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Hankel operators on Fock spaces

A. Perälä, A. Schuster and J. A. Virtanen

Abstract. We study Hankel operators on the weighted Fock spaces F^p_γ. The boundedness and compactness of these operators are characterized in terms of BMO and VMO, respectively. Along the way, we also study Berezin transform and harmonic conjugates on the plane. Our results are analogous to Zhu's characterization of bounded and compact Hankel operators on Bergman spaces of the unit disk.

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1. Introduction

Hankel operators have been studied for several decades in the setting of various analytic function spaces. Starting with Hankel matrices, which can be viewed as Hankel operators on Hardy spaces (see [9]), the field has expanded to Hankel operators on Bergman spaces, Dirichlet type spaces, Bergman and Hardy spaces of the unit ball in $\mathbb{C}^n$, of symmetric domains, and Fock spaces. In addition to being a beautiful and rapidly developing part of analysis, Hankel operators have a vast number of applications, which in the case of Hardy spaces are well known and recognized (see, e.g. [9]), while Hankel operators on Bergman and Fock spaces have found applications mainly in quantum mechanics.

We are interested in the basic properties of Hankel operators on Fock spaces, and in particular characterize their boundedness and compactness in terms of the (mean) oscillation of the generating symbols. In the Bergman space setting one is led to the space of bounded mean oscillation BMO^p_∂ and the space of vanishing mean oscillation VMO^p_∂ with respect to the Bergman metric. Due to K. Zhu [12], a characterization of bounded and compact Hankel operators has been known for

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two decades. It is natural to ask whether an analogous result carries over to Fock spaces. Indeed, the question was recently settled in [2] for the Hilbert Fock space F^2.

For a symbol f (satisfying suitable conditions), we define the Hankel operator H_f by

$$H_f = (I - P)M_f,$$

where P is a projection defined below in (1) and M_f is the operator of multiplication associated with f. In this paper we study Hankel operators on standard Fock spaces F^p_γ with $1 \le p < \infty$ and $\gamma > 0$. We will introduce spaces BMO^p and VMO^p (in the Euclidean metric) and obtain useful characterizations for these spaces. We prove decomposition theorems similar to those in [11, 12]; in particular, we show that these spaces can be characterized in terms of certain Gaussian integrals, where $\gamma > 0$ can be arbitrary.

Note that the John-Nirenberg theorem implies that the classical BMO and VMO spaces are independent of the parameter p. However, as in the case of the Bergman metric, the spaces BMO^p and VMO^p presented here depend on p.

2. The weighted Fock spaces

We will use the definitions from [7]. Let $\gamma > 0$ and $1 \le p < \infty$. The weighted Fock space F^p_γ consists of entire functions f such that

$$\|f\|^p_{p,\gamma} = \int_{\mathbb{C}} |f(z)|^p e^{-(\gamma p/2)|z|^2} dA(z) < \infty.$$

Here $dA(z) = dxdy$ is the standard Lebesgue area measure. Similarly, the space F^∞_γ consists of those entire f, for which

$$\|f\|_{\infty,\gamma} = \sup_{z\in\mathbb{C}} |f(z)| e^{-(\gamma/2)|z|^2}$$

is finite. The respective Lebesgue L^p_γ spaces and their norms are defined in an obvious way.

It is known that F^2_γ is a Hilbert space with inner product

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-\gamma|z|^2} dA(z).$$

Remark. The point-evaluation functionals $f \mapsto f(z)$ are bounded $F^p_\gamma \to \mathbb{C}$ are bounded and hence F^2_γ is known to possess reproducing kernels $K_z := K^\gamma_z$; $f(z) = \langle f, K_z\rangle$.

One immediate corollary is that norm convergence implies locally uniform convergence. In other words, if f_n and f are in F^p_γ and $\|f_n - f\|_{p,\gamma} \to 0$ as $n \to \infty$, then $f_n(z) \to f(z)$ uniformly on each compact subset of $\mathbb{C}$. Another corollary is that the space F^p_γ is *complete*; if $\{f_n\}$ is a Cauchy sequence in norm, then $f_n \to f$ in norm for some $f \in F^p_\gamma$.

The reproducing kernels K_z can be explicitly computed; $K_z(w) = e^{\gamma \bar{z} w}$. The Bergman projection $P := P_\gamma$ is given by

$$Pf(z) = \int_{\mathbb{C}} f(w) e^{\gamma z \bar{w}} e^{-\gamma |w|^2} dA(w). \tag{1}$$

It is known that $P : L^p_\gamma \to F^p_\gamma$ is bounded for every $\gamma > 0$ and $p \in [1, \infty]$. Proofs can be found in [5]. We will just write K_z and P, instead of K^γ_z and P_γ; the parameter γ will be clear from context.

A measurable function f is said to belong to $\tau^p = \tau^p_\gamma$ if and only if $fK_z \in L^p_\gamma$ for every $z \in \mathbb{C}$. This requirement is natural, since linear combinations of the kernel functions form a dense subset of F^p_γ. Henceforth, we will usually assume $f \in \tau^p$.

3. BMO and related spaces

For $0 < r < \infty$, let $D(z, r)$ be the Euclidean disk of radius r and center z. For $f \in L^1_{loc}, 0 < r < \infty, z \in \mathbb{C}$, let

$$\widehat{f}_r(z) = \frac{1}{\pi r^2} \int_{D(z,r)} f(w) dA(w).$$

Fix $0 < r < \infty$ and $p \geq 1$. Define BMO^p_r to be the set of L^p_{loc} integrable functions f such that

$$\|f\|_{BMO^p_r} = \sup_{z \in \mathbb{C}} \left\{ \frac{1}{\pi r^2} \int_{D(z,r)} |f(w) - \widehat{f}_r(z)|^p dA(w) \right\}^{\frac{1}{p}} < \infty.$$

Let BO_r be the set of continuous functions in $\mathbb{C}$ such that

$$\|f\|_{BO_r} = \sup_{z \in \mathbb{C}} \omega_r(f) < \infty,$$

where

$$\omega_r(f)(z) = \sup_{w \in D(z,r)} |f(z) - f(w)|.$$

Let BA^p_r be the set of functions f on $\mathbb{C}$ such that $\widehat{f}_r$ is bounded on $\mathbb{C}$.

Lemma 3.1. *Let $f \in L^p_{loc}$. Then $f \in BMO^p_r$ if and only if there is a constant $C > 0$ such that for every $z \in \mathbb{C}$ there is a constant λ_z such that*

$$\frac{1}{\pi r^2} \int_{D(z,r)} |f(w) - \lambda_z|^p dA(w) \leq C.$$

Proof. For the proof of the forward direction, let $\lambda_z = \widehat{f}_r(z)$. For the other direction, note that

$$\begin{aligned}
&\left\{\frac{1}{\pi r^2}\int_{D(z,r)} |f(w)-\widehat{f}_r(z)|^p dA(w)\right\}^{\frac{1}{p}}\\
\leq &\left\{\frac{1}{\pi r^2}\int_{D(z,r)} |f(w)-\lambda_z|^p dA(w)\right\}^{\frac{1}{p}}\\
+ &\left\{\frac{1}{\pi r^2}\int_{D(z,r)} |\lambda_z-\widehat{f}_r(z)|^p dA(w)\right\}^{\frac{1}{p}}\\
= &\left\{\frac{1}{\pi r^2}\int_{D(z,r)} |f(w)-\lambda_z|^p dA(w)\right\}^{\frac{1}{p}} + |\lambda_z-\widehat{f}_r(z)|.
\end{aligned}$$

But

$$\begin{aligned}
|\lambda_z-\widehat{f}_r(z)| &= |\frac{1}{\pi r^2}\int_{D(z,r)} (f(w)-\lambda_z) dA(w)|\\
&\leq \left\{\frac{1}{\pi r^2}\int_{D(z,r)} |\lambda_z-f(w)|^p dA(w)\right\}^{\frac{1}{p}}
\end{aligned}$$

Therefore

$$\left\{\frac{1}{\pi r^2}\int_{D(z,r)} |f(w)-\widehat{f}_r(z)|^p dA(w)\right\}^{\frac{1}{p}} \leq 2\left\{\frac{1}{\pi r^2}\int_{D(z,r)} |\lambda_z-f(w)|^p dA(w)\right\}^{\frac{1}{p}}.$$

□

LEMMA 3.2. *Let $s > r > 0$. Then $BMO_s^p \subset BMO_r^p$.*

Proof. Suppose $f \in BMO_s^p$ so that for every $z \in \mathbb{C}$ we have $\lambda_z \in \mathbb{C}$ such that

$$\sup_{z\in\mathbb{C}} \frac{1}{\pi s^2}\int_{D(z,s)} |f(w)-\lambda_z|^p dA(w) = C < \infty.$$

Now

$$\begin{aligned}
&\frac{1}{\pi r^2}\int_{D(z,r)} |f(w)-\lambda_z|^p dA(w)\\
\leq &\frac{s^2}{r^2}\frac{1}{\pi s^2}\int_{D(z,s)} |f(w)-\lambda_z|^p dA(w)\\
\leq &C\frac{s^2}{r^2}
\end{aligned}$$

for every $z \in \mathbb{C}$. □

LEMMA 3.3. *BO_r is independent of r.*

Proof. Let $r < s$. Then $\|f\|_{BO_r} \leq \|f\|_{BO_s}$.

Choose $N \in \mathbb{N}$ such that for any $w \in D(0,s)$, there exists a path $\{0 = z_1, z_2, \ldots, z_N = w\}$ in $D(0,s)$ such that $|z_{i-1} - z_i| < r$. Let now $z \in \mathbb{C}$. Then for any $w \in D(z,s)$, we have a path $\{z = \zeta_1, \zeta_2, \ldots, \zeta_N = w\}$, where $\zeta_i = z_i + z$, and $|\zeta_{i-1} - \zeta_i| < r$. Therefore

$$|f(z) - f(w)| \leq \sum_{i=1}^{N} |f(\zeta_{i-1}) - f(\zeta_i)| \leq N \sup\{w_r(f)(\zeta_i) : i\} \leq N\|f\|_{BO_r}.$$

We now take the supremum over all $w \in D(z,r)$ and then over all $z \in \mathbb{C}$ to obtain the desired result. $\square$

By the above lemma, we shall now refer to $BO = BO_1$.

LEMMA 3.4. *Let f be a continuous function on $\mathbb{C}$. Then $f \in BO$ if and only if there is a constant $C > 0$ such that*

$$|f(z) - f(w)| \leq C(|z-w| + 1)$$

for all $z, w \in \mathbb{C}$.

Proof. The backward direction is obviously true. For the forward direction, let $w, z \in \mathbb{C}$. If $f \in BO$, then

$$C \geq \sup_{\alpha \in \mathbb{C}} \omega_1(f)(\alpha) = \sup_{\alpha \in \mathbb{C}} \sup_{w \in D(\alpha,\beta) < 1} |f(\alpha) - f(\beta)|.$$

In other words, if $|z - w| \leq 1$, then $|f(z) - f(w)| \leq C \leq C(|z-w|+1)$. Suppose now that $|z - w| > 1$. Let $N = [|z-w|] + 1$, where $[x]$ is the greatest integer less than or equal to x. Let $z_0 = z$, z_1 be the point a distance of $|z-w|/N$ along the line from z to w. Let z_2 be the point a distance of $|z-w|/N$ along the line from z_1 to w, and so on, until $z_N = w$. Then

$$|f(z) - f(w)| \leq \sum_{i=1}^{N} |f(z_{i-1}) - f(z_i)| \leq N\|f\|_{BO} \leq \|f\|_{BO}(|z-w|+1).$$

$\square$

Let BA_r^p denote the space of all functions f on $\mathbb{C}$ such that

$$\|f\|_{BA_r^p} = \sup_{z \in \mathbb{C}} \left\{ \widehat{|f|^p}_r(z) \right\}^{\frac{1}{p}} < \infty.$$

In other words, $f \in BA_r^p$ if

$$\frac{1}{\pi r^2} \int_{D(z,r)} |f(w)|^p dA(w)$$

is bounded independently of $z \in \mathbb{C}$. The notion of BA_r^p is closely related to Carleson measures on Fock spaces, see [7], or [10] for more generality.

LEMMA 3.5. *Let $r > 0$. Then $f \in BA_r^p$ if and only if $M_f : F_\gamma^p \to L_\gamma^p$ is bounded.*

Proof. Let $d\mu(w) = |f(w)|^p dA(w)$. Then

$$\left\{\widehat{|f|^p}_r(z)\right\} = \frac{1}{\pi r^2}\int_{D(z,r)} |f(w)|^p dA(w).$$

Then

$$\mu(D(z,r)) = \int_{D(z,r)} d\mu(w) = \int_{D(z,r)} |f(w)|^p dA(w).$$

Of course this implies that μ is a Carleson measure if and only if $f \in BA^p_r$. But this means

$$\begin{aligned} &\int_{\mathbb{C}} |g(w)|^p e^{-(\gamma p/2)|w|^2} |f(w)|^p dA(w) \\ =&\int_{\mathbb{C}} |g(w)|^p e^{-(\gamma p/2)|w|^2} d\mu(w) \\ \leq& C\int_{\mathbb{C}} |g(w)|^p e^{-(\gamma p/2)|w|^2} dA(w) \end{aligned}$$

for all $g \in F^p_\gamma$. In other words, $M_f : F^p_\gamma \to L^p_\gamma$ is bounded. □

Lemma 3.6. *If $f \in BMO^p_{2r}$, then $\widehat{f}_r \in BO_r$.*

Proof. Let $f \in BMO^p_{2r}$, and suppose $|w - z| \leq r$. Then

$$\begin{aligned} |\widehat{f}_r(z) - \widehat{f}_r(w)| &\leq |\widehat{f}_r(z) - \widehat{f}_{2r}(z)| + |\widehat{f}_{2r}(z) - \widehat{f}_r(w)| \\ &= \left|\frac{1}{\pi r^2}\int_{D(z,r)} f(u)dA(u) - \frac{1}{\pi r^2}\int_{D(z,r)} \widehat{f}_{2r}(z)dA(u)\right| \\ &+ \left|\frac{1}{\pi r^2}\int_{D(w,r)} f(u)dA(u) - \frac{1}{\pi r^2}\int_{D(w,r)} \widehat{f}_{2r}(z)dA(u)\right| \\ &\leq \frac{1}{\pi r^2}\int_{D(z,r)} |f(u) - \widehat{f}_{2r}(z)|dA(u) \\ &+ \frac{1}{\pi r^2}\int_{D(w,r)} |f(u) - \widehat{f}_{2r}(z)|dA(u) \\ &\leq 4\frac{1}{4\pi r^2}\int_{D(z,2r)} |f(u) - \widehat{f}_{2r}(z)|dA(u) \\ &+ 4\frac{1}{4\pi r^2}\int_{D(z,2r)} |f(u) - \widehat{f}_{2r}(z)|dA(u) \\ &\leq 4\left\{\frac{1}{\pi 4r^2}\int_{D(z,2r)} |f(u) - \widehat{f}_{2r}(z)|^p dA(u)\right\}^{\frac{1}{p}} \\ &+ 4\left\{\frac{1}{\pi 4r^2}\int_{D(z,2r)} |f(u) - \widehat{f}_{2r}(z)|^p dA(u)\right\}^{\frac{1}{p}} \\ &\leq 8\|f\|_{BMO^p_{2r}}. \end{aligned}$$

The fourth line follows from the fact that $D(z,r) \subset D(z,2r)$ and $D(w,r) \subset D(z,r)$ and the fifth follows from Hölder's inequality. $\square$

Let $k_z = K_z/\|K_z\|_{2,\gamma}$, so that $k_z(w) = e^{\gamma\bar{z}w-(\gamma/2)|z|^2}$ denote the normalized reproducing kernel of F^2_γ. An easy calculation reveals that $k_z = k_z^\gamma$ is a unit vector on F^p_γ for every $p \in [1,\infty)$.

The Berezin transform (or the heat-transform) of a function f is given by

$$B_\gamma f(z) = \int_{\mathbb{C}} f(w)|k_z^\gamma(w)|^2 e^{-\gamma|w|^2} dA(w).$$

We will omit the γ, when it is clear form the context. In this case we just write Bf.

Lemma 3.7. *Let $f \in \tau^p$. Then the following are equivalent.*

(1) $f \in BA^p_r$;

(2) $\sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |f(z-w)|^p e^{-\gamma|w|^2} dA(w) \le C$ for some $\gamma > 0$;

(3) $\sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |f(z-w)|^p e^{-\gamma|w|^2} dA(w) \le C$ for all $\gamma > 0$.

Proof. By the definition of BA^p_r, $f \in BA^p_r$ if and only if $\int_{D(z,r)} |f(w)|^p dA(w) \le C$ if and only if $|f|^p dA$ is a Carleson measure for F^2_γ for some (and thus for every) $\gamma > 0$ if and only if the Berezin transform $B_\gamma|f|^p$ is bounded. But

$$\begin{aligned} B_\gamma|f|^p(z) &= \int_{\mathbb{C}} |k_z(w)|^2 e^{-\gamma|w|^2} |f(w)|^p dA(w) \\ &= \int_{\mathbb{C}} e^{-\gamma|z|^2+2\gamma\Re\bar{z}w-\gamma|w|^2} |f(w)|^p dA(w) \\ &= \int_{\mathbb{C}} e^{-\gamma|z-w|^2} |f(w)|^p dA(w) \\ &= \int_{\mathbb{C}} e^{-\gamma|w|^2} |f(z-w)|^p dA(w). \end{aligned}$$

$\square$

Note that by Lemmas 3.3 and 3.5, both BO_r and BA^p_r are independent of r. In fact, if we combine Lemmas 3.5 and 3.7, we obtain the following lemma.

Lemma 3.8. *Let $f \in \tau^p$. The following conditions are equivalent:*

(1) $f \in BA^p_r$;

(2) $\sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |f(z-w)|^p e^{-\gamma|w|^2} dA(w) < \infty$ for some (and thus all) $\gamma > 0$;

(3) $M_f : F^p_\gamma \to L^p_\gamma$ is bounded.

Lemma 3.9. *If $f \in BMO^p_{2r}$, then $f - \widehat{f}_r \in BA^p$.*

Proof. By assumption and Lemma 3.2, $f \in BMO_r^p$. Let $g = f - \widehat{f}_r$. Then

$$\begin{aligned}\left\{\widehat{|g|^p}_r(z)\right\}^{\frac{1}{p}} &= \left\{\frac{1}{\pi r^2}\int_{D(z,r)} |f(u) - \widehat{f}_r(u)|^p dA(u)\right\}^{\frac{1}{p}} \\ &\leq \left\{\frac{1}{\pi r^2}\int_{D(z,r)} |f(u) - \widehat{f}_r(z)|^p dA(u)\right\}^{\frac{1}{p}} \\ &+ \left\{\frac{1}{\pi r^2}\int_{D(z,r)} |\widehat{f}_r(z) - \widehat{f}_r(u)|^p dA(u)\right\}^{\frac{1}{p}} \\ &\leq \|f\|_{BMO_r^p} + \omega_r(\widehat{f}_r)(z).\end{aligned}$$

□

LEMMA 3.10. *Let $r > 0$. Then*

$$BMO_r^p \subset BO_r + BA_r^p.$$

Proof. Let $r = 2s$ and $f \in BMO_r^p = BMO_{2s}^p$. Then Lemmas 3.6 and 3.9 imply that $\widehat{f}_s \in BO_s$ and $f - \widehat{f}_s \in BA_s^p$. Therefore, $f = \widehat{f}_s + f - \widehat{f}_s \in BO_s + BA_s^p = BO_r + BA_r^p$. □

LEMMA 3.11. *If $f \in BMO_r^p$, then*

$$\int_{\mathbb{C}} |f(z-w) - B_\gamma f(z)|^p e^{-\gamma|w|^2} dA(w) \leq C$$

for all $z \in \mathbb{C}$ and $\gamma > 0$.

Proof. By Lemma 3.10, it is enough to show the inequality holds for $f \in BA^p$ and $f \in BO$. Suppose first that $f \in BA^p$. By Hölder,

$$|B_\gamma f(z)| \leq C \left\{\int_{\mathbb{C}} |f(z-w)|^p e^{-\gamma|w|^2} dA(w)\right\}^{\frac{1}{p}}.$$

Therefore

$$\begin{aligned}&\left\{\int_{\mathbb{C}} |f(z-w) - B_\gamma f(z)|^p e^{-\gamma|w|^2} dA(w)\right\}^{\frac{1}{p}} \\ \leq &\left\{\int_{\mathbb{C}} |f(z-w)|^p e^{-\gamma|w|^2} dA(w)\right\}^{\frac{1}{p}} + |B_\gamma f(z)| \\ \leq &(1 + C') \left\{\int_{\mathbb{C}} |f(z-w)|^p e^{-\gamma|w|^2} dA(w)\right\}^{\frac{1}{p}} \leq C,\end{aligned}$$

where the last inequality follows from Lemma 3.7.

Suppose next that $f \in BO$. Then

$$\begin{aligned}
&\int_{\mathbb{C}} |f(z-w) - B_\gamma f(z)|^p e^{-\gamma|w|^2} dA(w) \\
=&\int_{\mathbb{C}} |f(z-w) - \int_{\mathbb{C}} f(z-u) e^{-\gamma|u|^2} dA(u)|^p e^{-\gamma|w|^2} dA(w) \\
\leq & C \int_{\mathbb{C}} \int_{\mathbb{C}} |f(z-w) - f(z-u)|^p e^{-\gamma|w|^2} dA(w) e^{-\gamma|u|^2} dA(u).
\end{aligned}$$

Since $f \in BO$, Lemma 3.4 tells us that $|f(z-w) - f(z-u)|^p \leq C(|w-u|+1)^p$. Therefore, the last quantity in the last displayed equation is bounded above by

$$C^2 \int_{\mathbb{C}} \int_{\mathbb{C}} (|u-w|+1)^p e^{-\gamma|w|^2} dA(w) e^{-\gamma|u|^2} dA(u),$$

which is a constant. □

LEMMA 3.12. *Suppose there exists $\gamma > 0$ such that*

$$\int_{\mathbb{C}} |f(z-w) - B_\gamma f(z)|^p e^{-\gamma|w|^2} dA(w) \leq C$$

for all $z \in \mathbb{C}$. Then $f \in BMO_r^p$.

Proof. Let $z \in \mathbb{C}$ and fix $\gamma > 0$. Note that $e^{-\gamma|z-w|^2} \geq c > 0$ for $w \in D(z,r)$. Therefore

$$\begin{aligned}
& c \int_{D(z,r)} |f(w) - B_\gamma f(z)|^p dA(w) \\
\leq & \int_{D(z,r)} |f(w) - B_\gamma f(z)|^p e^{-\gamma|z-w|^2} dA(w) \\
\leq & \int_{\mathbb{C}} |f(w) - B_\gamma f(z)|^p e^{-\gamma|z-w|^2} dA(w) \\
= & \int_{\mathbb{C}} |f(z-w) - B_\gamma f(z)|^p e^{-\gamma|w|^2} dA(w) \leq C.
\end{aligned}$$

The result then follows from an application of Lemma 3.1. □

We now have proven that BMO_r^p is independent of r; in what follows, we will write $BMO^p = BMO_1^p$.

THEOREM 1. *Let $p \geq 1$. Then the following are equivalent:*

(1) $f \in BMO^p$;

(2) $f \in BO + BA^p$;

(3)

$$\sup_{z \in \mathbb{C}} \int_{\mathbb{C}} |f(z-w) - B_\gamma f(z)|^p e^{-\gamma|w|^2} dA(w) < \infty,$$

for some $\gamma > 0$;

(3')

$$\sup_{z\in\mathbb{C}}\int_{\mathbb{C}}|f(z-w)-B_\gamma f(z)|^p e^{-\gamma|w|^2}dA(w)<\infty,$$

for all $\gamma>0$*;*

(4) There is a constant C *and* $\gamma>0$ *such that for every* $z\in\mathbb{C}$*, there is a constant* λ_z *such that*

$$\int_{\mathbb{C}}|f(z-w)-\lambda_z|^p e^{-\gamma|w|^2}dA(w)\le C;$$

(4') For every $\gamma>0$ *there is a constant* C *such that for every* $z\in\mathbb{C}$*, there is a constant* λ_z *such that*

$$\int_{\mathbb{C}}|f(z-w)-\lambda_z|^p e^{-\gamma|w|^2}dA(w)\le C.$$

Proof. (1) $\Rightarrow$ (2) follows from Lemma 3.10. (2) $\Rightarrow$ (3′) follows from the proof of Lemma 3.11. Obviously (3′) $\Rightarrow$ (3) and (4′) $\Rightarrow$ (4). The proofs of (3) $\Leftrightarrow$ (4) and (3′) $\Leftrightarrow$ (4′) are similar to the proof of Lemma 3.1. (3) $\Rightarrow$ (1) follows from Lemma 3.12. □

Lemma 3.13. *If* $f\in BMO^p$*, then* $B_\gamma f\in BO$*, and* $f-B_\gamma f\in BA^p$ *for every* $\gamma>0$*.*

Proof. Fix $\gamma>0$. We have

$$\begin{aligned}|B_\gamma f(z)-\widehat{f}_r(z)| =&|B_\gamma f(z)-\frac{1}{\pi r^2}\int_{D(z,r)}f(w)dA(w)|\\ =&\left|\frac{1}{\pi r^2}\int_{D(z,r)}B_\gamma f(z)dA(w)-\frac{1}{\pi r^2}\int_{D(z,r)}f(w)dA(w)\right|\\ \le&\frac{1}{\pi r^2}\int_{D(z,r)}|f(w)-B_\gamma f(z)|dA(w)\\ \le&C\int_{\mathbb{C}}|f(z-w)-B_\gamma f(z)|e^{-\gamma|w|^2}dA(w).\end{aligned}$$

Here the last inequality follows from the proof of Lemma 3.12. Since $f\in BMO^p$, the last integral is finite. Thus $Bf-\widehat{f}_r$ is a bounded continuous function and so lies in $BO\cap BA^p$. By Lemma 3.6, $\widehat{f}_r\in BO$, so $B_\gamma f=B_\gamma f-\widehat{f}_r+\widehat{f}_r\in BO+BO=BO$. By Lemma 3.9, $f-\widehat{f}_r\in BA^p$, so $f-B_\gamma f=f-\widehat{f}_r+\widehat{f}_r-B_\gamma f\in BA^p+BA^p=BA^p$. □

4. Bounded Hankel operators

We begin with a short discussion of harmonic conjugates. If $f=u+iv$ is entire, then both u and v are harmonic. Conversely, given a harmonic $u:\mathbb{C}\to\mathbb{R}$, there exists a unique harmonic $v:\mathbb{C}\to\mathbb{R}$ such that $f=u+iv$ is entire and $v(0)=0$.

Lemma 4.1. *Let $u : \mathbb{C} \to \mathbb{R}$ be harmonic. If $u \in L^p_\gamma$ for $p \in (1, \infty)$ and $\gamma > 0$, then $v \in L^p_\gamma$.*

Proof. Looking at the proof of Theorem 4.1 of [6], one obtains for $r < 1$ a $C > 0$ such that

$$\int_0^{2\pi} |v(re^{i\theta})|^p d\theta \leq C \int_0^{2\pi} |u(re^{i\theta})|^p d\theta.$$

But if $r > 1$, consider the dilations $u_R(z) = u(Rz)$ and $v_R(z) = v(Rz)$ for large enough R. Of course, both u and u_R always belong to the hardy space h^p of the unit circle. Now,

$$\int_0^{2\pi} |v(re^{i\theta})|^p d\theta = \int_0^{2\pi} |v_R(se^{i\theta})|^p d\theta \leq C \int_0^{2\pi} |u_R(se^{i\theta})|^p d\theta = C \int_0^{2\pi} |u(re^{i\theta})|^p d\theta,$$

where R is chosen so that $s := r/R < 1$. Now, inevitably

$$\int_0^{2\pi} |v(re^{i\theta})|^p re^{-(p/2)r^2} d\theta \leq C \int_0^{2\pi} |u(re^{i\theta})|^p re^{-(p/2)r^2} d\theta.$$

The rest follows from evaluating the norms in polar coordinates.

□

Corollary 4.2. *Let $p \in (1, \infty)$ and $\gamma > 0$. Suppose $f = u + iv$ is entire and that $u \in L^p_\gamma$. Then $f \in F^p_\gamma$. Moreover, there exists $C > 0$ such that $\|f - f(0)\|_{p,\gamma} \leq \|u\|_{p,\gamma}$.*

In what follows, if the possible values of p are not indicated, we assume that $p \in (1, \infty)$.

Recall that the Bergman projection P is given by

$$Pg(z) = \int_{\mathbb{C}} g(w) e^{\gamma z \bar{w}} e^{-|w|^2} dA(w).$$

If $f \in \tau^p$, then the Hankel operator with symbol f is given for $g \in F^p_\gamma$ by

$$H_f g(z) = (I - P)(fg)(z).$$

Note that we can also write

$$H_f g(z) = \int_{\mathbb{C}} (f(z) - f(w)) g(w) e^{\gamma z \bar{w}} e^{-\gamma |w|^2} dA(w).$$

Lemma 4.3. *If $f \in BA^p$, then H_f is bounded on F^p_γ.*

Proof. By Lemma 3.8, M_f is bounded $F^p_\gamma \to L^p_\gamma$. Since P is bounded, we obtain the desired result. □

Lemma 4.4. *If $f \in BO$, then H_f is bounded on F^p_γ for every $p \in [1, \infty]$.*

Proof. By Lemma 3.4
$$|H_f(g)(z)| \leq C \int_{\mathbb{C}} (|z-w|+1)|e^{\gamma z\bar{w}} g(w)| e^{-\gamma|w|^2} dA(w).$$
There are $C > 0$ and $\epsilon > 0$ such that
$$|e^{\gamma z\bar{w}}| \leq C e^{(\gamma/2)|z|^2 + (\gamma/2)|w|^2 - \epsilon|z-w|}.$$

Therefore, we arrive at

$$\begin{aligned}
&|H_f(g)(z)|^p e^{-(p\gamma/2)|z|^2} \\
\leq & C e^{-((p-1)\gamma/2)|z|^2} \left| \int_{\mathbb{C}} (|z-w|+1)|g(w)| e^{-(\gamma/2)|w|^2} e^{-\epsilon|z-w|} dA(w) \right|^p \\
\leq & C e^{-((p-1)\gamma/2)|z|^2} \left\{ \int_{\mathbb{C}} |g(w)|^p e^{-(\gamma p/2)|w|^2} dA(w) \right\} \left\{ \int_{\mathbb{C}} |z-w|^q e^{-q\epsilon|z-w|} dA(w) \right\}^{p/q} \\
\leq & C e^{-((p-1)\gamma/2)|z|^2} \|g\|_{p,\gamma}^p.
\end{aligned}$$

If $1 < p < \infty$ and $1/p + 1/q = 1$, we get the desired result by integrating with respect to z.

If $p = 1$, we use the above reasoning together with Fubini and proceed as follows.
$$\begin{aligned}
&\int_{\mathbb{C}} |H_f(g)(z)|^p e^{-(\gamma/2)|z|^2} dA(z) \\
\leq & C \int_{\mathbb{C}} |g(w)| e^{-(\gamma/2)|w|^2} dA(w) \int_{\mathbb{C}} (|z-w|+1) e^{-\epsilon|z-w|} dA(z) \\
\leq & C \|g\|_{1,\gamma}.
\end{aligned}$$

By similar arguments, one can also show that
$$|H_f(g)(z)| e^{-(\gamma/2)|z|^2} \leq \int_{\mathbb{C}} (|z-w|+1) e^{-\epsilon|z-w|} dA(z) \|g\|_{\infty,\gamma} \leq C\|g\|_{\infty,\gamma}.$$
The result is now proven for all $p \in [1, \infty]$.

□

Theorem 2. *Let $f \in \tau^p$. Then $f \in BMO^p$ if and only if the operators H_f and $H_{\bar{f}}$ are both bounded $F^p_\gamma \to L^p_\gamma$.*

Proof. If $f \in BMO^p$, then so is $\bar{f}$ and it follows from the previous two lemmas that H_f and $H_{\bar{f}}$ are bounded.

Suppose now that H_f and $H_{\bar{f}}$ are both bounded. Without loss of generality, we may then assume that f is real-valued. Recall that $k_z(w) = e^{\gamma\bar{z}w - (\gamma/2)|z|^2}$ are unit vectors in F^p_γ and so we have $C > 0$ such that
$$\|fk_z - P(fk_z)\|_{p,\gamma} = \|H_f(k_z)\|_{p,\gamma} \leq C.$$

Note that $k_z(z-w) = 1/k_z(w)$ and

$$e^{-\gamma(p/2)|z-w|^2} = e^{-\gamma(p/2)|z|^2 - \gamma(p/2)|w|^2 + \gamma(p/2)z\bar{w} + \gamma(p/2)\bar{z}w}.$$

Thus, by a change of variables $w \mapsto z - w$, one obtains

$$\begin{aligned} C^p &\geq \|fk_z - P(fk_z)\|_{p,\gamma}^p \\ &= \int_{\mathbb{C}} |f(z-w) - e^{-(\gamma/2)|z|^2} P(fk_z)(z-w)|^p e^{-(p\gamma/2)|w|^2} dA(w) \end{aligned}$$

Setting $g_z(w) = e^{-(\gamma/2)|z|^2} P(fk_z)(z-w)$, one obtains

$$\sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |f(z-w) - g_z(w)|^p e^{-(p\gamma/2)|w|^2} dA(w) \leq C^p.$$

Since f is real-valued, then the imaginary part of g_z must belong L^p_γ and so

$$\|g_z - g_z(0)\|_{p,\gamma} \leq M$$

for every $z \in \mathbb{C}$ and some $M > 0$. Applying triangle inequality and the main theorem of the previous section with $\lambda_z = g_z(0)$, one sees that $f \in BMO^p$. $\square$

5. VMO and compact Hankel operators

In this section we study VMO and compactness of Hankel operators. The results and their proofs are completely analogous to the results of the previous two sections. A great deal of details is therefore omitted and left for the reader to verify.

Define VMO^p_r to be the set of L^p_{loc} integrable functions f such that

$$\lim_{z\to\infty} \frac{1}{\pi r^2} \int_{D(z,r)} |f(w) - \widehat{f}_r(z)|^p dA(w) = 0.$$

Let $VO_r \subset BO_r$ be the set of continuous functions in $\mathbb{C}$ such that

$$\lim_{z\to\infty} \omega_r(f) = 0.$$

Let VA^p_r be the set of functions f on $\mathbb{C}$ such that $\lim_{z\to\infty} \widehat{f}_r(z) = 0$. The space VA^p_r is related to the space of vanishing Carleson measures on Fock spaces, see [7], [10].

Similarly to the section 3, it can be shown that VO_r and VA^p_r are independent of r and we will write VO and VA^p, respectively. The following results are also analogous to the BMO-setting.

LEMMA 5.1. *Let $f \in VMO^p$. Then*

(1) $B_\gamma f \in BO$ for every $\gamma > 0$;
(2) $\widehat{f}_r \in BO$ for every $r > 0$;
(3) $f - B_\gamma f \in BA^p$ for every $\gamma > 0$;
(4) $f - \widehat{f}_r \in BA^p$ for every $r > 0$.

Theorem 3. *Let $p \geq 1$. Then the following are equivalent:*

(1) $f \in VMO^p$*;*

(2) $f \in VO + VA^p$*;*

(3)

$$\lim_{z\to\infty} \int_{\mathbb{C}} |f(z-w) - B_\gamma f(z)|^p e^{-\gamma|w|^2} dA(w) = 0,$$

for some $\gamma > 0$*;*

(3')

$$\lim_{z\to\infty} \int_{\mathbb{C}} |f(z-w) - B_\gamma f(z)|^p e^{-\gamma|w|^2} dA(w) = 0,$$

for all $\gamma > 0$*;*

(4) There is a $\gamma > 0$ *such that for every* $z \in \mathbb{C}$*, there is a constant* λ_z *such that*

$$\lim_{z\to\infty} \int_{\mathbb{C}} |f(z-w) - \lambda_z|^p e^{-\gamma|w|^2} dA(w) = 0;$$

(4') For every $\gamma > 0$ *and every* $z \in \mathbb{C}$*, there is a constant* λ_z *such that*

$$\lim_{z\to\infty} \int_{\mathbb{C}} |f(z-w) - \lambda_z|^p e^{-\gamma|w|^2} dA(w) = 0.$$

Theorem 4. *Let* $f \in \tau^p$*. Then the operators* H_f *and* $H_{\bar{f}}$ *are compact if and only if* $f \in VMO^p$*.*

Proof. Suppose first that $f \in VA^p$. But then $|f|^p dA$ is vanishing Carleson, so the multiplication operators M_f and $M_{\bar{f}}$ are compact $F^p_\gamma \to L^p_\gamma$. From the boundedness of the projection P, it follows that H_f and $H_{\bar{f}}$ are both compact.

If $f \in VO$, we refer to Lemma 5.1 of [2]. It follows that both H_f and $H_{\bar{f}}$ can be approximated in norm by Hankel operators with symbols having a compact support. Therefore, both operators are compact. In conclusion, we have shown that if $f \in VMO^p$, then H_f and $H_{\bar{f}}$ are compact.

As for the other direction. Note that $k_z \to 0$ weakly, as $z \to \infty$. But now

$$\|H_f k_z\|_{p,\gamma} \to 0 \text{ and } \|H_{\bar{f}} k_z\|_{p,\gamma} \to 0,$$

as $z \to \infty$. By reasoning similar to that in Theorem 2, it follows that

$$\int_{\mathbb{C}} |f(z-w) - g_z(0)|^p e^{-(\gamma p/2)|w|^2} dA(w) \to 0,$$

as $z \to \infty$, so $f \in VMO^p$. □

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A. Perälä
Department of Mathematics
University of Helsinki
Helsinki 00014
Finland
e-mail: `antti.i.perala@helsinki.fi`

A. Schuster
*Department of Mathematics
San Francisco State University
San Francisco, CA 94132
USA
*Department of Mathematics and Actuarial Science
The American University in Cairo
AUC Avenue PO Box 74
New Cairo Egypt 11835
e-mail: `schuster@sfsu.edu`

J. A. Virtanen
Department of Mathematics
University of Bristol
Bristol, England BS8 1TW
UK
e-mail: `virtanen@cims.nyu.edu`