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Competing Edge Networks

Mark Parsons[a,1,*], Peter Grindrod[a,2]

[a] *School of Mathematics and Physical Sciences, University of Reading, Whiteknights, PO Box 220 Reading, RG6 6AX, United Kingdom*

Abstract

We introduce a model for a pair of nonlinear evolving networks, defined over a common set of vertices, subject to edgewise competition. Each network may grow new edges spontaneously or through triad closure. Both networks inhibit the other's growth and encourage the other's demise. These nonlinear stochastic competition equations yield to a mean field analysis resulting in a nonlinear deterministic system. There may be multiple equilibria; and bifurcations of different types are shown to occur within a reduced parameter space. This situation models competitive peer-to-peer communication networks such as BlackBerry Messenger displacing SMS; or instant messaging displacing emails.

Keywords: Evolving networks, Edgewise competition, Mean field approximation, Bifurcation structure

1. Introduction

In this paper we consider an extension of modelling non-linear evolving edge networks introduced in [8], specifically by introducing a competing aspect to multiple networks' dynamics. For simplicity we shall consider two different types of edges, henceforth referred to as *Red* and *Blue* edges, acting upon

[*] Corresponding Author.
[1] Tel.: +44 118 931 8996. E-mail address: m.c.parsons@pgr.reading.ac.uk.
[2] Tel.: +44 118 931 8994. E-mail address: p.grindrod@reading.ac.uk.

the same group of nodes, where each edge type has its own discrete time dynamics, and series of adjacency matrices detailing its evolution. Since these edge types act upon the same group of nodes they may be superimposed onto a single graph, providing the different edge types are clearly differentiated. The work presented in this paper considers such a network where the different edge types are competing with one another, that is they negatively impact each other's growth. The specifics of our chosen model are outlined in Section 2.

Whilst the extension of tradition network theory from a stochastic to a dynamic setting has recently earnt much attention [1, 3, 4, 11, 9, 12, 13], the notion of competing networks remains largely unexplored. Competing networks can however be observed in many technological fields, for example one might consider a group of BlackBerry owners: the networks of BlackBerry Messenger usage and SMS texting amongst this group are seen to be competing with one another due to their similar functions. Here we see BlackBerry Messenger network partially displace the SMS network at user switch their method of communication, leading to a fall in SMS usage [2]. Public and social communication poses challenges to both commercial interests (mass customer industries such as telecommunications, retail, consumer goods, marketing, advertising and new media) and public interests (security, defence policy and opinion formation). Accordingly it is very timely to consider how one type of communication platform may displace another.

Such competing technologies are typically emergent, with one network, Red, boasting superior features (and hence its edge density will grow more rapidly) whereas the other, Blue, possessing a higher userbase (and hence has a higher initial edge density). We are interested in the equilibrium positions obtained by both networks (henceforth referred to as the *system*), and in Section 3 we introduce a mean field approximation of our system, to aid in locating these equilibria. We conclude that it is unlikely both networks would reach a high equilibrium position, since that would imply individual's node-node relations use both methods of communication, and instead argue that each other's presence negatively impacts each network. Section 4 shows that this causes either one network to be eliminated or both networks find a compromise at low edge density values, and we examine all possible equilibrium positions for the system, together with conditions for their existence.

Finally, in Section 5, we make observations concerning the system's equilibria in the case of highly asymmetric competition.

2. Competing Edge Dynamics

First we introduce some terminology to define our competing evolving networks.

Following [8, 7, 10] we define an evolving network, over discrete time steps indexed by $k = 1, 2, ...$, via a sequence of adjacency matrices, say $\{A_k\}$. We shall assume that all edges are undirected and we do not allow any edges connecting a vertex with itself. Thus all of our adjacency matrices lie in the set S_n of binary, symmetric, $n \times n$ matrices having zeros along their main diagonals. We assume the evolving network dynamic is first order in time: at the $(k+1)$th time step each edge in A_{k+1} will have a birth or death rate that is conditional on A_k. However no new vertices will enter, nor shall any existing vertices be permanently removed from the evolving network. At each time step the evolving network is thus a random network conditional on the evolving network at the previous time step, with a probability distribution $P(A_{k+1}|A_k)$, defined as A_{k+1} ranges over S_n.

We shall assume that presence of each each edge in A_{k+1} is determined independently of all other edges. This means that it is sufficient to specify the conditional expectation that each edge is present, given by

$$< A_{k+1}|A_k >= \sum_{A_{k+1} \in S_n} A_{k+1} P(A_{k+1}|A_k),$$

rather than dealing with full probability distribution. In fact for such edge-independent conditional random networks we may write

$$P(A_{k+1}|A_k) = \prod_{i<j} (< A_{k+1}|A_k >)_{ij}^{(A_{k+1})_{ij}} (1 - (< A_{k+1}|A_k >)_{ij})^{1-(A_{k+1})_{ij}},$$

demonstrating their equivalence.

Notice that since distinct edges may be conditionally dependent on some of the same information, it is possible for their appearance to be highly correlated over time, despite their independence.

Let the sequence $\{A_k\}$ within S_n denote a *Red* evolving network defined over a set of n vertices. Similarly let the sequence $\{B_k\}$ within S_n denote a *Blue* evolving network defined over the same set n vertices. Then, extending the above ideas, we will assume that both evolving networks have a first order edge-independent dynamic such that each network at each time step is a random network conditionally dependent upon both networks at the previous time step. Then such a competitive dynamic is completely determined by matrix equations of the form

$$\begin{aligned} < A_{k+1}|A_k, B_k > &= A_k \circ (\mathbf{1} - \Omega_A(A_k, B_k)) + (\mathbf{1} - A_k) \circ \Lambda_A(A_k, B_k)) \\ < B_{k+1}|A_k, B_k > &= B_k \circ (\mathbf{1} - \Omega_B(A_k, B_k)) + (\mathbf{1} - A_k) \circ \Lambda_B(A_k, B_k)). \end{aligned}$$

Here $\mathbf{1}$ denotes the adjacency matrix for the n-vertex clique (all ones except for the main diagonal); $\circ$ denotes the elementwise (Hadarmard) matrix product; $\Lambda_A(A_k, B_k)$ and $\Lambda_B(A_k, B_k)$ denote matrices of conditional edge birth probabilities ($P(\text{edge}_{ij} \in A_{k+1}|\text{edge}_{ij} \notin A_k) \in [0, 1]$); and $\Omega_A(A_k, B_k)$ and $\Omega_B(A_k, B_k)$ denote matrices of conditional edge death probabilities ($P(\text{edge}_{ij} \notin A_{k+1}|\text{edge}_{ij} \in A_k) \in [0, 1]$);

Now let us be more specific. We define our networks' individual edge birth rates to be based upon a triangulation mechanism [8] (where friends of friends are more likely to become friends, called triadic closure [5])[3], and also containing some antagonistic terms. We shall increase the probability of an existing Red edge dying if a Blue edge is also present between those two vertices, and vice-versa. We shall also decrease the probability of a Red edge being born is a Blue edge is already present between those two vertices, and vice-versa. Thus we consider

$$\begin{aligned} < A_{k+1}|A_k, B_k > &= A_k \circ (\mathbf{1}(1 - \omega_A) - \mu_A B_k) \\ &\quad +(\mathbf{1} - A_k) \circ (\delta_A + \epsilon_A {A_k}^2 - \gamma_A B_k) && (1) \\ < B_{k+1}|A_k, B_k > &= B_k \circ (\mathbf{1}(1 - \omega_B) - \mu_B A_k) \\ &\quad +(\mathbf{1} - B_k) \circ (\delta_B + \epsilon_B {B_k}^2 - \gamma_B A_k), && (2) \end{aligned}$$

where ω_A, ω_B, $\delta_A, \delta_B, \epsilon_A, \epsilon_B, \mu_A, \mu_B, \gamma_A$ and γ_B are all real constants in $(0, 1)$.

[3]These are networks that strive to achieve triadic closure where the edge dynamics between two vertices depends, amongst other values, on their current number of neighbours in common.

Notice that since both A_{k+1} and B_{k+1} are dependent upon A_k and B_k, there is therefore no 'first/late mover advantage' [6] for the Red or Blue network.

Figure 1 shows the evolution of various synthetic networks in terms of the edge density for the Red and Blue networks, where each simulation starts from the same initial pair of matrices, A_1 and B_1. Their evolution is modelled according to (1) and (2), with $n = 39$, and the same parameter values for both Red and Blue networks: $\omega = 1/25$, $\epsilon = 1/110$, $\mu = 1/17$, $\delta = 1/600$ and $\gamma = 1/600$. Notice that multiple apparently stable equilibria exist and that they are reachable from the same initial network pair at the first time step. This highlights the significance of identifying these equilibria for a given network, and motivates the analysis in the next section.

3. Mean Field Approximation

In order to identify and analyse the long term equilibria, we take the mean field approximation introduced in [8]. Symmetry of the dynamics implies there are no preferred vertices or edges (all edges satisfy the same rules since the birth and death rates have no explicit edge dependencies), so we assume that we may write $< A_k >\approx p_k\mathbf{1}$ and similarly $< B_k >\approx q_k\mathbf{1}$ where p_k and q_k represent the edge densities of the Red and Blue networks at the kth time step; and hence that these networks are approximated by Erdos-Renyi random graphs. Then the mean field approximation for the dynamics of this system is reduced to a nonlinear iteration of over the unit square:

$$\begin{aligned} p_{k+1} &= p_k(1-\omega_A-\mu_A q_k)+(1-p_k)(\delta_A+\epsilon_A(n-2){p_k}^2-\gamma_A q_k) \quad (3)\\ q_{k+1} &= q_k(1-\omega_B-\mu_B p_k)+(1-q_k)(\delta_B+\epsilon_B(n-2){q_k}^2-\gamma_B p_k). \quad (4)\end{aligned}$$

Notice that $0 \le p_k, q_k \le 1$ for all k, and our parameters should satisfy several constraints. In (3) we require:

(a) $(1-\omega_A-\mu_A q_k) \ge 0$, and hence, since $q_k \le 1$, we must have $\omega_A+\mu_A \le 1$;

(b) $(\delta_A+\epsilon_A(n-2){p_k}^2-\gamma_A q_k) \ge 0$, and hence, since $p_k \ge 0$ and $q_k \le 1$, we must have $\delta_A-\gamma_A \ge 0$;

(c) $(\delta_A+\epsilon_A(n-2){p_k}^2-\gamma_A q_k) \le 1$, and hence, since $p_k \le 1$ and $q_k \ge 0$, we must haven $\delta_A+\epsilon_A(n-2) \le 1$.

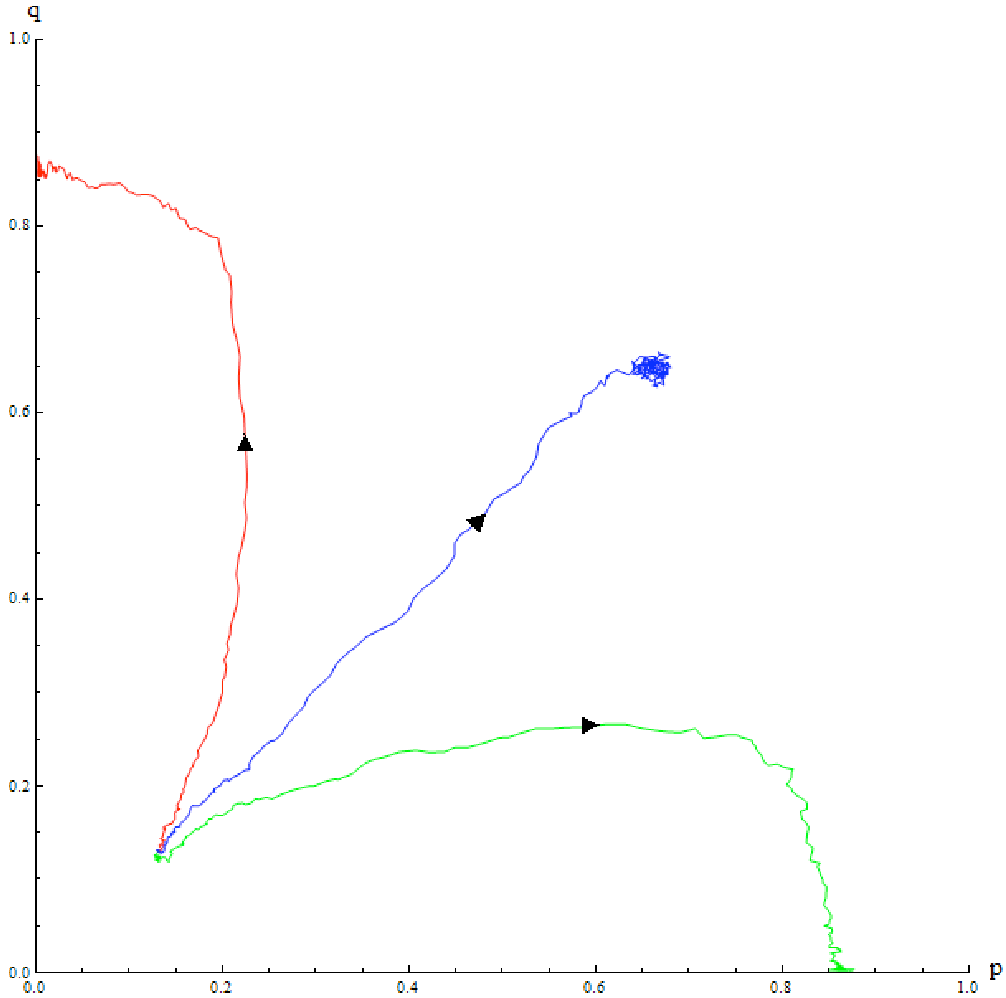

Figure 1: Three separate simulations of competing networks, modelled according to (1) and (2). In each case the edge densities of the competing networks are plotted against one another at each timestep. Notice that each simulation is performed with the same network parameter values and initial matrix pair, however evolve towards distinct network positions.

These three constraints hold similarly for the Blue network's parameters.

At equilibrium, where, say, $p_k = p$ and $q_k = q$ for all k, we may rearrange (3) and (4) into the following form,

$$q = \frac{\epsilon_A(n-2)(1-p)p^2 + (1-p)\delta_A - p\omega_A}{\gamma_A(1-p) + p\mu_A} = f_A(p), \tag{5}$$

$$p = \frac{\epsilon_B(n-2)(1-q)q^2 + (1-q)\delta_B - q\omega_B}{\gamma_B(1-q) + q\mu_B} = f_B(q), \tag{6}$$

where the functions f_A and f_B differ only in their (suppressed) parameter values.

The mean field approximation retains the nonlinear nature of the full stochastic iteration, but it is itself a deterministic iteration (over (p, q) space), since the stochastic evolution has been smoothed away by projecting the expected value of the adjacency matrix into its mean field representation. This approximation is likely to become unreliable where the original evolution is sensitive to small perturbations within the network structures (see [8] for further reading). This certainly would include situations where one or other network is very sparse and also where the pair are close to any unstable equilibrium or other regions of instability, for the mean field dynamics.

4. Identifying System Equilibria for Symmetrical Competition

Before locating the equilibria for our system we first make the following simplification that equalizes the competition: we shall assume that the parameter values for both the Red and Blue networks are equal, i.e., $\delta_A = \delta_B = \delta$ for every parameter in (3) and (4). Hence (5) and (6) become

$$q = f(p), \quad p = f(q), \tag{7}$$

where

$$f(y) = \frac{\epsilon(n-2)(1-y)y^2 + (1-y)\delta + y\omega}{\gamma(1-y) + y\mu}.$$

Now consider the one dimensional iteration defined on [0,1], indexed by $t = 1, 2, ...,$

$$y_{t+1} = f(y_t). \tag{8}$$

Then equilibria for the mean field iteration (3) and (4), necessarily satisfying (7), are represented by either fixed point equilibria, y^* say, for (8); or a period two, or "flip", solution for (8), say $y_1 = f(y_2)$, $y_2 = f(y_1)$ $(y_1 \neq y_2)$. The first case leads to a symmetrical equilibrium for (3) and (4), with $p = q = y^*$; the second case leads to a mirror image pair of non-symmetric equilibria for (3) and (4), with $(p, q) = (y_1, y_2)$ and (y_2, y_1).

Once an equilibrium is identified, its stability is determined by the spectral radius of the Jacobian obtained by linearizing (3) and (4) about that point, with asymptotic stability if and only if this is less than one.

Since the equations in (7) both lead to cubic polynomials, the condition that both equations are simultaneously satisfied is equivalent to locating the roots of a ninth order polynomial, where we are only concerned with real roots lying in the unit square. The constraints placed upon our parameter values do not exclude the existence of a full set of nine applicable roots, and such a case is shown in Figure 2. Here we take the parameter values: $n = 39$, $\omega = 1/25$, $\epsilon = 1/110$, $\mu = 1/17$, $\delta = 1/600$ and $\gamma = 1/600$.

Fixed points for (8) lead to a cubic equation, $p = f(p)$. So there is either one or three root(s) in [0,1].

We have

$$0 = p^3 + \left(\frac{\mu - \gamma}{\epsilon(n-2)} - 1\right) p^2 + \left(\frac{\omega + \gamma + \delta}{\epsilon(n-2)}\right) p - \frac{\delta}{\epsilon(n-2)} \qquad (9)$$

A cubic equation will have three real roots if, and only if, the cubic discriminant is positive. This implies a condition on the system parameters: writing $\hat{\epsilon} = \epsilon(n-2)$, we have

$$\begin{gathered} -4\delta^3\hat{\epsilon} + \delta^2((\gamma-\mu)^2 - 8\hat{\epsilon}^2 + 4\hat{\epsilon}(2\gamma - 5\mu - 3\omega)) + \\ (\gamma+\omega)^2((\gamma-\mu)^2 + \hat{\epsilon}^2 - 2\hat{\epsilon}(\gamma+\mu+2\omega)) - \\ 2\delta(2\hat{\epsilon}^3 + (\gamma-\mu)^2(\gamma - 2\mu - \omega) - \\ 2\hat{\epsilon}^2(2\gamma + 3\mu + 5\omega) + \hat{\epsilon}(\gamma^2 + 6\mu^2 + 11\mu\omega + 6\omega^2 + \gamma(-\mu+\omega))) \geq 0. \end{gathered}$$

When this is satisfied the three equilibrium solutions correspond to equilibria for (3) and (4) that lie along axis of reflection, $p = q$. In Figure 2 these roots correspond to points A, B and C.

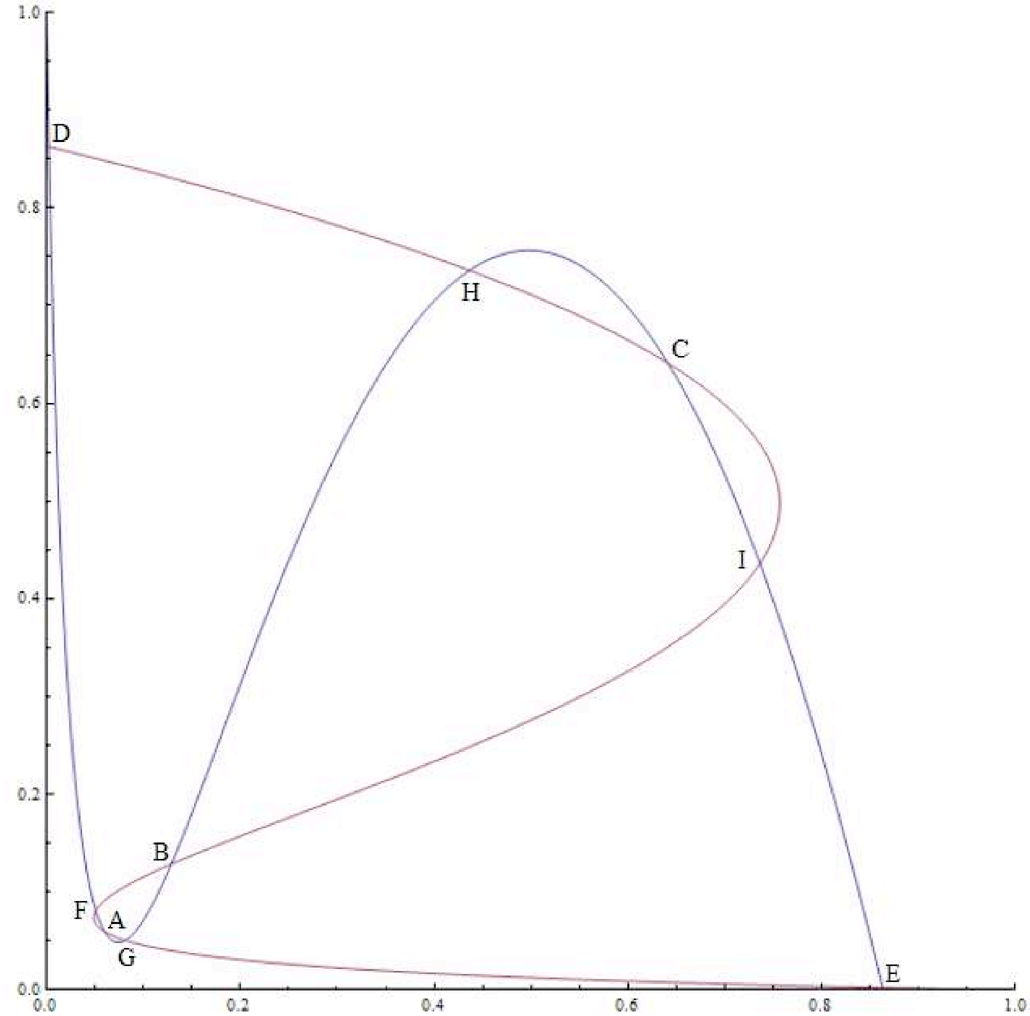

Figure 2: An example of fixed point curves for a mean field approximation of a competing edge network with specified parameters resulting in nine intersections.

The remaining non symmetric equilibria for (3) and (4) may be identified by considering two-periodic “flip-solutions” for (8). These may occur specifically when any fixed point for (8) (corresponding to a symmetric equilibrium for (3) and (4)) undergoes a flip bifurcation, when the system parameters change so that the slope of f goes from above to below -1 at that equilibrium, whence a pair of period-two solutions will be born. For (3) and (4) this event corresponds to a pitchfork bifurcation where two asymmetric equilibria are born as the stability of the corresponding symmetric equilibrium changes. We can locate such bifurcations in terms of all of the parameters by first solving (the cubic) for the equilibria, and then imposing the flip bifurcation condition that the derivative f' is exactly -1 there also. Hence we may determine those parameter surfaces for which such a pitchfork bifurcation of non symmetric equilibria occurs for (3) and (4).

This mechanism accounts for the pitchfork bifurcation at point A in Figure 2 (corresponding to a flip bifurcation for (8)) giving rise to the asymmetric equilibria at F and G, and similarly the pitchfork bifurcation at point C giving rise to the asymmetric equilibria at H and I.

Finally there remains the possible equilibrium points D and E, where one network dominates another. These points are stable and can only be annihilated by collisions with the equilibria at F and G respectively (that occur simultaneously by symmetry). In such a case the equilibrium at A would be the only survivor.

These observations combined imply that there exists a maximum of nine applicable roots and a minimum of one applicable root.

We have thus far provided tests to confirm if a system possesses nine applicable roots, however it is more useful to instead use these tests to locate parameter values which satisfy them. By first fixing all but two of our parameter values it is possible to determine which values the remaining parameters should take to guarantee[4] the existence of nine applicable roots. For example, let us fix $n = 39$, $\omega = 1/25$, $\delta = 1/600$ and $\gamma = 1/600$, as in Figure 2, and vary ϵ and μ. Then we are able to identify regions wherein we have three first order solutions, and two contours denoting where bifurcation of

[4]This is not always possible, depending on the other parameter values and due to the constraints placed upon the parameters.

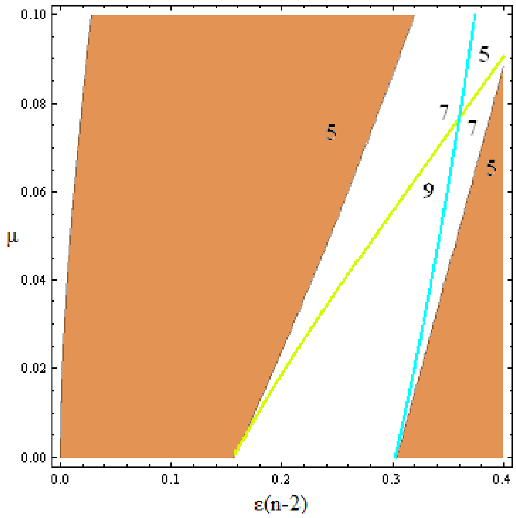

Figure 3: A bifurcation map for varying values of $\epsilon(n-2)$ (x-axis) and μ (y-axis). The white region indicates the existence of three first order solutions, whereas the orange shaded region has only one. The green curve is a contour denoting bifurcation at the lower first order root, where the pair of nonsymmetric roots persists to the left of the contour. Similarly the blue curve denotes bifurcation at the higher first order root, where the pair of nonsymmetric roots persist to the right of the contour.

the upper and lower first order solutions occur. Figure 3 demonstrates this idea.

Selecting ϵ and μ from the region satisfying all three conditions then guarantees the existence of nine applicable roots, and likewise we can determine the effects to our system's mean field approximated potential equilibria from small parameter changes in ϵ and μ. Figures 4 through 9 sample various ϵ and μ values within Figure 3 together with their associated mean field fixed points graph, and stability analysis of each root.

5. Non-Symmetric Competing Models

Whilst our previous assumption of symmetry between network parameter values greatly simplified our analysis, it is perhaps unreasonable of an assumption to make in practice. A new emergent network can only invade successfully with an existing established (high edge density) network if it possesses suitable parameter values and initial conditions.

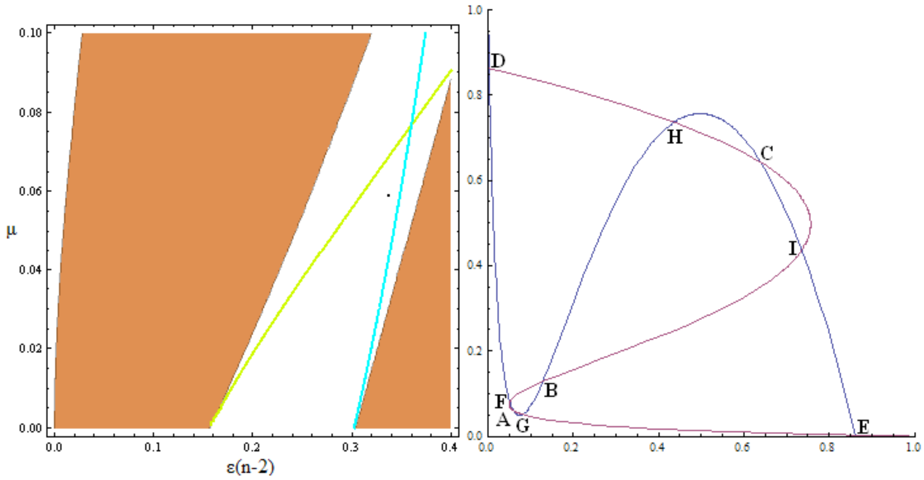

Figure 4: Parameter values chosen are $\epsilon(n-2) = 37/110$ and $\mu = 1/17$, resulting in an associated mean field fixed points graph with nine applicable roots. Of these roots, points A, C, D and E are found to be stable, whereas the others are unstable.

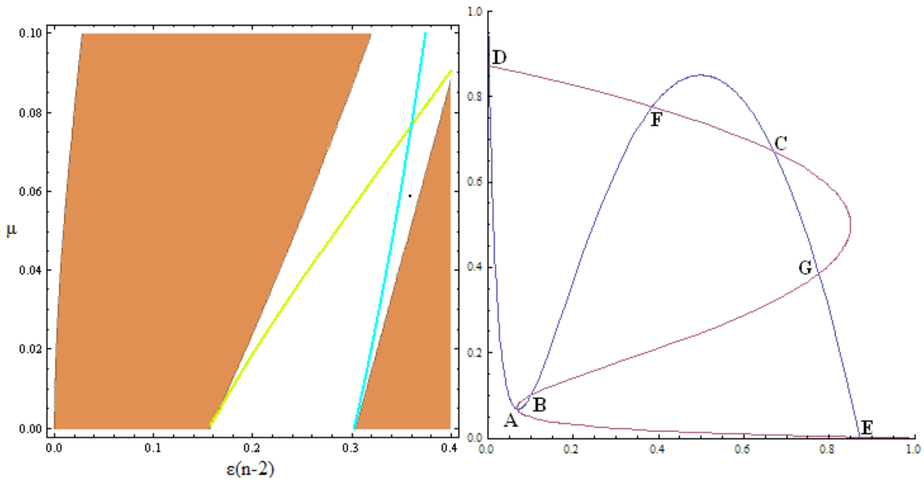

Figure 5: Parameter values chosen are $\epsilon(n-2) = 79/220$ and $\mu = 1/17$, resulting in an associated mean field fixed points graph with seven applicable roots. With respect to Figure 4, two roots are lost due to non-bifurcation of point A. Of these roots, points C, D and E are found to be stable, whereas the others are unstable.

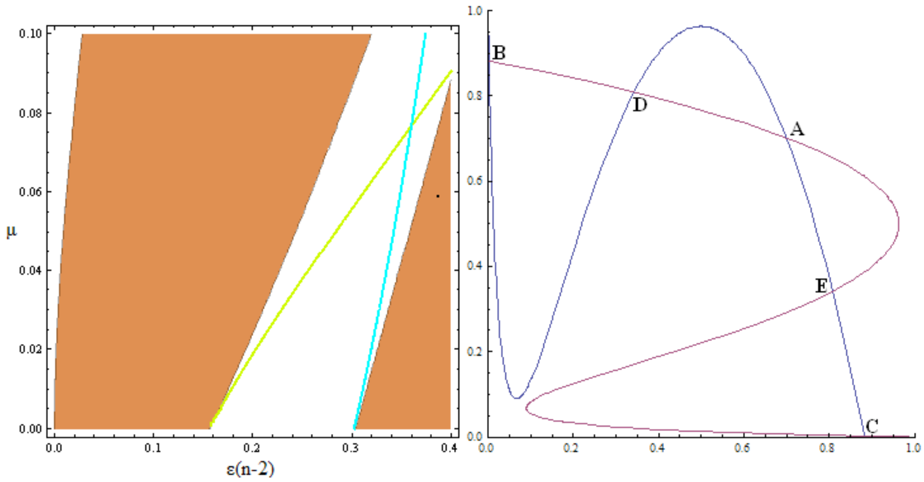

Figure 6: Parameter values chosen are $\epsilon(n-2) = 17/44$ and $\mu = 1/17$, resulting in an associated mean field fixed points graph with five applicable roots. With respect to Figure 5, two more roots are lost due to only possessing a single first order solution. Of these roots, points A, B and C are found to be stable, whereas the others are unstable.

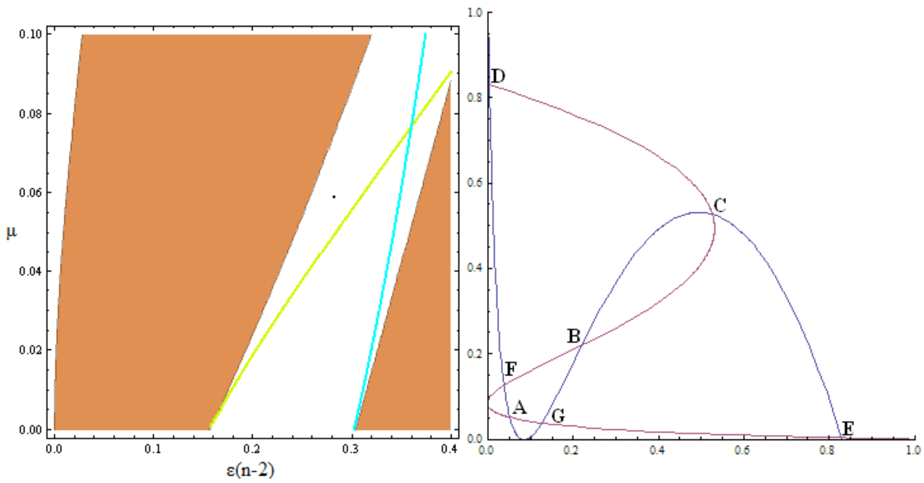

Figure 7: Parameter values chosen are $\epsilon(n-2) = 31/110$ and $\mu = 1/17$, resulting in an associated mean field fixed points graph with seven applicable roots. With respect to Figure 4, two roots are lost due to non-bifurcation of point C. Of these roots, points A, D and E are found to be stable, whereas the others are unstable.

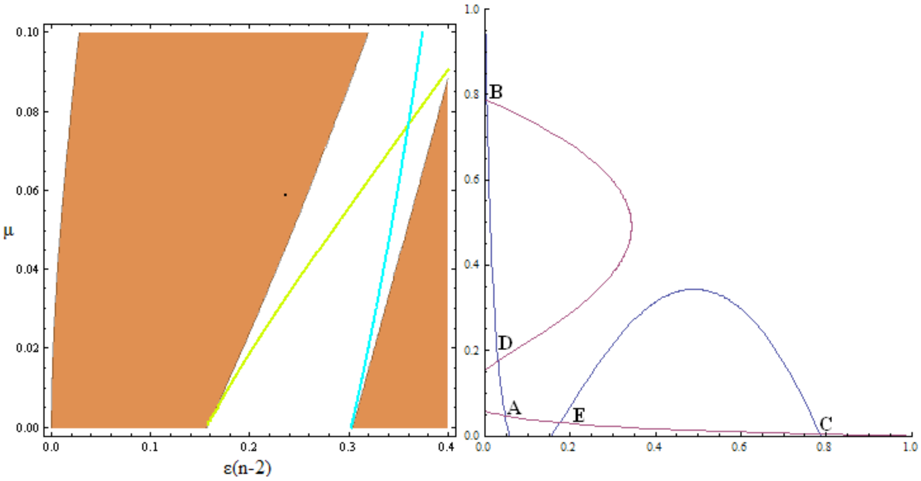

Figure 8: Parameter values chosen are $\epsilon(n-2) = 13/55$ and $\mu = 1/17$, resulting in an associated mean field fixed points graph with five applicable roots. With respect to Figure 7, two more roots are lost due to only possessing a single first order solution. Of these roots, points A, B and C are found to be stable, whereas the others are unstable.

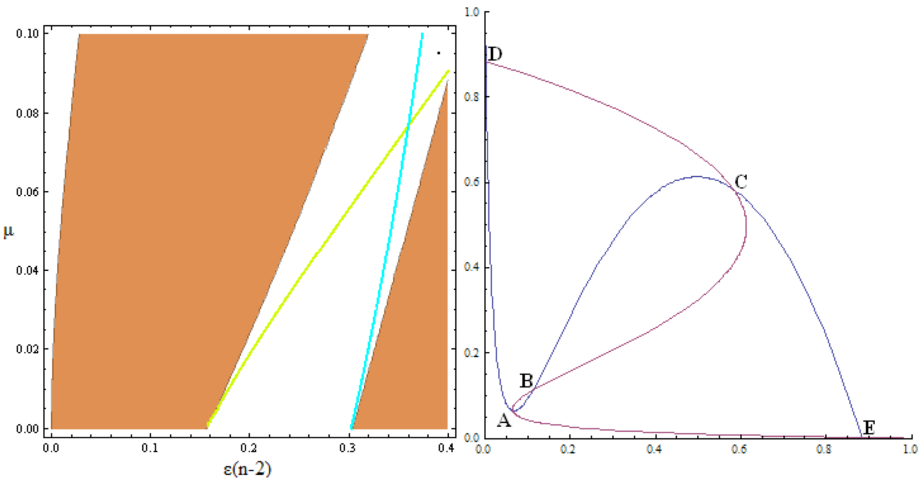

Figure 9: Parameter values chosen are $\epsilon(n-2) = 43/110$ and $\mu = 2/21$, resulting in an associated mean field fixed points graph with five applicable roots. With respect to Figure 4, two roots are lost due to non-bifurcation of point A and a further two roots are lost due to non-bifurcation of point C. Of these roots, points D and E are found to be stable, whereas the others are unstable.

Given $f_B(f_A(p)) - p = 0$, from (5) and (6), this occurs where p is the root of a ninth order polynomial $r(p)$, say. Then by direct calculation

$$r(0) = {\delta_A}^2 \epsilon_B(\delta_A - \gamma_A) + \gamma_A(\delta_B(\delta_A - \gamma_A) + \delta_A \omega_B). \tag{10}$$

Recall both that all parameters are positive and $\gamma_A \geq \delta_A$: hence $r(0) > 0$. Notice also that,

$$\begin{aligned} r(1) \;\; = \;\; & -{\mu_A}^3(\delta_B - \gamma_B) - {\mu_A}^2 \omega_A(\delta_B - \gamma_B) - {\mu_A}^2 \mu_B \omega_A \\ & -\epsilon_B \mu_A {\omega_A}^2 - \epsilon_B {\omega_A}^3 - {\mu_A}^2 \omega_A \omega_B. \end{aligned} \tag{11}$$

Then similarly $r(1) < 0$.

It follows that there exists an odd number of applicable roots satisfying (5) and (6) in $[0, 1]$, even without equality between network parameters by the intermediate value theorem. We would expect this to be the case since applicable roots can only be lost in pairs, through the coalescence of two roots or a pitchfork bifurcation.

6. Discussion

In this paper we have introduced a model for networks that compete edge-wise over a fixed set of vertices. Both networks inhibit the other's growth (through lower edge birth rates) and encourage the other's demise (through greater edge death rates).

The nonlinear stochastic competition equations yield to a mean field analysis that results in an associated nonlinear deterministic system. This in turn indicates there may be multiple dynamic steady states; regions of stability, with some sensitivity to the stochastic details found close to unstable equilibria; and a sensitivity to sparse initial conditions.

The applications we have in mind are situations where one peer-to-peer communication network competes and gradually displaces the other. For example where the emergence of BlackBerry Messenger has created a competing network against SMS messages, resulting in a decreased edge density for SMS communication over this userbase [2]. Our analysis illustrates how the ultimate fate of such competitions may depend upon early sensitive and stochastic behaviour.

Acknowledgements

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[1] C. Avin, M. Koucky, and Z. Lotker, *How to explore a fast-changing world (cover time of a simple random walk on evolving graphs)*, ICALP 08: Proceedings of the 35th international colloquium on Automata, Languages and Programming, Part I, Berlin, Heidelberg, 2008, Springer-Verlag, pp. 121-132.

[2] T. Bradshaw, *Instant messaging forecast to hit texting*, Financial Times, April 24, 2011

[3] A. E. F. Clementi, C. Macci, A. Monti, Francesco Pasquale and R. Silvestri, *Flooding time in edge-Markovian dynamic graphs*, Proceedings of the 27th Annual ACM SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC08), ACM Press, 2008, pp. 213222.

[4] A. E. F. Clementi, F. Pasquale, A. Monti, and R. Silvestri, *Information spreading in stationary Markovian evolving graphs*, Proceedings of the 2009 IEEE International Symposium on Parallel and Distributed Processing, Washington, DC, USA, 2009, IEEE Computer Society, pp. 112.

[5] S. Goodreau, J. A. Kitts, and M. Morris, *Birds of a feather or friend of a friend? Using exponential random graph models to investigate adolescent friendship networks*, Demography, 46 (2009), pp. 103126.

[6] R. A. Kerin, P. R. Varadarajan and R. A. Peterson, *First-Mover Advantage: A Synthesis, Conceptual Framework, and Research Propositions*, The Journal of Marketing, Vol. 56, No. 4 (Oct., 1992), pp. 33-52.

[7] P. Grindrod and D. J. Higham, *Models for evolving networks: with applications in telecommunication and online activities*, IMA Journal of Management Mathematics, first published online: doi:10.1093/imaman/dpr001 (2011).

[8] P. Grindrod, D. J. Higham and M. Parsons, *Bistable Evolving Networks*, Internet Mathematics (submitted) 2011.

[9] P. Grindrod and M. C. Parsons, *Social Networks: Evolving Graphs with Memory Dependent Edges*, Physica A: Statistical Mechanics and its Applications 390 (2011), pp. 3970-3981

[10] P. Grindrod, M. C. Parsons, D. J. Higham and E. Estrada, *Communicability Across Evolving Networks*, Phys. Rev. E, 83 (2011), p. 046120.

[11] P. Holme, *Network reachability of real-world contact sequences*, Physical Review E, 71 (2005).

[12] J. Tang, M. Musolesi, C. Mascolo, V. Latora, and V. Nicosia, *Analysing information flows and key mediators through temporal centrality metrics*, SNS 10: Proceedings of the 3rd Workshop on Social Network Systems, New York, NY, USA, 2010, ACM, pp. 1-6.

[13] J. Tang, S. Scellato, M. Musolesi, C. Mascolo, and V. Latora, S*mallworld behavior in time-varying graphs*, Physical Review E, 81 (2010), p. 05510.