The Dirichlet-to-Neumann map for the elliptic sine Gordon

Article

Accepted Version

Fokas, A. S. and Pelloni, B. (2012) The Dirichlet-to-Neumann map for the elliptic sine Gordon. Nonlinearity, 25 (4). 1011. ISSN 1361-6544 doi: https://doi.org/10.1088/0951-7715/25/4/1011 Available at http://centaur.reading.ac.uk/26388/

The Dirichlet-to-Neumann map for the elliptic sine-Gordon equation

A.S. Fokas* and B. Pelloni**

*Department of Applied Mathematics and Theoretical Physics
Cambridge University
Cambridge CB3 0WA, UK.
t.fokas@damtp.cam.ac.uk

**Department of Mathematics
University of Reading
Reading RG6 6AX, UK
b.pelloni@reading.ac.uk

February 3, 2012

Abstract

We analyse the Dirichlet problem for the elliptic sine Gordon equation in the upper half plane. We express the solution $q(x, y)$ in terms of a Riemann-Hilbert problem whose jump matrix is uniquely defined by a certain function $b(\lambda)$, $\lambda \in \mathbb{R}$, explicitly expressed in terms of the given Dirichlet data $g_0(x) = q(x, 0)$ and the unknown Neumann boundary value $g_1(x) = q_y(x, 0)$, where $g_0(x)$ and $g_1(x)$ are related via the global relation $\{b(\lambda) = 0,\ \lambda \geq 0\}$. Furthermore, we show that the latter relation can be used to characterise the Dirichlet to Neumann map, i.e. to express $g_1(x)$ in terms of $g_0(x)$. It appears that this provides the first case that such a map is explicitly characterised for a nonlinear integrable *elliptic* PDE, as opposed to an *evolution* PDE.

1 Introduction

A new method for analysing boundary value problems, extending ideas of the so-called inverse scattering transform method, was introduced in [17], see also [18, 23]. This method has already been applied to linear evolution PDEs on the half-line and on the finite interval [8, 12, 14, 24, 29, 46, 47, 48], to integrable nonlinear evolution PDEs on the half-line and on the finite interval [4, 5, 20, 22, 28, 35, 37, 39, 40, 41, 42], to linear and integrable nonlinear evolution PDEs in two space variables [36, 44], to linear elliptic PDEs in the interior of a convex polygon [1, 2, 7, 9, 10, 11, 13, 32, 38, 53, 54] and to the two prototypical integrable nonlinear elliptic PDEs in two dimensions, namely the elliptic sine-Gordon [45, 49] and the Ernst equation [43].
The new method has the following advantages: (a) For linear PDEs, it yields an explicit integral representation involving appropriate transforms of boundary values. Similarly, for integrable nonlinear PDEs in two or three dimensions, it expresses the solution in terms of either a Riemann-Hilbert or a d-bar problem; these problems are uniquely defined in terms of appropriate nonlinear transforms of boundary values, called *spectral functions.* These functions are defined in the Fourier space, i.e. they do not involve the independent variables

of the PDE. This crucial feature of the new method makes it possible to obtain useful asymptotic information about the solution even before characterising the spectral functions in terms of the given boundary conditions [27].

(b) For a large class of linear boundary value problems, it is possible to eliminate the transforms of the unknown boundary values and to obtain an effective integral representation defined only in terms of the given boundary conditions [23]. This is achieved by analysing a certain equation, called the *global relation*, which couples the transforms of all relevant boundary values. Similarly, for certain nonlinear boundary value problems, called *linearisable* [20, 22], it is possible, by analysing the global relation, to express the spectral functions in terms of the given boundary conditions. Thus, for linearisable boundary conditions, the new method yields a representation which is as effective as the classical representation for the Cauchy problem obtained by the inverse scattering transform method.

(c) For general linear boundary value problems, the global relation provides a novel approach for characterising the unknown boundary values in terms of the given boundary conditions, i.e. for characterising the generalised Dirichlet to Neumann map. Novel numerical techniques for computing this map are presented in [16, 31, 33, 50, 52]. Similarly, for integrable nonlinear *evolution* PDEs, the global relation yields an effective characterisation of the generalised Dirichlet to Neumann map. This important development was first achieved in [6] by employing the so-called Gelfand-Levitan-Marchenko (GLM) representation, and was implemented numerically in [56]. It was later realised that this approach also provides a direct characterisation of the spectral functions in terms of the given boundary conditions, avoiding the GLM representation [21, 55]. However, it must be noted that for non-linearisable boundary value problems, the above characterisation involves a *nonlinear* equation.

In this paper we implement step (c) above for the elliptic sine-Gordon equation formulated in the half plane. To our knowledge, this is the first instance that the Dirichlet to Neumann map is characterised for an integrable *elliptic* (as opposed to *evolution*) PDE. Actually, although the relevant characterisation shares conceptual similarities with the method used for evolution PDEs, it does involve novel elements. In particular, while for evolution PDEs it suffices to analyse a single eigenfunction, in the case of the elliptic sine Gordon equation formulated in the upper half plane it is necessary to analyse two different eigenfunctions and to combine the resulting expressions.

The sine Gordon equation posed in the upper half plane was considered in [49], but the Dirichlet to Neumann map (Theorem 5.1) was not derived. Furthermore, Theorem 3.1, which proves that a formula based on the solution of a certain Riemann-Hilbert problem solves the given problem *provided* that the global relation, namely $\{b(\lambda) = 0,\ \lambda \geq 0\}$, is valid, was not derived in [49].

2 Preliminaries

The elliptic sine-Gordon

$$q_{xx} + q_{yy} = \sin q, \qquad x, y \in \mathbb{R}, \tag{2.1}$$

is the compatibility condition of the following Lax pair:

$$\Psi_x(x,y,\lambda) + \omega(\lambda)[\sigma_3, \Psi](x,y,\lambda) = Q(x,y,\lambda)\Psi(x,y,\lambda), \tag{2.2}$$

$$\Psi_y(x,y,\lambda) + \Omega(\lambda)[\sigma_3, \Psi](x,y,\lambda) = iQ(x,y,-\lambda)\Psi(x,y,\lambda), \tag{2.3}$$

where $\lambda \in \mathbb{C}$ and

$$\sigma_3 = diag(1,-1), \quad \omega(\lambda) = \frac{1}{4i}\left(\lambda - \frac{1}{\lambda}\right), \quad \Omega(\lambda) = \frac{1}{4}\left(\lambda + \frac{1}{\lambda}\right), \tag{2.4}$$

$$Q(x,y,\lambda) = \frac{1}{4}\begin{pmatrix} \frac{i}{\lambda}(1-\cos q(x,y)) & -\frac{1}{\lambda}\sin q(x,y) + iq_x(x,y) + q_y(x,y) \\ \frac{1}{\lambda}\sin q(x,y) + iq_x(x,y) + q_y(x,y) & \frac{i}{\lambda}(\cos q(x,y) - 1) \end{pmatrix}. \tag{2.5}$$

Equations (2.2 and 2.3) can be written in the form

$$d\left[\mathrm{e}^{(\omega(\lambda)x+\Omega(\lambda)y)\widehat{\sigma_3}}\Psi(x,y,\lambda)\right] = \mathrm{e}^{(\omega(\lambda)x+\Omega(\lambda)y)\hat{\sigma}_3}W(x,y,\lambda), \tag{2.6}$$

where the differential form W is given by

$$W(x,y,\lambda) = (Q(x,y,\lambda)dx + iQ(x,y,-\lambda)dy)\,\Psi(x,y,\lambda) \tag{2.7}$$

and the action of $\widehat{\sigma_3}$ on a matrix A is defined by

$$\widehat{\sigma_3}A = [\sigma_3, A],$$

hence

$$\mathrm{e}^{\widehat{\sigma_3}x}A = \mathrm{e}^{\sigma_3 x}A\mathrm{e}^{-\sigma_3 x} = \begin{pmatrix} A_{11} & \mathrm{e}^{2x}A_{12} \\ \mathrm{e}^{-2x}A_{21} & A_{22} \end{pmatrix}.$$

The definitions of $\Omega(\lambda)$ and $\omega(\lambda)$ yield

$$Re(\omega(\lambda)) > 0, \quad for\ Im\lambda > 0; \qquad Re(\Omega(\lambda)) > 0, \quad for\ Re\lambda > 0. \tag{2.8}$$

Equation (2.6) implies that if $Q(x,y)$ is a solution of equation (2.1) in a given simply connected domain D, then the following equation characterises a function $\Psi_j(x,y,\lambda)$ which solves both equations (2.2) and (2.3):

$$\Psi_j(x,y,\lambda) = I + \int_{(x_j,y_j)}^{(x,y)} \mathrm{e}^{-(\omega(\lambda)(x-\xi)+\Omega(\lambda)(y-\eta))\widehat{\sigma_3}}W_j(\xi,\eta,\lambda), \quad (x,y),\ (x_j,y_j) \in\ D, \tag{2.9}$$

where (x_j, y_j) is a fixed point in D and W_j is the differential form defined by (2.7) with Ψ replaced by Ψ_j and (x,y) replaced by (ξ,η). It is shown in [19] that if D is the interior of a convex polygon, then the proper choice for (x_j, y_j) is the collection of all the vertices of the polygon.

3 The elliptic sine-Gordon equation on the half plane

We consider equation (2.1) posed in the upper half plane, i.e. in the domain $\{-\infty < x < \infty,\ 0 < y < \infty\}$. We first review the results of [49].
In this case, the simply connected domain D consist of a degenerate polygon with two corners, $((-\infty, y)$ and $(\infty, y))$, namely it is the half plane $y \geq 0$.

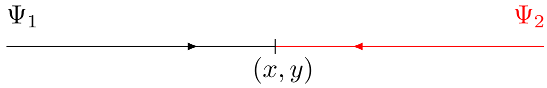

Figure 1: The functions Ψ_1, Ψ_2.

Let Ψ_1, Ψ_2 denote the solutions of (2.6) associated with these two corners, see figure 1. Then

$$\Psi_1(x,y,\lambda) = I + \int_{-\infty}^{x} \mathrm{e}^{-\omega(\lambda)(x-\xi)\widehat{\sigma_3}} \left(Q\Psi_1\right)(\xi,y,\lambda)d\xi, \tag{3.1}$$

$$\Psi_2(x,y,\lambda) = I - \int_{x}^{\infty} \mathrm{e}^{-\omega(\lambda)(x-\xi)\widehat{\sigma_3}} \left(Q\Psi_2\right)(\xi,y,\lambda)d\xi, \tag{3.2}$$

$$x\in\mathbb{R},\ \ 0\le y<\infty, \quad \lambda\in\mathbb{R}.$$

The first of equations (2.8) implies that the first and the second column vectors of Ψ_1 are analytic in λ for $Im\lambda<0$ and $Im\lambda>0$ respectively. The function Ψ_2 has the opposite analyticity. This remark justifies the following notations:

$$\Psi_1 = (\Psi_1^-, \Psi_1^+),\ \ \lambda\in(\mathbb{C}^-,\mathbb{C}^+), \qquad \Psi_2 = (\Psi_2^+, \Psi_2^-),\ \ \lambda\in(\mathbb{C}^+,\mathbb{C}^-). \tag{3.3}$$

Using the fact that Ψ_1^+ and Ψ_2^+ are analytic fucntions of λ for $Im\lambda>0$, whereas Ψ_1^- and Ψ_2^- are analytic functions of λ for $Im\lambda<0$, it is possible to formulate a Riemann-Hilbert (RH) problem with a jump along the real axis of the complex λ plane. In order to compute the jump matrix of the RH problem, we use the fact that any two solutions of the Lax pair (2.2)-(2.3) are related by a matrix of the form $[\exp(-(\omega(\lambda)x+\Omega(\lambda)y)\widehat{\sigma_3})]R(\lambda)$. Hence

$$\Psi_2(x,y,\lambda) = \Psi_1(x,y,\lambda)\mathrm{e}^{-(\omega(\lambda)x+\Omega(\lambda)y)\widehat{\sigma_3}}R(\lambda), \quad \lambda\in\mathbb{R}, \quad x\in\mathbb{R},\ \ 0\le y<\infty. \tag{3.4}$$

Evaluating this equation at the point $\{y=0,\ x\to-\infty\}$, we find

$$R(\lambda) = I - \int_{-\infty}^{\infty} \mathrm{e}^{\omega(\lambda)\xi\hat{\sigma}_3} \left(Q\Psi_2\right)(\xi,0,\lambda)d\xi, \quad \lambda\in\mathbb{R}. \tag{3.5}$$

Introducing the notation

$$m(x,\lambda) = \Psi_2(x,0,\lambda), \qquad \lambda\in(\mathbb{C}^+,\mathbb{C}^-), \quad x\in\mathbb{R},$$

it follows that the spectral function $R(\lambda)$ is uniquely determined in terms of the solution of the linear Volterra integral equation

$$m(x,\lambda) = I - \int_{x}^{\infty} \mathrm{e}^{-\omega(\lambda)(x-\xi)\widehat{\sigma_3}} Q_0(\xi,\lambda)m(\xi,\lambda)d\xi,\ \ \lambda\in(\mathbb{C}^+,\mathbb{C}^-),\ \ x\in\mathbb{R}, \tag{3.6}$$

where Q_0 is defined in terms of the boundary values $q(x,0)$ and $q_y(x,0)$ by

$$Q_0(x,\lambda)=\frac{1}{4}\left(\begin{array}{cc} \frac{i}{\lambda}(1-\cos q(x,0)) & -\frac{1}{\lambda}\sin q(x,0)+iq_x(x,0)+q_y(x,0) \\ \frac{1}{\lambda}\sin q(x,0)+iq_x(x,0)+q_y(x,0) & \frac{i}{\lambda}(\cos q(x,0)-1) \end{array}\right),$$

$$x\in\mathbb{R},\ \lambda\in\mathbb{C}. \tag{3.7}$$

The symmetry properties of Q_0 imply certain symmetry relations for $m(x,\lambda)$ and hence for $R(\lambda)$. This implies that $R(\lambda)$ has the following form:

$$R(\lambda)=\left(\begin{array}{cc} a(\lambda) & b(-\lambda) \\ b(\lambda) & a(-\lambda) \end{array}\right), \qquad \lambda\in\mathbb{R}, \tag{3.8}$$

where $a(\lambda)$ has an analytic continuation for $Im\lambda>0$ and $b(\lambda)$ is defined for $\lambda\in\mathbb{R}$.

The global relation

The global relation yields the following important equations:

$$a(\lambda)=1, \quad Im\lambda\geq 0; \qquad b(\lambda)=0, \quad Re\lambda\geq 0. \tag{3.9}$$

These relations were derived in [49]. We give here an alternative derivation by introducing an additional solution of equation (2.6) associated with the vertex (x,∞), namely we consider the function Ψ_3 defined by

$$\Psi_3(x,y,\lambda)=I-i\int_y^\infty \mathrm{e}^{-\Omega(\lambda)(y-\eta)\widehat{\sigma_3}}Q(x,\eta,-\lambda)\Psi_3(x,\eta,\lambda)d\eta, \ \ x\in\mathbb{R},\ 0<y<\infty, \ \ \lambda\in\mathbb{R}. \tag{3.10}$$

The second of equation (2.8) implies that the first and the second column vectors of Ψ_3 are analytic in λ for $Re\lambda\geq 0$ and $Re\lambda\leq 0$ respectively. This fact justifies the notation

$$\Psi_3=(\Psi_3^R,\Psi_3^L), \quad (Re\lambda\geq 0, Re\lambda\leq 0). \tag{3.11}$$

The functions Ψ_2 and Ψ_3 satisfy the same boundary condition at infinity and the same equation, hence they are equal in their respective domain of analyticity, i.e.

$$\Psi_2^+(x,y,\lambda)=\Psi_3^R(x,y,\lambda), \quad x\in\mathbb{R},\ 0<y<\infty, \quad 0\leq\arg(\lambda)\leq\frac{\pi}{2}. \tag{3.12}$$

Similarly, the functions Ψ_1 and Ψ_3 satisfy the same boundary condition at infinity and the same equation, hence they are equal in their respective domain of analyticity, i.e.

$$\Psi_1^-(x,y,\lambda)=\Psi_3^R(x,y,\lambda), \quad x\in\mathbb{R},\ 0<y<\infty, \quad \frac{3\pi}{2}\leq\arg(\lambda)\leq 2\pi. \tag{3.13}$$

We denote by $m^+=(m_1,m_2)^\tau$ the first column vector of $m(x,\lambda)$ given by (3.6), i.e.

$$m^+(x,\lambda)=\left(\begin{array}{c} m_1 \\ m_2 \end{array}\right)=\Psi_2^+(x,0,\lambda), \quad x\in\mathbb{R},\ \lambda\in\mathbb{C}^+. \tag{3.14}$$

Then, evaluating equation (3.12) at $y = 0$ we find

$$m^{+}(x,\lambda) = \Psi_3^R(x,0,\lambda), \quad x \in \mathbb{R}, \ \ 0 \le \arg(\lambda) \le \frac{\pi}{2}. \tag{3.15}$$

Letting $x \to -\infty$ in (3.15) we find

$$a(\lambda) = 1, \quad 0 \le \arg(\lambda) \le \frac{\pi}{2}; \qquad b(\lambda) = 0, \quad \lambda \ge 0. \tag{3.16}$$

The analytic continuation of the first equation implies the first of equations (3.9). Similarly, we denote by $n^{-} = (n_1, n_2)^{\tau}$ the first column vector of $\Psi_1(x,0,\lambda)$:

$$n^{-}(x,\lambda) = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \Psi_1^{-}(x,0,\lambda), \qquad x \in \mathbb{R}, \ \ \lambda \in \mathbb{C}^{-}. \tag{3.17}$$

Then evaluating the equation (3.13) at $y = 0$ we find

$$n^{-}(x,\lambda) = \Psi_3^R(x,0,\lambda), \quad x \in \mathbb{R}, \quad \frac{3\pi}{2} \le \arg(\lambda) \le 2\pi. \tag{3.18}$$

Summary

The above discussion suggests the following steps for solving the Dirichlet problem for the elliptic sine-Gordon equation:
(i) Given Dirichlet data $q(x,0) = g_0(x)$, characterise the unknown Neumann boundary value $q_y(x,0)$ by the requirement that the spectral functions $a(\lambda)$, $b(\lambda)$ satisfy the constraints (3.16). These functions are defined as follows:

$$\begin{aligned} a(\lambda) &= 1 - \frac{1}{4}\int_{-\infty}^{\infty}\left\{\frac{i}{\lambda}(1-\cos g_0(\xi))m_1(\xi,\lambda) + \left[-\frac{1}{\lambda}\sin g_0(\xi) + i\dot{g_0}(\xi) + q_y(\xi,0)\right] m_2(\xi,\lambda)\right\} d\xi, \\ &\quad Im\lambda \ge 0, \end{aligned} \tag{3.19}$$

$$\begin{aligned} b(\lambda) &= -\frac{1}{4}\int_{-\infty}^{\infty} \mathrm{e}^{-2\omega(\lambda)\xi}\left\{\frac{i}{\lambda}(\cos g_0(\xi) - 1)m_2(\xi,\lambda) + \left[\frac{1}{\lambda}\sin g_0(\xi) + i\dot{g_0}(\xi) + q_y(\xi,0)\right] m_1(\xi,\lambda)\right\} d\xi, \\ &\quad \lambda \in \mathbb{R}, \end{aligned} \tag{3.20}$$

where $(m_1, m_2)^{\tau}$ satisfies the first column vector of the ODE (3.6) with $q(x,0)$ replaced by $g_0(x)$, and $\dot{g_0}(x)$ denotes the derivative of $g_0(x)$.
(ii) After characterising $b(\lambda)$ in terms of the given data $g_0(x)$, solve the RH problem with the jump matrix uniquely defined in terms of $b(\lambda)$. This RH problem is given by (3.27) below, and is equivalent to the following equation

$$\begin{aligned} &(\Psi_1^{-}(x,y,\lambda), -\Psi_1^{+}(x,y,\lambda)) = I + \\ &\frac{1}{2\pi i}\int_{-\infty}^{0}(\Psi_1^{-}(x,y,l), \Psi_1^{+}(x,y,l))\begin{pmatrix} 0 & b(-l)\mathrm{e}^{-2(\omega(l)x+\Omega(l)y)} \\ b(l)\mathrm{e}^{2(\omega(l)x+\Omega(l)y)} & 0 \end{pmatrix}\frac{dl}{l-\lambda}, \quad \lambda \notin \mathbb{R}. \end{aligned} \tag{3.21}$$

Equation (2.2) implies

$$iq_x(x,y) + q_y(x,y) = -\frac{1}{\pi}\int_{-\infty}^{0} b(l)(\Psi_1^{+})_2(x,y,l)\mathrm{e}^{2\omega(l)x+2\Omega(l)y}dl, \quad x \in \mathbb{R}, \ \ 0 < y < \infty, \tag{3.22}$$

where $(\Psi_1^{+})_2$ denotes the second component of the vector Ψ_1^{+}.

The existence theorem

The above procedure outlines how, under the assumption of existence, a solution can be constructed. We now prove that, assuming the given boundary conditions are such that the associated spectral functions $a(\lambda)$, $b(\lambda)$ satisfy the global relation and have sufficiently small norm, the problem admits a unique solution.

Theorem 3.1 *Let the functions $g_0(x)$, $g_1(x)$ be such that $g_0 - 2\pi m \in \mathbf{H}^1(\mathbb{R})$, $m \in \mathbb{Z}$, and $g_1(x) \in \mathbf{H}^1(\mathbb{R})$.*
Define $a(\lambda)$ and $b(\lambda)$ by

$$\begin{aligned} a(\lambda) &= 1 - \frac{1}{4}\int_{-\infty}^{\infty}\left\{\frac{i}{\lambda}(1-\cos g_0(\xi))m_1(\xi,\lambda) + \left[-\frac{1}{\lambda}\sin g_0(\xi) + i\dot{g_0}(\xi) + g_1(\xi)\right]m_2(\xi,\lambda)\right\}d\xi, \\ & \qquad Im\lambda \geq 0, \end{aligned} \tag{3.23}$$

$$\begin{aligned} b(\lambda) &= -\frac{1}{4}\int_{-\infty}^{\infty} \mathrm{e}^{-2\omega(\lambda)\xi}\left\{\frac{i}{\lambda}(\cos g_0(\xi) - 1)m_2(\xi,\lambda) + \left[\frac{1}{\lambda}\sin g_0(\xi) + i\dot{g_0}(\xi) + g_1(\xi)\right]m_1(\xi,\lambda)\right\}d\xi, \\ & \qquad \lambda \in \mathbb{R}, \end{aligned} \tag{3.24}$$

where $(m_1(x,\lambda),\ m_2(x,\lambda))$ denotes the solution of the following system of ODEs:

$$\begin{cases} (m_1)_x = \frac{i}{\lambda}[1-\cos g_0(x)]m_1 - [\frac{1}{\lambda}\sin g_0(x) - i\dot{g_0}(x) - g_1(x)]m_2, \\ (m_2)_x + 2\omega(\lambda)m_2 = [\frac{1}{\lambda}\sin g_0(x) + i\dot{g_0}(x) + g_1(x)]m_1 - \frac{i}{\lambda}[1-\cos g_0(x)]m_2, \\ \lim_{x\to\infty}(m_1,m_2) = (1,0), \qquad x\in\mathbb{R},\ \lambda\in\mathbb{C}^+. \end{cases} \tag{3.25}$$

Assume that, given $g_0(x)$, there exists a function $g_1(x)$ such that $a(\lambda)$, $b(\lambda)$ satisfy the following constraints:

$$a(\lambda) = 1,\ 0 \leq arg(\lambda) \leq \pi, \qquad b(\lambda) = 0,\ \lambda \geq 0. \tag{3.26}$$

Define the following Riemann-Hilbert problem in terms of $b(\lambda)$:

$$\Psi^-(x,y,\lambda) = \Psi^+(x,y,\lambda)J(x,y,\lambda), \qquad \lambda\in\mathbb{R}, \quad \Psi = I + O\left(\frac{1}{\lambda}\right),\ \lambda\to\infty, \tag{3.27}$$

where

$$J = \begin{pmatrix} 1 & b(-\lambda)\mathrm{e}^{-\theta(x,y,\lambda)} \\ -b(\lambda)\mathrm{e}^{\theta(x,y,\lambda)} & 1 \end{pmatrix} \qquad \theta(x,y,\lambda) = 2(\omega(\lambda)x + \Omega(\lambda)y). \tag{3.28}$$

If the $\mathbf{H}^1$ norm of the data $g_0(x)$, $g_1(x)$ is sufficiently small, the above Riemann-Hilbert problem admits a unique solution $\Psi(x,y,\lambda)$.
Let the function $q(x,y)$, $x\in\mathbb{R}$, $0<y<\infty$, be defined in terms of this unique solution by

$$iq_x + q_y = -\lim_{\lambda\to\infty}(i\lambda\Psi)_{12}, \quad \cos q(x,y) = 1 - \lim_{\lambda\to\infty} 4i\lambda\left(\frac{\partial\Psi}{\partial x}\right)_{22} - 2\lim_{\lambda\to\infty}(\lambda\Psi)_{12}^2. \tag{3.29}$$

Then $q(x,y)$ solves the elliptic sine-Gordon equation (2.1) in the half plane $y>0$, and furthermore

$$q(x,0) = g_0(x), \qquad q_y(x,0) = g_1(x), \quad x\in\mathbb{R}. \tag{3.30}$$

Proof
The unique solvability of the Riemann-Hilbert problem (3.27) is a consequence of the so-called *vanishing lemma*, which states that the problem corresponding to the same jump matrix, but with the condition $\Psi = O\left(\frac{1}{\lambda}\right)$ at infinity, admits only the trivial solution. The validity of the lemma follows from our small norm assumption.
The proof that $q(x,y)$ defined by (3.29) is a solution of equation (2.1) is a standard application of ideas of the so-called dressing method, see [23].
To prove (3.30), we note that the map $(a(\lambda), b(\lambda)) \to (g_0(x), g_1(x))$ which is the inverse of the map defined by (3.24), is given by

$$\begin{aligned} \cos g_0(x) &= 1 - \lim_{\lambda\to\infty} 4i\lambda \left(\frac{\partial M(x,\lambda)}{\partial x}\right)_{22} - 2\lim_{\lambda\to\infty} (\lambda M(x,\lambda))_{12}^2, \\ i\dot{g}_0(x) + g_1(x) &= -i \lim_{\lambda\to\infty} (\lambda M(x,\lambda))_{12}, \qquad x\in\mathbb{R},\ \lambda\in\mathbb{C}^-, \end{aligned} \tag{3.31}$$

where M is the solution of the Riemann-Hilbert problem

$$M^-(x,\lambda) = M^+(x,\lambda)J^{(x)}(x,\lambda), \quad \lambda\in\mathbb{R}, \qquad M = I + O\left(\frac{1}{\lambda}\right),\ |\lambda|\to\infty, \tag{3.32}$$

with

$$J^{(x)}(x,\lambda) = \begin{pmatrix} 1 & \frac{b(-\lambda)}{a(-\lambda)}\mathrm{e}^{-2\omega(\lambda)x} \\ -\frac{b(\lambda)}{a(\lambda)}\mathrm{e}^{2\omega(\lambda)x} & \frac{1}{a(\lambda)a(-\lambda)} \end{pmatrix}, \qquad \lambda\in\mathbb{R},\ \ x\in\mathbb{R}. \tag{3.33}$$

If the constraints (3.26) are satisfied, then the jump matrix of the above Riemann-Hilbert problem satisfies

$$J^{(x)}(x,\lambda) = J(x,0,\lambda), \tag{3.34}$$

where J is given by (3.27). Evaluating equations (3.27) and (3.28) at $y=0$ and comparing the resulting equations with equations (3.32) and (3.33), where $J^{(x)}$ is given by (3.34), we conclude that $\Psi(x,0,\lambda) = M(x,\lambda)$. Hence, evaluating equation (3.29) at $y=0$ and comparing the resulting expression with equations (3.31) we find equations (3.30).
QED

4 The linear limit

In the linear limit,

$$(\Psi_1^+)_2 \to 1, \quad m_1 \to 1, \quad m_2 \to 0, \ \ \sin g_0 \to g_0,$$

thus equation (3.24) becomes

$$b(\lambda) = -\frac{1}{4}\int_{-\infty}^{\infty} \mathrm{e}^{-2\omega(\lambda)\xi}\left[\chi(\xi) + \frac{g_0(\xi)}{\lambda}\right] d\xi, \quad \lambda\in\mathbb{R}, \tag{4.1}$$

where $\chi(\xi)$ is defined by

$$\chi(\xi) = i\dot{g}_0(\xi) + q_y(\xi,0), \quad \xi\in\mathbb{R}. \tag{4.2}$$

Equation (3.22) becomes

$$iq_x(x,y) + q_y(x,y) = -\frac{1}{\pi}\int_{-\infty}^{0} b(l)\mathrm{e}^{2\omega(l)x+2\Omega(l)y}dl, \qquad x \in \mathbb{R},\ \ 0 < y < \infty. \tag{4.3}$$

Equation (4.3) implies

$$q(x,y) = -\frac{1}{\pi}\int_{-\infty}^{0} \frac{b(l)}{l}\mathrm{e}^{2\omega(l)x+2\Omega(l)y}dl, \qquad x \in \mathbb{R},\ \ 0 < y < \infty. \tag{4.4}$$

Indeed, applying the operator $i\partial_x + \partial_y$ to equation (4.4) and using

$$2i\omega(\lambda) + 2\Omega(\lambda) = \lambda,$$

we find (4.3).
It can immediately be verified that equation (4.4) satisfies the modified Helmholtz equation

$$q_{xx} + q_{yy} = q, \qquad x \in \mathbb{R},\ \ 0 < y < \infty.$$

The expression for $b(\lambda)$ given by equation (4.1) involves the unknown function $q_y(x,0)$. Thus, in order to obtain an effective solution of the linear problem, we must use the global relation $\{b(\lambda) = 0, \lambda > 0\}$ to eliminate this unknown boundary value. In this respect we note that the transformation $\lambda \to -\frac{1}{\lambda}$ leaves $\omega(\lambda)$ invariant. Using this transformation in the global relation $\{b(\lambda) = 0, \lambda > 0\}$, we find

$$\int_{-\infty}^{\infty} \mathrm{e}^{-2\omega(\lambda)\xi}\left[\chi(\xi) - \lambda g_0(\xi)\right] d\xi = 0, \quad \lambda \leq 0. \tag{4.5}$$

Multiplying this equation by $\frac{1}{4}$ and adding the resulting equation to equation (4.1) with $\lambda \leq 0$, we find

$$b(\lambda) = -\Omega(\lambda)\int_{-\infty}^{\infty} \mathrm{e}^{-2\omega(\lambda)\xi} g_0(\xi)d\xi, \quad \lambda \leq 0. \tag{4.6}$$

In summary, the solution of the Dirichlet problem of the modified Helmholtz equation in the upper half plane is given by equation (4.4), where $b(\lambda)$ is given explicitly in terms of the Dirichlet boundary condition $q(x,0) = g_0(x)$ by equation (4.6).

The above construction of $b(\lambda)$ for the modified Helmholtz equation illustrates the following remarkable fact that occurs in a large class of linear boundary value problems: by utilising certain invariant properties in the complex λ plane (in our example, $\lambda \to -\frac{1}{\lambda}$), it is possible to *eliminate* the unknown boundary values, i.e. it is possible to compute the spectral functions *without* the need to compute the unknown boundary values.
For nonlinearisable integrable boundary value problems, it is *not* possible to eliminate the unknown boundary values; however, it *is* possible to characterise these unknown functions in terms of the given boundary conditions by analysing the global relation. The relevant construction needed here is presented in section 5. In order to motivate this construction we present next two different approaches. In both approaches, the global relation (4.5) is used to obtain the following formula:

$$\chi(x) = \frac{1}{\pi}\int_{-\infty}^{0} \Omega(\lambda)\mathrm{e}^{2\omega(\lambda)x}\left(\int_{-\infty}^{\infty} \mathrm{e}^{-2\omega(\lambda)\xi} g_0(\xi)d\xi\right) d\lambda, \quad x \in \mathbb{R}. \tag{4.7}$$

This formula expresses the unknown Neumann boundary values in terms of the given Dirichlet boundary condition.
Before deriving this equation, we note that the right hand side of (4.7) coincides with the expression obtained by evaluating equation (4.3) at $y = 0$.
In the first approach, we consider the global relation $\{b(\lambda) = 0,\ \lambda \geq 0\}$, i.e. the equation

$$\int_{-\infty}^{\infty} \mathrm{e}^{-2\omega(\lambda)\xi} \left[\chi(\xi) + \frac{g_0(\xi)}{\lambda} \right] d\xi = 0, \quad \lambda \geq 0. \tag{4.8}$$

We multiply (4.8) by $(1 + \frac{1}{\lambda^2})\mathrm{e}^{2\omega(\lambda)x}$ and integrate the resulting equation with respect to λ along the positive real axis, to obtain

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \left(1 + \frac{1}{\lambda^2}\right) \left[\int_{-\infty}^{\infty} \mathrm{e}^{2\omega(\lambda)(x-\xi)} \chi(\xi) d\xi \right] d\lambda = \\ -&\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{1}{\lambda} \left(1 + \frac{1}{\lambda^2}\right) \mathrm{e}^{2\omega(\lambda)x)} \left[\int_{-\infty}^{\infty} \mathrm{e}^{-2\omega(\lambda)\xi} g_0(\xi) d\xi \right] d\lambda. \end{aligned} \tag{4.9}$$

Using in the left and the right hand sides of equation (4.9) the change of variables, respectively

$$\Lambda = \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right), \qquad l = -\frac{1}{\lambda},$$

equation (4.9) becomes equation (4.7).
In the second approach, we use the linear limit of the more general global relation (3.15) (recall that the global relation $\{b(\lambda) = 0,\ \lambda \geq 0\}$ is the particular case of equation (3.15) when $x \to -\infty$). The linear limit of equation (3.15) yields

$$\begin{aligned} I - \int_{x}^{\infty} \mathrm{e}^{-\omega(\lambda)(x-\xi)\widehat{\sigma_3}} Q_0^{(L)}(\xi, \lambda) d\xi \quad &= \quad I - i \int_{0}^{\infty} \mathrm{e}^{\Omega(\lambda)\eta\widehat{\sigma_3}} Q^{(L)}(\xi, \eta, -\lambda) d\eta, \\ &\qquad x \in \mathbb{R}, 0 < arg(\lambda) < \frac{\pi}{2}, \end{aligned} \tag{4.10}$$

where $Q_0^{(L)}$ denotes the linear limit of the expression Q_0 defined in (3.7) and $Q^{(L)}$ denotes the linear limit of the expression for Q defined in (2.5), i.e.

$$Q_0^{(L)}(\xi, \lambda) = \frac{1}{4} \begin{pmatrix} 0 & \chi(\xi) - \frac{g_0(\xi)}{\lambda} \\ \chi(\xi) + \frac{g_0(\xi)}{\lambda} & 0 \end{pmatrix}, \quad \xi \in \mathbb{R},\ \lambda \in \mathbb{C} \tag{4.11}$$

and

$$\begin{aligned} Q^{(L)}(x, \eta, -\lambda) = \frac{1}{4} &\begin{pmatrix} 0 & iq_x(x,\eta) + q_y(x,\eta) + \frac{q(x,\eta)}{\lambda} \\ iq_x(x,\eta) + q_y(x,\eta) - \frac{q(x,\eta)}{\lambda} & 0 \end{pmatrix}, \\ &x \in \mathbb{R},\ \ 0 < \eta < \infty,\ \ \lambda \in \mathbb{C}. \end{aligned} \tag{4.12}$$

The (21) component of the matrix equation (4.10) yields

$$\int_{x}^{\infty} \left(\chi(\xi) + \frac{g_0(\xi)}{\lambda} \right) \mathrm{e}^{-2\omega(\lambda)(\xi - x)} d\xi = i \int_{0}^{\infty} \left(iq_x(x,\eta) + q_y(x,\eta) - \frac{q(x,\eta)}{\lambda} \right) \mathrm{e}^{-2\Omega(\lambda)\eta} d\eta,$$

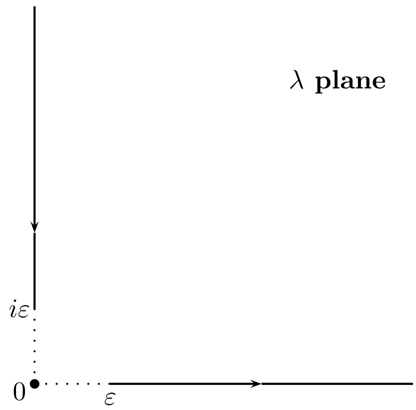

Figure 2: The contour ∂D_1^ε

$$x \in \mathbb{R}, \quad 0 < \eta < \infty, \quad 0 \leq arg(\lambda) \leq \frac{\pi}{2}. \tag{4.13}$$

Before solving this equation for $\chi(x)$, we note that the global relation (4.8) is a particular case of equation (4.13). Indeed, the exponentials appearing in the left and the right hand sides of equation (4.9) are bounded as $|\lambda| \to \infty$ and as $|\lambda| \to 0$, if $0 \leq arg(\lambda) \leq \pi$ and $-\frac{\pi}{2} \leq arg(\lambda) \leq \frac{\pi}{2}$ respectively. Thus equation (4.13) is well defined for $0 \leq arg(\lambda) \leq \frac{\pi}{2}$. However, if $x \to -\infty$, the exponential on the left hand side of (4.13) is bounded only for $\lambda \in \mathbb{R}$, thus in this case equation (4.13) is valid only for $\lambda \geq 0$; then letting $x \to -\infty$ in (4.13) we find equation (4.8).

We multiply equation (4.13) by $1 + \frac{1}{\lambda^2}$, subtract from both sides of the resulting equation the term $\frac{2ig_0(x)}{\lambda^2} + \frac{4iq_y(x,0)}{\lambda}$ and then integrate with respect to λ along the curve ∂D_1 defined as follows (see figure 2):

$$\partial D_1 = \lim_{\varepsilon \to 0} \partial D_1^\varepsilon = \{|\lambda| e^{i0} \cup |\lambda| e^{\frac{i\pi}{2}}, \ \varepsilon \leq |\lambda| < \infty\}. \tag{4.14}$$

The resulting equation can be simplified using the following results:

(i)
$$\int_{\partial D_1} \left(1 + \frac{1}{\lambda^2}\right) \left[\int_x^\infty \chi(\xi) e^{\frac{i}{2}(\lambda - \frac{1}{\lambda})(\xi - x)} d\xi\right] d\lambda = 2\pi\chi(x), \quad x \in \mathbb{R}. \tag{4.15}$$

Indeed, we use Cauchy's theorem in the domain bounded by the union of ∂D_1^ε and of the circular arcs $C^{(\varepsilon)}$ and $C^{(\infty)}$ (see figure 3)

$$C^{(\varepsilon)} = \{\varepsilon e^{i\theta}, \ 0 \leq \theta \leq \frac{\pi}{2}\}, \quad C^{(\infty)} = \lim_{R \to \infty} C^{(R)}, \ C^{(R)} = \{R e^{i\theta}, \ 0 \leq \theta \leq \frac{\pi}{2}\}.$$

The integrand of the λ-integral in (4.15) is an analytic function in λ, with poles at the origin and at infinity. Integration by parts yields the term $\frac{2i\chi(x)}{\lambda} \frac{\lambda^2+1}{\lambda^2-1}$, thus

$$\left(1 + \frac{1}{\lambda^2}\right) \int_x^\infty \chi(\xi) e^{\frac{i}{2}(\lambda - \frac{1}{\lambda})(\xi - x)} d\xi \sim \begin{cases} \frac{2i\chi(x)}{\lambda}, & |\lambda| \to \infty, \\ -\frac{2i\chi(x)}{\lambda}, & |\lambda| \to 0, \end{cases} \quad x \in \mathbb{R}.$$

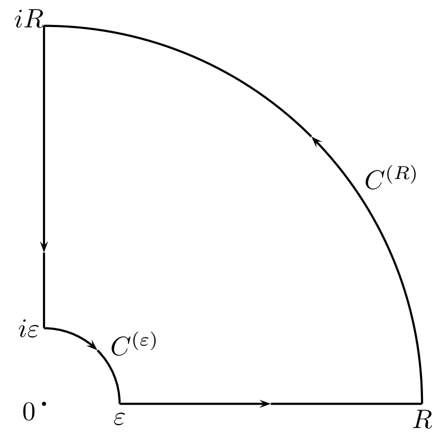

Figure 3: The arcs $C^{(\varepsilon)}$ and $C^{(R)}$

Thus Cauchy's theorem yields

$$\int_{\partial D_1}\left(1+\frac{1}{\lambda^2}\right)\left[\int_x^\infty \chi(\xi)\mathrm{e}^{\frac{i}{2}(\lambda-\frac{1}{\lambda})(\xi-x)}d\xi\right]d\lambda - 2i\chi(x)\int_{\pi/2}^0 id\theta + 2i\chi(x)\int_0^{\pi/2} id\theta = 0, \quad x\in\mathbb{R},$$

i.e. equation (4.15).

(ii)

$$\int_{\partial D_1}\left(1+\frac{1}{\lambda^2}\right)i\left[\int_0^\infty (iq_x(x,\eta)+q_y(x,\eta))\mathrm{e}^{-\frac{1}{2}(\lambda+\frac{1}{\lambda})\eta}d\eta\right]d\lambda = 0, \quad x\in\mathbb{R}. \tag{4.16}$$

The integrand of the λ-integral in (4.16) is again an analytic function of λ with poles at the origin and at infinity. Integration by parts yields the term $\frac{2i\chi(x)}{\lambda}$, thus

$$\left(1+\frac{1}{\lambda^2}\right)i\int_0^\infty (iq_x(x,\eta)+q_y(x,\eta))\mathrm{e}^{-\frac{1}{2}(\lambda+\frac{1}{\lambda})\eta}d\eta \sim \begin{cases} \frac{2i\chi(x)}{\lambda}, & |\lambda|\to\infty, \\ \frac{2i\chi(x)}{\lambda}, & |\lambda|\to 0, \end{cases} \quad x\in\mathbb{R},$$

and Cauchy's theorem yields

$$\int_{\partial D_1}\left(1+\frac{1}{\lambda^2}\right)i\left[\int_0^\infty (iq_x(x,\eta)+q_y(x,\eta))\mathrm{e}^{-\frac{1}{2}(\lambda+\frac{1}{\lambda})\eta}d\eta\right]d\lambda + 2i\chi(x)\int_{\pi/2}^0 id\theta + 2i\chi(x)\int_0^{\pi/2} id\theta = 0,$$

hence equation (4.16).

(iii)

$$\int_{\partial D_1}\left[\frac{1}{\lambda}\left(1+\frac{1}{\lambda^2}\right)(-i)\int_0^\infty q(x,\eta)\mathrm{e}^{-\frac{1}{2}(\lambda+\frac{1}{\lambda})\eta}d\eta + \frac{2ig_0(x)}{\lambda^2} + \frac{4iq_y(x,0)}{\lambda+\lambda^3}\right]d\lambda = 0, \quad x\in\mathbb{R}. \tag{4.17}$$

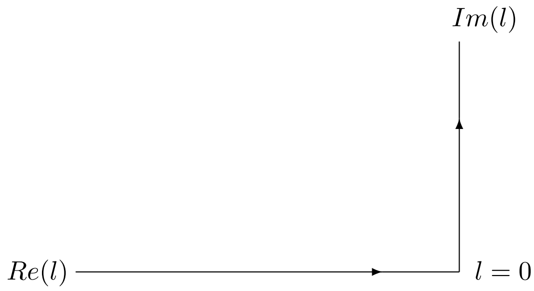

Figure 4: The contour ∂D_2 in the l- plane

Indeed, integrating by parts twice, we find

$$\frac{1}{\lambda}\left(1+\frac{1}{\lambda^2}\right)(-i)\int_0^\infty q(x,\eta)\mathrm{e}^{-\frac{1}{2}(\lambda+\frac{1}{\lambda})\eta}d\eta+\frac{2ig_0(x)}{\lambda^2}+\frac{4iq_y(x,0)}{\lambda+\lambda^3}\sim\begin{cases} O\left(\frac{1}{\lambda^2}\right), & |\lambda|\to\infty, \\ O(1), & |\lambda|\to 0, \end{cases} \quad x\in\mathbb{R}.$$

Hence the integrand of the λ integral on the right hand side of (4.17) is analytic and bounded in D_1 and equation (4.17) follows.

(iv) Define $G(\lambda)$ by the integral below; integrating by parts twice we find

$$G(\lambda)=\frac{1}{\lambda}\left(1+\frac{1}{\lambda^2}\right)\int_x^\infty g_0(\xi)\mathrm{e}^{\frac{i}{2}(\lambda-\frac{1}{\lambda})(\xi-x)}d\xi+\frac{2ig_0(x)}{\lambda^2}+\frac{4iq_y(x,0)}{\lambda+\lambda^3}\sim\begin{cases} O\left(\frac{1}{\lambda^2}\right), & |\lambda|\to\infty, \\ O\left(\frac{1}{\lambda}\right), & |\lambda|\to 0. \end{cases} \tag{4.18}$$

Thus the only singularity of the function $G(\lambda)$ in D_1 a simple pole at the origin, and its integral around ∂D_1 is bounded.

Using equations (4.15)-(4.18) we find

$$2\pi\chi(x)=-\int_{\partial D_1}\left[\frac{1}{\lambda}\left(1+\frac{1}{\lambda^2}\right)\mathrm{e}^{2\omega(\lambda)x}\left(\int_x^\infty g_0(\xi)\mathrm{e}^{-2\omega(\lambda)\xi}d\xi\right)+\frac{2ig_0(x)}{\lambda^2}+\frac{4iq_y(x,0)}{\lambda+\lambda^3}\right]d\lambda, \quad x\in\mathbb{R}. \tag{4.19}$$

Letting $l=-\frac{1}{\lambda}$, equation (4.19) becomes

$$2\pi\chi(x)=4\int_{\partial D_2}\left[\Omega(l)\mathrm{e}^{2\omega(l)x}\left(\int_x^\infty g_0(\xi)\mathrm{e}^{-2\omega(l)\xi}d\xi\right)-\frac{ig_0(x)}{2}+\frac{ilq_y(x,0)}{l^2+1}\right]dl, \quad x\in\mathbb{R}, \tag{4.20}$$

where ∂D_2 is the oriented boundary of the second quadrant of the complex l-plane shown in figure 4.

We now repeat the above computations starting with the linear limit of equation (3.18), namely the equation

$$I+\int_{-\infty}^x \mathrm{e}^{-\omega(\lambda)(x-\xi)\widehat{\sigma_3}}Q_0^{(L)}(x,\lambda)d\xi=I-i\int_0^\infty \mathrm{e}^{\Omega(\lambda)\eta\widehat{\sigma_3}}Q^{(L)}(\xi,\eta,\lambda)d\eta, \tag{4.21}$$

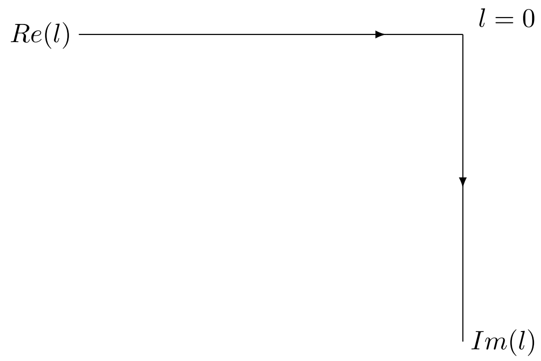

Figure 5: The contour ∂D_3 in the l- plane

valid for $\frac{3\pi}{2} \leq arg(\lambda) \leq 2\pi$.
The (21) component of the matrix equation (4.21) yields

$$\int_{-\infty}^{x} \left(\chi(\xi) + \frac{g_0(\xi)}{\lambda}\right) \mathrm{e}^{-2\omega(\lambda)(\xi - x)} d\xi = -i \int_{0}^{\infty} \left(iq_x(x,\eta) + q_y(x,\eta) - \frac{q(x,\eta)}{\lambda}\right) \mathrm{e}^{-2\Omega(\lambda)\eta} d\eta,$$

$$x \in \mathbb{R}, \quad 0 < \eta < \infty, \quad \frac{3\pi}{2} \leq arg(\lambda) \leq 2\pi, \quad x \in \mathbb{R}. \tag{4.22}$$

Multiplying this equation by $1 + \frac{1}{\lambda^2}$ and subtracting the leading order terms at the origin, using computation analogous to those presented in (i)-(iv) above, we find

$$2\pi\chi(x) = -\int_{\partial D_4} \left[\frac{1}{\lambda}\left(1 + \frac{1}{\lambda^2}\right) \mathrm{e}^{2\omega(\lambda)x} \left(\int_{-\infty}^{x} g_0(\xi) \mathrm{e}^{-2\omega(\lambda)\xi} d\xi\right) - \frac{2ig_0(x)}{\lambda^2} - \frac{4iq_y(x,0)}{\lambda + \lambda^3}\right] d\lambda,$$

$$x \in \mathbb{R}, \tag{4.23}$$

where ∂D_4 is the boundary of the fourth quadrant in the complex λ plane. Hence using the change of variables $l = -\frac{1}{\lambda}$,

$$2\pi\chi(x) = 4\int_{\partial D_3} \left[\Omega(l)\mathrm{e}^{2\omega(l)x} \left(\int_{-\infty}^{x} g_0(\xi)\mathrm{e}^{-2\omega(l)\xi} d\xi\right) + \frac{ig_0(x)}{2} - \frac{ilq_y(x,0)}{l^2+1}\right] dl, \quad x \in \mathbb{R}, \tag{4.24}$$

where ∂D_3 is the oriented boundary of the third quadrant of the complex l-plane shown in figure 5.
Adding equations (4.20) and (4.24) and deforming the contours along the positive and the negative imaginary axis to contours along the positive real axis we find

$$\chi(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[\Omega(l)\mathrm{e}^{2\omega(l)x} \left(\int_{-\infty}^{\infty} g_0(\xi)\mathrm{e}^{-2\omega(l)\xi} d\xi\right)\right] dl, \quad x \in \mathbb{R}. \tag{4.25}$$

Using the global relation $\{b(\lambda) = 0,\ \lambda \geq 0\}$, equation (4.25) reduces to equation (4.7).

5 The Dirichlet to Neumann map

In this section, with slight abuse, we use the following notation:

$$\chi(x,y) = iq_x(x,y) + q_y(x,y), \qquad \chi(x) = i\dot{g}_0(x) + q_y(x,0). \tag{5.1}$$

Our aim is to characterise uniquely the unknown function $\chi(x)$ (hence the Neumann datum $g_1(x) = q_y(x,0)$) in terms of the given function $g_0(x)$. We prove the following result:

Theorem 5.1 *Let $q(x,y)$ satisfy the elliptic sine-Gordon equation (2.1), for $-\infty < x < \infty$, $0 < y < \infty$, with prescribed Dirichlet boundary condition (within 2π multiples)*

$$q(x,0) = g_0(x), \quad g_0(x) - 2\pi m \in \mathbf{H}^1(\mathbb{R}) \text{ (some } m \in \mathbb{Z}). \tag{5.2}$$

The Neumann boundary value is characterised by

$$\begin{aligned} & q_y(x,0)\cos\frac{g_0(x)}{2} = -i\dot{g_0}(x)\cos\frac{g_0(x)}{2} \qquad\qquad x \in \mathbb{R} \\ + \quad & \frac{1}{\pi}\int_{\partial\mathbb{R}} \Omega(l)\left\{\int_x^\infty \left[\sin g_0(\xi) m_1(\xi, -\frac{1}{l}) + i(\cos g_0(\xi) - 1) m_2(\xi, -\frac{1}{l})\right] \mathrm{e}^{-2\omega(l)(\xi - x)} d\xi \right. \\ + \quad & \left. \int_{-\infty}^x \left[\sin g_0(\xi) n_1(\xi, -\frac{1}{l}) + i(\cos g_0(\xi) - 1) n_2(\xi, -\frac{1}{l})\right] \mathrm{e}^{-2\omega(l)(\xi - x)} d\xi \right\} d\lambda, \end{aligned} \tag{5.3}$$

where the vectors $(m_1(x,\lambda), m_2(x,\lambda))$ and $(n_1(x,\lambda), n_2(x,\lambda))$ satisfy the ODEs

$$\begin{cases} (m_1)_x = \frac{i}{\lambda}(1 - \cos g_0(x)) m_1 - [\frac{1}{\lambda}\sin g_0(x) - \chi(x)] m_2 \\ (m_2)_x + 2\omega(\lambda) m_2 = [\frac{1}{\lambda}\sin g_0(x) + \chi(x)] m_1 - \frac{i}{\lambda}(1 - \cos g_0(x)) m_2 \end{cases} \quad \lim_{x\to\infty}(m_1, m_2) = (1,0)$$
$$x \in \mathbb{R}, \ \lambda \in \mathbb{C}^+,$$

$$\begin{cases} (n_1)_x = \frac{i}{\lambda}(1 - \cos g_0(x)) n_1 - [\frac{1}{\lambda}\sin g_0(x) - \chi(x)] n_2 \\ (n_2)_x + 2\omega(\lambda) n_2 = [\frac{1}{\lambda}\sin g_0(x) + \chi(x)] n_1 - \frac{i}{\lambda}(1 - \cos g_0(x)) n_2 \end{cases} \quad \lim_{x\to-\infty}(n_1, n_2) = (1,0)$$
$$x \in \mathbb{R}, \ \lambda \in \mathbb{C}^-.$$

Proof:
To prove this result, we consider the global relation (3.15). The (21) element of this relation gives

$$-\frac{i}{4}\int_0^\infty \mathrm{e}^{-2\Omega(\lambda)\eta}\left[[\chi(x,\eta) - \frac{1}{\lambda}\sin q(x,\eta)](\Psi_3^R)_{11} - \frac{i}{\lambda}(\cos q(x,\eta) - 1)(\Psi_3^R)_{21}\right] d\eta =$$

$$-\frac{1}{4}\int_x^\infty \mathrm{e}^{-2\omega(\lambda)(\xi - x)}\left[\chi(\xi) m_1(\xi,\lambda) + \frac{1}{\lambda}\sin g_0(\xi) m_1(\xi,\lambda) \quad + \quad \frac{i}{\lambda}(\cos g_0(\xi) - 1) m_2(\xi,\lambda)\right] d\xi,$$
$$x \in \mathbb{R}, \qquad 0 \le \arg\lambda \le \frac{\pi}{2}. \tag{5.4}$$

Multiplying equation (5.4) by $1+\frac{1}{\lambda^2}$, and integrating the resulting expression with respect to λ along the boundary of D_1 defined by (4.14), we obtain

$$\begin{aligned}
& \frac{i}{4}\int_{\partial D_1}\left(1+\frac{1}{\lambda^2}\right)\int_0^\infty \mathrm{e}^{-2\Omega(\lambda)\eta}\left[\chi(x,\eta)(\Psi_3^R)_{11}(x,\eta,\lambda)\right]d\eta d\lambda \\
& -\frac{i}{4}\int_{\partial D_1}\frac{1}{\lambda}\left(1+\frac{1}{\lambda^2}\right)\int_0^\infty \mathrm{e}^{-2\Omega(\lambda)\eta}\left[(\sin q(\Psi_3^R)_{11}+i(\cos q-1)(\Psi_3^R)_{21}\right](x,\eta,\lambda)d\eta d\lambda \\
= & \frac{1}{4}\int_{\partial D_1}\left(1+\frac{1}{\lambda^2}\right)\int_x^\infty (iq_x(\xi,0)+q_y(\xi,0))m_1(\xi,\lambda)\mathrm{e}^{-2\omega(\lambda)(\xi-x)}d\xi d\lambda \\
& +\frac{1}{4}\int_{\partial D_1}\frac{1}{\lambda}\left(1+\frac{1}{\lambda^2}\right)\int_x^\infty \left[\sin g_0(\xi)m_1(\xi,\lambda)+i(\cos g_0(\xi)-1)m_2(\xi,\lambda)\right]\mathrm{e}^{-2\omega(\lambda)(\xi-x)}d\xi d\lambda.
\end{aligned}$$

The above equation is the analogue of the equation resulting from (4.13) in the linear case. We can simplify this equation using the following facts:

(i) In the first quadrant of the λ complex plane, the function $\Psi_3^R(x,0,\lambda)$, $x\in\mathbb{R}$, has the following asymptotic behaviour:

$$(\Psi_3^R)_{11}(x,0,\lambda)=m_1(x,\lambda)\quad\sim\quad\begin{cases}1+\frac{A(x)}{\lambda}+O\left(\frac{1}{\lambda^2}\right), & |\lambda|\to\infty,\\ \cos\frac{g_0(x)}{2}+O(1), & |\lambda|\to 0,\end{cases}\tag{5.5}$$

$$(\Psi_3^R)_{21}(x,0,\lambda)=m_2(x,\lambda)\quad\sim\quad\begin{cases}-\frac{i\chi(x)}{2\lambda}+O\left(\frac{1}{\lambda^2}\right), & |\lambda|\to\infty,\\ i\sin\frac{g_0(x)}{2}+O(1), & |\lambda|\to 0,\end{cases}\tag{5.6}$$

where $A(x)$ satisfies

$$A_x(x)=\frac{1}{4}(1-\cos g_0(x))-\frac{1}{2}\chi(x)^2,\qquad \lim_{x\to\infty}A(x)=1.$$

We give a proof of this statement in the Appendix.

(ii)

$$\frac{i}{4}\int_{\partial D_1}\left(1+\frac{1}{\lambda^2}\right)\int_0^\infty \mathrm{e}^{-2\Omega(\lambda)\eta}\left[\chi(x,\eta)(\Psi_3^R)_{11}(x,\eta,\lambda)\right]d\eta d\lambda=\frac{\pi}{4}\chi(x)(1-\cos\frac{g_0(x)}{2}).\tag{5.7}$$

Integrating by parts, we find that the asymptotic behaviour of the integrand of the left hand side of (5.7) is

$$\frac{i\chi(x)}{2\lambda}(\Psi_3^R)_{11}(x,0,\lambda)\sim\begin{cases}\frac{i\chi(x)}{2\lambda} & \lambda\to\infty\\ \frac{i\chi(x)}{2\lambda}\cos\frac{g_0(x)}{2}, & \lambda\to 0.\end{cases}$$

Calculating the residues at infinity and at the origin, we find (5.7).

(iii) For $x\in\mathbb{R}$,

$$\frac{1}{4}\int_{\partial D_1}\left(1+\frac{1}{\lambda^2}\right)\int_x^\infty \chi(\xi)m_1(\xi,\lambda)\mathrm{e}^{-2\omega(\lambda)(\xi-x)}d\xi d\lambda=\frac{\pi}{4}\chi(x)(1+\cos\frac{g_0(x)}{2}). \quad (5.8)$$

Integrating by parts, we find that the asymptotic behaviour of the integrand of the left hand side of (5.8) is

$$\frac{i}{2\lambda}\frac{\lambda^2+1}{\lambda^2-1}\chi(x)m_1(x,\lambda)\sim\begin{cases}\frac{i\chi(x)}{2\lambda} & |\lambda|\to\infty\\ -\frac{i\chi(x)}{2\lambda}\cos\frac{g_0(x)}{2}, & |\lambda|\to 0,\end{cases}\qquad x\in\mathbb{R},\quad \lambda\in\mathbb{C}^+.$$

Calculating the residues at infinity and at the origin, we find (5.7).

(iv)

$$\int_{\partial D_1}\left\{\left(-\frac{i}{4}\right)\frac{1}{\lambda}\left(1+\frac{1}{\lambda^2}\right)\int_0^\infty \mathrm{e}^{-2\Omega(\lambda)\eta}\left[(\sin q(\Psi_3^R)_{11}+i(\cos q-1)(\Psi_3^R)_{21}\right](x,\eta,\lambda)d\eta\right.$$
$$\left.+\frac{i}{\lambda^2}\sin\frac{g_0(x)}{2}+\frac{iq_y(x,0)}{\lambda+\lambda^3}\cos\frac{g_0(x)}{2}\right\}d\lambda=0,\qquad x\in\mathbb{R}. \quad (5.9)$$

Integrating by parts and using the asymptoptic behaviour of Ψ_3^R, as well as the identity

$$\sin g_0(x)\cos\frac{g_0(x)}{2}+i(\cos g_0(x)-1)i\sin\frac{g_0(x)}{2}=2\sin\frac{g_0(x)}{2},$$

we find that the integrand of the left hand side of (5.9) is of order $O\left(\frac{1}{\lambda^2}\right)$ at infinity and $O(1)$ at the origin, It follows that the integrand is analytic and bounded in D_1, hence its integral around the boundary vanishes.

In summary, we find

$$\begin{aligned}\frac{\pi}{4}\chi(x)(1-\cos\frac{g_0(x)}{2}) &= \frac{\pi}{4}\chi(x)(1+\cos\frac{g_0(x)}{2}) \quad (5.10)\\ +\frac{1}{4}\int_{\partial D_1}\left\{\frac{1}{\lambda}\left(1+\frac{1}{\lambda^2}\right)\int_x^\infty[\sin g_0(\xi)m_1(\xi,\lambda)\right. &+ i(\cos g_0(\xi)-1)m_2(\xi,\lambda)]\,\mathrm{e}^{-2\omega(\lambda)(\xi-x)}d\xi\\ +\frac{i}{\lambda^2}\sin\frac{g_0(x)}{2} &+ \left.\frac{iq_y(x,0)}{\lambda+\lambda^3}\cos\frac{g_0(x)}{2}\right\}d\lambda,\quad x\in\mathbb{R}.\end{aligned}$$

Simplifying and using the change of variables $l=-\frac{1}{\lambda}$, we finally obtain

$$\frac{\pi}{2}\chi(x)\cos\frac{g_0(x)}{2}=\int_{\partial D_2}\left\{\Omega(l)\int_x^\infty\left[\sin g_0(\xi)m_1(\xi,-\frac{1}{l})+i(\cos g_0(\xi)-1)m_2(\xi,-\frac{1}{l})\right]\mathrm{e}^{-2\omega(l)(\xi-x)}d\xi\right.$$
$$\left.-i\sin\frac{g_0(x)}{2}+\frac{ilq_y(x,0)}{1+l^2}\cos\frac{g_0(x)}{2}\right\}dl,\quad x\in\mathbb{R}. \quad (5.11)$$

We now consider the global relation (3.18). Repeating the construction where we now use equation (3.18) in the fourth quadrant, we find

$$\frac{\pi}{2}\chi(x)\cos\frac{g_0(x)}{2} = \int_{\partial D_4}\left\{\Omega(l)\int_{-\infty}^{x}\left[\sin g_0(\xi)n_1(\xi,-\frac{1}{l}) + i(\cos g_0(\xi)-1)n_2(\xi,-\frac{1}{l})\right]\mathrm{e}^{-2\omega(l)(\xi-x)}d\xi\right.$$
$$\left. + i\sin\frac{g_0(x)}{2} - \frac{ilq_y(x,0)}{1+l^2}\cos\frac{g_0(x)}{2}\right\}dl, \quad x\in\mathbb{R}. \tag{5.12}$$

Adding (5.12) and (5.11) and deforming the appropriate contours to $\mathbb{R}$, we find

$$\chi(x)\cos\frac{g_0(x)}{2} = \frac{1}{\pi}\int_{\partial\mathbb{R}}\Omega(l)\left\{\int_{x}^{\infty}\left[\sin g_0(\xi)m_1(\xi,-\frac{1}{l}) + i(\cos g_0(\xi)-1)m_2(\xi,-\frac{1}{l})\right]\mathrm{e}^{-2\omega(l)(\xi-x)}d\xi\right.$$
$$\left. + \int_{-\infty}^{x}\left[\sin g_0(\xi)n_1(\xi,-\frac{1}{l}) + i(\cos g_0(\xi)-1)n_2(\xi,-\frac{1}{l})\right]\mathrm{e}^{-2\omega(l)(\xi-x)}d\xi\right\}d\lambda. \tag{5.13}$$

Using the definition of $\chi(x)$ this is equation (5.3).
QED

Remark 5.1 The linear limit of equation (5.13) is equation (4.7). Indeed, in the linear limit

$$m_1,\, n_1 \to 1, \quad m_2,\, n_2 \to 0, \;\; \sin g_0 \to g_0, \quad \cos g_0, \cos\frac{g_0}{2} \to 1,$$

so that (5.13) becomes

$$\chi(x) = \frac{1}{\pi}\int_{\partial\mathbb{R}}\Omega(l)\left\{\int_{x}^{\infty} g_0(\xi)\mathrm{e}^{-2\omega(l)(\xi-x)}d\xi + \int_{-\infty}^{x} g_0(\xi)\mathrm{e}^{-2\omega(l)(\xi-x)}d\xi\right\}d\lambda,$$

which is (4.7).

6 Conclusions

We have derived the Dirichlet to Neumann map for an elliptic nonlinear boundary value problem, namely the Dirichlet problem for the elliptic sine-Gordon equation posed in a half plane, see Theorem 5.1. This nonlinear map characterises the spectral functions in terms of the given boundary conditions only. The derivation is based on the general ideas of the analogous derivation for the case of integrable evolution equations, but it also contains novel steps, in particular it involves the analysis of *all* eigenfunctions of the direct problem. This is conceptually justified by the fact that, contrary to the case of evolution equations, there is no part of the boundary where *all* boundary values are prescribed. Hence, the details of the derivation of the Dirichlet to Neumann map are technically more challenging and, to our knowledge, they are presented for a nonlinear elliptic problem for the first time.

Acknowledgements

This research was partially supported by EPSRC. ASF would like to express his gratitude to the Guggenheim Foundation, USA.

Appendix

In this appendix we derive the asymptotic formulas (5.5) and (5.6) for the vector $m^+(x,\lambda)$, $x \in \mathbb{R}$, $\lambda \in \mathbb{C}^+$, defined in (3.15).
Recall that the components (m_1, m_2) of $m^+(x,\lambda)$ satisfy the system of ODEs (see (3.25)

$$(m_1)_x = \frac{1}{4}\left[\frac{i}{\lambda}(1-\cos q_0(x))m_1 + (\chi(x) - \frac{\sin q_0(x)}{\lambda})m_2\right] \tag{6.1}$$

$$(m_2)_x - 2\omega(\lambda)m_2 = \frac{1}{4}\left[(\chi(x) + \frac{\sin q_0(x)}{\lambda})m_1 - \frac{i}{\lambda}(1-\cos q_0(x))m_2\right], \tag{6.2}$$

$$x \in \mathbb{R}, \quad \lambda \in \mathbb{C}^+,$$

where $\chi(x)$ is given by (5.1).

The asymptotic behaviour of $m^+(x,\lambda)$ as $|\lambda| \to \infty$

Substituting in (6.1)-(6.2) the ansatz

$$m_1(x,y,\lambda) = 1 + \frac{A(x)}{\lambda} + O\left(\frac{1}{\lambda^2}\right), \qquad m_2(x,y,\lambda) = \frac{a(x)}{\lambda} + O\left(\frac{1}{\lambda^2}\right),$$

$$as \;\; |\lambda| \to \infty, \; 0 < arg(\lambda) < \frac{\pi}{2},$$

and equating powers of $\frac{1}{\lambda}$, we find

$$a(x) = -\frac{i}{2}\chi(x), \qquad A_x(x) = \frac{i}{4}(1-\cos g_0(x)) - \frac{i}{2}\chi(x)^2, \qquad x \in \mathbb{R},$$

hence the behaviour as $|\lambda| \to \infty$ in (5.5) and (5.6).

The asymptotic behaviour of $m^+(x,\lambda)$ as $|\lambda| \to 0$

Note that $m^+(x,\lambda)$, $x \in \mathbb{R}$, $\lambda \in \mathbb{C}^+$, is the first column vector of the matrix $\Psi_2(x,0,\lambda)$ defined by (3.2). This matrix is bounded for $\lambda \in (\mathbb{C}^+, \mathbb{C}^-)$.
Let

$$\Psi_2(x,0,\lambda) = \varphi(x,\lambda)\mathrm{e}^{\omega(\lambda)x\sigma_3}.$$

Since Ψ_2 satisfies (2.2), the matrix φ satisfies

$$\varphi_x = (Q(x,0,\lambda) - \omega(\lambda)\sigma_3)\varphi, \qquad \lim_{x\to\infty} \varphi(x,\lambda)\mathrm{e}^{\omega(\lambda)x\sigma_3} = I.$$

with $Q(x,y,\lambda)$ given by (2.5). Thus

$$\varphi_x = -\frac{i}{4\lambda}\tilde{Q}(x)\varphi + O(1), \qquad as \;\; |\lambda| \to 0, \;\; \lambda \in (\mathbb{C}^+, \mathbb{C}^-),$$

where

$$\tilde{Q}(x) = \begin{pmatrix} \cos g_0(x) & -i\sin g_0(x) \\ i\sin g_0(x) & -\cos g_0(x) \end{pmatrix}. \tag{6.3}$$

Noting that

$$\tilde{Q} = D\sigma_3 D^{-1}, \qquad D = \begin{pmatrix} \cos\frac{g_0(x)}{2} & i\sin\frac{g_0(x)}{2} \\ i\sin\frac{g_0(x)}{2} & \cos\frac{g_0(x)}{2} \end{pmatrix},$$

we find

$$(D^{-1}\varphi)_x = -\frac{i}{4\lambda}\sigma_3 D^{-1}\varphi + O(1).$$

Solving this ODE, using the boundary condition at $x \to \infty$, we find

$$\varphi(x,\lambda) = \left[\begin{pmatrix} \cos\frac{g_0(x)}{2} & i\sin\frac{g_0(x)}{2} \\ i\sin\frac{g_0(x)}{2} & \cos\frac{g_0(x)}{2} \end{pmatrix} + O(1)\right] \mathrm{e}^{\frac{1}{4i\lambda}x\sigma_3}, \qquad |\lambda| \to 0,\ \lambda \in (\mathbb{C}^+, \mathbb{C}^-). \tag{6.4}$$

From equation (6.4) it follows that

$$m^+(x,\lambda) = \begin{pmatrix} \cos\frac{g_0(x)}{2} \\ i\sin\frac{g_0(x)}{2} \end{pmatrix} + O(1), \qquad as\ |\lambda| \to 0,\ \ \lambda \in C^+,\ \ x \in \mathbb{R}, \tag{6.5}$$

hence the behaviour as $|\lambda| \to 0$ in (5.5) and (5.6).

References

[1] Y.Antipov and A.S. Fokas, The modified Helmholtz equation in a semi-strip, *Math. Proc. Camb. Phil Soc.* (2005) **138**.

[2] D. ben-Avraham and A.S. Fokas, The Modified Helmholtz Equation in a Triangular Domain and an Application to Diffusion-limited Coalescence, *Phys. Rev. E* (2001) **64**.

[3] J.L. Bona, and A.S. Fokas, Initial-Boundary-Value Problems for Linear and Integrable Nonlinear Dispersive Partial Differential Equations, *Nonlinearity* (2008) **21**.

[4] A. Boutet de Monvel, A.S. Fokas and D. Shepelsky, The Modified KdV Equation on the Half-Line, *J. of the Inst. of Math. Jussieu* (2004) **3**.

[5] A. Boutet de Monvel, A.S. Fokas and D. Shepelsky, Integrable Nonlinear Evolution Equations on a Finite Interval, *Communications in Mathematical Physics* (2006) **263**.

[6] A. Boutet de Monvel, A. S. Fokas and D. Shepelsky, Analysis of the Global Relation for the Nonlinear Schrdinger Equation on the Half-line, *Lett Math Phys* (2003) **65**.

[7] D. Crowdy and A.S. Fokas, Explicit Integral Solutions for the Plane Elastostatic Semi-Strip, *Proc. R. Soc. London A* (2004) **460**.

[8] G Dassios, What nonlinear methods offered to linear problems? The Fokas transform method, *Int. J. Nonl. Sci* (2007) **42**.

[9] G . Dassios and M. Doschoris, Asymmetric Stokes' flow in a spherical shell revisited via the Fokas method. Part I: Irrotational flow, *Math Mech in Appl Sci* (2011) **34**.

[10] G . Dassios and M. Doschoris, On the global relation and the Dirichelt-to-Neumann correspondence, *Stud Appl Math* (2010) **126**.

[11] G. Dassios and A.S. Fokas, The Basic Elliptic Equations in an Equilateral Triangle, *Proc. R.Soc. Lond. A* (2005) **461**.

[12] B. Deconinck, T. Trogdon and V. Vasan, Solving linear partial differential equations, (submitted).

[13] M. Doschoris, Limiting values of the Gegenbauer functions on the cut, *Integral transforms and special functions* (2011), **1**.

[14] G.M. Dujardin, Asymptotics of linear initial boundary value problems with periodic boundary data on the half-line and finite intervals, *Proc. Royal Soc. Series A* (2009) **465**.

[15] L. D. Fadeev and L.A.Takhtajan *Hamiltonian methods in the theory of solitons*, Springer Verlag, 2007.

[16] N. Flyer and B Fornberg, A numerical implementation of Fokas boundary integral approach: Laplace's equation in a polygonal domain, *Proc Royal Soc Lond A* (2011) **467**.

[17] A.S. Fokas, A unified transform method for solving linear and certain nonlinear PDE's, *Proc. Royal Soc. Series A* (1997) **453**.

[18] A.S.Fokas, On the integrability of linear and nonlinear partial differential equations , *J. Math. Phys.* (2000) **41**.

[19] A.S.Fokas, Two Dimensional Linear PDEs in a Convex Polygon, *Proc. R. Soc. A* (2001) **457**.

[20] A.S. Fokas, Integrable nonlinear evolution equations on the half-line, *Comm Math Phys* (2002) **230**.

[21] A.S.Fokas, The generalized Dirichlet-to-Neumann map for certain nonlinear evolution PDEs , *Comm.Pure Appl. Math.* (2005) **58**.

[22] A.S. Fokas, Linearizable initial boundary value problems for the sine-Gordon equation on the half line, *Nonlin.* (2004) **17**.

[23] A.S. Fokas, *A Unified Approach to Boundary Value Problems*, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM (2008).

[24] A.S. Fokas, A new transform method for evolution partial differential equations, *IMA J. of Appl. Math.* (2002) **67**.

[25] A.S. Fokas and A.A. Kapaev, A Riemann-Hilbert Approach to the Laplace Equation, *J. Math. Anal. and Appl.*(2000) **251**.

[26] A.S. Fokas and A.A. Kapaev, On a Transform Method for the Laplace Equation in a Polygon, *IMA J. Appl. Math.* (2003) **68** .

[27] A.S. Fokas and A.R. Its, The linearization of the initial-boundary value problem of the nonlinear Schrödinger equation, SIAM J. Math. Anal. (1996). **27**.

[28] A.S.Fokas, A.R. Its and L-Y Sung, The nonlinear Schrödinger equation on the half-line, *Nonlinearity* (2005) **18**.

[29] A.S. Fokas and B. Pelloni, A transform method for linear evolution PDEs on a finite interval, *IMA J. of Appl. Math.* (2005) **70**.

[30] A.S. Fokas and B. Pelloni, Boundary value problems for the elliptic sine-Gordon equation in a semi-strip, *J Nonlin Sci* (submitted)

[31] A.S. Fokas and E.A. Spence, Novel Analytical and Numerical Methods for Elliptic Boundary Value Problems, in "Highly Oscillatory Problems", Cambridge University Press (2009).

[32] A.S. Fokas and M. Zyskin, The Fundamental Differential Form and Boundary Value Problems, *Quart. J. Mech. Appl. Math.* (2002)**55**.

[33] S. R. Fulton, A. S. Fokas and C. A. Xenophontos, An analytical method for linear elliptic PDEs and its numerical implementation, *J of Comp Appl Math* (2004) **167**.

[34] E.S. Gutshabash and V.D.Lipovskii, Boundary value problem for the two-dimensional elliptic sine-Gordon equation and its applications to the theory of the stationary Josephson effect, *J. Math. Sciences* (1994) **68**.

[35] K. Kalimeris, Explicit soliton asymptotics for the nonlinear Schrödinger equation on the half line, *J Nonlin Math Phys* (2010) **17**.

[36] K. Kalimeris, Initial and boundary value problems in two and three dimensions, PhD Thesis, University of Cambridge (2011).

[37] S. Kamvissis, Semiclassical nonlinear Schrödinger equation on the half line, *J. Math. Phys* (2003) **44**.

[38] A.R. Its, E. Its and J. Kaplunov, Riemann-Hilbert approach to the elastodynamic equation: part I, *Lett Math Phys* (2010) **96**.

[39] J Lenells, The derivative nonlinear Schrödinger equation on the half line, *Physica D* (2008) **237** 3008-3019.

[40] J Lenells, The solution of the global relation for the derivative nonlinear Schrödinger equation on the half line, *Physica D* (2011) **240**.

[41] J. Lenells and A.S. Fokas, An integrable generalisation of the nonlinear Schrödinger equation on the half-line and solitons, *Inv. Prob.* (2009) **25** .

[42] J. Lenells and A.S. Fokas, Explicit Soliton Asymptotics for the Korteweg-de-Vries Equation on the Half-Line, *Nonlin* (2010) **23**.

[43] J. Lenells and A.S. Fokas,, Boundary-value problems for the stationary axisymmetric Einstein equations: a rotating disc *Nonlinearity* (2011) **24**.

[44] D. Mantzavinos and A.S.Fokas, The Kadomtsev-Petviashvili II equation on the half-plane, *Physica D*, (2010) **240**.

[45] B Pelloni, Spectral analysis of the elliptic sine-Gordon equation in the quarter plane, *Theor Math Phys.* (2009) **160**.

[46] B Pelloni, Linear and nonlinear generalized Fourier transform, *Phil. Trans. Royal Soc. London A* (2006) **364**.

[47] B Pelloni, The spectral representation of two-point boundary value problems for linear evolution equations, *Proc. R. Soc. A* (2005) **461**.

[48] B. Pelloni, Well-posed boundary value problems for linear evolution equations on a finite interval, *Math. Proc. Camb. Phil. Soc.* (2004), **136**.

[49] B, Pelloni and D. Pinotsis, The elliptic sine-Gordon equation in a half plane *Nonlinearity* (2010) **23**.

[50] A. Sifalakis, A.S. Fokas, S.R. Fulton, and Y. G. Saridakis, The Generalized Dirichlet-. Neumann Map for Linear Elliptic PDEs and its Numerical implementation, *J. Comp. Appl. Math.* (2009) **219**.

[51] D.A. Smith, *Spectral theory of ordinary and partial linear differential operators on finite intervals* PhD Thesis, University of Reading , 2001.

[52] S. Smitheman, E. Spence and A.S. Fokas, A spectral collocation method for the Laplace and modified Helmholtz equations in a convex polygon, *IMA J Numer Anal* (2010) **30**.

[53] E. Spence and A.S. Fokas, A new transform method I: domain-dependent fundamenta solution and integral representations, *Proc Royal Soc Lond A* (2010) **466**.

[54] E. Spence and A.S. Fokas, A new transform method II: the global relation and boundary-calue problems in polar coordinates, *Proc Royal Soc Lond A* (2010) **466**.

[55] P.A. Treharne and A.S. Fokas, The Generalized Dirichlet to Neumann map for the KdV Equation on the Half-Line, *J. Nonlinear Science* (2008) **18**.

[56] C. Zheng, Exact nonreflecting boundary conditions for one-dimensional cubic nonlinear Schrödinger equation, *J Comput Phys* (2006) **215**.

[57] C. Zheng, Numerical Simulation of a Modified KdV Equation On the Whole Real Axis, *Numer. Math.* (2006) 1**105**.