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New results and open problems on Toeplitz operators in Bergman spaces

A. Perälä, J. Taskinen and J. A. Virtanen

Abstract. We discuss some of the recent progress in the field of Toeplitz operators acting on Bergman spaces of the unit disk, formulate some new results, and describe a list of open problems — concerning boundedness, compactness and Fredholm properties — which was presented at the conference "Recent Advances in Function Related Operator Theory" in Puerto Rico in March 2010.

Contents

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1. Introduction

Toeplitz operators form one of the most significant classes of concrete operators because of their importance both in pure and applied mathematics and in many other sciences, such as economics, (mathematical) physics, and chemistry. Despite their simple definition, Toeplitz operators exhibit a very rich spectral theory and employ several branches of mathematics.

Let X be a function space and let P be a projection of X onto some closed subspace Y of X. Then the Toeplitz operator $T_a : X \to Y$ with symbol a is defined by $T_a f = P(af)$. The two most widely understood cases are when Y is either a Bergman space or a Hardy space; more recently Toeplitz operators have been also studied in many other function spaces, such as Fock, Besov, Harmonic-Bergman, and bounded mean oscillation types of spaces; see, e.g., [10, 8, 26, 33].

We are interested in the case when Toeplitz operators are acting on Bergman spaces A^p of the unit disk, which consists of all analytic functions in $L^p := L^p(\mathbb{D})$ (with area measure). For Toeplitz operators on Bergman spaces of other types of domains, such as the unit ball, bounded symmetric domains, pseudo-convex domains, see [3, 7, 12, 13]. The Bergman projection $P : L^p \to A^p$ has the following integral presentation

$$Pf(z) = \int\limits_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^2} dA(w) = \int\limits_{\mathbb{D}} f(w) K_z(w) dA(w) \,, \tag{1.1}$$

where dA denotes the normalized area measure on $\mathbb{D}$ and K_z is the Bergman kernel. The properties of Toeplitz operators we are interested in are Fredholmness, compactness, and boundedness when the symbols are in general (matrix-valued) functions in L^1_{loc} or distributions. We focus on Bergman spaces A^p when $1 < p < \infty$, except for Section 5 in which we briefly discuss Toeplitz operators on the space A^1.

2. Bounded Toeplitz operators

2.1. Locally integrable symbols. Clearly the Toeplitz operator T_a is bounded on A^p with $1 < p < \infty$ when $a \in L^\infty$. The real difficulty lies in determining when Toeplitz operators with unbounded symbols are bounded. One of the first results was Luecking's characterization (see [18]) which states the Toeplitz operator $T_a : A^2 \to A^2$ with a *nonnegative* symbol $a \in L^1$ is bounded if and only if the average $\hat{a}_r$ of a is bounded; here the average of a is defined by

$$\hat{a}_r(z) = |B(z,r)|^{-1} \int\limits_{B(z,r)} a(w)\, dA(w),$$

where $B(z,r)$ denotes the Bergman disk at z with radius r.

A complete description of bounded Toeplitz operators with radial symbols was found by Grudsky, Karapetyants, and Vasilevski (see [31]), that is, they showed that $T_a : A^2 \to A^2$ with a radial symbol a is bounded if and only if $\sup_{m\in\mathbb{Z}_+} |\gamma_a(m)| < \infty$, where

$$\gamma_a(m) = (m+1) \int_0^1 a(\sqrt{r}) r^m dr.$$

This result is derived from the observation that in the radial case the Toeplitz operator is unitarily equivalent to a multiplication operator on the sequence space ℓ^2. More precisely, T_a is a Taylor coefficient multiplier. Since the monomials z^n do not form an unconditional Schauder basis in A^p for $p \neq 2$, it is hard to provide an analogous result for the more general case. However, a partial generalization to the case $p \neq 2$ was very recently found in [20].

Another useful tool for dealing with Toeplitz operators is the Berezin transform, defined by

$$B(f)(z) = (1-|z|^2)^2 \int_{\mathbb{D}} \frac{f(w)}{|1-z\bar{w}|^4} dA(w)\,. \tag{2.1}$$

Zorboska (see [41]) observed that Luecking's result can be used to deal with a large class of unbounded symbols, and showed that when a is of bounded mean oscillation, that is, when $\sup_{z\in\mathbb{D}} MO^1_r(a)(z) < \infty$, where

$$MO^p_r(a)(z) := \left(\frac{1}{|B(z,r)|} \int_{B(z,r)} |a(w) - \hat{a}_r(z)|^p dA(w) \right)^{1/p}, \tag{2.2}$$

the Toeplitz operator $T_a : A^2 \to A^2$ is bounded if and only if $B(a)$ is bounded.

All the results above only deal with the Hilbert space case and it was not until recently that results in A^p spaces were established. Indeed, denote by $\mathcal{D}$ the family that consists of the sets $D := D(r,\theta)$ defined by

$$D = \left\{ \rho e^{i\phi} \mid r \leq \rho \leq 1 - \tfrac{1}{2}(1-r)\ ,\ \theta \leq \phi \leq \theta + \pi(1-r) \right\} \tag{2.3}$$

for all $0 < r < 1$, $\theta \in [0, 2\pi]$. Given $D = D(r,\theta) \in \mathcal{D}$ and $\zeta = \rho e^{i\phi} \in D$, denote

$$\hat{a}_D(\zeta) := \frac{1}{|D|} \int_r^\rho \int_\theta^\phi a(\varrho e^{i\varphi}) \varrho d\varphi d\varrho.$$

Two of the authors showed (see [29]) that if $a \in L^1_{\text{loc}}$ and if there is a constant C such that

$$|\hat{a}_D(\zeta)| \leq C \tag{2.4}$$

for all $D \in \mathcal{D}$ and all $\zeta \in D$, then the Toeplitz operator $T_a : A^p \to A^p$ is well defined and bounded for all $1 < p < \infty$, and there is a constant C such that

$$\|T_a; A^p \to A^p\| \le C \sup_{D\in\mathcal{D},\zeta\in D} |\hat{a}_D(\zeta)|. \tag{2.5}$$

Note that not all such symbols are in L^1. We also remark that if a is nonnegative, the condition in (2.4) is equivalent to Luecking's condition, and thus the preceding theorem shows Luecking's result holds true also for Toeplitz operators on A^p with $1 < p < \infty$. Further, using this corollary, one can show that Zorboska's result can be generalized to the case $1 < p < \infty$; we leave out the details here and only note that the proof is similar to that of Zorboska's.

The fundamental question remains open, that is, find a sufficient and necessary condition for Toeplitz operators to be bounded on A^2.

2.2. Distributional symbols. We next consider the case of symbols that are distributions, which leads to a natural generalization of the cases in which symbols are functions (as above) or measures (see, e.g., [40]). Since

$$w \mapsto f(w)(1 - z\bar{w})^{-2}$$

is obviously smooth, whenever f is smooth, it is not difficult to define Toeplitz operators for compactly supported distributions. Indeed, if a is such a distribution, then for $f \in A^p$, we have

$$T_a f(z) = \langle f(w)(1 - z\bar{w})^{-2}, a\rangle_w,$$

where the dual bracket is taken with respect to the pairing $\langle C^\infty, (C^\infty)^*\rangle$. Observe that compactly supported distributions always generate compact Toeplitz operators. A characterization of finite rank Toeplitz operators can be found in [1].

On the other hand, it seems difficult to define Toeplitz operators for arbitrary distributional symbols because

$$w \mapsto f(w)(1 - z\bar{w})^{-2}$$

fails to be a compactly supported test function, unless f is one. In particular, the only such $f \in A^p$ is the zero function.

In [25] we showed that symbols in a weighted Sobolev space $W_\nu^{-m,\infty}(\mathbb{D})$ of negative order generate bounded Toeplitz operators on A^p. More precisely, let $\nu(z) = 1 - |z|^2$ and for $m \in \mathbb{N}$, denote by $W_\nu^{m,1} := W_\nu^{m,1}(\mathbb{D})$ the weighted Sobolev space consisting of measurable functions f on $\mathbb{D}$ such that the distributional derivatives satisfy

$$\|f; W_\nu^{m,1}\| := \sum_{|\alpha|\le m} \int_{\mathbb{D}} |D^\alpha f(z)| \nu(z)^{|\alpha|} dA(z) < \infty. \tag{2.6}$$

Here we use the standard multi-index notation, which is explained in detail in [25]. Since the subspace $C_0^\infty := C_0^\infty(\mathbb{D})$ is dense in $W_\nu^{m,1}$ (see [25]), we can describe the dual space, that is, for $m \in \mathbb{N}$ we denote by $W_\nu^{-m,\infty} :=$

$W_\nu^{-m,\infty}(\mathbb{D})$ the (weighted Sobolev) space consisting of distributions a on $\mathbb{D}$ which can be written in the form

$$a = \sum_{0\le|\alpha|\le m} (-1)^{|\alpha|} D^\alpha b_\alpha, \tag{2.7}$$

where $b_\alpha \in L^\infty_{\nu^{-|\alpha|}} := L^\infty_{\nu^{-|\alpha|}}(\mathbb{D})$, i.e.,

$$\|b_\alpha; L^\infty_{\nu^{-|\alpha|}}\| := \operatorname*{ess\,sup}_{\mathbb{D}} \nu(z)^{-|\alpha|}|b_\alpha(z)| < \infty. \tag{2.8}$$

Here every b_α is considered as a distribution like a locally integrable function, and the identity (2.7) contains distributional derivatives. Note that the representation (2.7) need not be unique in general. Hence, we define the norm of a by

$$\|a\| := \|a; W_\nu^{-m,\infty}\| := \inf \max_{0\le|\alpha|\le m} \|b_\alpha; L^\infty_{\nu^{-|\alpha|}}\|, \tag{2.9}$$

where the infimum is taken over all representations (2.7).

Suppose that

$$a \in W_\nu^{-m,\infty} \subset \mathcal{D}' \tag{2.10}$$

for some m. By Theorem 3.1 of [25], the Toeplitz operator T_a, defined by the formula

$$T_a f(z) = \sum_{0\le|\alpha|\le m} \int_{\mathbb{D}} \left(D_\zeta^\alpha \frac{f(\zeta)}{(1-z\bar\zeta)^2}\right) b_\alpha(\zeta) dA(\zeta) \quad , \; f \in A^p, \tag{2.11}$$

is well defined and bounded $A^p \to A^p$ for all $1 < p < \infty$. The resulting operator is independent of the choice of the representation (2.7). Moreover, there is a constant $C > 0$ such that

$$\|T_a : A^p \to A^p\| \le C\|a; W_\nu^{-m,\infty}\|. \tag{2.12}$$

We remark that when $\mathbb{D}$ is considered as a subset of $\mathbb{R}^2$ and $f(w)(1-z\bar w)^{-2}$ a real-analytic function, we can even consider Toeplitz operators with symbols that are arbitrary hyperfunctions on $\mathbb{D} \subset \mathbb{R}^2$. This obviously makes it possible to define Toeplitz operators for distributions of arbitrary order as well, since hyperfunctions generalize distributions. We restrict our hyperfunction considerations in this paper to the following example; for further details about hyperfunctions, see [15, 22].

Example 1. Consider the (not necessarily continuous) linear form h on C^∞ defined by

$$h : f \mapsto \sum_\alpha a_\alpha (D^\alpha f)(0)/\alpha!.$$

Suppose also that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|a_\alpha| \le C_\varepsilon \varepsilon^{|\alpha|}$. This functional represents a hyperfunction, which is a distribution if and only if the sum is finite. However, assuming that a_α tends to zero rapidly enough as $\alpha \to \infty$ ($a_\alpha = |\alpha|!^{-5}$ will do), it is easy to see that if h_m is defined by

$$h_m : f \mapsto \sum_{|\alpha| \le m} a_\alpha (D^\alpha f)(0)/\alpha!,$$

then the associated finite rank Toeplitz operators T_{h_m} converge in norm to a compact operator, which indicates that one could extend the theory of Toeplitz operators even beyond distributional symbols.

In what follows we restrict to the case $a \in L^1_{\mathrm{loc}}$ is radial, i.e. $a(z) = a(|z|)$. This is motivated by two facts. First, the sufficient conditions in (2.4) and (2.10) can be formulated in a more simple way, suitable for radial symbols. Second, we are able to clarify the relation of the two conditions: Proposition 7 shows that in the radial case (2.4) is weaker than (2.10).

For all $r \in]0,1[=: \mathbb{I}$ denote $I := I(r) = [r, 1-(1-r)/2]$ and

$$\hat{a}_I(\rho) = \frac{1}{1-r} \int_r^\rho a(\varrho)\varrho d\varrho\,, \tag{2.13}$$

where $\rho \in I$.

Lemma 2. *For a radial $a \in L^1_{\mathrm{loc}}$, (2.4) is equivalent to the existence of a constant $C > 0$ such that*

$$|\hat{a}_I(\rho)| \le C \tag{2.14}$$

for all $r \in \mathbb{I}$ and $\rho \in I(r)$.

Proof. Assume that a satisfies (2.14). Let $r \in]0,1[$ and $\theta \in [0, 2\pi]$ be given, and let ϕ be such that $\theta \le \phi \le \theta + \pi(1-r)$; see (2.3). We have

$$\left| \int_r^\rho \int_\theta^\phi a(\varrho e^{i\varphi})\varrho d\varphi d\varrho \right| = (\phi - \theta) \left| \int_r^\rho a(\varrho)\varrho d\varrho \right| \le C(\phi-\theta)(1-r) \le C'|D|,$$

where D is as in (2.3). Notice that $|D|$ is proportional to $(1-r)^2$. On the contrary, if a satisfies (2.4), we can deduce from the radiality of a

$$\left| \int_r^\rho a(\varrho)\varrho d\varrho \right| = \left| \int_r^\rho \frac{1}{\pi(1-r)} \int_\theta^{\theta+\pi(1-r)} a(\varrho e^{i\varphi})\varrho d\varphi d\varrho \right| \le \frac{C|D|}{1-r} \le C'(1-r). \quad \square$$

The main result on the boundedness of Toeplitz operators in [29] now gives the following fact.

Corollary 3. *If the symbol $a \in L^1_{\mathrm{loc}}$ is radial and satisfies (2.14), then the Toeplitz operator $T_a : A^p \to A^p$ is bounded.*

It also follows from (2.5) that in the presence of (2.14), the bound

$$\|T_a : A^p \to A^p\| \le C \sup |\hat{a}_I(\rho)|$$

holds, where the supremum is taken over all intervals I and $\rho \in I$.

We next consider radial distributional symbols. We choose an approach which in the beginning only comprises distributions on $\mathbb{D}\setminus\{0\}$, see Remark 5 for a discussion. Let us define the weight function $\mu(r) = r(1-r^2)$, $r \in \mathbb{I}$, and the Sobolev space $W_\mu^{m,1}(\mathbb{I})$, $m \in \mathbb{N}$, which consists of measurable functions $f : \mathbb{I} \to \mathbb{C}$ such that the distributional derivatives of f up to order m are locally integrable functions $\mathbb{I} \to \mathbb{C}$ and satisfy

$$\|f; W_\mu^{m,1}\| := \sum_{j=0}^{m} \int_0^1 \Big|\frac{d^j f(r)}{dr^j}\Big| \mu(r)^j dr < \infty\,. \tag{2.15}$$

Moreover, by $W_\mu^{-m,\infty}(\mathbb{I})$ we denote the space of distributions on $\mathbb{I}$ which can be written, using distributional derivatives, in the form

$$a = \sum_{j=0}^{m} (-1)^j \frac{d^j b_j(r)}{dr^j} \tag{2.16}$$

for some functions $b_j \in L^\infty_{\mu^j}(\mathbb{I})$; the spaces have the norms

$$\|b_j; L^\infty_{\mu^{-j}}\| := \operatorname*{ess\,sup}_{r\in\mathbb{I}} \mu^{-j}(r)|b_j(r)| \quad , \quad \|a; W_\mu^{-m,\infty}\| := \inf \max_{j\le m} \|b_j; L^\infty_{\mu^{-j}}\|,$$

where the infimum is taken over all representations (2.16).

Lemma 4. *The dual of $W_\mu^{m,1}(\mathbb{I})$ is isometric to $W_\mu^{-m,\infty}(\mathbb{I})$ with respect to the dual paring*

$$\langle f, a\rangle = \sum_{j=0}^{m} \int_0^1 \frac{d^j f(r)}{dr^j} b_j dr. \tag{2.17}$$

Here $f \in W_\mu^{m,1}(\mathbb{I})$, $a \in W_\mu^{-m,\infty}(\mathbb{I})$, and the representation (2.16) applies.

This can be proven in the same way as the general case in Section 2 of [25]. Notice that the representation (2.16) is not unique, but the value of the right hand side of (2.17) is. See [25] for further details.

If $b : \mathbb{D} \to \mathbb{C}$ is a smooth function, the chain rule implies $\partial_r b(re^{i\theta}) := \partial b(re^{i\theta})/\partial r = (D^{(1,0)}b(z))\cos\theta + (D^{(0,1)}b(z)) i \sin\theta$, where $z = re^{i\theta}$ and the multi-index notation is used for partial derivatives; now more generally,

$$\partial_r^j b(re^{i\theta}) = \sum_{l=0}^{j} c_{j,l} (D^{(j-l,l)}b)(z)(\cos\theta)^{j-l}(\sin\theta)^l, \tag{2.18}$$

where $c_{j,l}$ are positive constants. Given $a \in W_\mu^{-m,\infty}(\mathbb{I})$ as in (2.16) the correct extension of it as a distribution on $\mathbb{D}\setminus\{0\}$ is given by the formula

$$\langle \varphi, a\rangle = \sum_{j=0}^{m} \int_0^{2\pi} \int_0^1 b_j(r) \frac{\partial^j r\varphi(re^{i\theta})}{\partial r^j} dr d\theta, \tag{2.19}$$

where φ is an arbitrary compactly supported C^∞-test function on $\mathbb{D} \setminus \{0\}$. The reason is that if $a \in C^m$, (2.19) equals $\int_{\mathbb{D}} a(|z|)\varphi(z)dA(z)$, by (2.16) and an integration by parts in the variable r. Moreover, by (2.18), (2.19) also equals

$$\sum_{j=0}^{m}\sum_{l=0}^{j}\int_{\mathbb{D}} b_j(z)c_{j,l}D^{(j-l,l)}\Big(\varphi(z)(\cos\theta)^{j-l}(\sin\theta)^l\Big)dA(z), \tag{2.20}$$

where $b_j(z) := b_j(|z|)$. Note that it does not matter that the functions $\cos\theta$ and $\sin\theta$ are not smooth at the origin because of the support of φ.

Remark 5. It was necessary to define the weight μ such that it vanishes also at 0. Otherwise, the simple duality relation of the Sobolev spaces presented above would fail, and in practise this would lead to unnecessary technical complications in the partial integration above (especially in the substitutions at 0).

The present approach leads to the drawback that the Dirac measure of $0 \in \mathbb{D}$, or any of its derivatives, are not included in the symbol class of the next theorem. However, this is not at all serious, since the results of [25] show that all distributions with compact support inside $\mathbb{D}$ automatically define compact Toeplitz operators.

Theorem 6. *If $a \in W_\mu^{-m,\infty}(\mathbb{I})$, then the Toeplitz operator T_a defined by the formula*

$$T_a f(z) = \sum_{j=0}^{m}\int_0^{2\pi}\int_0^1 b_j(r)\frac{\partial^j}{\partial r^j}\frac{rf(re^{i\theta})}{(1-zre^{-\theta})^2}drd\theta, \tag{2.21}$$

where the functions b_j are as in (2.16), is well defined and bounded $A^p \to A^p$.

We also get the bound $\|T_a : A^p \to A^p\| \le C\|a; W_\mu^{-m,\infty}\|$.

Proof. Referring to the notation of [25], the identities (2.16) and (2.18), or alternatively, (2.19) and (2.20), imply that T_a coincides with the Toeplitz operator T_A on the disk in the sense of [25], where

$$A := \sum_{j=0}^{m}\sum_{l=0}^{j}(-1)^j b_{j,l} \quad , \quad b_{j,l} := c_{j,l}\big(D^{(j-l,l)}b_j(z)\big)(\cos\theta)^{j-l}(\sin\theta)^l \tag{2.22}$$

and the partial derivatives are in the sense of distributions (on the disk). Comparing to (2.7)–(2.9) and taking into account the definition of the space $W_\mu^{-m,\infty}(\mathbb{I}) \ni a$, we see that $A \in W_\nu^{-m,\infty}(\mathbb{D})$, and $T_a = T_A$ is bounded $A^p \to A^p$; see Theorem 3.1 of [25]. □

In particular any a such that the supports of all b_j are contained in some interval $[0, R]$ with $R < 1$ defines a compact Toeplitz operator on A^p, by [25, Proposition 4.1].

The motivation of the definition (2.21) is that it is obviously much simpler in the radially symmetric case, just due to the use of polar coordinates. Another motivation is the following observation which clarifies the relation of the sufficient conditions in (2.4) and (2.10) for radial symbols: the latter condition is weaker.

Proposition 7. *If the radial symbol* $a \in L^1_{\text{loc}}(\mathbb{D})$ *satisfies* (2.4), *then* $a \in W_\nu^{-1,\infty}(\mathbb{D})$*; in particular* a *satisfies* (2.10).

Proof. Since any compactly supported function in $L^1_{\text{loc}}(\mathbb{D})$ satisfies (2.10), we may assume that the support of a is outside the disk $\{|z| \le 1/2\}$. Moreover, we may assume by Lemma 2 that a satisfies (2.14), and finally, by the proof of Theorem 6, it will be enough to show that the restriction of a to $\mathbb{I}$ belongs to the Sobolev space $W_\mu^{-1,\infty}(\mathbb{I})$.

First, let $r \in \mathbb{I}$ and denote, for all $n \in \mathbb{N}$, $r_n = 1 - 2^{-n}$. Keeping in mind that a is only locally integrable, we define

$$\int_r^1 a(\varrho)d\varrho := \int_r^{r_N} a(\varrho)d\varrho + \sum_{n=N}^{\infty} \int_{r_n}^{r_{n+1}} a(\varrho)d\varrho,$$

where $N = N(r) \in \mathbb{N}$ is the unique number such that $r \in]r_{N-1}, r_N]$. This sum converges, since the formulas (2.13) and (2.14) imply

$$\sum_{n=N}^{\infty} \left| \int_{r_n}^{r_{n+1}} a(\varrho)d\varrho \right| \le \sum_{n=N}^{\infty} C2^{-n} = C2^{-N+1} \tag{2.23}$$

for any N. Let $\psi : \mathbb{I} \to [0,1]$ be a C^∞-function which is increasing, equal to 0 in $]0, 1/8]$ and equal to 1 in $[1/4, 1[$. We define

$$b_0(r) = \psi'(r) \int_r^1 a(\varrho)d\varrho,$$

$$b_1(r) = \psi(r) \int_r^1 a(\varrho)d\varrho.$$

The identity (2.16) follows from the assumptions made on the supports of a and ψ:

$$b_0(r) - \frac{db_1(r)}{dr} = \psi(r)a(r) = a(r).$$

We need to show that $b_j \in L^\infty_{\mu^{-j}}$. Let us first consider b_1. Due to the choice of ψ we have $b_1(r) \le cr$ for small r, and it remains to show that $|b_1(r)| \le C(1-r)$ for r close to 1. But assuming $r > 1/2$ and choosing N as in (2.23), we have $|\int_r^{r_N} a| \le C2^{-N}$, and hence an estimate similar to (2.23) implies

$$|b_1(r)| = \left| \int_r^1 a(\varrho) d\varrho \right| \le C 2^{-N} \le C'(1-r).$$

This estimate, the fact that $a \in L^1_{\mathrm{loc}}(\mathbb{I})$, and the compactness of the support of ψ' clearly also imply that b_0 is a bounded function. $\square$

3. Compact Toeplitz operators

3.1. Locally integrable symbols. For bounded symbols, a compactness criterion (in terms of the Berezin transform) for Toeplitz operators on A^p is well known, see, e.g., Suárez's recent description of compact operators in the Toeplitz algebra generated by bounded symbols in [27] and references therein for previous results concerning finite sums of finite products of Toeplitz operators.

For general symbols, the results in the previous section can be reformulated for compactness by replacing the condition "be bounded" by "vanishes on the boundary." For example, for a positive symbol a in L^1, the Toeplitz operator T_a is compact on A^p $(1 < p < \infty)$ if and only if $B(a)(z) \to 0$ as $|z| \to 1$ (see [18, 29]); for further details about compactness of Toeplitz operators with several classes of (locally) integrable symbols, see the articles we referred to in Section 2. We'd like to mention one generalization provided by Zorboska (see [41]), that is, if $f \in L^1$, if T_f is bounded on A^2, and if

$$\sup_{z\in\mathbb{D}} \|T_{f\circ\varphi_z} 1; L^p\| < \infty \quad \text{and} \quad \sup_{z\in\mathbb{D}} \|T_{\bar{f}\circ\varphi_z} 1; L^p\| < \infty \tag{3.1}$$

for some $p > 3$, where $\varphi_z(w) = (z-w)(1-\bar{z}w)^{-1}$, then T_f is compact on A^2 whenever $B(f)(z) \to 0$ as $|z| \to 1$. She also posed a question of whether this result remains true when (3.1) holds for some $p > 2$.

As in the case of boundedness, the most fundamental question remains open: find a sufficient and necessary condition for Toeplitz operators with L^1 symbols to be compact on A^2.

Regarding compact Toeplitz operators, it is worth noting that Luecking [19] showed that there are no nontrivial finite rank Toeplitz operators on A^2 with bounded symbols; observe that his proof actually covers Toeplitz operators on any space of analytic polynomials. It would also be interesting to find out whether there are nontrivial finite rank Toeplitz operators on the Bloch space $\mathcal{B} = \{f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}} |f'(z)|\,(1-|z|)^2 < \infty\}$.

3.2. Distributional symbols. Note first that all distributions $a \in \mathcal{D}'$ with compact support belong to the Sobolev space $W_\nu^{-m,\infty}$ and generate Toeplitz operators on A^p via (2.11); see [25]. In the same article it was also shown that if we make no assumption that the symbol a be compactly supported, then T_a is still compact provided that a has a representation (2.7) such that the functions b_α satisfy

$$\lim_{r\to 1} \operatorname*{ess\,sup}_{|z|\geq r} \nu(z)^{-|\alpha|}|b_\alpha(z)| = 0\,. \tag{3.2}$$

Let us look at radial symbols as in the previous section.

Lemma 8. *For a radial $a \in L^1_{\mathrm{loc}}$ the condition*

$$\lim_{r\to 1} \sup_{\rho\in I(r)} |\hat{a}_I(\rho)| = 0 \tag{3.3}$$

is equivalent to the compactness condition in [29]*, that is,*

$$\lim_{d(D)\to 0} \sup_{\xi\in D} |\hat{a}_D(\zeta)| = 0. \tag{3.4}$$

Proof. Proceed as in the proof of Lemma 2 and note that $1 - r \to 0$ if and only if $d(D) \to 0$, which happens if and only if $|D| \to 0$. ☐

Theorem 9. *Suppose that $a \in W_\mu^{-m,\infty}(\mathbb{I})$ has a representation*

$$a = \sum_{j=0}^{m} (-1)^j \frac{d^j b_j(r)}{dr^j}$$

where each b_j satisfies

$$\operatorname{ess} \lim_{s\to 1} \sup_{r\in(s,1)} \mu(r)^{-j}|b_j(r)| = 0,$$

then T_a is compact.

Proof. Since the symbol a can be seen as a distribution that satisfies (3.4), an application of Theorem 6 completes the proof. ☐

For the following result, see the comment preceding Proposition 7.

Proposition 10. *If the radial symbol $a \in L^1_{\mathrm{loc}}(\mathbb{D})$ satisfies (3.3), then $a \in W_\nu^{-1,\infty}(\mathbb{D})$ satisfies the condition of the preceding theorem.*

Proof. We proceed as in the proof of Proposition 7 and write a using b_0 and b_1. The function b_0 is obviously compactly supported. To deal with b_1, we just note that given $\varepsilon > 0$, we can pick N such that

$$\left| \int_{r_n}^{r_{n+1}} a(\varrho) d\varrho \right| \leq \varepsilon 2^{-n}$$

for $n \geq N - 1$. Arguing along the lines of the proof of Proposition 7 we see that $|(1 - r)b_1(r)| \leq \varepsilon$, when r is close enough to 1. This proves that the representation is as desired. ☐

3.3. The Berezin transform. Recall that, for an operator T on A^2, the Berezin transform of T at the point $z \in \mathbb{D}$ is defined by

$$\tilde{T}(z) = \langle T k_z, k_z \rangle,$$

where k_z is the normalized reproducing kernel $k_z = K_z/\|K_z\|_2$ and K_z is the kernel in (1.1). Also recall (2.1). For a distribution $a \in W_\nu^{-m,\infty}$, we define

$$\tilde{a}(z) = \langle T_a k_z, k_z \rangle = \langle |k_z(w)|^2, a \rangle_w = \langle 1, a \circ \varphi_z \rangle_w,$$

where $\langle \cdot, \cdot \rangle_w$ stands for the dual bracket of the pair $\langle W_\nu^{m,1}, W_\nu^{-m,\infty} \rangle$ and φ_z is the disk automorphism $w \mapsto (z-w)/(1-\bar{z}w)$, which interchanges the origin and z; also note that the expression $a \circ \varphi_z$ is defined by its action on $W_\nu^{m,1}$ by

$$\langle f(w), a \circ \varphi_z \rangle_w = \langle (f \circ \varphi_z) |\varphi_z'|^2(w), a \rangle_w.$$

For $f = 1$, all of the above definitions are the same.

Since the functions k_z converge to 0 weakly as z approaches $\mathbb{T}$, it is clear that the compactness of T_a implies $\tilde{a}(z)$ vanishes on the boundary. On the other hand, in [25], we gave a sufficient condition for compactness, that is, if $a \in \mathcal{D}'$ is in $W_\nu^{-m,\infty}$ for some m, then T_a is compact provided that a has a representation (2.7) such that the functions b_α satisfy

$$\lim_{r \to 1} \operatorname*{ess\,sup}_{|z| \geq r} \nu(z)^{-|\alpha|} |b_\alpha(z)| = 0. \tag{3.5}$$

This condition is by no means related to the Berezin transform and it would be useful to shed light to the relevance of the Berezin transform in the study of compact Toeplitz operators generated by distributions.

4. Fredholm properties

Fredholm theory is often very useful in connection with applications, and this is indeed the case with Toeplitz operators; see, e.g., [5, 6, 31]. Let X be a Banach space and let T be a bounded operator on X. Then T is Fredholm if

$$\alpha := \dim \ker T \quad \text{and} \quad \beta := \dim(X/T(X))$$

are both finite, in which case the index of T is $\operatorname{Ind} T = \alpha - \beta$. For further details of Fredholm theory, see, e.g., [23].

In addition to the scalar-valued symbols, we also discuss the matrix-valued case. For that, recall that if X is a Banach space and we set $X_N = \{(f_1, \ldots, f_N) : f_k \in X\}$, then X_N is also a Banach space with the norm

$$\|(f_1, \ldots, f_N); X_N\| := \|f_1; X\| + \ldots + \|f_N; X\|$$

(or with any equivalent norm). Note each $A \in \mathcal{L}(X_N)$ can be expressed as an operator matrix $(A_{ij})_{i,j=1}^N$ in $\mathcal{L}(X_{N \times N})$.

The Fredholm properties of Toeplitz operators with continuous matrix-valued symbols are well understood (see [11] for the Hilbert space case and [24] for the general case). The case of scalar-valued symbols in the

Douglas algebra $C(\overline{\mathbb{D}}) + H^\infty(\mathbb{D})$ was dealt with in [9], however their treatment included no formula for the index. A formula for the index can be found in [24], which also deals with matrix-valued symbols in the Douglas algebra and shows that Fredholmness can be reduced to the scalar-valued case; however, finding an index formula remains an open problem even in the Hilbert space case when the symbols are matrix-valued.

The situation is similar with the so called Zhu class $L^\infty \cap VMO$, that is, Fredholmness of T_a with a matrix-valued symbol in the Zhu class can be reduced to the scalar-valued case, while the index computation remains open; for further details, see [24].

A treatment on the Fredholm properties of Toeplitz operators on A^2 with scalar-valued piecewise continuous symbols can be found in Vasilevski's recent book [31]. Roughly speaking, the essential spectrum is obtained in this case by joining the jumps of the symbol and adding them to continuous parts to get a closed continuous curve. What happens in A^p is not known, but we suspect that the value of p affects the way one should join the jumps; indeed, in the Hardy space case (which is of course in many ways different from the A^2 case), one joins the jumps by lines when $p = 2$ while in other cases by curves whose curvature is determined by the value of p. Further one could also try to establish Fredholm theory for Toeplitz operators on A^2 with matrix-valued piecewise continuous symbols, which is an extremely important part of the theory of Toeplitz operators on Hardy spaces.

We finish this section by mentioning a result which deals with a symbols class that contains *unbounded* symbols, see [29]. Suppose that $a \in VMO^1$ satisfies (2.4) and that for some $\delta > 0$, $C > 0$,

$$|\hat{a}_D(\zeta)| \geq C$$

for all $D \in \mathcal{D}$ with $d(D) \leq \delta$, for all $\zeta \in D$. Then T_a is Fredholm, and there is a positive number $R < 1$ such that

$$\operatorname{Ind} T_a = -\operatorname{ind}(B(a)\restriction s\mathbb{T}) = -\operatorname{ind}(\hat{a}_r \restriction s\mathbb{T})$$

for any $s \in [R, 1)$, where $h\restriction s\mathbb{T}$ stands for the restriction of h into the set $s\mathbb{T}$.

5. Toeplitz and Hankel operators acting on the Bergman space A^1

Here the extra difficulty is caused by the fact that the Bergman projection is no longer bounded and bounded symbols no longer generate bounded Toeplitz operators. In order to deal with some of these difficulties, let us recall the logarithmically weighted versions of BMO spaces: we say that a function $f \in L^1$ is in $BMO^p_{\log}$ if

$$\sup_{z\in\mathbb{D}} \mathcal{W}(z) MO^p_r(f)(z) < \infty \quad , \text{ where } \mathcal{W}(z) := 1 + \log \frac{1}{1-|z|}$$

(recall (2.2) for the definition of $MO^p_r f$). The $VMO^p_{\log}$ space is defined similarly.

Zhu was the first one to consider this case and showed that if $a \in L^\infty \cap BMO^2_{\log}$, then T_a is bounded on A^1; see [36]. More recently two of the authors established the following useful norm estimates (see [28])

$$\|T_a : A^1 \to A^1\| \le C_1 \|a\|, \quad \|H_a : A^1 \to L^1\| \le C_2 \|a\|,$$

where $\|a\| = \|a; L^\infty\| + \|a; BMO_{\log}\|$. Wu, Zhao, and Zorboska [33] proved that for $a \in L^\infty$, the Toeplitz operator $T_{\bar{a}}$ is bounded on A^1 if and only if $P(a)$ belongs to the logarithmic Bloch space

$$L\mathcal{B} = \left\{ f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} \log(1-|z|^2)^{-1} \left|f'(z)\right| (1-|z|)^2 < \infty \right\}.$$

Let us look at Hankel operators and their compactness on L^1. The case of continuous $VMO_{\log}$ symbols was recently considered in [28]. For more general symbols, recall Zhu's result that states H_a and $H_{\bar{a}}$ are both bounded on L^p with $1 < p < \infty$ if and only if $a \in BMO^p$. It is natural to ask whether Hankel operators are compact on L^1 with $BMO_{\log}$ symbols.

Using the decompositions

$$BMO^p_{\log} = BO_{\log} + BA^p_{\log} \quad \textit{and} \quad VMO^p_{\log} = VO_{\log} + VA^p_{\log};$$

cf. $BMO^p = BO + BA^p$ and $VMO^p = VO + VA^p$, where

$$BO = \left\{ f \in C(\mathbb{D}) : \sup_{z \in \mathbb{D}} \sup_{w \in D(z,r)} |f(z) - f(w)| < \infty \right\}$$

and

$$BA^p = \left\{ f \in L^p : \sup_{z \in \mathbb{D}} \widehat{|f|^p_r}(z) < \infty \right\},$$

two of the authors [30] recently showed that if $a \in BO_{\log} \cap L^\infty$, then $T_a : A^1 \to A^1$ is bounded; and if $a \in BO_{\log} \cap L^\infty + BA^1_{\log}$, then $T_a : A^1 \to A^1$ is bounded. In the same article, also "logarithmic versions" of the boundedness and compactness results of [29] were considered. As a consequence, they also derived that if $a \in BMO^1_{\log}$ is such that $a = f + g$ with $f \in BO_{\log} \cap L^\infty$ and $g \in BA^1_{\log}$, then the Hankel operator $H_a : A^1 \to L^1$ is bounded. The problem whether $H_a : A^1 \to L^1$ is bounded for every $a \in BMO^1_{\log}$ remains open.

Concerning the Fredholm properties, things get even more complicated and there are only very few results; we mention a recent result (see [28]). Let $a \in C(\overline{\mathbb{D}}) \cap VMO_{\log}$. Then T_a is Fredholm on A^1 if and only if $a(t) \neq 0$ for any $t \in \mathbb{T}$, in which case

$$\operatorname{Ind} T_a = -\operatorname{ind} a_r.$$

We can also prove an analogous result for Toeplitz operators with matrix-valued symbols.

Theorem 11. *Let a be a matrix-valued symbol with $a_{jk} \in C(\overline{\mathbb{D}}) \cap VMO_{\log}$. Then the Toeplitz operator T_a is Fredholm on A^1_N if and only if $\det a(t) \neq 0$ for any $t \in \mathbb{T}$, in which case $\operatorname{Ind} T_a = -\operatorname{ind} \det a_r$.*

Proof. Since Toeplitz operators with continuous $VMO_{\log}$ symbols commute modulo compact operators (use the compactness of Hankel operators—see [28] or [30]) and each T_a with $a \in C(\overline{\mathbb{D}}) \cap VMO_{\log}$ can be approximated by Fredholm Toeplitz operators with symbols in the same algebra $C(\overline{\mathbb{D}}) \cap VMO_{\log}$ (see the proof of Theorem 14 in [28]), it is not difficult to see that the matrix-valued case can be reduced to the scalar case (see Chapter 1 of [16]). □

6. Summary of open problems

We summarize the open problems discussed in the previous sections.

Problem 1. Find a sufficient and necessary condition for Toeplitz operators with L^1, or L^1_{loc}, or distributional symbols to be bounded on Bergman spaces A^p (for $1 < p < \infty$ or at least for $p = 2$). Notice that for locally integrable and thus for L^1-symbols, the condition (2.10) makes very well sense, and in view of Proposition 7 it is to be expected that (2.10) is weaker than (2.4). We in particular ask, if (2.10) is also a necessary condition for the boundedness of $T_a : A^p \to A^p$, $1 < p < \infty$, say, for $a \in L^1$.

Problem 2. Generalize the results on boundedness, compactness and Fredholmness of Toeplitz operators on A^2 with radial symbols to other Bergman spaces A^p.

Problem 3. Find a necessary and sufficient condition for Toeplitz operators with L^1_{loc} (or even distributional) symbols to be compact on Bergman spaces A^p (at least in the case $p = 2$).

Problem 4. Generalize Zorboska's result on compactness to other Bergman spaces A^p.

Problem 5. Find an index formula for Fredholm Toeplitz operators on A^p with matrix-valued symbols in $C(\overline{\mathbb{D}}) + H^\infty(\mathbb{D})$ (at least in the case $p = 2$). Also consider the index when the symbols are matrix-valued in the Zhu class $L^\infty \cap VMO$.

Problem 6. Extend Fredholm theory of Toeplitz operators on A^2 with piecewise continuous symbols to other Bergman spaces A^p. Also consider matrix-valued piecewise continuous symbols.

Problem 7. Determine when Hankel operators are bounded and compact on L^1, in order to extend Fredholm theory of Toeplitz operators on the Bergman space A^1.

Problem 8. Assume that $a \in W_\nu^{-m,\infty}$ with $\tilde{a}(z) \to 0$ as $|z| \to 1$. Find out whether the function a can then be represented in the following form

$$a = \sum_{\alpha \leq m} (-1)^{|\alpha|} D^\alpha b_\alpha, \tag{6.1}$$

where

$$\operatorname{ess} \lim_{r \to 1} \sup_{r<|z|<1} |b_\alpha(z)| \nu^{-|\alpha|}(z) = 0.$$

An affirmative answer implies that T_a is compact on A^p, and hence provides a sufficient and necessary condition for T_a to be compact.

Problem 9. If $a \in W_\nu^{-m,\infty}$ is of the form (6.1) with $b_\alpha \nu^{-|\alpha|} \in C(\overline{\mathbb{D}})$, does if follow that $\tilde{a} \in C(\overline{\mathbb{D}})$? A positive answer would be useful in the study of Fredholm properties of Toeplitz operators with distributional symbols.

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Department of Mathematics and Statistics, P.O.Box 68, University of Helsinki, 00014 Helsinki, Finland
antti.i.perala@helsinki.fi

Department of Mathematics and Statistics, P.O.Box 68, University of Helsinki, 00014 Helsinki, Finland
jari.taskinen@helsinki.fi

Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, N.Y. 10012
virtanen@courant.nyu.edu

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