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ASYMPTOTIC BEHAVIOR AT INFINITY OF SOLUTIONS OF MULTIDIMENSIONAL SECOND KIND INTEGRAL EQUATIONS

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ABSTRACT. We consider second kind integral equations of the form $x(s) - \int_\Omega k(s,t)x(t)\,dt = y(s)$ (abbreviated $x - Kx = y$), in which Ω is some unbounded subset of $\mathbf{R}^n$. Let X_p denote the weighted space of functions x continuous on $\overline{\Omega}$ and satisfying $x(s) = O(|s|^{-p})$, $s \to \infty$. We show that if the kernel $k(s,t)$ decays like $|s-t|^{-q}$ as $|s-t| \to \infty$ for some sufficiently large q (and some other mild conditions on k are satisfied), then $K \in B(X_p)$ (the set of bounded linear operators on X_p), for $0 \leq p \leq q$. If also $(I-K)^{-1} \in B(X_0)$, then $(I-K)^{-1} \in B(X_p)$ for $0 \leq p < q$, and $(I-K)^{-1} \in B(X_q)$ if further conditions on k hold. Thus, if $k(s,t) = O(|s-t|^{-q})$, $|s-t| \to \infty$, and $y(s) = O(|s|^{-p})$, $s \to \infty$, the asymptotic behavior of the solution x may be estimated as $x(s) = O(|s|^{-r})$, $|s| \to \infty$, $r := \min(p,q)$. The case when $k(s,t) = \kappa(s-t)$, so that the equation is of Wiener-Hopf type, receives especial attention. Conditions, in terms of the symbol of $I-K$, for $I-K$ to be invertible or Fredholm on X_p are established for certain cases (Ω a half-space or cone). A boundary integral equation, which models three-dimensional acoustic propagation above flat ground, absorbing apart from an infinite rigid strip, illustrates the practical application and sharpness of the above results. This integral equation models, in particular, road traffic noise propagation along an infinite road surface surrounded by absorbing ground. We prove that the sound propagating along the rigid road surface eventually decays with distance at the same rate as sound propagating above the absorbing ground.

1. Introduction. We consider integral equations of the form

$$x(s) - \int_\Omega k(s,t)x(t)\,dt = y(s), \qquad s \in \overline{\Omega}, \tag{1.1}$$

where Ω is some unbounded open subset of $\mathbf{R}^n$, dt is n-dimensional Lebesgue measure and $x, y \in X$, the Banach space of bounded and continuous functions on $\overline{\Omega}$. We abbreviate (1.1) in operator form as

$$x - Kx = y \tag{1.2}$$

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where the operator K is defined as

$$K\psi(s) = \int_\Omega k(s,t)\psi(t)\,dt, \qquad s \in \overline{\Omega}. \tag{1.3}$$

Let $k_s(t) = k(s,t)$, $s,t \in \overline{\Omega}$. We suppose throughout that $0 \in \overline{\Omega}$, $k_s \in L_1(\Omega)$ for each $s \in \overline{\Omega}$, and that k satisfies the following assumptions:

A. $\sup_{s\in\overline{\Omega}} ||k_s||_1 = \sup_{s\in\overline{\Omega}} \int_\Omega |k(s,t)|\,dt < \infty$.

B. For all $s \in \overline{\Omega}$, $\int_\Omega |k(s,t) - k(s',t)|\,dt \to 0$ as $s' \to s$ with $s' \in \overline{\Omega}$.

These hypotheses imply that $K \in B(X)$, the set of bounded linear operators on X, with norm $||K|| = \sup_{s\in\Omega} ||k_s||_1$, and that if $S \subset X$ is bounded, then KS is bounded and equicontinuous, but, since Ω is unbounded, do not imply that K is compact.

For $p \geq 0$, let $w_p(s) = (1+|s|)^p$ and let X_p denote the weighted space $X_p := \{x \in X : ||x||_\infty^p := ||w_p x||_\infty < \infty\}$ ($||\cdot||_\infty$ denotes the supremum norm on X). Then $x \in X_p$ if and only if x is continuous on $\overline{\Omega}$ and $x(s) = O(|s|^{-p})$ as $|s| \to \infty$, uniformly in s.

We are concerned in this paper to develop sufficient conditions on the kernel k (in addition to A and B) to ensure that $K \in B(X_p)$, the space of bounded linear operators on X_p, for $p > 0$, and conditions which ensure that $(I-K)^{-1} \in B(X_p)$ or, at least, that $I - K$ is Fredholm as an operator on X_p. A main result is that if k satisfies A and B and $|k(s,t)| \leq |\kappa(s-t)|$, $s,t \in \overline{\Omega}$, where κ is locally integrable and $\kappa(s) = O(|s|^{-q})$ as $|s| \to \infty$, for some sufficiently large q, then $K \in B(X_p)$, $0 \leq p \leq q$. If also $I - K$ is Fredholm as an operator on X, then it is Fredholm as an operator on X_p for $0 \leq p < q$. With the help of further conditions on the kernel k we are able to sharpen these latter results to include the case $p = q$.

In terms of the integral equation (1.1), these results help us to bound the asymptotic behavior at infinity of the solution x: if $(I-K)^{-1} \in B(X_p)$, $p > 0$, and $y(s) = O(|s|^{-p})$, $|s| \to \infty$, uniformly in s, then $x(s) = O(|s|^{-p})$, $|s| \to \infty$, uniformly in s. In the case of a pure convolution kernel, $k(s,t) = \kappa(s-t)$, our results show that if $\kappa(s) = O(|s|^{-q})$ and $y(s) = O(|s|^{-p})$, $|s| \to \infty$, then $x(s) = O(|s|^{-r})$, $|s| \to \infty$, where $r = \min(p,q)$.

Our results and methods of proof generalize and extend previous results for integral equations on the real line and on the half line in Chandler-Wilde [**7,8**].

A main step in the argument is to show that, with the assumptions we make on the kernel k, $K - K^{(p)}$ is a compact operator on X for $0 \le p < q$, where $K^{(p)} := w_p K(1/w_p)$. In Section 2, preliminary to the main results, we present sets of sufficient conditions on k which ensure that K is a compact operator on X. These conditions are of some interest in their own right.

Section 3 presents the main results of the paper. In Section 4 we consider further the important case when $k(s,t) = \kappa(s-t)$, $s, t \in \overline{\Omega}$, so that (1.1) is an equation of Wiener-Hopf type. Illustrating the results of Section 3 we give sufficient conditions, in terms of the behavior of κ at infinity and the symbol of the operator $I-K$, for $(I-K)^{-1} \in B(X_p)$ in the case in which Ω is the whole or half space, and for $I - K$ to be Fredholm on X_p in the more general case when Ω is a cone, extending the results of [**9, 15**] to weighted function spaces.

In Section 5 we illustrate the general results of Sections 3 and 4 by a boundary integral equation in acoustics of Wiener-Hopf type which models acoustic scattering by an infinite rigid strip set in an impedance plane. In particular, this models sound propagation from a motor vehicle along a road which is surrounded by sound absorbing ground. Using the results of Section 4 we are able to show that, at least if the road is not too wide, the sound level eventually decays with distance at the same rate along the rigid road surface as it does over the absorbing ground. This application in Section 5 also illustrates the sharpness of the results obtained in Section 3.

Throughout, we shall use the following notation. Define, for $A > 0$ and $s \in \mathbf{R}^n$,

$$B_A(s) := \{u \in \mathbf{R}^n : |s-u| < A\}. \tag{1.4}$$

Also, for $A > 0$, let

$$\Omega_A := \Omega \cap B_A(0),$$

and let K_A denote the "finite section" approximation to K, defined by

$$K_A \psi(s) := \int_{\Omega_A} k(s,t)\psi(t)\, dt, \qquad s \in \overline{\Omega}. \tag{1.5}$$

2. Conditions for compactness. Various conditions for the compactness of the integral operator K in the case $n = 1$ and $\Omega = \mathbf{R}^+$

are given in Anselone and Sloan [**1, 2**]. We generalize and modify these results to provide conditions for the compactness of K for arbitrary $\Omega \subset \mathbf{R}^n$.

Our first result (cf. Anselone and Sloan [**1**]) is that K is compact if k satisfies A and B and the following additional assumption:

C. $\int_\Omega |k(s,t)|\, dt \to 0$ as $|s| \to \infty$ with $s \in \overline{\Omega}$, uniformly in s.

Lemma 2.1. *If k satisfies* A, B *and* C*, then K is a compact operator on X.*

Proof. Suppose that ψ_n is a bounded sequence in X, and let $\chi_n = K\psi_n$. Since K maps bounded sets onto bounded equicontinuous sets, χ_n is bounded and equicontinuous on the whole of $\overline{\Omega}$. Thus, by the Arzela-Ascoli theorem applied to successive regions $\overline{\Omega}_1, \overline{\Omega}_2, \dots$, and a diagonal argument, χ_n has a subsequence $\phi_m = \chi_{n_m}$ which converges uniformly on $\overline{\Omega}_A$ for every $A > 0$. Now, for any integers n and m,

$$||\phi_m - \phi_n|| \le \sup_{s\in\Omega_A} |\phi_m(s) - \phi_n(s)| + \sup_{s\notin\Omega_A} |\phi_m(s) - \phi_n(s)|.$$

For all $\varepsilon > 0$ the second term is less than $\varepsilon/2$ for A sufficiently large by Assumption C. Also, for all $A > 0$, the first term is less than $\varepsilon/2$ for all sufficiently large n and m. Thus, ϕ_m is a Cauchy sequence and, since X is a Banach space, is convergent. We have shown that the image of every bounded sequence has a convergent subsequence, so that K is compact. $\square$

For the case $\Omega = \mathbf{R}^+$, Chandler-Wilde [**8**] shows that if the integral operator K is compact then k satisfies A, B and the following additional assumption:

D. $||K - K_A|| = \sup_{s\in\overline{\Omega}} \int_{\Omega\setminus\Omega_A} |k(s,t)|\, dt \to 0$ as $A \to \infty$.

Since the subspace of compact operators is closed in $B(X)$, K is compact if k satisfies A, B, and D and if K_A is compact for all $A > 0$. This is the case if the kernel of K_A satisfies C for all $A > 0$, i.e., if the following assumption (cf. Atkinson [**3**]) is satisfied:

E. For all $A > 0$, $\int_{\Omega_A} |k(s,t)|\, dt \to 0$ as $|s| \to \infty$ with $s \in \overline{\Omega}$, uniformly in s.

Thus

$$\text{A}, \text{B}, \text{D}, \text{E} \Longrightarrow K \text{ compact.} \tag{2.1}$$

From our final lemma (cf. Anselone and Sloan [**2**]) it follows that also

$$\text{A}, \text{B}, \text{D}, \text{F} \Longrightarrow K \text{ compact,} \tag{2.2}$$

where F is the following condition:

F. For all $A > 0$ there exists $C > 0$ such that, for all $s \in \overline{\Omega} \backslash \Omega_C$, $k(s, \cdot)$ is continuous in Ω_A uniformly in s.

Lemma 2.2. *If k satisfies* A, B *and* F*, then K_A is compact on X for all $A > 0$.*

Proof. For any subset G of Ω let $BC(\overline{G})$ denote the Banach space of bounded and continuous functions on $\overline{G}$. Suppose that k satisfies F and write K_A as

$$K_A = \tilde{E}\tilde{K}_A + \hat{E}\hat{K}_A \tag{2.3}$$

where $\tilde{K}_A : X \to BC(\overline{\Omega}_C)$ and $\hat{K}_A : X \to BC(\overline{\Omega \backslash \Omega_C})$ are defined by

$$\tilde{K}_A \psi(s) := \int_{\Omega_A} k(s,t)\psi(t)\, dt, \qquad s \in \overline{\Omega}_C,$$
$$\hat{K}_A \psi(s) := \int_{\Omega_A} k(s,t)\psi(t)\, dt, \qquad s \in \overline{\Omega \backslash \Omega_C},$$

and the extension operators $\tilde{E} : BC(\overline{\Omega}_C) \to L_\infty(\Omega)$ and $\hat{E} : BC(\overline{\Omega \backslash \Omega_C}) \to L_\infty(\Omega)$ are defined by

$$\tilde{E}\psi(s) := \begin{cases} \psi(s), & s \in \Omega_C, \\ 0, & s \in \overline{\Omega} \backslash \Omega_C, \end{cases}$$
$$\hat{E}\psi(s) := \begin{cases} \psi(s), & s \in \overline{\Omega} \backslash \Omega_C, \\ 0, & s \in \Omega_C. \end{cases}$$

Since k satisfies A and B, $\tilde{K}_A$ maps bounded sets in X onto bounded equicontinuous sets in $BC(\overline{\Omega}_C)$. Thus, by the Arzela-Ascoli theorem, $\tilde{K}_A$ is compact.

Choose a sequence of subdivisions of $\Omega_A = \cup_{i=1}^n \Omega_A^{i,n}$, such that the measurable sets $\Omega_A^{i,n}$ are disjoint and their diameters satisfy $\max_{1\le i\le n}\{\mathrm{diam}\,\Omega_A^{i,n}\} \to 0$ as $n \to \infty$. Now select points $t_{i,n} \in \Omega_A^{i,n}$ and consider the sequence of operators $\hat{K}_A^{(n)} : X \to BC(\overline{\Omega\backslash\Omega_C})$, defined by

$$\hat{K}_A^{(n)}\phi(s) := \sum_{i=1}^n k(s,t_{i,n}) \int_{\Omega_A^{i,n}} \phi(t)\,dt = \int_{\Omega_A} k_n(s,t)\phi(t)\,dt$$

where $k_n(s,t) := k(s,t_{i,n})$, $s \in \overline{\Omega\backslash\Omega_C}$, $t \in \Omega_A^{i,n}$, $i = 1,\dots,n$. Then each $\hat{K}_A^{(n)}$ is bounded and compact since it has a finite dimensional range. Since k satisfies F, for all $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that

$$|k(s,t) - k_n(s,t)| \le \frac{\varepsilon}{\int_{\Omega_A} dt}, \qquad s \in \overline{\Omega}\backslash\Omega_C,\ n \ge N(\varepsilon),$$

so that $||\hat{K}_A - \hat{K}_A^{(n)}|| \le \varepsilon$ for $n \ge N(\varepsilon)$. Thus, $\hat{K}_A$ is the limit of a norm convergent sequence of compact operators and so is compact.

We have shown that $\tilde{K}_A$ and $\hat{K}_A$ are compact and, clearly, $\tilde{E}$ and $\hat{E}$ are bounded. Thus, from (2.3), we see that K_A is compact as an operator from X onto $L_\infty(\Omega)$. But K_A maps X onto X and any sequence in X convergent in $L_\infty(\Omega)$ is convergent in X. Thus, K_A is compact also as an operator from X onto X. □

3. Asymptotic behavior at infinity. Let

$$Q(\Omega) := \left\{ q \in [0,\infty) : G_q := \sup_{s\in\overline{\Omega}} \int_\Omega (1+|s-t|)^{-q}\,dt < \infty \right\}, \tag{3.1}$$

and let $q^* = q^*(\Omega) := \inf Q(\Omega)$. Then $0 \le q^* \le n$; for example, $q^* = n$ if $\Omega = \mathbf{R}^n$, $q^* = 1$ if $\Omega = \{t = (t_1,t_2,\dots,t_n) : t_1 \in \mathbf{R}, |t_2|,\dots,|t_n| < 1\}$, and $q^* = 0$ if Ω has finite measure.

The following assumption is stronger than Assumption A and imposes a bound on the rate of decay of the kernel as $|s-t| \to \infty$:

A$'$. $|k(s,t)| \le |\kappa(s-t)|$, $s,t \in \overline{\Omega}$, where κ is locally integrable on $\mathbf{R}^n$ and, for some $q > q^*$, $\kappa(s) = O(|s|^{-q})$ as $|s| \to \infty$, uniformly in s.

If k satisfies A$'$, then, for some $M, C > 0$ and all $s, t \in \overline{\Omega}$,

$$|k(s,t)| \leq |\kappa(s-t)| \leq \frac{M}{(1+|s-t|)^q}, \qquad |s-t| > C, \tag{3.2}$$

and, since $q > q^*$ and κ is locally integrable,

$$||\kappa||_\Omega := \sup_{s\in\overline{\Omega}} \int_\Omega |\kappa(s-t)|\, dt < \infty. \tag{3.3}$$

Note that equation (1.1) is equivalent to

$$x_p(s) - \int_\Omega k^{(p)}(s,t)x_p(t)\, dt = y_p(s), \qquad s \in \overline{\Omega}, \tag{3.4}$$

where $x_p := w_p x$, $y_p := w_p y$ and $k^{(p)}(s,t) := (w_p(s)/w_p(t))k(s,t)$. Defining the integral operator $K^{(p)}$ by (1.3) with k replaced by $k^{(p)}$ we may abbreviate (3.4) as

$$x_p - K^{(p)}x_p = y_p. \tag{3.5}$$

From the equivalence of equations (1.1) and (3.4) and the observation that, for $\psi \in X$ and $p \geq 0$,

$$K^{(p)}\psi = w_p K(\psi/w_p), \qquad K\psi = (1/w_p)K^{(p)}(w_p\psi), \tag{3.6}$$

it follows straightforwardly that

$$K \in B(X_p) \Longleftrightarrow K^{(p)} \in B(X), \tag{3.7}$$

$$(I-K)^{-1} \in B(X_p) \Longleftrightarrow (I-K^{(p)})^{-1} \in B(X). \tag{3.8}$$

Also notice that

$$I - K \text{ injective on } X \Longrightarrow I-K \text{ injective on } X_p \tag{3.9}$$
$$\Longleftrightarrow I - K^{(p)} \text{ injective on } X. \tag{3.10}$$

Before we proceed with the first theorem the following technical lemmas are required.

Lemma 3.1. *For $q > q^*$*

$$\int_{\Omega\setminus B_{|s|^{1/2}}(s)} (1+|s-t|)^{-q}\,dt \longrightarrow 0 \quad \text{as } |s|\to\infty$$

with $s \in \overline{\Omega}$, uniformly in s.

Proof. Let $\varepsilon = (q-q^*)/2 > 0$. Then (see (3.1) and the definition of q^*) $G_{q^*+\varepsilon}$ is finite. Also, for all $s, t \in \overline{\Omega}$,

$$(1+|s-t|)^{-q} = (1+|s-t|)^{-(q^*+\varepsilon)}(1+|s-t|)^{-\varepsilon}$$

and

$$\begin{aligned}\int_{\Omega\setminus B_{|s|^{1/2}}(s)} &(1+|s-t|)^{-q}\,dt \\ &\le (1+|s|^{1/2})^{-\varepsilon}\int_{\Omega\setminus B_{|s|^{1/2}}(s)} (1+|s-t|)^{-(q^*+\varepsilon)}\,dt \\ &\le G_{q^*+\varepsilon}(1+|s|^{1/2})^{-\varepsilon} \longrightarrow 0\end{aligned}$$

as $|s|\to\infty$, uniformly in s. □

Lemma 3.2. *For $\alpha, \beta \ge 0$, $\alpha+\beta > q^*$, define*

$$f_{\alpha\beta}(s) := \int_\Omega (1+|t|)^{-\alpha}(1+|s-t|)^{-\beta}\,dt, \qquad s\in\overline{\Omega}.$$

Then

$$F_{\alpha\beta} := \sup_{s\in\overline{\Omega}} |f_{\alpha\beta}(s)| < \infty,$$

and, moreover, if $\alpha,\beta > 0$, $|f_{\alpha\beta}(s)| \to 0$ as $|s|\to\infty$ with $s\in\overline{\Omega}$, uniformly in s.

Proof. Since $\alpha+\beta > q^*$,

$$G_{\alpha+\beta} = \sup_{s\in\overline{\Omega}}\int_\Omega (1+|s-t|)^{-(\alpha+\beta)}\,dt < \infty.$$

Clearly $f_{\alpha\beta}(s) \le G_{\alpha+\beta}$ if either $\alpha = 0$ or $\beta = 0$.

Suppose now that $\alpha > 0$, $\beta > 0$. Define the functions

$$
\begin{aligned}
H_s^1(t) &:= \min\left\{1, \left(\frac{1+|t|}{1+|s-t|}\right)^{\beta}\right\},\\
H_s^2(t) &:= \min\left\{1, \left(\frac{1+|s-t|}{1+|t|}\right)^{\alpha}\right\}.
\end{aligned}
$$

Then

$$
(1+|t|)^{-\alpha}(1+|s-t|)^{-\beta} = \begin{cases} (1+|t|)^{-(\alpha+\beta)} H_s^1(t), & |s-t| \ge |t|,\\ (1+|s-t|)^{-(\alpha+\beta)} H_s^2(t), & |s-t| \le |t|, \end{cases}
$$

and hence

$$
f_{\alpha\beta}(s) \le I_1(s) + I_2(s)
$$

where

$$
\begin{aligned}
I_1(s) &:= \int_\Omega (1+|t|)^{-(\alpha+\beta)} H_s^1(t)\, dt, \qquad s \in \overline{\Omega},\\
I_2(s) &:= \int_\Omega (1+|s-t|)^{-(\alpha+\beta)} H_s^2(t)\, dt, \qquad s \in \overline{\Omega}.
\end{aligned}
$$

Now $H_s^1(t), H_s^2(t) \le 1$, $s, t \in \overline{\Omega}$, so

$$
I_j(s) \le G_{\alpha+\beta}, \qquad j = 1, 2,
$$

so that $F_{\alpha\beta} \le 2G_{\alpha+\beta}$. Also,

$$
\begin{aligned}
I_1(s) &= \int_{\Omega \setminus B_{|s|^{1/2}}(0)} (1+|t|)^{-(\alpha+\beta)} H_s^1(t)\, dt\\
&\quad + \int_{\Omega \cap B_{|s|^{1/2}}(0)} (1+|t|)^{-(\alpha+\beta)} H_s^1(t)\, dt\\
&\le \int_{\Omega \setminus B_{|s|^{1/2}}(0)} (1+|t|)^{-(\alpha+\beta)}\, dt\\
&\quad + \left(\frac{1+|s|^{1/2}}{1+|s|-|s|^{1/2}}\right)^{\beta} G_{\alpha+\beta} \longrightarrow 0
\end{aligned}
$$

as $|s| \to \infty$ with $s \in \overline{\Omega}$, uniformly in s. Similarly,

$$\begin{aligned} I_2(s) &= \int_{\Omega \setminus B_{|s|^{1/2}}(s)} (1+|s-t|)^{-(\alpha+\beta)} H_s^2(t)\, dt \\ &\quad + \int_{\Omega \cap B_{|s|^{1/2}}(s)} (1+|s-t|)^{-(\alpha+\beta)} H_s^2(t)\, dt \\ &\leq \int_{\Omega \setminus B_{|s|^{1/2}}(s)} (1+|s-t|)^{-(\alpha+\beta)}\, dt \\ &\quad + \left(\frac{1+|s|^{1/2}}{1+|s|-|s|^{1/2}} \right)^{\alpha} G_{\alpha+\beta} \longrightarrow 0 \end{aligned}$$

as $|s| \to \infty$ with $s \in \overline{\Omega}$, uniformly in s, by Lemma 3.1. □

Our first main result is that A′ and B are sufficient conditions to ensure that $K \in B(X_p)$ for $0 \leq p \leq q$.

Theorem 3.3. *If k satisfies* A′ *and* B *and* $0 \leq p \leq q$, *then* $k^{(p)}$ *satisfies* A *and* B *and* $K \in B(X_p)$, $K^{(p)} \in B(X)$.

Proof. For $s, t \in \mathbf{R}^n$,

$$\begin{aligned} \frac{w_p(s)}{w_p(t)} &= \left(1 + \frac{|s|-|t|}{1+|t|}\right)^p \\ &\leq 2^p \left(1 + \left(\frac{|s-t|}{1+|t|}\right)^p\right). \end{aligned} \tag{3.11}$$

From the above inequality and equation (3.2),

$$|k^{(p)}(s,t)| \leq \begin{cases} 2^p M\{(1+|s-t|)^{-q} + (1+|t|)^{-p}(1+|s-t|)^{p-q}\}, & |s-t| > C, \\ (1+C)^p |\kappa(s-t)|, & |s-t| \leq C. \end{cases} \tag{3.12}$$

Hence we have, for $s \in \overline{\Omega}$, where $f_{\alpha\beta}$ and $F_{\alpha\beta}$ are defined in Lemma 3.2,

$$\begin{aligned} \int_{\Omega} |k^{(p)}(s,t)|\, dt &\leq (1+C)^p ||\kappa||_{\Omega} + 2^p\{f_{0,q}(s) + f_{p,q-p}(s)\} \\ &\leq (1+C^p) ||\kappa||_{\Omega} + 2^p\{F_{0,q} + F_{p,q-p}\}. \end{aligned}$$

Thus, $k^{(p)}$ satisfies Assumption A. To show that $k^{(p)}$ satisfies Assumption B, note that

$$\left|\frac{w_p(s)}{w_p(t)}k_s(t) - \frac{w_p(s')}{w_p(t)}k_{s'}(t)\right| \le |w_p(s)k_s(t) - w_p(s')k_{s'}(t)|$$

since $|w_p(t)| \ge 1$. Hence

$$\int_\Omega |k^{(p)}(s,t) - k^{(p)}(s',t)|\,dt \le |w_p(s) - w_p(s')|\,||k_s||_1 + |w_p(s)|\,||k_s - k_{s'}||_1.$$

But w_p is continuous and k satisfies Assumptions A and B so $k^{(p)}$ satisfies Assumption B. The rest of the lemma follows from the equivalence in (3.7). $\square$

The next theorem shows that, under the same assumptions, $K - K^{(p)}$ is in fact compact for $0 \le p < q$.

Theorem 3.4. *If Assumptions* A$'$ *and* B *are satisfied by* k *and* $0 \le p < q$, *then*

$$\int_\Omega |k(s,t) - k^{(p)}(s,t)|\,dt \longrightarrow 0$$

as $|s| \to \infty$ *with* $s \in \overline{\Omega}$, *uniformly in* s, *so that* $K - K^{(p)}$ *is a compact operator on* X.

Proof. Define

$$F(s) := \int_\Omega |k(s,t) - k^{(p)}(s,t)|\,dt.$$

We have immediately from Assumption A$'$ that, for $s \in \overline{\Omega}$,

$$\begin{aligned} F(s) &\le \int_\Omega \left|\frac{w_p(s)}{w_p(t)} - 1\right| |\kappa(s-t)|\,dt \\ &\le F_1(s) + F_2(s) \end{aligned} \tag{3.13}$$

where

$$F_1(s) := \int_{\Omega\setminus B_{|s|^{1/2}}(s)} \left|\frac{w_p(s)}{w_p(t)} - 1\right| |\kappa(s-t)|\,dt,$$

$$F_2(s) := \int_{\Omega\cap B_{|s|^{1/2}}(s)} \left|\frac{w_p(s)}{w_p(t)} - 1\right| |\kappa(s-t)|\,dt.$$

From (3.2) and (3.11), for $|s|$ sufficiently large, we have

$$F_1(s) \le (2^p+1)M \int_{\Omega\setminus B_{|s|^{1/2}}(s)} (1+|s-t|)^{-q}\,dt + 2^p M |f_{p,q-p}(s)| \longrightarrow 0$$

as $|s| \to \infty$ with $s \in \overline{\Omega}$, uniformly in s, by Lemmas 3.1 and 3.2. Let

$$\begin{aligned} c_p(s) &:= \sup_{t\in B_{|s|^{1/2}}(s)} \left| 1 - \frac{w_p(s)}{w_p(t)} \right| \\ &= \left(\frac{1+|s|}{1+|s|-|s|^{1/2}} \right)^p - 1. \end{aligned} \tag{3.14}$$

Then

$$F_2(s) \le c_p(s) ||\kappa||_\Omega \to 0$$

as $|s| \to \infty$, uniformly in s. We have just shown that $F(s) \to 0$ as $|s| \to \infty$, i.e., that $k - k^{(p)}$ satisfies Assumption C. Hence $K - K^{(p)}$ is compact from Theorem 3.3 and Lemma 2.1. □

Theorem 3.5. *If k satisfies* A$'$ *and* B, $0 \le p < q$, *and* $(I-K)^{-1} \in B(X)$, *then* $(I-K^{(p)})^{-1} \in B(X)$ *and* $(I-K)^{-1} \in B(X_p)$.

Proof. Suppose that Assumptions A$'$ and B are satisfied by k and that $(I-K)^{-1}$ in $B(X)$. Then, for $0 \le p < q$, $K^{(p)} \in B(X)$ and $K^{(p)} - K$ is a compact operator on X, by Theorems 3.3 and 3.4, respectively. Moreover, from (3.10), $I - K^{(p)}$ is injective on X. Thus, $(I-K^{(p)})^{-1} \in B(X)$ since $I - K^{(p)} = (I-K) + (K - K^{(p)})$ is the sum of an invertible operator and a compact operator and so satisfies the Fredholm alternative. That $(I-K)^{-1} \in B(X_p)$ then follows from (3.8). □

As a corollary to the above theorem, we have

Corollary 3.6. *Suppose that the conditions of the previous theorem are satisfied and that $y \in X_p$ for some $0 \le p < q$. Then equation* (1.1) *has a unique solution $x \in X_p$ and*

$$|x(s)| \le C_p ||y||_\infty^p (1+|s|)^{-p}, \qquad s \in \overline{\Omega}, \tag{3.15}$$

where C_p denotes the norm of $(I-K)^{-1} \in B(X_p)$.

The previous results do not extend as they stand to the case $p = q$, since A$'$ and B are not sufficient conditions on k to ensure that $K - K^{(q)}$ is compact; see [**8**]. We now examine the case $p = q$ further. Define

$$\bar{k}_s(t) = \bar{k}(s,t) := \frac{w_q(s-t)}{w_q(t)} k(s,t), \tag{3.16}$$

and the operator $\overline{K}$, with kernel $\bar{k}$, by (1.3) with $K(k)$ replaced by $\overline{K}(\bar{k})$.

Lemma 3.7. *If k satisfies* A$'$ *and* B*, then $\bar{k}$ satisfies Assumptions* A, B *and* D.

Proof. Let $\Omega_A^* := \Omega \backslash B_A(0)$. Recalling the inequality (3.2), for $A \geq 0$ and $s \in \overline{\Omega}$ we have

$$\begin{aligned}\int_{\Omega_A^*} |\bar{k}(s,t)|\,dt &\leq \int_{\Omega_A^* \cap B_C(s)} \frac{w_q(s-t)}{w_q(t)} |\kappa(s-t)|\,dt \\ &\quad + M \int_{\Omega_A^* \backslash B_C(s)} \frac{dt}{w_q(t)} \\ &\leq \left(\frac{1+C}{1+A}\right)^q ||\kappa||_\Omega + M \int_{\Omega_A^*} \frac{dt}{w_q(t)}.\end{aligned}$$

Thus $\bar{k}$ satisfies Assumptions A and D. Further, since k satisfies Assumption B, so does $\bar{k}$ (cf. proof of Theorem 3.3). □

We now show that $K - K^{(q)} + \overline{K}$ is a compact operator.

Theorem 3.8. *If k satisfies Assumptions* A$'$ *and* B*, then*

$$\int_\Omega |k(s,t) - k^{(q)}(s,t) + \bar{k}(s,t)|\,dt \to 0 \tag{3.17}$$

as $|s| \to \infty$ with $s \in \overline{\Omega}$, uniformly in s, so that $K - K^{(q)} + \overline{K}$ is a compact operator on X.

Proof. Since k satisfies A$'$, from (3.13) and (3.14),

$$\int_{\Omega\cap B_{|s|^{1/2}}(s)} |k(s,t)-k^{(q)}(s,t)|\,dt \le c_q(s)||\kappa||_\Omega \longrightarrow 0 \tag{3.18}$$

as $|s|\to\infty$. Thus, and since $\bar{k}$ satisfies Assumption D, it remains only to show that

$$I_1(s) := \int_{\Omega\setminus B_{|s|^{1/2}}(s)} |k(s,t)|\,dt \longrightarrow 0, \qquad |s|\to\infty,$$
$$I_2(s) := \int_{\Omega\setminus B_{|s|^{1/2}}(s)} |k^{(q)}(s,t)-\bar{k}(s,t)|\,dt \longrightarrow 0, \qquad |s|\to\infty.$$

By (3.2) and Lemma 3.1, for $|s|$ sufficiently large,

$$I_1(s) \le M\int_{\Omega\setminus B_{|s|^{1/2}(s)}} (1+|s-t|)^{-q}\,dt \longrightarrow 0, \qquad |s|\to\infty. \tag{3.19}$$

Also

$$\begin{aligned} I_2(s) &= \int_{\Omega\setminus B_{|s|^{1/2}}(s)} |k(s,t)|\frac{w_q(s-t)}{w_q(t)}\left|1-\frac{w_q(s)}{w_q(s-t)}\right| dt \\ &\le M\int_{\Omega\setminus B_{|s|^{1/2}}(s)} \left|1-\frac{w_q(s)}{w_q(s-t)}\right|\frac{dt}{w_q(t)} \\ &\le J_1(s)+J_2(s) \end{aligned} \tag{3.20}$$

where

$$J_1(s) := M\int_{\Omega\cap B_{|s|^{1/2}}(0)} \left|1-\frac{w_q(s)}{w_q(s-t)}\right|\frac{dt}{w_q(t)}$$

and

$$J_2(s) := M\int_{\Omega\setminus(B_{|s|^{1/2}}(s)\cup B_{|s|^{1/2}}(0))} \left|1-\frac{w_q(s)}{w_q(s-t)}\right|\frac{dt}{w_q(t)}.$$

Now

$$J_1(s) \le Mc_q(s)\int_\Omega \frac{dt}{w_q(t)} \longrightarrow 0 \tag{3.21}$$

as $|s| \to \infty$, uniformly in s. Further, from (3.11),

$$\frac{w_q(s)}{w_q(t)} \leq 2^q\left(1 + \frac{w_q(s-t)}{w_q(t)}\right),$$

so

$$\left|1 - \frac{w_q(s)}{w_q(s-t)}\right| \frac{1}{w_q(t)} \leq \frac{(2^q+1)}{w_q(t)} + \frac{2^q}{w_q(s-t)}$$

so that

$$\begin{aligned} J_2(s) &\leq (2^q+1)\int_{\Omega\setminus B_{|s|^{1/2}}(0)} \frac{dt}{w_q(t)} \\ &\quad + 2^q \int_{\Omega\setminus B_{|s|^{1/2}}(s)} \frac{dt}{w_q(s-t)} \longrightarrow 0 \end{aligned} \tag{3.22}$$

as $|s| \to \infty$ with $s \in \overline{\Omega}$, uniformly in s, by Lemma 3.1. □

Define $\overline{K}_A$, a "finite section" version of the operator $\overline{K}$, by

$$\overline{K}_A\psi(s) = \int_{\Omega_A} \bar{k}(s,t)\psi(t)\,dt, \quad s \in \overline{\Omega}. \tag{3.23}$$

We have the following extension of Theorem 3.5 to the case $p = q$.

Theorem 3.9. *Suppose that k satisfies* A$'$ *and* B*, that $(I-K)^{-1} \in B(X)$ and that $\overline{K}_A$ is compact for all $A > 0$. Then $(I-K)^{-1} \in B(X_q)$ and $(I-K^{(q)})^{-1} \in B(X)$.*

Proof. Since $\overline{K}_A$ is compact for all $A > 0$ and, by Lemma 3.7, $\bar{k}$ satisfies A, B and D, $\overline{K}$ is compact. Thus, and by Theorem 3.8, $K - K^{(q)}$ is compact. The result follows as in the proof of Theorem 3.5. □

From Lemmas 2.1 and 2.2, $\overline{K}_A$ is compact for all $A > 0$ if $\bar{k}$ satisfies E or F. Applying Lemmas 2.1 and 2.2, we obtain the following additional criterion for compactness of $\overline{K}_A$, utilized in Sections 4 and 5.

Lemma 3.10. *Suppose that k satisfies* A$'$ *and* B *and that, for every $A > 0$, $w_q(s-t)k(s,t) = k^*(s,t) + o(1)$ as $|s| \to \infty$ with $s \in \overline{\Omega}$,* *uniformly in* s *and* t *for* $t \in \Omega_A$*, and that* k^* *is continuous and bounded on* $\overline{\Omega} \times \overline{\Omega}$ *and satisfies* F*. Then* $\overline{K}_A$ *is compact for all* $A > 0$ *so that* $\overline{K}$ *is compact.*

Proof. Define $\bar{k}_1(s,t) = k^*(s,t)/w_q(t)$, $\bar{k}_2(s,t) = \bar{k}(s,t) - \bar{k}_1(s,t)$, and let $\overline{K}_{1,A}$, $\overline{K}_{2,A}$ denote the integral operators defined by (1.3) with k replaced by $\bar{k}_1$ and $\bar{k}_2$, respectively. Then it is easy to see that $\bar{k}_1$ satisfies A, B and F, so that, by Lemma 2.2, $\overline{K}_{1,A}$ is compact for all $A > 0$. Hence, and by Lemma 3.7, $\bar{k}_2$ satisfies A and B. Moreover, $\bar{k}_2(s,t) = (w_q(s-t)k(s,t) - k^*(s,t))/w_q(t) \to 0$ as $s \to \infty$ with $s \in \overline{\Omega}$, uniformly in s and t for $t \in \Omega_A$, so $\bar{k}_2$ also satisfies E. Thus, $\overline{K}_{2,A}$ is compact and so $\overline{K}_A = \overline{K}_{1,A} + \overline{K}_{2,A}$ is compact for all $A > 0$. □

Combining Theorem 3.9 and Lemma 3.10, we have the following extension of Corollary 3.6.

Corollary 3.11. *Suppose that the conditions of the previous lemma are satisfied, that* $(I - K)^{-1} \in B(X)$*, and that* $y \in X_p$ *for some* $0 \le p \le q$*. Then equation* (1.1) *has a unique solution* $x \in X_p$*, and this solution satisfies the inequality* (3.15)*.*

The above theorems, 3.5 and 3.9, give conditions for the invertibility of $I - K$ in the weighted space X_p. In cases where we do not know that $(I - K)^{-1} \in B(X)$, these results do not apply, but we may still be able to obtain information about the Fredholm properties of $I - K$. For $p \ge 0$, let $\phi(X_p) \subset B(X_p)$ denote the set of all Fredholm operators on X_p (see, e.g., Jörgens [**12**] for definitions). We have the following result:

Theorem 3.12. *Suppose that* k *satisfies* A$'$ *and* B *and that* $I - K \in \phi(X_p)$ *for some* p *in the range* $0 \le p < q$*. Then* $I - K \in \phi(X_p)$ *for all* $0 \le p < q$*, and the index of* $I - K$ *is the same in each of these spaces.*

Proof. Note that the inverse operations of multiplication by w_p and multiplication by $1/w_p$ are isometric isomorphisms from X_p to X and from X to X_p, respectively. Thus, each of these operations is a Fredholm operator of index zero. It therefore follows from (3.6) and a standard result on the composition of Fredholm operators (see, e.g., [**12**, Theorem 5.6]) that

$$I - K \in \phi(X_p) \Longleftrightarrow I - K^{(p)} \in \phi(X) \tag{3.24}$$

and that if $I - K \in \phi(X_p)$, then the indices of $I - K \in \phi(X_p)$ and $I - K^{(p)} \in \phi(X)$ are the same.

Now suppose that $I-K \in \phi(X_p)$ for some p with $0 \leq p < q$. Then, by the above remarks, $I-K^{(p)} \in \phi(X)$ with the same index, and since, by Theorem 3.4, $K - K^{(p)}$ is compact, it follows (see [**12**, Theorem 5.12]) that $I - K \in \phi(X)$ with the same index. Reversing this argument we can show that, if $I - K \in \phi(X)$, then, for any p with $0 \leq p < q$, $I - K \in \phi(X_p)$ with the same index. The result follows. □

Remark 3.13. The above result depends on the compactness of $K - K^{(p)}$. If the conditions of Lemma 3.10 are satisfied, then $K - K^{(q)}$ is also compact and Theorem 3.12 holds with the range $0 \leq p < q$ extended to $0 \leq p \leq q$.

4. Wiener-Hopf integral equations. We consider the important special case when $k(s,t) = \kappa(s-t)$, $s, t \in \overline{\Omega}$, so that (1.1) is an equation of Wiener-Hopf type (see, e.g., [**9, 15, 12, 13, 16, 6**]). The conditions of the previous section simplify somewhat in this special case as the following lemma illustrates.

Lemma 4.1. *If $k(s,t) = \kappa(s-t)$, $s, t \in \overline{\Omega}$, κ is locally integrable, and, for some $q > q^*$, $\kappa(s) = O(|s|^{-q})$ as $|s| \to \infty$, uniformly in s, then k satisfies* A$'$ *and* B.

Proof. Clearly k satisfies A$'$. To see that k satisfies B note that, for all $A > 0$,

$$\begin{aligned}\int_{\Omega \cap B_{A+1}(s)} |k(s,t) - k(s',t)|\, dt &\leq \int_{B_{A+1}(s)} |\kappa(s-t) - \kappa(s'-t)|\, dt \\ &= \int_{B_{A+1}(0)} |\kappa(t) - \kappa(t-(s'-s))|\, dt \\ &\longrightarrow 0\end{aligned}$$

as $s' \to s$, since κ is locally integrable. Also, for some $M, C > 0$ and all $s, t \in \overline{\Omega}$, κ satisfies (3.2). Thus, for $|s' - s| \le 1$ and $A \ge C$,

$$\begin{aligned}\int_{\Omega\setminus B_{A+1}(s)} &|k(s,t) - k(s',t)|\, dt \\ &\le M \int_{\Omega\setminus B_{A+1}(s)} \left(\frac{1}{(1+|s-t|)^q} + \frac{1}{(1+|s'-t|)^q}\right) dt \\ &\le 2M \sup_{s\in\overline{\Omega}} \int_{\Omega\setminus B_A(s)} \frac{dt}{(1+|s-t|)^q} \longrightarrow 0\end{aligned}$$

as $A \to \infty$ by Lemma 3.1. $\square$

Note further that if k satisfies the conditions of Lemma 4.1 and also

$$|s|^q \kappa(s) = \kappa^*(s) + o(1) \quad \text{as } |s| \to \infty, \quad \text{uniformly in } s, \tag{4.1}$$

with κ^* bounded and uniformly continuous on $\mathbf{R}^n$, then the conditions of Lemma 3.10 are satisfied.

Throughout the remainder of the section we suppose that $k(s,t) = \kappa(s-t)$, $s, t \in \overline{\Omega}$, with κ locally integrable.

Combining the above lemma and remark with Theorems 3.5, 3.9 and Lemma 3.10, we obtain

Theorem 4.2. *Suppose that, for some $q > q^*$, $\kappa(s) = O(|s|^{-q})$ as $|s| \to \infty$, uniformly in s. Then $K \in B(X_p)$, $0 \le p \le q$. If also $(I-K)^{-1} \in B(X)$, then $(I-K)^{-1} \in B(X_p)$, $0 \le p < q$. If, moreover, (4.1) holds, then $(I-K)^{-1} \in B(X_q)$.*

In the specific cases $\Omega = \mathbf{R}^n$ and $\Omega = \mathbf{R}^n_+ := \{(s_1, \dots, s_n) \in \mathbf{R}^n : s_n > 0\}$, necessary and sufficient conditions for $(I-K)^{-1} \in B(X)$ are known in terms of the Fourier symbol of the operator $I - K$, defined by

$$\hat{\phi}(\xi) = 1 - \int_{\mathbf{R}^n} e^{i\xi . s} \kappa(s)\, ds, \qquad \xi \in \mathbf{R}^n$$

($\xi . s$ is the scalar product of ξ and s). If $\kappa \in L_1(\mathbf{R}^n)$ and either (a) $\Omega = \mathbf{R}^n$, or (b) $\Omega = \mathbf{R}^n_+$ and $n \ge 2$, then $(I-K)^{-1} \in B(X)$ if and only if

$$\hat{\phi}(\xi) \ne 0, \qquad \xi \in \mathbf{R}^n \tag{4.2}$$

(see [**12, 9**]). (Note that if $\Omega = \mathbf{R}^n$ or $\mathbf{R}^n_+$, then $q^* = n$ so that $\kappa \in L_1(\mathbf{R}^n)$ if the conditions of Lemma 4.1 are satisfied.) Thus, as a corollary of the previous theorem, we have the following extension of the results of Wiener [**12**, page 340] and of Goldenstein and Gohberg [**9**] to the weighted spaces X_p.

Theorem 4.3. *Suppose that $\Omega = \mathbf{R}^n$ or $\mathbf{R}^n_+$ (with $n \geq 2$ in the second case). Suppose further that, for some $q > n$, $\kappa(s) = O(|s|^{-q})$ as $|s| \to \infty$, uniformly in s, and that* (4.2) *holds. Then $(I-K)^{-1} \in B(X_p)$, $0 \leq p < q$. If, moreover,* (4.1) *holds, then also $(I-K)^{-1} \in B(X_q)$.*

Note that the case $\Omega = \mathbf{R}_+$ is excluded from the above result. Sufficient conditions for the invertibility of $I - K$ on X_p in the case $\Omega = \mathbf{R}_+$ are given in [**8**].

For a larger class of regions Ω the nonvanishing of the symbol $\hat{\phi}$, while not known to guarantee the invertibility of $I - K$ on X, still ensures that $I - K$ is Fredholm. For example, this is the case if Ω is a connected open conic set, provided the boundary of Ω, except at the point 0, is a smooth surface (the case if Ω is a circular cone, etc.).

Combining these observations with Theorem 3.12 and Remark 3.13, we have the following extension of the results of Simonenko [**15**] to the weighted space X_p.

Theorem 4.4. *Suppose that $\Omega \subset \mathbf{R}^n$, $n \geq 2$, is a connected open conic set, and that the boundary of Ω, except at the point* 0, *is a smooth (C^1) surface. Suppose also that, for some $q > n$, $\kappa(s) = O(|s|^{-q})$ as $|s| \to \infty$, uniformly in s, and that* (4.2) *holds. Then $I - K \in \phi(X_p)$, $0 \leq p < q$, and has index zero in each of these spaces. If, moreover,* (4.1) *holds, then also $I - K \in \phi(X_q)$ with index zero.*

Proof. Simonenko [**15**] established that if (4.2) holds, then $I - K \in \phi(L_p(\Omega))$ for $1 \leq p < \infty$, and this result is established also for $p = \infty$ in [**5**]. Since K is a continuous mapping from $L_\infty(\Omega)$ to the closed subspace X, it is easy to see that $I - K \in \phi(L_\infty(\Omega)) \Rightarrow I - K \in \phi(X)$. Thus $I - K \in \phi(X)$. Further, Simonenko [**15**] shows that $I - K$ has index zero as an operator on $L_p(\Omega)$, $1 \leq p < \infty$, and the homotopy argument he uses applies equally to $I - K$ as an operator on X. Thus, $I - K \in \phi(X)$ with index zero. The result now follows from Theorem 3.12 and Remark 3.13. $\square$

5. An application in acoustics. Consider the following boundary value problem for the Helmholtz equation in the half space $\mathbf{R}^3_+ := \{s = (s_1, s_2, s_3) \in \mathbf{R}^3 : s_3 > 0\}$:

$$(5.1) \qquad \begin{gathered} \Delta u + u = F \quad \text{in } \mathbf{R}^3_+, \\ \frac{\partial u}{\partial n} + i\beta u = 0 \quad \text{on } \mathbf{R}^2 = \partial \mathbf{R}^3_+, \\ u \text{ satisfies the Sommerfeld radiation condition.} \end{gathered}$$

In (5.1) the functions $\beta \in L_\infty(\mathbf{R}^2)$ and F are supposed given, with $F \in L_2(\mathbf{R}^3_+)$ compactly supported and $\Re\mathrm{e}\, \beta \geq 0$.

Let $\Omega \subset \mathbf{R}^2 = \partial \mathbf{R}^3_+$ denote the strip $\{(s_1, s_2) : 0 < s_1 < d, s_2 \in \mathbf{R}\}$. Define β by

$$(5.2) \qquad \beta(s) = \begin{cases} 0, & s \in \Omega, \\ \beta_c, & s \in \mathbf{R}^2 \backslash \Omega, \end{cases}$$

where $\beta_c \in \mathbf{C}$ with $\Re\mathrm{e}\, \beta_c > 0$.

The boundary value problem (5.1) models outdoor sound propagation from the source region (the support of F) over a flat ground plane. In this context $|u|$ is the amplitude of the pressure fluctuation due to the sound wave and β the relative surface admittance of the ground plane; its value $\beta(s)$ at a particular point $s \in \mathbf{R}^2$ depends on the frequency of the sound source and on local properties of the ground at that point [**4**]. Where $\beta = 0$, the ground is perfectly rigid, while where $\Re\mathrm{e}\, \beta > 0$ the ground is energy absorbing. Thus, the choice (5.2) models a rigid infinite strip (Ω) in an otherwise homogeneous energy absorbing plane. In particular, the boundary value problem (5.1) with β given by (5.2) is a good model of sound propagation above a long straight road (the rigid strip Ω) surrounded by absorbing ground (for example, grassland) [**11**].

An interesting practical question is at what rate the sound generated by a motor vehicle decays with distance along the road. Using the results of the previous section, we shall show that, at least if the width of the road is not too large, for an observer on the road surface, while the decay in the sound pressure with distance may initially be the same as above, a completely rigid ground ($O(|s|^{-1})$), the decay with distance must ultimately be that for a completely absorbing ground ($O(|s|^{-2})$). To the best of our knowledge, this result has not previously been established.

The Green's function $G_{\beta_c}(s,t)$ which satisfies (5.1) with $F(s) = \delta(s-t)$ and $\beta(s) = \beta_c$, $s \in \mathbf{R}^2$, is given by [**17**]

$$
\begin{aligned}
G_{\beta_c}(s,t) = &-\frac{e^{i|s-t|}}{4\pi|s-t|} - \frac{e^{i|s-t'|}}{4\pi|s-t'|} \\
&+ \frac{i\beta_c}{2\pi} e^{i|s-t'|} \int_0^\infty \frac{e^{-|s-t'|u}\, du}{\sqrt{u^2 - 2i(1+\beta_c\gamma)u - (\gamma+\beta_c)^2}} \\
&+ \delta \frac{\beta_c}{2} H_0^{(1)}(r\sqrt{1-\beta_c^2}) e^{-i\beta_c(s_3+t_3)},
\end{aligned}
\tag{5.3}
$$

for $s = (s_1, s_2, s_3)$, $t = (t_1, t_2, t_3) \in \mathbf{R}^3_+$, $s \neq t$. Here $t' = (t_1, t_2, -t_3)$ denotes the image of t in the ground plane, $r = \sqrt{(s_1-t_1)^2 + (s_2-t_2)^2}$, $\gamma = (s_3+t_3)/|s-t'|$, and δ is given by

$$
\delta = H[-\Im m \beta_c] H[\Re e\, \{-\beta_c(s_3+t_3) + \sqrt{1-\beta_c^2}\, r - |s-t'|\}],
$$

$$
H[u] = \begin{cases} 1, & u > 0, \\ 0, & u \leq 0. \end{cases}
$$

In (5.3), $\Re e\,(\sqrt{1-\beta_c^2}) \geq 0$ and the branch cut for the square root in the integrand should be chosen so that the square root depends continuously on u and takes the value $i(\gamma+\beta_c)$ at $u = 0$.

From (5.3) it is easy to see, using Watson's lemma and the asymptotic behavior of the Hankel function $H_0^{(1)}$ for large argument in the case $\Re e\, \beta_c > 0$ that, for any constant $C > 0$, as $|s-t| \to \infty$ with $s_3 + t_3 \leq C$,

$$
G_{\beta_c}(s,t) \begin{cases} \sim -\exp(i|s-t|)/2\pi|s-t|, & \beta_c = 0, \\ = O(|s-t|^{-2}), & \Re e\, \beta_c > 0, \end{cases}
\tag{5.4}
$$

uniformly in s and t. A full asymptotic expansion for G_{β_c} in the limit $|s-t| \to \infty$ is given in [**14**]. The asymptotic result (5.4) illustrates the faster decay rate over absorbent ground ($\Re e\, \beta_c > 0$) than over rigid ground ($\beta_c = 0$).

Applying Green's theorem to u and G_{β_c} in $\mathbf{R}^3_+$, the boundary value problem (5.1), with β given by (5.2), can be reformulated as the following boundary integral equation for x, the restriction of u to $\overline{\Omega}$ [**11**].

$$x(s) = y(s) + i\beta_c \int_\Omega g_{\beta_c}(s-t)x(t)\,dt, \qquad s \in \overline{\Omega}, \tag{5.5}$$

with, for $s = (s_1, s_2) \in \mathbf{R}^2$,

$$y(s) := \int_{\mathbf{R}^3_+} G_{\beta_c}((s_1, s_2, 0), t)F(t)\,dt, \tag{5.6}$$

$$\begin{aligned} g_{\beta_c} &:= G_{\beta_c}((s_1, s_2, 0), 0) \\ &= -\frac{e^{i|s|}}{2\pi|s|} + \frac{i\beta_c}{2\pi} e^{i|s|} \int_0^\infty \frac{e^{-|s|u}\,du}{\sqrt{u^2 - 2iu - \beta_c^2}} \\ &\quad + \delta \frac{\beta_c}{2} H_0^{(1)}(|s|\sqrt{1-\beta_c^2}). \end{aligned} \tag{5.7}$$

Equation (5.5), a convolution equation on the strip Ω, is identical to equation (1.1) if we define

$$k(s,t) := \kappa(s-t) := i\beta_c g_{\beta_c}(s-t), \qquad s, t \in \overline{\Omega}. \tag{5.8}$$

From (5.4), since $\Re e\, \beta_c > 0$, it follows that $y(s) = O(|s|^{-2})$, as $|s| \to \infty$, uniformly in s. Further it can be seen from (5.3), or more conveniently from an inverse Hankel transform representation of $G_{\beta_c}(s,t)$ [**10**] that $G_{\beta_c}(.,t)$ is continuous in $\overline{\mathbf{R}}_3^+$ except at t. Thus, $y \in X_p$ for $0 \le p \le 2$. Also, from (5.7) it can be seen that g_{β_c} is continuous in $\mathbf{R}^2$ except for an integrable singularity at 0. Thus k and κ satisfy the conditions of Lemma 4.1 with $q = 2$, so that k satisfies A$'$ and B.

We have shown that k satisfies the conditions of Theorem 3.3 so that, where K is the integral operator (1.3) with k given by (5.8), we have

Theorem 5.1. *For $0 \le p \le 2$, $K \in B(X_p)$ and $K^{(p)} \in B(X)$.*

To obtain similar mapping properties for the inverse operator $(I - K)^{-1}$ and establish the asymptotic behavior of the solution x of equation (5.5), we need first that $(I-K)^{-1} \in B(X)$. Now it is easily seen that

$$
\begin{aligned}
||K|| &= \sup_{s\in\overline{\Omega}} \int_\Omega |k(s,t)|\,dt \\
&\leq 2|\beta_c| \int_\Omega |g_{\beta_c}(t)|\,dt.
\end{aligned}
\tag{5.9}
$$

Thus $||K|| < 1$ provided d (the width of Ω) is sufficiently small. Thus, we obtain

Theorem 5.2. *Provided d is sufficiently small so that the righthand side of* (5.9) *is* < 1, $(I-K)^{-1} \in B(X)$ *so that equation* (5.5) *has a unique bounded continuous solution x.*

As $y \in X_p$ for $0 \leq p \leq 2$ and k satisfies A$'$ and B with $q = 2$, we can combine Theorems 3.6 and 5.2 to immediately obtain that also $(I-K)^{-1} \in B(X_p)$ and that

$$x(s) = O(|s|^{-p}), \qquad s \to \infty,$$

for $0 \leq p < 2$. To sharpen this result, note that, from Rawlins [**14**],

$$g_{\beta_c}(s) \sim a e^{i|s|}|s|^{-2}, \qquad a := -i/(2\pi\beta_c^2), \tag{5.10}$$

as $s \to \infty$, uniformly in $s \in \mathbf{R}^2$. Thus (4.1) is satisfied with $\kappa^*(s) := ae^{i|s|}$. We can therefore apply Theorem 4.2 to obtain

Theorem 5.3. *If the condition of the previous theorem is satisfied, then* $(I-K)^{-1} \in B(X_p)$, $0 \leq p \leq 2$, *so that the solution of equation* (5.5) *satisfies* $x(s) = O(|s|^{-2})$ *as* $|s| \to \infty$, *uniformly in s.*

This example we have given, of practical interest, also serves to illustrate the sharpness of the results we have obtained in Section 3. From (5.10) it follows that if $\psi \in X$ is compactly supported, then

$$K\psi(s) \sim a \int_\Omega e^{-i\hat{s}.t}\psi(t)\,dt e^{i|s|}|s|^{-2}, \tag{5.11}$$

as $|s| \to \infty$, uniformly in $\hat{s} = s/|s|$, where $\hat{s}.t$ denotes the scalar product of $\hat{s}$ and t. In general, the integral on the righthand side of (5.11) will not vanish for any $s \in \Omega$ so that $K\psi \notin X_p$ for $p > 2$ even though $\psi \in X_p$ for all $p > 0$. Thus, $K \notin B(X_p)$ for $p > 2$.

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