

**Definition 1.2** An *outer g-prime system* is a pair of functions  $\Pi, N$  with  $\Pi \in S_0^+$  and  $N \in S_1^+$  such that  $N = \exp_* \Pi$ .

Of course, if  $\Pi \in S_0^+$ , then  $\exp_* \Pi \in S_1^+$ , so  $\Pi$  determines a g-prime system (with  $N = \exp_* \Pi$ ). On the other hand if  $N \in S_1^+$ , then  $N = \exp_* \Pi$  for some  $\Pi \in S_0$  by 1.1(b), but  $\Pi$  need not be increasing. If  $\Pi$  is increasing, then we say  $N$  *determines* an outer g-prime system. The above definition is somewhat more general than the usual ‘generalised primes’, since we have not mentioned the equivalent of the prime counting function  $\pi(x)$ .

**Definition 1.3** A *g-prime system* is an outer g-prime system for which there exists  $\pi \in S_0^+$  such that

$$\Pi(x) = \sum_{k=1}^{\infty} \frac{1}{k} \pi(x^{1/k}).$$

we say  $N$  *determines a g-prime system* if there exists such an increasing  $\pi \in S_0$ .

*Remarks.*

(a) As such,  $\pi(x)$  is given by

$$\pi(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \Pi(x^{1/k}).$$

In fact this sum always converges for  $\Pi \in S^+$  (since  $\Pi(x^{1/k})$  decreases with  $k$  and  $\sum_{k=1}^{\infty} \frac{\mu(k)}{k}$  converges). But of course  $\pi$  need not be increasing.

(b) A g-prime system is *discrete* if  $\pi$  is a step function with integer jumps. In this case the g-primes are the discontinuities of  $\pi$  and the step is the multiplicity.

(c) An outer g-prime system is *continuous* if  $N$  (and hence  $\Pi$  – see below) is continuous in  $(1, \infty)$ .

(d) For an outer g-prime system  $(\Pi, N)$ , let  $\psi = \Pi_L$  (i.e.  $\psi(x) = \int_1^x \log t d\Pi(t)$ ) denote the *generalised Chebyshev function*.

Note that  $\psi \in S_0^+$ , and that  $N = \exp_* \Pi$  is equivalent to  $\psi * N = N_L$  (see [3] and [5]).

If  $N$  determines a g-prime system and  $N(x) = cx + O(x(\log x)^{-\gamma})$  for some  $\gamma > 3/2$ , then by Beurling’s Prime Number Theorem<sup>4</sup>(see [2] or [4]),  $\psi(x) \sim x$ . Also  $\psi_1(x) = \log x + \kappa + o(1)$  for some constant  $\kappa$ , where  $\psi_1(x) = \int_1^x \frac{1}{t} d\psi(t)$ .

(e) Applying 1.2(c) to outer g-primes shows that  $D_{N_L} = D_N D_\psi$ . But  $D_{N_L} = D_N \setminus \{1\}$ , so  $D_N \setminus \{1\} = D_N D_\psi$ .

#### Theorem 1.1

Let  $(\Pi, N)$  be an outer g-prime system. Then

(a)  $\Delta\Pi \leq \Delta N$ . In particular,  $\Pi$  is continuous at the points of continuity of  $N$ .

(b)  $(\Pi_J, N_J)$  is an outer g-prime system.

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<sup>4</sup>This is usually formulated for g-prime systems, but actually proved for outer g-prime systems. No use of  $\pi(x)$  being increasing is made, only that of  $\Pi(x)$ .

*Proof.* (a) Apply  $\Delta$  to both sides of  $\psi * N = N_L$  and use  $\Delta N_L(x) = \Delta N(x) \log x$ . Thus

$$\Delta N(x) \log x = \Delta(\psi * (N_J + N_C))(x) = \Delta(\psi * N_J)(x) \geq \Delta\psi(x),$$

since  $N$  has a jump of 1 at 1. But  $\Delta\psi(x) = \Delta\Pi(x) \log x$ , so  $\Delta\Pi \leq \Delta N$  and (a) follows.

(b) Take the jump function of both sides of the equation  $\psi * N = N_L$ . Thus  $(\psi * N)_J = (N_L)_J$ . By 1.2(a) and (d) this is  $\psi_J * N_J = (N_J)_L$ . Since  $N_J$  and  $\psi_J$  are increasing, this implies  $(\Pi_J, N_J)$  forms a g-prime system. □

Theorem 1.1 gives a useful necessary condition for  $N \in S_1^+$  to determine a g-prime system; namely that  $N_J$  must determine a g-prime system. Of course, this is no use if  $N$  is continuous, in which case  $N_J = i$  — the identity w.r.t.  $*$ .

Finally, we remark that if  $N$  is continuously differentiable on  $(1, \infty)$ , then so is  $\psi$  and  $\psi' = N'_L - N' * \psi_1$ . The proof follows 1.1(d) with  $f = N$  and  $g = \psi$ , so that  $(f * g)' = N'_L$ . The first integral on the RHS of (1.1) then tends to  $f' * g_1 = N' * \psi_1$ , while the second integral lies between

$$\frac{N(1)}{h} \int_x^{x+h} d\psi(t) \quad \text{and} \quad \frac{N(1+h)}{h} \int_x^{x+h} d\psi(t).$$

Since  $N$  is right-continuous at 1, it follows that  $\frac{\psi(x+h) - \psi(x)}{h}$ , must therefore tend to a limit as  $h \rightarrow 0^+$ . Similarly, for  $h \rightarrow 0^-$ .

In the same way,  $N \in T$  implies  $\psi \in T$ .

## §2. Continuous g-prime systems with $N(x) - cx$ periodic.

Suppose now that  $N \in S_1$  and  $N(x) = cx - R(x)$  where  $R(x)$  is periodic for some  $c > 0$ . Extend  $R$  to the whole real line by periodicity. Thus  $R$  is right continuous, locally of bounded variation, and  $R(1) = c - 1$ .

In what follows we shall always write  $N = \exp_* \Pi$  where  $\Pi \in S_0$ .

### Theorem 2.1

Let  $N(x) = cx - R(x) \in S_1^+$ , where  $R$  is continuously differentiable and periodic, and  $c > 0$ . Then  $\Pi$  is increasing if and only if  $R$  is constant; i.e.  $N(x) = cx + 1 - c$  for  $x \geq 1$ .

*Proof.* If  $R$  is constant, then  $N(x) = cx + 1 - c$  ( $x \geq 1$ ) and  $\hat{N}(s) = 1 + \frac{c}{s-1}$ . Thus  $\hat{\psi}(s) = -\hat{N}'(s)/\hat{N}(s) = \frac{1}{s-1} - \frac{1}{s+c-1}$ , which implies  $\psi'(x) = 1 - x^{-c} \geq 0$ . Hence  $\Pi$  is increasing.

For the converse, let  $R$  be non-constant and suppose, for a contradiction, that  $\Pi$  is increasing. Equivalently, suppose that  $\psi' \geq 0$ . Differentiate the relation  $N_L = \psi * N$ , using 1.1(d). Thus for  $x > 1$ ,

$$N'(x) \log x = (N' * \psi_1)(x) + \psi'(x) \geq (N' * \psi_1)(x), \quad (2.1)$$

where  $\psi_1(x) = \int_1^x \frac{1}{t} d\psi(t)$ . Since  $N' = c - R'$ , this becomes

$$R'(x) \log x - (R' * \psi_1)(x) \leq c \log x - c\psi_1(x).$$

By Beurling's PNT, the righthand side tends to a limit as  $x \rightarrow \infty$ , so for some constant  $A$  and all  $x > 1$ ,

$$R'(x) \log x - (R' * \psi_1)(x) \leq A \quad (2.2)$$

Let  $P$  be a period of  $R$ . Extend  $R$  to  $\mathbb{R}$  by periodicity. By continuity and periodicity of  $R'$  there exists  $x_0 \in [0, P]$  such that

$$R'(x_0) = \max_{x \in \mathbb{R}} R'(x).$$

Furthermore, for  $\delta > 0$  sufficiently small, the set of points  $x$  in  $[0, P]$  for which  $R'(x) \leq R'(x_0) - \delta$  contains an interval, say  $[\alpha, \beta]$  with  $0 < \alpha < \beta < P$ . (If not then  $R'$  is constant which forces  $R$  constant.) Let



$x = nP + x_0$  in (2.2) where  $n \in \mathbb{N}$ . Since  $\log(nP + x_0) = \psi_1(nP + x_0) + O(1)$  and  $R'$  has period  $P$ , (2.2) can be written as

$$\int_{1-}^{nP+x_0} R'(x_0) - R'\left(P\left\{\frac{nP+x_0}{tP}\right\}\right) d\psi_1(t) \leq A. \quad (2.3)$$

(A different constant  $A$ .) Note that the integrand is non-negative. Furthermore, the integrand is at least  $\delta$  for  $t \in [\frac{nP+x_0}{kP+\beta}, \frac{nP+x_0}{kP+\alpha}]$  for each positive integer  $k \leq n$ .

Let  $K$  be a fixed positive integer less than  $n$ . Thus the LHS of (2.3) is at least

$$\sum_{k=1}^K \int_{\frac{nP+x_0}{kP+\beta}}^{\frac{nP+x_0}{kP+\alpha}} \delta d\psi_1(t) = \delta \sum_{k=1}^K \left( \psi_1\left(\frac{nP+x_0}{kP+\alpha}\right) - \psi_1\left(\frac{nP+x_0}{kP+\beta}\right) \right).$$

As  $n \rightarrow \infty$ , the  $k^{\text{th}}$ -term in the sum tends to  $\log(\frac{kP+\beta}{kP+\alpha}) = -\log(1 - \frac{\beta-\alpha}{kP+\beta}) \geq \frac{\beta-\alpha}{kP+\beta}$ . Thus

$$\liminf_{n \rightarrow \infty} \int_{1-}^{nP+x_0} R'(x_0) - R'\left(P\left\{\frac{nP+x_0}{tP}\right\}\right) d\psi_1(t) \geq \delta(\beta - \alpha) \sum_{k=1}^K \frac{1}{kP+\beta} \geq \delta' \log K$$

for some  $\delta' > 0$ . This is true for every  $K \geq 1$  so the lefthand side of (2.3) cannot be bounded. This contradiction proves the theorem.  $\square$

*Remark.* (i) We see that  $N(x) = cx + 1 - c$  determines an outer g-prime system for every  $c > 0$ . What about g-prime systems; i.e. for which values of  $c$  is  $\pi$  increasing? We show in the appendix that this happens for  $0 < c \leq \lambda$  and fails for  $c > \lambda$  for some  $\lambda > 2$ .

(ii) The proof of Theorem 2.1 can be readily extended to the case where  $R$  is absolutely continuous and  $R'(x)$  has a maximum value, say at  $x = x_0$  and the set

$$\{x \in [0, P] : R'(x) \leq R'(x_0) - \delta\}$$

contains an interval, for some  $\delta > 0$ .

In particular this shows that none of the functions  $N_\lambda$  with  $\lambda > 1$  (as defined in [7], section 3) form part of a g-prime system, except of course when  $\rho_\lambda = 0$ . (To recall:  $N_\lambda(x) = x - R_\lambda(x)$  for  $x \geq 1$  and zero otherwise, where  $R_\lambda(x)$  is periodic with period 1 and defined for  $0 \leq x < 1$  by  $R_\lambda(x) = \rho_\lambda(\zeta(1-\lambda, 1-x) - \zeta(1-\lambda))$ . Here  $\rho_\lambda$  is a continuous function of  $\lambda$  with  $\rho_1 = 1$ .)

For  $\lambda > 2$ , this follows from Theorem 2.1 since  $R_\lambda$  is continuously differentiable and non-constant. For  $1 < \lambda \leq 2$ , this follows on noting that  $R_\lambda$  is absolutely continuous and  $R'_\lambda$  is maximum at  $0+$ .

### §3. G-prime systems with $N(x) - cx$ periodic and finitely many discontinuities

Suppose now that  $N$  has discontinuities (other than at 1). To check whether  $N$  comes from a g-prime system we consider its jump function  $N_J$ . By Theorem 1.1, a necessary condition that  $N$  determines a g-prime system is that  $N_J$  does.

Our strategy for determining the possible  $N$  will be as follows. Writing  $N = N_J + N_C$ , we first show by extending Theorem 2.1 that we must have  $N_C(x) = a(x-1)$  for some  $a \geq 0$ . Then we show that the discontinuities must occur at the (rational) integers and that the period, say  $P$ , is an integer. Writing  $a_n$  for the jump at  $n$  we therefore have  $a_{n+P} = a_n$  for  $n \geq 2$ . Next we show that  $a_{1+P} = a_1$  is forced, so  $a_n$  is truly periodic. Using a result of Saias and Weingartner [8] on Dirichlet series with periodic coefficients, we deduce that (i)  $a_n$  must be even (mod  $P$ ) and (ii) that  $a_n$  is multiplicative. We are then in a position to deduce  $N_C \equiv 0$  (i.e.  $N$  is a step function) and determine exactly which arise from g-prime systems.

First we extend Theorem 2.1 to members of  $T$ .

#### Theorem 3.1

Let  $N(x) = cx - R(x) \in T$ , where  $R$  is periodic and such that  $\Pi$  is increasing. Then  $N(x) = N_J(x) + a(x-1)$  for some  $a \geq 0$ .

*Proof.* We proceed as in the proof of Theorem 2.1 but with  $R'_+$  in place of  $R'$ . Now (2.1) becomes

$$N'_+(x) \log x = (N'_+ * \psi_1)(x) + (N_{J,1} * \psi'_+)(x) \geq (N'_+ * \psi_1)(x),$$

and (2.2) still holds with  $R'$  replaced by  $R'_+$ . If  $R'_+$  is not constant, then as before, we can find an  $x_0 \in [0, P]$  which maximises  $R'_+$  and for which  $R'_+(x) \leq R'_+(x_0) - \delta$  holds throughout some interval for some (sufficiently small)  $\delta > 0$ . We obtain a contradiction as before and hence  $N'_+$  is constant.

But  $N$  has finitely many discontinuities in bounded intervals, so  $N'_+ = (N_C)'_+$ . So  $N'_+ \equiv a$  implies (since  $N_C$  is continuous) that  $N_C(x) = a(x-1)$ , using  $N_C(1) = 0$ . Since  $N_C$  is increasing, we must have  $a \geq 0$ . □

Later on, we shall see that the only possible value of  $a$  is 0.

### Notation

Let  $\lambda$  denote the total jump of  $N$  per interval of length  $P$ ; i.e.  $N_J(x+P) - N_J(x) = \lambda$  for  $x \geq 1$ . Thus  $N_J(x) = \frac{\lambda}{P}x + O(1)$  and, by integration by parts,  $(N_J)_L(x) = \frac{\lambda}{P}x \log x + O(x)$ . Note that  $\lambda = 0$  implies  $N$  is continuous, while  $\lambda = cP$  implies  $N = N_J$ .

For the following,  $D_N$  denotes the set of discontinuities of  $N$  in  $(0, \infty)$  and  $D_N^* = D_N \cap (1, P+1]$ . We suppose that  $D_N^*$  is a finite, but non-empty, set.

### Proposition 3.2

Let  $D_N^*$  have  $k$  elements. Suppose  $\alpha \in D_N$  such that  $\alpha$  is irrational. Then there are at most  $k^2$  numbers  $\beta \in D_N$  such that  $\alpha\beta \in D_N$ .

*Proof.* Suppose, for a contradiction, that there are  $l > k^2$  numbers  $\beta \in D_N$  such that  $\alpha\beta \in D_N$ . Let  $D_N^* = \{c_1, \dots, c_k\}$ . Each  $\beta$  is of the form  $nP + c_i$ . There are  $k$  choices for  $c_i$  so some  $c_{i_0}$  will appear at least  $k+1$  times. (If not and all appear at most  $k$  times, then there can be at most  $k^2$  such numbers  $\beta$ .)

Thus we have (at least)  $k+1$  equations

$$\alpha(nP + c_{i_0}) = mP + c_j,$$

with (possibly different)  $m, n \in \mathbb{N}$  and some  $c_j \in D_N^*$ . As  $D_N^*$  has only  $k$  elements, at least one  $c_j$  must occur twice; i.e. there exist positive integers  $n_1, n_2, m_1, m_2$  such that

$$\alpha(n_1P + c_{i_0}) = m_1P + c_{j_0} \quad \text{and} \quad \alpha(n_2P + c_{i_0}) = m_2P + c_{j_0}.$$

Note that  $n_1 \neq n_2$  and  $m_1 \neq m_2$  otherwise they are not genuinely different equations. Subtracting these two gives

$$\alpha(n_2 - n_1) = m_2 - m_1,$$

and  $\alpha$  is rational — a contradiction. □

### Proposition 3.3

$D_N$  contains only rational numbers and  $P$  is rational.

*Proof.* By 1.2(a) and Theorem 1.1,

$$(N_J)_L(x) = (N_J * \psi_J)(x) = \sum_{\substack{\alpha\beta \leq x \\ \alpha, \beta \in D_N}} \Delta N(\alpha) \Delta \psi(\beta). \quad (3.1)$$

Since  $(N_J)_L(x) = \frac{\lambda}{P}x \log x + O(x)$  and  $D_\psi D_N = D_{N_L} = D_N \setminus \{1\}$ , we may rewrite (3.1) as

$$\sum_{\alpha \leq x} \Delta N(\alpha) \sum_{\substack{\beta \leq x/\alpha \\ \text{s.t. } \alpha\beta \in D_N}} \Delta \psi(\beta) = \frac{\lambda}{P}x \log x + O(x). \quad (3.2)$$

For  $\alpha$  irrational, by Proposition 3.2 there are at most  $k^2$  possible  $\beta$ s for which  $\alpha\beta \in D_N$ , where  $k = |D_N^*|$ . For each such  $\beta$ ,  $\Delta\psi(\beta) \leq \Delta N(\beta) \log \beta \leq C \log \beta$  for some  $C$ . Hence the inner sum on the left of (3.2) is at most  $Ck^2 \log(x/\alpha)$ . Thus the contribution of irrational  $\alpha$  to the LHS of (3.2) is less than

$$Ck^2 \sum_{\alpha \leq x} \Delta N(\alpha) \log \frac{x}{\alpha} = Ck^2 \int_{1-}^x \log \frac{x}{t} dN_J(t) = Ck^2 \int_1^x \frac{N_J(t)}{t} dt = O(x).$$

Hence

$$\sum_{\substack{\alpha \leq x \\ \alpha \text{ rational}}} \Delta N(\alpha) \sum_{\substack{\beta \leq x/\alpha \\ \text{s.t. } \alpha\beta \in D_N}} \Delta\psi(\beta) = \frac{\lambda}{P} x \log x + O(x). \quad (3.3)$$

But the LHS of (3.3) is (using Beurling's PNT for  $\psi_J(x)$ )

$$\sum_{\substack{\alpha \leq x \\ \alpha \text{ rational}}} \Delta N(\alpha) \psi_J\left(\frac{x}{\alpha}\right) \sim x \sum_{\substack{\alpha \leq x \\ \alpha \text{ rational}}} \frac{\Delta N(\alpha)}{\alpha}. \quad (3.4)$$

Now the function

$$N_{J,\mathbb{Q}}(x) \stackrel{\text{def}}{=} \sum_{\substack{\alpha \leq x \\ \alpha \text{ rational}}} \Delta N(\alpha) = \frac{\mu}{P} x + O(1)$$

for some  $\mu \leq \lambda$  by periodicity. ( $\mu$  is the jump per interval of length  $P$  from the rational discontinuities.) The RHS of (3.4) is therefore

$$x \int_1^x \frac{1}{t} dN_{J,\mathbb{Q}}(t) = x \int_1^x \frac{N_{J,\mathbb{Q}}(t)}{t^2} dt + O(x) = \frac{\mu}{P} x \log x + O(x).$$

It follows that  $\mu = \lambda$  and there are no irrational numbers in  $D_N$ .

Finally,  $\alpha \in D_N$  with  $\alpha > 1$  implies  $\alpha + P \in D_N$  by periodicity. As  $D_N$  contains only rationals, this forces  $P$  rational. □

### Proposition 3.4

$D_N \subset \mathbb{N}$  and  $P \in \mathbb{N}$ .

*Proof.* Since  $D_N \setminus \{1\} = D_{\psi * N} = D_{\psi} D_N$ , if  $\alpha \in D_{\psi}$  then  $\alpha\beta \in D_N$  for every  $\beta \in D_N$ . In particular (using  $D_{\psi} \subset D_N$ )  $\alpha \in D_{\psi}$  implies  $\alpha^n \in D_N$  for every  $n \in \mathbb{N}$ . By periodicity,  $\alpha^n - kP \in D_N$  for every integer  $k$  provided  $\alpha^n - kP \geq 1$ .

Now write  $\alpha = r/s$  and  $P = t/u$  where  $r, s, t, u \in \mathbb{N}$  and  $(r, s) = (t, u) = 1$ . For  $D_N^*$  to be finite, the numbers  $1 + P\{\frac{\alpha^n - 1}{P}\}$  ( $n = 1, 2, 3, \dots$ ) (take  $k = \lfloor \frac{\alpha^n - 1}{P} \rfloor$  above) must repeat themselves infinitely often; i.e. for infinitely many values of  $n$ ,

$$\alpha^n - kP = \alpha^{n_0} - k_0 P$$

for some integers  $k, k_0$ , and  $n_0$ . As such,

$$P = \frac{\alpha^n - \alpha^{n_0}}{k - k_0} = \frac{\left(\frac{r}{s}\right)^n - \left(\frac{r}{s}\right)^{n_0}}{k - k_0} = \frac{t}{u}.$$

Multiplying through by  $(k - k_0)us^{n_0}$  shows that  $s^{n-n_0}|ur^n$  for infinitely many  $n$ . But  $(r, s) = 1$ , so  $s^{n-n_0}|u$  for infinitely many  $n$ . This is only possible if  $s = 1$ ; i.e.  $\alpha \in \mathbb{N}$ . Hence  $D_{\psi} \subset \mathbb{N}$ .

It follows that  $D_{\Pi} \subset \mathbb{N}$  also, and  $D_{\Pi * k} \subset \mathbb{N}$  for every positive integer  $k$ . Since  $N = \sum_{k=0}^{\infty} \Pi^{*k}/k!$ , it follows that  $D_N \subset \mathbb{N}$  also.

Finally,  $m \in D_N$  with  $m > 1$  implies  $m + P \in D_N$  by periodicity. Since  $D_N \subset \mathbb{N}$ , this implies  $P \in \mathbb{N}$ . □

#### §4. Determining the jumps

Now that we have established the discontinuities are at the integers, it remains to determine the possible jumps. Write  $a_n = \Delta N(n)$  and  $c_n = \Delta \psi(n)$ . Thus  $a_1 = 1$  and  $a_{n+P} = a_n$  for  $n > 1$ . The equation  $\Delta N_L = (\Delta N) * \psi_J$  translates as

$$a_n \log n = \sum_{d|n} c_d a_{n/d}. \quad (4.1)$$

Thus  $c_1 = 0$ , for a prime  $p$ ,  $c_p = a_p \log p$ , while for distinct primes  $p$  and  $q$ , we have (after some calculation)  $c_{pq} = (a_{pq} - a_p a_q) \log pq$ .

Next we show that  $a_n$  is truly periodic ( $a_{n+P} = a_n$  for  $n \geq 1$ ). For the proof, let  $\langle \mathbb{P}_{r,P} \rangle$  denote the set of numbers of the form  $p_1 \dots p_k$  where the  $p_i$  are distinct primes, all congruent to  $r \pmod{P}$ . Here  $r$  is coprime to  $P$ . Each such set is infinite by Dirichlet's Theorem on primes in arithmetic progressions.

#### Proposition 4.1

$a_{P+1} = 1$ .

*Proof.* First we prove that  $a_{P+1} = 0$  or  $1$ .

Let  $p_1, \dots, p_k$  be distinct primes all of the form  $1 \pmod{P}$ , with  $k \geq 3$ . Let  $n = p_1 \dots p_k$ , which is also  $1 \pmod{P}$ . Note that for every  $d|n$ ,  $d = 1 \pmod{P}$ , so that  $a_d = a_{P+1}$  if  $d > 1$ . In particular  $c_{p_i p_j} = a_{P+1}(1 - a_{P+1}) \log p_i p_j$  for any  $1 \leq i, j \leq k$  with  $i \neq j$ . Since  $c_n \geq 0$ , (4.1) implies

$$a_{P+1} \log n \geq \sum_{1 \leq i < j \leq k} c_{p_i p_j} a_{n/p_i p_j} = a_{P+1}^2 (1 - a_{P+1}) \sum_{1 \leq i < j \leq k} \log p_i p_j = a_{P+1}^2 (1 - a_{P+1}) (k-1) \log n.$$

This is impossible for  $k$  sufficiently large unless  $a_{P+1}$  equals 0 or 1.

Next we show that  $a_{P+1} = 0$  implies  $a_n = 0$  for all  $n > 1$ , and hence that  $N_J(x) = 1$  for  $x \geq 1$  — i.e. the continuous case.

By induction. Suppose  $a_{P+1} = 0$  and that  $a_n = 0$  for all  $n > 1$  such that  $\Omega(n) < k$ , some  $k \geq 1$ . (It is vacuously true for  $k = 1$ .) Then  $a_{nr} = 0$  for all such  $n$  and all  $r \equiv 1 \pmod{P}$ , by periodicity. In particular, if we take  $r \in \langle \mathbb{P}_{1,P} \rangle$ . Note that this implies  $c_{nr} = 0$  also for such  $n$  and  $r$ .

Now let  $n$  be such that  $\Omega(n) = k$ . Then, with  $r \in \langle \mathbb{P}_{1,P} \rangle$  such that  $(n, r) = 1$ ,

$$a_{nr} \log nr = \sum_{d|nr} c_d a_{nr/d} = \sum_{d_1|n} \sum_{d_2|r} c_{d_1 d_2} a_{nr/d_1 d_2}.$$

Now  $d_2 \in \langle \mathbb{P}_{1,P} \rangle$  also, so by assumption,  $c_{d_1 d_2} = 0$  if  $\Omega(d_1) < k$ . Hence only the terms with  $\Omega(d_1) = k$  give a contribution; i.e. only if  $d_1 = n$ . Also  $a_{nr} = a_n$  by periodicity. Thus

$$a_n \log nr = \sum_{d_2|r} c_{n d_2} a_{r/d_2} = c_{nr}, \quad (4.2)$$

since only the term with  $d_2 = r$  makes  $a_{r/d_2}$  non-zero.

Now consider  $a_{n^2 r}$  with  $n$  and  $r$  as above. We have

$$a_{n^2 r} \log n^2 r \geq \sum_{d|r} c_{n d} a_{nr/d}.$$

Using (4.2) and noting that  $a_{n^2 r} = a_{n^2}$ , we therefore have<sup>6</sup>

$$a_{n^2} \log n^2 r \geq a_n^2 \sum_{d|r} \log nd = \frac{a_n^2}{2} d(r) \log n^2 r.$$

<sup>5</sup>As usual,  $\Omega(n)$  denotes the total number of prime factors of  $n$ ;  $\omega(n)$  denotes the number of distinct prime factors of  $n$ .

<sup>6</sup>Using  $2 \sum_{d|n} \log kd = d(n) \log k^2 n$ .

i.e.  $2a_{n^2} \geq a_n^2 d(r)$  for all  $r \in \langle \mathbb{P}_{1,P} \rangle$  such that  $(n, r) = 1$ . But  $r$  can be chosen such that  $d(r)$  is arbitrarily large, and we have a contradiction if  $a_n > 0$ . Thus  $a_n = 0$  is forced.

Hence by induction,  $a_n = 0$  for all  $n > 1$ . □

Thus, for the discontinuous case,  $\hat{N}_J(s)$  is a Dirichlet series with purely periodic coefficients. Further, if  $N_J$  determines a g-prime system, then  $\hat{N}_J$  has no zeros in<sup>7</sup>  $H_1$ . Now we use the main result of Saias and Weingartner ([8], Corollary): *Let  $F$  be a Dirichlet series with periodic coefficients. Then  $F$  does not vanish in  $H_1$  if and only if  $F = PL_\chi$ , where  $P$  is a Dirichlet polynomial with no zeros in  $H_1$  and  $\chi$  is a Dirichlet character.*

Thus  $\hat{N}_J = PL_\chi$  for some Dirichlet polynomial  $P$  and Dirichlet character  $\chi$ . We shall see below that the positivity of the coefficients of  $\hat{N}_J$  implies that  $\chi$  must be a principal character, showing that we actually have  $\hat{N}_J = Q\zeta$  for some Dirichlet polynomial  $Q$ .

**Proposition 4.2**

$\hat{N}_J(s) = Q(s)\zeta(s)$  where  $Q$  is a Dirichlet polynomial with no zeros in  $H_1$ . Furthermore,  $a_n$  is even (mod  $P$ ); i.e.  $a_n = a_{(n,P)}$ , and  $Q(s) = \sum_{d|P} \frac{q(d)}{d^s}$  for some  $q(d)$ .

*Proof.* From above,  $\hat{N}_J(s) = P(s)L_\chi(s)$ , where  $P(s) = \sum_{n=1}^N b_n n^{-s}$  say. Extend  $b_n$  so that  $b_n = 0$  for  $n > N$ . By inversion,

$$b_n = \sum_{d|n} \mu(d)\chi(d)a_{n/d} = 0 \quad \text{for } n > N.$$

In particular, for every prime  $p > N$ ,  $a_p = \chi(p)$ . A simple induction on  $\Omega(n)$  shows that more generally,  $a_n = \chi(n)$  whenever all the prime factors of  $n$  are greater than  $N$ . Consequently, for all such  $n$ ,  $a_n = 0$  or 1 (since  $a_n \geq 0$  while  $\chi(n) = 0$  or a root of unity).

Now let  $p > \max\{N, P\}$  be prime. Then  $p \equiv r \pmod{P}$  for some  $r$  with  $(r, P) = 1$ . Let  $n = p^{\phi(P)}$ . Then  $n \equiv r^{\phi(P)} \equiv 1 \pmod{P}$  and hence

$$1 = a_1 = a_n = \chi(n) = \chi(p^{\phi(P)}) = \chi(p)^{\phi(P)}.$$

But  $\chi(p) = 0$  or 1, so  $\chi(p) = 1$  for all sufficiently large  $p$ .

This implies  $\chi$  must be a principal character. For suppose  $\chi$  is a character modulo  $m$ . Let  $(r, m) = 1$ . For a sufficiently large prime  $p$  in each residue class  $r \pmod{m}$ ,  $1 = \chi(p) = \chi(r)$  by periodicity. Thus  $\chi(r) = 1$  whenever  $(r, m) = 1$ ; i.e.  $\chi$  is principal. Thus

$$\hat{N}_J(s) = P(s)L_{\chi_0}(s) = P(s)\zeta(s) \prod_{p|m} \left(1 - \frac{1}{p^s}\right) = Q(s)\zeta(s),$$

where  $Q$  is again a Dirichlet polynomial, non-zero in  $H_1$ . Denoting the coefficients of  $Q$  by  $q(n)$ , we see that  $q(1) = 1$ ,  $q(n) = 0$  for  $n$  sufficiently large, and

$$a_n = \sum_{d|n} q(d).$$

To show  $a_n$  is even (mod  $P$ ), we first show that for  $d|P$ ,  $a_{pd} = a_d$  for all sufficiently large primes  $p$ . It is true for  $d = 1$ , so suppose it is true if  $\Omega(d) < k$ , for some  $k \geq 1$ .

Let  $d|P$  such that  $\Omega(d) = k$ . Let  $p$  be prime and sufficiently large so that  $(p, d) = 1$  and  $q(pd) = 0$ . Then

$$\begin{aligned} 0 = q(pd) &= \sum_{c|pd} \mu(c)a_{pd/c} = \sum_{c|d} \mu(c)a_{pd/c} + \sum_{c|d} \mu(pc)a_{d/c} \\ &= a_{pd} + \sum_{\substack{c|d \\ c > 1}} \mu(c)a_{pd/c} - \sum_{c|d} \mu(c)a_{d/c} = a_{pd} - a_d \end{aligned}$$

<sup>7</sup>For  $\theta \in \mathbb{R}$ ,  $H_\theta$  denotes the half-plane  $\{s \in \mathbb{C} : \Re s > \theta\}$ .

since  $a_{pd/c} = a_{d/c}$  as  $\Omega(d/c) < k$  in the first sum.

Let  $d = (n, P)$ . Then  $(\frac{n}{d}, \frac{P}{d}) = 1$  and there exist arbitrarily large primes  $p$  congruent to  $\frac{n}{d} \pmod{\frac{P}{d}}$ . For such primes  $p$ ,  $pd \equiv n \pmod{P}$ , and by periodicity  $a_n = a_{pd} = a_d$  for  $p$  sufficiently large. Thus  $a_n = a_{(n, P)}$ .

As a result, we can write

$$\hat{N}_J(s) = \sum_{d|P} \sum_{\substack{n=1 \\ (n, P)=d}}^{\infty} \frac{a_n}{n^s} = \sum_{d|P} \frac{a_d}{d^s} \sum_{\substack{m=1 \\ (m, P/d)=1}}^{\infty} \frac{1}{m^s} = \sum_{d|P} \frac{a_d}{d^s} \prod_{p|P/d} \left(1 - \frac{1}{p^s}\right) \zeta(s) = Q(s)\zeta(s),$$

which shows that  $q(n)$  is supported on the divisors of  $P$ . □

Next we show that  $a_n$  is multiplicative.

**Theorem 4.3**

$a_n$  is multiplicative.

*Proof.* Equivalently, we show  $q(n)$  is multiplicative. Let the period be  $P = p_1^{m_1} \dots p_k^{m_k}$ . Write

$$Q(s) = \sum_{d|P} \frac{q(d)}{d^s} = \exp\left\{\sum_{n=1}^{\infty} \frac{t(n)}{n^s}\right\},$$

for some  $t(n)$ , where  $t(1) = 0$ . Since  $\hat{N}_J(s) = \exp\{\sum_{n=1}^{\infty} \frac{b_n}{n^s}\}$  for some  $b_n \geq 0$ , Proposition 4.2 implies that  $t(n) = b_n \geq 0$  for  $n$  not a prime power. The aim is to show that  $t(n) = 0$  for such  $n$ .

Since the  $q(n)$  are supported on the divisors of  $P$ ,  $t(n)$  is supported on the set  $\{p_1^{n_1} \dots p_k^{n_k} : n_1, \dots, n_k \in \mathbb{N}_0\}$ .

For each  $p|P$  let

$$Q_p(s) = \sum_{r=0}^{\infty} \frac{q(p^r)}{p^{rs}}$$

(This is a polynomial in  $p^{-s}$ .) Then

$$\prod_{p|P} Q_p(s) = \exp\left\{\sum_{n \text{ prime power}} \frac{t(n)}{n^s}\right\},$$

where the sum is over prime powers only. Now define  $T_1(s)$  and  $t_1(n)$  by

$$\frac{Q(s)}{\prod_{p|P} Q_p(s)} = \exp\{T_1(s)\} = \exp\left\{\sum_{n=1}^{\infty} \frac{t_1(n)}{n^s}\right\}. \quad (4.3)$$

i.e.  $t_1(n) = t(n)$  for  $n$  not a prime power and zero otherwise.

If the Dirichlet series for  $T_1(s)$  converges everywhere, then the result follows. For the LHS of (4.3) is then entire and of order 1 while if  $t_1(n_0) > 0$  for some  $n_0 > 1$ , then the RHS of (4.3) is, for negative  $s$ , at least  $e^{t_1(n_0)n_0^{-s}}$ , which has infinite order. The contradiction implies  $T_1$  is identically zero and  $Q = \prod_p Q_p$ .

Suppose then that the series for  $T_1$  has a finite abscissa of convergence, say  $-\beta$ . Since the coefficients are non-negative,  $-\beta$  must be a singularity of the function; i.e.  $-\beta$  must be a zero of one of the  $Q_p(s)$ . (As we shall see later,  $Q_p(s) \neq 0$  in  $H_0$ , so  $\beta \geq 0$ , but we do not require to know this at this stage.)

We can write down the ‘spatial extension’ of (4.3). We can think of this as substituting  $z_i = p_i^{-s}$ . For  $p$  prime, let  $\tilde{Q}_p(z) = \sum_{r=0}^{\infty} q(p^r)z^r$ , so that  $\tilde{Q}_p(p^{-s}) = Q_p(s)$ . Now define

$$\tilde{Q}(z_1, \dots, z_k) = \sum_{b_1, \dots, b_k \geq 0} q(p_1^{b_1} \dots p_k^{b_k}) z_1^{b_1} \dots z_k^{b_k},$$

(the series is of course finite) and similarly for  $\tilde{T}_1$ . Then (4.3) becomes

$$\frac{\tilde{Q}(z_1, \dots, z_k)}{\tilde{Q}_{p_1}(z_1) \dots \tilde{Q}_{p_k}(z_k)} = \exp\left\{\tilde{T}_1(z_1, \dots, z_k)\right\} = \exp\left\{\sum_{n_1, \dots, n_k \geq 0} t_1(p_1^{n_1} \dots p_k^{n_k}) z_1^{n_1} \dots z_k^{n_k}\right\} \quad (4.4)$$

Since (4.3) holds for  $\sigma > -\beta$ , (4.4) holds in the domain  $\{(z_1, \dots, z_k) : |z_1| < p_1^\beta, \dots, |z_k| < p_k^\beta\}$ .

Let  $r$  be the smallest positive integer such that  $t_1(n) = 0$  whenever  $\omega(n) < r$ . (Thus  $2 \leq r \leq k$ ). Put  $z_{r+1}, \dots, z_k = 0$ . Then (4.4) becomes

$$\frac{\tilde{Q}(z_1, \dots, z_r)}{\tilde{Q}_{p_1}(z_1) \dots \tilde{Q}_{p_r}(z_r)} = \exp\left\{\sum_{n_1, \dots, n_r \geq 0} t_1(p_1^{n_1} \dots p_r^{n_r}) z_1^{n_1} \dots z_r^{n_r}\right\} \quad (4.5)$$

where we identified  $\tilde{Q}(z_1, \dots, z_r)$  with  $\tilde{Q}(z_1, \dots, z_r, 0, \dots, 0)$ . Without loss of generality, we may assume that the numerator and denominator of the left-hand side of (4.5) have no common factors. (If there are any, cancel them, and apply the argument to what remains.)

Let  $z_i = x_i$  ( $i = 1, \dots, r$ ) be real and positive. Take logs of (4.5) and differentiate with respect to each of the variables  $x_1, \dots, x_r$ . This gives

$$\sum_{n_1, \dots, n_r \geq 0} n_1 \dots n_r t_1(p_1^{n_1} \dots p_r^{n_r}) x_1^{n_1} \dots x_r^{n_r} = \frac{\partial^r}{\partial x_1 \dots \partial x_r} \log \tilde{Q}(x_1, \dots, x_r) = \frac{P(x_1, \dots, x_r)}{\tilde{Q}(x_1, \dots, x_r)^r}, \quad (4.6)$$

for some polynomial  $P$ . The crucial point here is that the polynomials  $\tilde{Q}_p$  have all disappeared.

Now,  $\tilde{Q}_p(p^\beta) = 0$  for some  $p|P$ , say  $p = p_1$ . Fix  $x_2, \dots, x_r$  and let  $x_1 \rightarrow p_1^\beta$  through real values from below. If  $\tilde{Q}(p_1^\beta, x_2, \dots, x_r) \neq 0$ , then the RHS of (4.6) remains bounded, and hence (since  $t_1(n) \geq 0$ ), the series

$$\sum_{n_1, \dots, n_k \geq 1} n_1 \dots n_r t_1(p_1^{n_1} \dots p_r^{n_r}) p_1^{n_1 \beta} x_2^{n_2} \dots x_r^{n_r} \quad \text{converges} \quad (4.7)$$

while the LHS of (4.5) tends to infinity, so

$$\sum_{n_1, \dots, n_r \geq 0} t_1(p_1^{n_1} \dots p_r^{n_r}) p_1^{n_1 \beta} x_2^{n_2} \dots x_r^{n_r} \quad \text{diverges.} \quad (4.8)$$

But (4.7) and (4.8) are in contradiction since in (4.8) we actually require  $n_1, \dots, n_r \geq 1$  (if any  $n_j = 0$ , there is no contribution to the sum as  $\omega(p_1^{n_1} \dots p_r^{n_r}) < r$ ).

Thus this forces  $\tilde{Q}(p_1^\beta, x_2, \dots, x_r) = 0$  for every  $x_i$  ( $i = 2, \dots, r$ ) in some interval, and hence for all such  $x_i$ , since  $\tilde{Q}$  is a polynomial. But this implies  $(x_1 - p_1^\beta)$  is a factor of both  $\tilde{Q}(x_1, \dots, x_r)$  and  $\tilde{Q}_{p_1}(x_1)$  — a contradiction. Hence  $T_1$  is identically zero and the result follows.  $\square$

### Determining $a$ for which $N_J(x) + a(x - 1)$ is a $g$ -prime system

The problem thus reduces to determining  $Q_p(s)$ . We shall see in Theorem 4.4 that the zeros of  $Q_p(s)$  all have real part less than or equal to zero. We use this fact to deduce that the only permissible value of  $a$  is 0.

For, using this fact, the zeros of  $Q$  then all lie in  $\mathbb{C} \setminus H_0$ . In particular, in  $H_0$ , the zeros of  $\hat{N}_J$  are precisely the zeros of  $\zeta$  and hence  $\hat{N}_J$  has no real positive zeros. Indeed,  $Q(\sigma) > 0$  for  $\sigma > 0$  since  $Q(\sigma)$  is real and non-zero here and as  $\sigma \rightarrow \infty$ ,  $Q(\sigma) \rightarrow 1$ . Thus  $\hat{N}_J(\sigma) < 0$  for  $0 < \sigma < 1$ . Also  $\hat{N}(\sigma) = \hat{N}_J(\sigma) - \frac{a}{1-\sigma} < 0$  for  $\sigma \in (0, 1)$ .

Now  $N = N_J + N_C$  and  $\psi = \psi_J + \psi_C$  and by assumption  $\psi_C$  is increasing. (Here  $N_C(x) = a(x - 1)$ , so that  $\hat{N}_C(s) = \frac{a}{s-1}$ .) Thus

$$\hat{\psi}_C(s) = \hat{\psi}(s) - \hat{\psi}_J(s) = \frac{\hat{N}_J'(s)}{\hat{N}_J(s)} - \frac{\hat{N}'(s)}{\hat{N}(s)},$$

since  $(\Pi_J, N_J)$  and  $(\Pi, N)$  are g-prime systems. Note that  $\hat{\psi}_C \neq -\hat{N}_C'/\hat{N}_C$  as  $(\Pi_C, N_C)$  is not a g-prime system (indeed  $N_C(1) = 0$ ).

Both  $\psi(s)$  and  $\psi_J(s)$  are meromorphic functions, holomorphic in  $\overline{H_1} \setminus \{1\}$ , with simple poles at  $s = 1$  and residue 1. Thus  $\psi_C(s)$  has a removable singularity at 1 and poles at the zeros of  $\hat{N}$  and  $\hat{N}_J$ .

Landau's Oscillation Theorem (cf. [3], p.137) applied to  $\hat{\psi}_C$  implies that  $\hat{\psi}_C$  has a singularity at its abscissa of convergence, say  $\theta$ . Of course  $\theta < 1$  must be a zero of  $\hat{N}$  or  $\hat{N}_J$ . But neither  $\hat{N}$  nor  $\hat{N}_J$  has real positive zeros, so  $\theta \leq 0$ . But then  $\hat{\psi}_C$  must be holomorphic in  $H_0$ , implying that  $\hat{N}$  and  $\hat{N}_J$  have the same zeros here; i.e. all the non-trivial Riemann zeros. But at each such zero, say  $\rho$ ,  $\hat{N}_C(\rho) = 0$  also. This is impossible as  $\hat{N}_C$  has no zeros, except if  $a = 0$ .

Hence  $a = 0$  is forced and  $N = N_J$ .

### Criteria for g-primes

We have  $\hat{N}(s) = Q(s)\zeta(s) = \exp\{T(s) + \log \zeta(s)\} = \exp\{\hat{\Pi}(s)\}$ . Thus

$$\hat{\Pi}(s) = \sum_{n=1}^{\infty} \frac{\Lambda_1(n) + t(n)}{n^s}.$$

For  $\Pi$  to be increasing, the coefficients of  $\hat{\Pi}$  must be non-negative; i.e.  $\Lambda_1(n) + t(n) \geq 0$  for all  $n \in \mathbb{N}$ . As  $t(n)$  is supported on the powers of the prime divisors of  $P$ , we have:

$$\Pi \text{ is increasing} \iff t(p^k) \geq -\frac{1}{k} \text{ for } p|P \text{ and } k \in \mathbb{N}. \quad (*)$$

Note that  $t(p) = q(p) = a_p - 1 \geq -1$  for  $p$  prime, so  $(*)$  is satisfied for  $k = 1$ .

Turning now to  $\pi(x)$ ,  $N$  determines g-primes if  $\pi$  is increasing where  $\pi(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \Pi(x^{1/k})$ . But

$$\hat{\pi}(s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \hat{\Pi}(ks) = \sum_p \frac{1}{p^s} + \sum_{k,n \geq 1} \frac{\mu(k)t(n)}{kn^{ks}} = \sum_{n=1}^{\infty} \frac{\pi_n}{n^s},$$

say, for some coefficients  $\pi_n$ . Thus  $\pi$  is increasing if and only if  $\pi_n \geq 0$  for all  $n$ . Now  $\pi_1 = 0$  and  $\pi_p = 1 + t(p) \geq 0$  for  $p$  prime, while  $\pi_n = 0$  for  $n$  not a prime power. Hence

$$\pi \text{ is increasing} \iff \sum_{d|n} \frac{\mu(d)}{d} t(p^{n/d}) \geq 0 \text{ for } n \geq 2 \text{ and } p|P. \quad (**)$$

To deal with these criteria, it is useful to write them in terms of the zeros of  $\tilde{Q}_p$ .

### The zeros of $\tilde{Q}_p$

Let  $p|P$  and let  $k$  be the degree of  $\tilde{Q}_p$ . Then  $\tilde{Q}_p$  has  $k$  zeros  $\lambda_1, \dots, \lambda_k$ . Letting  $\mu_r = 1/\lambda_r$  gives  $\tilde{Q}_p(z) = (1 - \mu_1 z) \dots (1 - \mu_k z)$  and

$$\log \tilde{Q}_p(z) = \sum_{r=1}^k \log(1 - \mu_r z) = - \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{r=1}^k \mu_r^n \right) z^n.$$

Since  $\log \tilde{Q}_p(z) = \sum_{r=1}^{\infty} t(p^r) z^r$ , equating coefficients gives

$$t(p^n) = -\frac{1}{n} \sum_{r=1}^k \mu_r^n.$$

Hence  $(*)$  is satisfied for a prime  $p|P$  if and only if

$$\tau_n := \sum_{r=1}^k \mu_r^n \leq 1 \text{ for } n \in \mathbb{N}. \quad (\dagger)$$



Turning to (\*\*), let  $s_n(w) = \sum_{d|n} \mu(d)w^{n/d}$  for  $w \in \mathbb{C}$ . Then  $\sum_{d|n} \frac{\mu(d)}{d} t(p^{n/d}) = -\frac{1}{n} \sum_{r=1}^k s_n(\mu_r)$  and (\*\*) is satisfied for a prime  $p|P$  if and only if

$$\sum_{r=1}^k s_n(\mu_r) \leq 0 \quad \text{for } n \geq 2. \quad (\dagger\dagger)$$

**Theorem 4.4**

Let  $\tilde{Q}_p$ ,  $k$  and  $\mu_1, \dots, \mu_k$  be as above. For  $k = 1$ ,  $(\dagger)$  is satisfied if and only if  $|\mu_1| \leq 1$ . For  $k > 1$ , if  $(\dagger)$  is satisfied, then  $|\mu_r| < 1$  for all  $r$ .

*Proof.* For  $k = 1$  this is trivial so assume  $k > 1$  and that  $(\dagger)$  is satisfied. The numbers  $\mu_1, \dots, \mu_k$  are either real or occur in complex conjugate pairs. Denote the real ones by  $\mu_1, \dots, \mu_l$  and the complex ones by  $\nu_1 e^{\pm i\theta_1}, \dots, \nu_m e^{\pm i\theta_m}$  where  $\nu_r > 0$  and  $0 < \theta_r < \pi$ . Thus  $(\dagger)$  becomes

$$\tau_n = \mu_1^n + \dots + \mu_l^n + 2(\nu_1^n \cos n\theta_1 + \dots + \nu_m^n \cos n\theta_m) \leq 1. \quad (4.9)$$

Assume without loss of generality that  $|\mu_1| \geq \dots \geq |\mu_l|$  and  $\nu_1 \geq \dots \geq \nu_m$ . If  $|\mu_1| \geq 1$ , then  $\mu_1^{2n} \geq 1$  and (4.9) implies

$$\nu_1^{2n} \cos 2n\theta_1 + \dots + \nu_m^{2n} \cos 2n\theta_m \leq 0 \quad \text{for all } n \in \mathbb{N}.$$

Suppose  $\nu_1 = \dots = \nu_q > \nu_{q+1}$  for some  $q \leq m$ , then this involves

$$\cos 2n\theta_1 + \dots + \cos 2n\theta_q \leq \frac{a}{A^n} \quad (n \in \mathbb{N}) \quad (4.10)$$

for some  $a$  and  $A > 1$ . But this is impossible as we show below.

Thus if any  $\mu_r$  is real, then  $|\mu_r| < 1$ . Now suppose  $\nu_1 = \dots = \nu_q > \nu_{q+1}$  and  $\nu_1 \geq 1$ . Then (4.9) implies

$$\cos 2n\theta_1 + \dots + \cos 2n\theta_q \leq \frac{1}{2} + \frac{a}{A^n} \quad (n \in \mathbb{N}) \quad (4.11)$$

for some  $a$  and  $A > 1$ . We show this is impossible, which in turn implies (4.10) is impossible.

Let  $\phi_r = \theta_r/\pi$ . By Dirichlet's Theorem (see [6], p.170), the numbers  $n\phi_1, \dots, n\phi_q$  can be made arbitrarily close to  $q$  integers simultaneously; i.e. given  $\varepsilon > 0$ , there exists  $n \in \mathbb{N}$  such that  $|n\phi_r - K_r| < \varepsilon$  for  $r = 1, \dots, q$  and integers  $K_r$ . Thus, for some  $|\delta_r| < \varepsilon$

$$\cos 2n\theta_r = \cos 2\pi n\phi_r = \cos 2\pi(K_r + \delta_r) = \cos 2\pi\delta_r > \cos 2\pi\varepsilon,$$

which can be made as close to 1 as we please. The inequalities (4.10) and (4.11) are impossible and hence  $\nu_r < 1$  for all  $r$ . □

To deal with  $(\dagger\dagger)$  we require the following.

**Lemma 4.5**

- (a) Let  $w \in \mathbb{R}$ . Then  $s_n(w) \leq 0$  for all  $n > 1$  if and only if  $w = 0$  or  $1$ .
- (b) Let  $w_1, \dots, w_k$  be non-zero complex numbers of modulus less than one, and symmetric about  $\mathbb{R}$ ; i.e.  $\overline{w_i} = w_j$  for some  $j$ . Then  $s_n(w_1) + \dots + s_n(w_k)$  changes sign infinitely often.

*Proof.* (a) For  $p$  prime,  $s_p(w) = w^p - w > 0$  for  $w > 1$ , while for  $p$  an odd prime,  $s_{2p}(w) = w^{2p} - w^p - w^2 + w > 0$  whenever  $w < -1$  for  $p$  sufficiently large. This leaves  $-1 \leq w \leq 1$ . For  $w = 1$ ,  $s_n(w) = 0$  for  $n > 1$ , while for  $w = -1$ ,  $s_n(w) = 0$  for  $n > 2$  and  $s_2(-1) = 2$ , so it narrowly fails in this case. For  $w = 0$  the result holds trivially.

Now suppose  $-1 < w < 1$ ,  $w \neq 0$ . Consider the entire function defined by the Dirichlet series

$$H_w(s) = \sum_{n=1}^{\infty} \frac{w^n}{n^s}.$$

Note that

$$\frac{H_w(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{s_w(n)}{n^s}.$$

Now if  $s_n(w)$  is ultimately of one sign, then the abscissa of convergence of this series must be a singularity of  $H_w/\zeta$ . This singularity must be real, and there can be no others further to the right. But the first real singularity (furthest to the right) is at  $-2$ , so  $H_w$  must be zero at all the complex zeros of  $\zeta$ . This is a contradiction as  $H_w$ , being bounded in any strip, has at most  $O(T)$  zeros up to height  $T$  here.

(b) This time

$$\frac{H_{w_1}(s) + \cdots + H_{w_k}(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{s_n(w_1) + \cdots + s_n(w_k)}{n^s}.$$

If  $s_n(w_1) + \cdots + s_n(w_k)$  is ultimately of one sign, then the abscissa of convergence is a singularity of the LHS. Each  $H_{w_i}$  is entire, so the first real singularity occurs at  $-2$ . As in (a), this gives a contradiction.  $\square$

*Proof of Theorem A.* By Lemma 4.5(b), if  $k > 1$ ,  $(\dagger\dagger)$  cannot be satisfied (for then  $|\mu_r| < 1$  for all  $r$ ). So, for  $\pi$  to be increasing, we require  $k = 1$ ; i.e.  $\tilde{Q}_p(z) = 1 + q(p)z$ . Hence  $\mu_1 = -q(p)$  and  $(\dagger\dagger)$  holds if and only if  $s_n(\mu_1) = s_n(-q(p)) \leq 0$  for  $n \geq 2$ . By (a) of Lemma 4.5, this only happens if  $q(p) = 0$  or  $-1$ . Thus

$$\hat{N}(s) = \zeta(s) \prod_{p|P} \left(1 + \frac{q(p)}{p^s}\right) = \zeta(s) \prod_{i=1}^l \left(1 - \frac{1}{p_i^s}\right)$$

for some prime divisors  $p_1, \dots, p_l$  of  $P$ .  $\square$

### Outer g-prime systems with $N(x) - cx$ periodic

The condition in Theorem 4.4 does not allow us to determine which coefficients  $a_n$  will lead to outer g-prime systems as they are only necessary and not sufficient. Instead we use the relation

$$kq(p^k) = \sum_{r=1}^k rt(p^r)q(p^{k-r}) \quad (4.12)$$

which follows directly from  $Q = e^T$ . This allows us to calculate  $t(p^k)$  explicitly in special cases. Suppose  $\tilde{Q}_p$  has degree 1. Then  $q(p^r) = 0$  for  $r > 1$  and (4.12) gives  $kt(p^k) = -(k-1)t(p^{k-1})q(p)$  for  $k \geq 2$ . Thus

$$t(p^k) = \frac{(-1)^{k-1}q(p)^k}{k}.$$

As a result,  $(*)$  holds if and only if  $(-q(p))^k \leq 1$  for all  $k$ , which is easily seen to be equivalent to  $-1 \leq q(p) \leq 1$  for all  $p|P$  (i.e.  $0 \leq a_p \leq 2$ ). In particular, we have proven:

### Theorem C

Let  $N \in T$  be such that  $N(x) - cx$  has squarefree period  $P$ . Then  $N$  determines an outer g-prime system if and only if

$$N(x) = \sum_{d|P} q(d) \left\lfloor \frac{x}{d} \right\rfloor,$$

where  $q(\cdot)$  is multiplicative,  $q(p) \in [-1, 1]$ , and  $c = \prod_{p|P} (1 + q(p)/p)$ .

For example, the outer g-prime systems for which  $N(x) - cx$  has period 6 are given by

$$N(x) = [x] + \lambda \left\lfloor \frac{x}{2} \right\rfloor + \mu \left\lfloor \frac{x}{3} \right\rfloor + \lambda\mu \left\lfloor \frac{x}{6} \right\rfloor,$$

where  $(\lambda, \mu \in [-1, 1])$  and  $(1 + \lambda/2)(1 + \mu/3) = c$ .

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### APPENDIX – When does $N(x) = cx + 1 - c$ determine a g-prime system?

From the proof of Theorem 2.1 we saw that  $\psi'(x) = 1 - x^{-c}$  for  $x \geq 1$ . Thus  $\psi$  (equivalently  $\Pi$ ) is increasing for every  $c \geq 0$ . What about  $\pi$ ? Let  $\theta = \pi_L$  be the generalization of Chebyshev's  $\theta$ -function. Then  $\theta(x) = \sum_{n=1}^{\infty} \mu(n)\psi(x^{1/n})$  so that

$$\theta'(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} x^{\frac{1}{n}-1} \psi'(x^{\frac{1}{n}}) = \frac{1}{x} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} (x^{\frac{1}{n}} - x^{\frac{1-c}{n}}).$$

Let  $f$  be the entire function

$$f(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} (e^{\frac{z}{n}} - 1) = \sum_{k=1}^{\infty} \frac{z^k}{k! \zeta(k+1)}.$$

Then  $e^x \theta'(e^x) = f(x) - f((1-c)x)$  and  $\theta$  is increasing if and only if

$$f(x) \geq f((1-c)x) \quad \forall x \geq 0. \tag{A_c}$$

For  $0 \leq c \leq 2$  this is easily seen to hold as

$$f(x) - f((1-c)x) = \sum_{k=1}^{\infty} \frac{(1 - (1-c)^k) x^k}{k! \zeta(k+1)}$$

and the coefficients are all non-negative if (and only if)  $0 \leq c \leq 2$ .

Now consider  $c > 2$ . It is clear that (A<sub>c</sub>) holds for all  $c > 2$  (actually for  $c \geq 1$ ) if and only if

$$f(-x) \leq 0 \quad \text{for } x \geq 0. \tag{B}$$

For if (B) is true, then since  $(1-c)x \leq 0$ , we have

$$f((1-c)x) \leq 0 \leq f(x)$$

and (A<sub>c</sub>) holds. Conversely, assume (A<sub>c</sub>) holds for all  $c > 2$ . Suppose, for a contradiction, that  $f(-x_0) > 0$  for some  $x_0 > 0$ . Then

$$0 < f(-x_0) = f\left((1-c) \cdot \frac{x_0}{c-1}\right) \leq f\left(\frac{x_0}{c-1}\right)$$

for every  $c > 2$ . This is false for  $c$  sufficiently large as the RHS can be arbitrarily close to zero. Thus (B) is true.

However, we show that (B) is false, and hence that  $(A_c)$  fails for some  $c > 2$ .

**Theorem A1**

There exists  $\lambda > 2$  such that for  $c \leq \lambda$ ,  $\pi$  is increasing, while for  $c > \lambda$ ,  $\pi$  is not increasing.

*Proof.* Clearly, if  $(A_c)$  holds for some  $c = c_0 > 1$ , then it holds for all smaller  $c$ , since  $(A_c)$  is equivalent to

$$f(-y) \leq f\left(\frac{y}{c-1}\right) \quad \forall y \geq 0 \tag{A'_c}$$

and  $f$  is increasing on  $(0, \infty)$ . Also, if  $(A'_c)$  holds for all  $c < c_1$ , then by continuity of  $f$ , it holds for  $c = c_1$ . Now we show (B) is false.

Starting from the formula<sup>8</sup>  $\frac{1}{2\pi i} \int_{(-1,0)} \Gamma(s)x^{-s} ds = e^{-x} - 1$  ( $x > 0$ ) we have

$$\frac{1}{2\pi i} \int_{(-1,0)} \frac{\Gamma(s)}{\zeta(1-s)} x^{-s} ds = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cdot \frac{1}{2\pi i} \int_{(-1,0)} \Gamma(s) \left(\frac{x}{n}\right)^{-s} ds = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} (e^{-x/n} - 1) = f(-x),$$

using the absolute and uniform convergence of the Dirichlet series for  $1/\zeta(1-s)$ . Changing the variable gives

$$f(-x) = \frac{1}{2\pi i} \int_{(1,2)} \frac{\Gamma(1-s)}{\zeta(s)} x^{s-1} ds.$$

By Mellin inversion

$$\frac{\Gamma(1-s)}{\zeta(s)} = \int_0^{\infty} \frac{f(-x)}{x^s} dx \tag{1 < \sigma < 2.}$$

Hence

$$\int_1^{\infty} \frac{f(-x)}{x^s} dx = \frac{\Gamma(1-s)}{\zeta(s)} - \int_0^1 \frac{f(-x)}{x^s} dx = \frac{\Gamma(1-s)}{\zeta(s)} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k! \zeta(k+1)(k+1-s)}.$$

Since the LHS converges and is holomorphic in  $H_1$ , the singularities at  $2, 3, 4, \dots$  on the RHS are all removable, as is the singularity at  $s = 1$ .

Suppose now that  $f(-x)$  is ultimately of one sign. Then the abscissa of convergence of the LHS Mellin transform must be a (real) singularity of the function. But the first real singularity occurs at  $-2$  (zero of  $\zeta$ ). This is a contradiction as there are singularities at the non-trivial zeros of  $\zeta$  to the right of this. Thus  $f(-x)$  cannot ultimately be of one sign; i.e.  $f$  changes sign infinitely often in  $(-\infty, 0)$  and has infinitely many zeros here.

Thus  $(A'_c)$  fails for some  $c \geq 2$  and hence all larger  $c$ . Let  $\lambda$  denote the supremum of those  $c$  for which  $(A'_c)$  holds. Thus  $(A'_c)$  holds for  $c \leq \lambda$  and fails for  $c > \lambda$ .

Finally,  $\lambda > 2$  since  $f(\frac{y}{\lambda-1}) \geq f(-y)$  for all  $y \geq 0$  with equality for some  $y > 0$  (or  $\lambda$  would not be optimal) and this is false for  $\lambda = 2$ . □

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<sup>8</sup>Here  $\int_{(\alpha,\beta)}$  means  $\lim_{T \rightarrow \infty} \int_{\sigma-iT}^{\sigma+iT}$  for any  $\sigma \in (\alpha, \beta)$ .