

# *Nonlinear saturation of baroclinic instability: part III: bounds on the energy*

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## Nonlinear Saturation of Baroclinic Instability. Part III: Bounds on the Energy

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### ABSTRACT

Rigorous upper bounds are derived on the saturation amplitude of baroclinic instability in the two-layer model. The bounds apply to the eddy energy and are obtained by appealing to a finite amplitude conservation law for the disturbance pseudoenergy. These bounds are to be distinguished from those derived in Part I of this study, which employed a pseudomomentum conservation law and provided bounds on the eddy potential enstrophy. The bounds apply to conservative (inviscid, unforced) flow, as well as to forced-dissipative flow when the dissipation is proportional to the potential vorticity.

Bounds on the eddy energy are worked out for a general class of unstable westerly jets. In the special case of the Phillips model of baroclinic instability, and in the limit of infinitesimal initial eddy amplitude, the bound states that the eddy energy cannot exceed  $\epsilon\beta^2/6F$ , where  $\epsilon = (U - U_{\text{crit}})/U_{\text{crit}}$  is the relative supercriticality. This bound captures the essential dynamical scalings (i.e., the dependence on  $\epsilon$ ,  $\beta$ , and  $F$ ) of the saturation amplitudes predicted by weakly nonlinear theory, as well as exhibiting remarkable quantitative agreement with those predictions, and is also consistent with heuristic baroclinic adjustment estimates.

### 1. Introduction

In Part I of this study (Shepherd 1988a), rigorous upper bounds were derived on the nonlinear saturation of baroclinic instability in a two-layer quasigeostrophic fluid. These bounds were obtained through the use of a Liapunov (normed) stability theorem, which is the finite amplitude generalization of the well-known Charney–Stern theorem. This stability theorem is based on conservation of the disturbance pseudomomentum and proves stability in the potential enstrophy norm; the rigorous upper bounds that were obtained from this theorem provide bounds on the growth of the potential enstrophy of the nonzonal part of the flow. Comparison of the bounds with the weakly nonlinear theory of Pedlosky (1970) for the special case of the Phillips (1954) model of baroclinic instability, showed very similar parameter dependences as well as remarkable quantitative agreement.

While potential enstrophy is an important measure of disturbance amplitude, especially because it can be associated (at small amplitude at least) with meridional particle displacements, more emphasis has traditionally been placed on the energy. The distinction between energy and potential enstrophy is certainly not pedantic since for a continuous system, such as a fluid, all norms are not equivalent. For example, the temporary amplification of nonmodal disturbances (e.g., Farrell 1982

et seq.) is quite different when viewed from the perspective of different norms: in the canonical case of linear Couette flow (e.g., Shepherd 1985 and refs.), the energy norm can amplify by an arbitrarily large amount (depending on the initial condition), while the enstrophy norm cannot amplify at all.

It was pointed out in Shepherd [1988a, Eq. (8.2)] that the potential enstrophy saturation bounds can be turned into energy bounds based on the finite width of the channel. However, these latter bounds can be expected to be very weak (i.e., much too large) when the Rossby deformation radius is small compared with the channel width. It is, therefore, of interest to see whether more useful upper bounds on the energy may be obtained by some other route. It turns out that there is another Liapunov stability theorem available for two-layer quasigeostrophic flow, namely the finite amplitude generalization of the Fjørtoft theorem, alternatively the quasigeostrophic version of Arnol'd's (1965, 1966) first stability theorem (Holm et al. 1985; Swaters 1986; McIntyre and Shepherd 1987, appendix B). This stability theorem is based on conservation of the disturbance pseudoenergy and proves stability in a norm consisting of a weighted combination of energy and potential enstrophy.

At first sight, the above-mentioned theorem would not appear to be too promising for the purpose at hand. First, stability requires the basic flow to satisfy  $U/Q_y < 0$ , where  $U$  is the zonal velocity and  $Q_y$  is the meridional gradient of potential vorticity; and as Andrews (1984) has noted this is not the usual situation for atmospherically relevant flows—including all the clas-

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sical models of baroclinic instability. Second, even when the stability criterion is satisfied, Liapunov stability is not provable in the energy norm (cf. McIntyre and Shepherd 1987). Fortunately, however, it turns out that neither of these difficulties is fatal when it comes to obtaining saturation bounds on the eddy energy in baroclinic instability, at least within the context of the two-layer model. In this paper, explicit derivations of such bounds will be presented, based on pseudoenergy conservation.

The plan of the paper is as follows. The governing equations for conservative (inviscid, unforced) flow are reviewed in section 2, in order to fix notation. In section 3 a brief derivation of the pseudoenergy conservation law is presented for disturbances to steady basic states, as well as the basis for its use in the nonlinear saturation theory. The theory is applied in section 4 to the classical case of the Phillips (1954) model of baroclinic instability. The rigorous upper bounds obtained for that case are then compared with the weakly nonlinear theories of Pedlosky (1970) and Warn and Gauthier (1989) (section 5) and with heuristic baroclinic adjustment estimates (section 6). Finally in section 7 the theory is worked out for a general class of unstable westerly jets. The paper concludes with a discussion.

**2. Governing equations for the two-layer model**

The system to be considered in this paper is the simplest possible one capable of representing baroclinic instability: the two-layer model. A thorough treatment of the system can be found in Pedlosky (1987, sections 6.16 and 7.9), whose notation is largely followed here. This is also the system treated by Shepherd (1988a), though in the present case the possibility of different layer depths is allowed. The flow is governed by material conservation of quasigeostrophic potential vorticity in each layer, namely

$$\frac{D_i P_i}{Dt} \equiv \frac{\partial P_i}{\partial t} + \partial(\Phi_i, P_i) = 0, \quad i = 1, 2, \quad (2.1)$$

where  $i = 1$  and  $i = 2$  refer, respectively, to the upper and lower layers,  $\Phi_i$  is the geostrophic streamfunction,  $\partial(f, g) \equiv f_x g_y - f_y g_x$  is the two-dimensional Jacobian operator, and

$$P_i = \nabla^2 \Phi_i + (-1)^i F_i (\Phi_1 - \Phi_2) + f + \beta y \quad (2.2)$$

is the potential vorticity in each layer. Here  $\nabla^2$  is the two-dimensional Laplacian operator,  $f$  is a constant representative of the midlatitude value of the Coriolis parameter,  $\beta$  is a constant representative of the midlatitude value of the meridional gradient of the Coriolis parameter, and  $F_i$  is the internal rotational Froude number for each layer (a measure of the static stability). Let the layer depths be given by  $D_1$  and  $D_2$ ; it follows from the definition of  $F_i$  that

$$D_1 F_1 = D_2 F_2. \quad (2.3)$$

The domain is taken to be infinite in the zonal coordinate  $x$  but bounded in the meridional coordinate  $y$ , with boundary conditions at the channel walls of no normal flow:

$$\frac{\partial \Phi_i}{\partial x} = 0 \quad \text{at } y = 0, 1, \quad (2.4a)$$

and conservation of circulation:

$$\frac{d}{dt} \frac{\partial \bar{\Phi}_i}{\partial y} \Big|_{y=0} \equiv - \frac{d \Sigma_i^0}{dt} = 0, \quad \frac{d}{dt} \frac{\partial \bar{\Phi}_i}{\partial y} \Big|_{y=1} \equiv \frac{d \Sigma_i^1}{dt} = 0; \quad (2.4b)$$

the overbar refers to a zonal average

$$\bar{f} \equiv \lim_{X \rightarrow \infty} \frac{1}{2X} \int_{-X}^{+X} f(x) dx, \quad (2.5)$$

which is presumed to be well defined for all variables.

The system described by (2.1), (2.4) possesses certain integral invariants. It is straightforward to verify, using (2.3), that the (kinetic plus available potential) energy

$$\mathcal{E}[P_i, \Sigma_i^j] = \int_0^1 \frac{1}{2} \{ D_1 |\nabla \Phi_1|^2 + D_2 |\nabla \Phi_2|^2 + D_1 F_1 (\Phi_1 - \Phi_2)^2 \} dy \quad (2.6)$$

is conserved in time. The notation  $\mathcal{E}[P_i, \Sigma_i^j]$  is shorthand for  $\mathcal{E}[P_1, P_2, \Sigma_1^0, \Sigma_1^1, \Sigma_2^0, \Sigma_2^1]$  and makes explicit the fact that  $\mathcal{E}$  is a functional of the potential vorticity of the flow and the circulation along the channel walls: that is, given  $P_1, P_2$ , and  $\Sigma_1^0, \Sigma_1^1, \Sigma_2^0, \Sigma_2^1$ , one can invert the relation (2.2) to determine  $\Phi_1, \Phi_2$ , and hence, the energy  $\mathcal{E}$ . It is, furthermore, obvious from (2.1) and (2.4a) that the spatial integral of any function of  $P_1$  or  $P_2$  is also conserved in time. Together with conservation of circulation (2.4b), this implies that functionals  $\mathcal{E}$  of the form

$$\mathcal{E}[P_i, \Sigma_i^j] = \int_0^1 \{ \overline{C_1(P_1)} + \overline{C_2(P_2)} \} dy + c_1^0 \Sigma_1^0 + c_1^1 \Sigma_1^1 + c_2^0 \Sigma_2^0 + c_2^1 \Sigma_2^1 \quad (2.7)$$

are conserved in time, for any choice of the functions  $C_1$  and  $C_2$ , and for any constants  $c_i^j$ . These invariants  $\mathcal{E}$  and  $\mathcal{E}$  will be used in the next section to construct the pseudoenergy relative to a steady basic state.

**3. Pseudoenergy and nonlinear saturation bounds**

Introduce a basic state  $\{\Phi_i, P_i, \Sigma_i^j\} = \{\Psi_i, Q_i, \Gamma_i^j\}$  consisting of a steady zonal flow  $U_i(y) = -d\Psi_i/dy$ , with associated potential vorticity

$$Q_i = - \frac{dU_i}{dy} + (-1)^i F_i (\Psi_1 - \Psi_2) + f + \beta y. \quad (3.1)$$

Since the basic state  $\{\Psi_i, Q_i, \Gamma_i^j\}$  is a steady solution of the dynamics, it follows from (2.1) that

$$\partial(\Psi_i, Q_i) = 0 \Leftrightarrow \Psi_i = \Psi_i(Q_i); \quad (3.2)$$

that is,  $\Psi_i$  is necessarily a function of  $Q_i$ . This functional dependence is rather trivial in the present case of a zonally symmetric basic flow, because  $\Psi_i$  and  $Q_i$  can obviously be related to each other through their mutual dependence on  $y$ .

Now consider a flow  $\{\Phi_i, P_i, \Sigma_i^j\}$  consisting of a disturbance  $\{\psi_i, q_i, \gamma_i^j\}$  to the above basic state, namely

$$\Phi_i = \Psi_i + \psi_i, \quad P_i = Q_i + q_i, \quad \Sigma_i^j = \Gamma_i^j + \gamma_i^j, \quad (3.3)$$

with

$$q_i = \nabla^2 \psi_i + (-1)^i F_i(\psi_1 - \psi_2),$$

$$\gamma_i^0 = -\left. \frac{\partial \bar{\psi}_i}{\partial y} \right|_{y=0}, \quad \gamma_i^1 = \left. \frac{\partial \bar{\psi}_i}{\partial y} \right|_{y=1}. \quad (3.4a,b,c)$$

The (disturbance) pseudoenergy relative to the basic state  $\{\Psi_i, Q_i, \Gamma_i^j\}$  is given by (Holm et al. 1985; McIntyre and Shepherd 1987)

$$\mathcal{A}[P_i, \Sigma_i^j, Q_i, \Gamma_i^j] \equiv \mathcal{E}[P_i, \Sigma_i^j] - \mathcal{E}[Q_i, \Gamma_i^j] + \mathcal{C}[P_i, \Sigma_i^j] - \mathcal{C}[Q_i, \Gamma_i^j], \quad (3.5)$$

where  $\mathcal{E}$  is given by (2.6), and  $\mathcal{C}$  is the specific functional given by (2.7) for the choices

$$C_i(\eta) = D_i \int \Psi_i(\eta) d\eta, \quad c_i^0 = -D_i \Psi_i|_{y=0},$$

$$c_i^1 = -D_i \Psi_i|_{y=1}, \quad (3.6)$$

with  $\Psi_i(\cdot)$  denoting the two functions (for  $i = 1, 2$ ) defined by (3.2). [In contrast to the treatment in McIntyre and Shepherd (1987), here the possibility of nonzero disturbance circulations  $\gamma_i^j$  is explicitly included; this generalization is essential for the saturation-bound calculations.] Now,  $\mathcal{A}$  is made up of two exact dynamical invariants,  $\mathcal{E}[P_i, \Sigma_i^j]$  and  $\mathcal{C}[P_i, \Sigma_i^j]$ , and two constants,  $\mathcal{E}[Q_i, \Gamma_i^j]$  and  $\mathcal{C}[Q_i, \Gamma_i^j]$ ; therefore,  $\mathcal{A}$  is itself an exact dynamical invariant:

$$\mathcal{A}(t) = \mathcal{A}(0) \quad \text{for all } t. \quad (3.7)$$

After some manipulation (see Appendix for details), (3.5) can be cast in the form

$$\mathcal{A} = \int_0^1 \left\{ \frac{1}{2} (D_1 |\nabla \psi_1|^2 + D_2 |\nabla \psi_2|^2 + D_1 F_1 (\psi_1 - \psi_2)^2) + D_1 \int_0^{q_1} [\Psi_1(Q_1 + \tilde{q}) - \Psi_1(Q_1)] d\tilde{q} + D_2 \int_0^{q_2} [\Psi_2(Q_2 + \tilde{q}) - \Psi_2(Q_2)] d\tilde{q} \right\} dy. \quad (3.8)$$

The nature of the pseudoenergy construction guaran-

tees that  $\mathcal{A}$  is of quadratic order in disturbance amplitude, a fact that is made explicit in the representation (3.8). Moreover, it is clear that if

$$\frac{d\Psi_i}{dQ_i} \geq 0, \quad (3.9)$$

then  $\mathcal{A} > 0$  for all possible disturbances. Since  $\mathcal{A}$  is a dynamical invariant, this suggests that basic flows satisfying (3.9) are stable (Holm et al. 1985; Swaters 1986; McIntyre and Shepherd 1987): to prove Liapunov stability  $d\Psi_i/dQ_i$  must be not only positive but bounded away from zero and infinity. This is the well-known extension of Arnol'd's (1965, 1966) first stability theorem to quasigeostrophic baroclinic flow. In the case of zonally symmetric basic flows, the condition (3.9) takes the form

$$\frac{U_i(y)}{Q_i(y)} \leq 0, \quad (3.10)$$

where  $Q_i'(y) \equiv dQ_i/dy$ , which is recognized as Fjortoft's sufficient condition for stability [e.g., Pedlosky 1987, Eq. (7.3.31a)]—though it should be emphasized that the use of the exact (nonlinear) invariant (3.8) demonstrates that the stability is valid for *finite amplitude* disturbances, indeed for disturbances of any magnitude whatsoever.

The stability theorem described above will now be used to derive rigorous saturation bounds on instabilities, as in Shepherd (1988a), but here using the pseudoenergy rather than the pseudomomentum. To do this we reverse the logic of the foregoing derivation and consider the total flow rather than the basic flow as being given. In particular, suppose we are given an initial condition at  $t = 0$ , that is,  $\{\Phi_i^{(0)}, P_i^{(0)}, \Sigma_i^j\}$ , whose zonally averaged component is known (perhaps from a normal-mode analysis) to be unstable. Then, for any choice of a stable basic flow  $\{\Psi_i, Q_i, \Gamma_i^j\}$  satisfying (3.9), the associated pseudoenergy  $\mathcal{A}[P_i, \Sigma_i^j, Q_i, \Gamma_i^j]$  is conserved in time. Its initial value is simply

$$\mathcal{A}(0) = \mathcal{A}[P_i^{(0)}, \Sigma_i^j, Q_i, \Gamma_i^j], \quad (3.11)$$

which is a functional of the given initial condition and the choice of the stable basic flow. In order to derive a bound on the amplitude of the instability, we must bound this amplitude in terms of  $\mathcal{A}(t)$  and thereby  $\mathcal{A}(0)$ .

To do this, first note that since  $\Psi_i'(Q_i) \geq 0$ ,

$$\frac{1}{2} \Psi_i'(Q_i)|_{\min q_i^2} \leq \int_0^{q_i} [\Psi_i(Q_i + \tilde{q}) - \Psi_i(Q_i)] d\tilde{q}, \quad (3.12)$$

where the minimum will be considered to be taken over both layers. [In fact, (3.12) applies to each layer separately, with the minimum taken in that layer, but we shall be content with the global minimum.] It is then clear from (3.8) and (3.12) that

$$\int_0^1 \frac{1}{2} \{ D_1 |\overline{\nabla \psi_1}|^2 + D_2 |\overline{\nabla \psi_2}|^2 + D_1 F_1 \overline{(\psi_1 - \psi_2)^2} + \Psi'_i(Q_i)|_{\min}(D_1 \overline{q_1^2} + D_2 \overline{q_2^2}) \} dy \leq \mathcal{A}(t). \quad (3.13)$$

The inequality (3.13) is still not what we want, because the definition of the disturbance  $q_i$  depends on the choice of the stable basic flow. However, in order to bound the instability we seek a constraint on the nonzonal (or eddy) component of the total flow  $\{\Phi'_i, P'_i\}$ , and evidently

$$\Phi'_i = \psi'_i, \quad P'_i = q'_i \quad (3.14)$$

because the stable basic flow is zonally symmetric. The left-hand side of (3.13) is quadratic and is thus separable into its zonally averaged and nonzonal components, both of which are necessarily positive; hence, using (3.14) we may write

$$\int_0^1 \frac{1}{2} \{ D_1 |\overline{\nabla \Phi'_1}|^2 + D_2 |\overline{\nabla \Phi'_2}|^2 + D_1 F_1 \overline{(\Phi'_1 - \Phi'_2)^2} + \Psi'_i(Q_i)|_{\min}(D_1 \overline{P_1'^2} + D_2 \overline{P_2'^2}) \} dy \leq \mathcal{A}(t). \quad (3.15)$$

Taken together with (3.7), the inequality (3.15) provides a rigorous upper bound on a weighted combination of the energy and potential enstrophy of the nonzonal component of the flow, the upper bound consisting of the initial pseudoenergy (3.11). This may be turned into a bound on the eddy energy alone by using

$$\int_0^1 \frac{1}{2} \{ D_1 |\overline{\nabla \Phi'_1}|^2 + D_2 |\overline{\nabla \Phi'_2}|^2 + D_1 F_1 \overline{(\Phi'_1 - \Phi'_2)^2} \} \times dy \leq \pi^{-2} \int_0^1 \frac{1}{2} \{ D_1 \overline{P_1'^2} + D_2 \overline{P_2'^2} \} dy \quad (3.16)$$

[cf. Shepherd 1988a, Eq. (8.2)]. Combining (3.16) with (3.15) and (3.7) then yields

$$\int_0^1 \frac{1}{2} \{ D_1 |\overline{\nabla \Phi'_1}|^2 + D_2 |\overline{\nabla \Phi'_2}|^2 + D_1 F_1 \overline{(\Phi'_1 - \Phi'_2)^2} \} dy \leq \frac{\mathcal{A}(0)}{(1 + \pi^2 \Psi'_i(Q_i)|_{\min})}. \quad (3.17)$$

This is the major result of this section. The right-hand side of (3.17) is a functional of the initial flow and the stable basic flow; for any basic flow satisfying (3.9) it provides a rigorous upper bound on the eddy energy, thereby providing a bound on the nonlinear saturation of the instability. For a given initial flow, then, one may evaluate the right-hand side of (3.17) for a variety of possible choices of stable basic flow and choose the smallest such value as the optimal bound.

#### 4. Application to the Phillips model

The theory of the previous section is first applied to the specific case of the Phillips (1954) model of baro-

clinic instability, for which the initial zonal flow has no meridional shear. The initial condition is, thus, taken to consist of a zonal flow of the form

$$-\frac{d\bar{\Phi}_1^{(0)}}{dy} = \frac{\beta}{F_2} (1 + \epsilon) + u_0, \quad -\frac{d\bar{\Phi}_2^{(0)}}{dy} = u_0, \quad (4.1)$$

plus a nonzonal (eddy) perturbation  $\{\Phi_i^{(0)}, P_i^{(0)}\}$ . It was shown by Phillips (1954) that the flow (4.1) is unstable for  $\epsilon > 0$  (provided the channel is not too narrow); we therefore refer to  $\epsilon$  as the (relative) supercriticality, which we regard as given. In the absence of friction the problem is Galilean invariant, so the constant  $u_0$  is arbitrary; we keep it free for the time being. (We could equally well set  $u_0 = 0$  and use a combined pseudoenergy–pseudomomentum invariant to obtain the nonlinear saturation bounds; the two approaches are, for this problem, entirely equivalent.)

Now introduce a subcritical (stable) basic flow,

$$U_1 = \frac{\beta}{F_2} (1 - \delta) + u_0, \quad U_2 = u_0, \quad (4.2)$$

where  $\delta$  is a free parameter, with corresponding potential vorticity gradients

$$\frac{dQ_1}{dy} = \beta + \beta \frac{F_1}{F_2} (1 - \delta), \quad \frac{dQ_2}{dy} = \beta - \beta(1 - \delta). \quad (4.3)$$

For this basic flow, we have

$$\Psi'_1(Q_1) = -\frac{U_1}{Q'_1(y)} = -\frac{\beta(1 - \delta) + u_0 F_2}{\beta[F_2 + F_1(1 - \delta)]},$$

$$\Psi'_2(Q_2) = -\frac{U_2}{Q'_2(y)} = -\frac{u_0}{\beta\delta}, \quad (4.4)$$

which in both cases are constants (independent of  $y$ ). It is convenient at this stage to change variables from  $u_0$  to

$$\xi = -\frac{\beta(1 - \delta) + u_0 F_2}{\beta[F_2 + F_1(1 - \delta)]}, \quad (4.5)$$

in terms of which (4.4) becomes

$$\Psi'_1(Q_1) = \xi,$$

$$\Psi'_2(Q_2) = \frac{(1 - \delta)}{F_2 \delta} + \left(1 + \frac{F_1}{F_2} (1 - \delta)\right) \frac{\xi}{\delta}. \quad (4.6)$$

For this basic flow to be stable by the criterion (3.9), we evidently require the free parameters  $\delta$  and  $\xi$  to satisfy

$$\xi \geq 0, \quad \frac{(1 - \delta)}{F_2 \delta} + \left(1 + \frac{F_1}{F_2} (1 - \delta)\right) \frac{\xi}{\delta} \geq 0. \quad (4.7)$$

Now we seek to determine the right-hand side of (3.17) for the initial and basic flows described above, and minimize it over all  $\delta, \xi$  being consistent with (4.7).

Since  $\Psi'_i(Q_i)$  is a constant in each layer, the integrals over  $\tilde{q}$  in (3.8) evaluate exactly to

$$\int_0^{q_i} [\Psi_i(Q_i + \tilde{q}) - \Psi_i(Q_i)] d\tilde{q} = \frac{1}{2} \Psi'_i(Q_i) q_i^2. \quad (4.8)$$

At  $t = 0$ , the disturbance  $(\psi_i, q_i)$  is given by

$$\psi_1 = \Phi_1^{(0)} - \Psi_1 = -\frac{\beta}{F_2} (\epsilon + \delta) \left( y - \frac{1}{2} \right) + \Phi_1^{(0)}, \quad (4.9a)$$

$$\psi_2 = \Phi_2^{(0)} - \Psi_2 = \Phi_2^{(0)}, \quad (4.9b)$$

$$q_1 = P_1^{(0)} - Q_1 = \beta \frac{F_1}{F_2} (\epsilon + \delta) \left( y - \frac{1}{2} \right) + P_1^{(0)}, \quad (4.9c)$$

$$q_2 = P_2^{(0)} - Q_2 = -\beta(\epsilon + \delta) \left( y - \frac{1}{2} \right) + P_2^{(0)}, \quad (4.9d)$$

after using (4.1)–(4.3). [The integration constant in  $y$  is chosen, as described in Shepherd (1988a, §4), in order to obtain the tightest bounds.] We can now substitute (4.6), (4.8), and (4.9) into the expression (3.8) for  $\mathcal{A}$ , separating zonal mean from eddy contributions, to determine

$$\begin{aligned} \mathcal{A}(0) = & \int_0^1 \frac{1}{2} \left\{ D_1 \frac{\beta^2}{F_2^2} (\epsilon + \delta)^2 + D_1 F_1 \frac{\beta^2}{F_2^2} (\epsilon + \delta)^2 \right. \\ & \times \left( y - \frac{1}{2} \right)^2 + D_1 \xi \beta^2 \frac{F_1^2}{F_2^2} (\epsilon + \delta)^2 \left( y - \frac{1}{2} \right)^2 \\ & + D_2 \left[ \frac{(1 - \delta)}{F_2 \delta} + \left( 1 + \frac{F_1}{F_2} (1 - \delta) \right) \frac{\xi}{\delta} \right] \\ & \times \beta^2 (\epsilon + \delta)^2 \left( y - \frac{1}{2} \right)^2 \Big\} dy + E'_0 + \xi Z'_{0,1} \\ & + \left[ \frac{(1 - \delta)}{F_2 \delta} + \left( 1 + \frac{F_1}{F_2} (1 - \delta) \right) \frac{\xi}{\delta} \right] Z'_{0,2}, \end{aligned} \quad (4.10)$$

where

$$\begin{aligned} E'_0 = & \int_0^1 \frac{1}{2} \{ D_1 |\nabla \Phi_1^{(0)}|^2 + D_2 |\nabla \Phi_2^{(0)}|^2 \\ & + D_1 F_1 \overline{(\Phi_1^{(0)} - \Phi_2^{(0)})^2} \} dy \end{aligned} \quad (4.11)$$

is the initial eddy energy, and

$$Z'_{0,i} = \int_0^1 \frac{1}{2} D_i \overline{(P_i^{(0)})^2} dy \quad (4.12)$$

are the initial eddy potential enstrophies in each layer. After performing the integration in  $y$ , the expression (4.10) reduces to

$$\begin{aligned} \mathcal{A}(0) = & \frac{\beta^2 (\epsilon + \delta)^2}{24 F_2^2} \left\{ 12 D_1 + D_1 F_1 + D_1 F_1^2 \xi \right. \\ & + D_2 F_2^2 \left[ \frac{(1 - \delta)}{F_2 \delta} + \left( 1 + \frac{F_1}{F_2} (1 - \delta) \right) \frac{\xi}{\delta} \right] \Big\} + E'_0 \\ & + \xi Z'_{0,1} + \left[ \frac{(1 - \delta)}{F_2 \delta} + \left( 1 + \frac{F_1}{F_2} (1 - \delta) \right) \frac{\xi}{\delta} \right] Z'_{0,2}. \end{aligned} \quad (4.13)$$

Finally, it can be seen from (4.6) that, since  $\xi \geq 0$ , we have  $\Psi'_1(Q_1) \leq \Psi'_2(Q_2)$ , unless  $\delta > 1$ . No useful bound is possible for  $\delta > 1$ ; so we simply take

$$\Psi'_i(Q_i)|_{\min} = \Psi'_i(Q_i) = \xi. \quad (4.14)$$

The expressions (4.13) and (4.14) can then be substituted into the right-hand side of (3.17) to determine an upper bound on the eddy energy, which is a function of  $\delta$  and  $\xi$ .

The goal is now to minimize this bound over all  $\delta$  and  $\xi$ , consistent with (4.7). It turns out that the  $\xi$  dependence is easy to deal with: unless  $\delta = O(1)$  in which case—as we shall see—no useful constraint is possible, it is straightforward to show that the right-hand side of (3.17) is an increasing function of  $\xi$ . Hence we take  $\xi = 0$ , which is the minimum consistent with (4.7). This choice corresponds to  $U_1 = 0$ . The range of  $\delta$  consistent with (4.7) is then

$$0 < \delta \leq 1. \quad (4.15)$$

With  $\xi = 0$ , the right-hand side of (3.17) is just  $\mathcal{A}(0)$ , which simplifies to

$$\frac{\beta^2 (\epsilon + \delta)^2}{24 F_2^2} \left( 12 D_1 + \frac{D_1 F_1}{\delta} \right) + E'_0 + \frac{(1 - \delta)}{F_2 \delta} Z'_{0,2}, \quad (4.16)$$

after using (2.3). Seeking the local minimum of (4.16), setting  $\partial \mathcal{A}(0) / \partial \delta = 0$  leads to the cubic equation

$$\begin{aligned} 24 D_1 \delta^3 + D_1 (24 \epsilon + F_1) \delta^2 - D_1 F_1 \epsilon^2 \\ - \frac{24 F_2}{\beta^2} Z'_{0,2} = 0. \end{aligned} \quad (4.17)$$

One can then substitute the root of (4.17) that gives the minimum of (4.16).

An important special case occurs when the initial eddy amplitude is infinitesimal; this is the limit  $E'_0 \rightarrow 0$ ,  $Z'_{0,i} \rightarrow 0$ . In this limit the cubic (4.17) factors to

$$(\epsilon + \delta)(24 D_1 \delta^2 + D_1 F_1 \delta - D_1 F_1 \epsilon) = 0, \quad (4.18)$$

with roots

$$\delta = -\epsilon, \quad \delta = \frac{F_1}{48} \left[ -1 \pm \left( 1 + \frac{96 \epsilon}{F_1} \right)^{1/2} \right]. \quad (4.19)$$

It can be seen that the root in (4.19) corresponding to the local minimum of (4.16) is the largest one, which

is the one corresponding to the positive square root in the second choice. Ideally one would substitute this root into (4.16). However, it is clear that for  $\epsilon \ll F_1/96$  this root is well approximated by  $\delta \approx \epsilon$ , and since we are only interested in rough bounds this will suffice in the present context. [Recall that the bound is valid for any choice of  $\delta$  satisfying (4.15), not just the optimal choice.] So taking  $\delta = \epsilon$  across the full range, we obtain

$$\frac{1}{6} \frac{D_1 F_1}{F_2^2} \beta^2 \left( 1 + \frac{12}{F_1} \epsilon \right) \epsilon, \quad (4.20)$$

which represents a rigorous upper bound on the eddy energy for all  $\epsilon \leq 1$ . Note in particular that the bound goes to zero like  $\epsilon$  as the supercriticality  $\epsilon$  goes to zero.

The above bound can only be considered nontrivial if it is less than the total amount of energy in the system, which is clearly also a rigorous upper bound on the eddy energy. For the limit  $E'_0 \rightarrow 0$ ,  $Z'_{0,t} \rightarrow 0$  the total energy at  $t = 0$  is given by

$$\begin{aligned} \mathcal{E} &= \mathcal{E}[\bar{F}_i^{(0)}, \Sigma^j_1] = \int_0^1 \frac{1}{2} \left\{ D_1 \left[ \frac{\beta}{F_2} (1 + \epsilon) + u_0 \right]^2 \right. \\ &\quad \left. + D_2 u_0^2 + D_1 F_1 \frac{\beta^2}{F_2^2} (1 + \epsilon)^2 \left( y - \frac{1}{2} \right)^2 \right\} dy \\ &= \frac{1}{2} \left( (D_1 + D_2) u_0^2 + 2 D_1 \frac{\beta}{F_2} u_0 (1 + \epsilon) \right. \\ &\quad \left. + D_1 \frac{\beta^2}{F_2^2} (1 + \epsilon)^2 + \frac{1}{12} D_1 F_1 \frac{\beta^2}{F_2^2} (1 + \epsilon)^2 \right). \end{aligned}$$

However, this is not a Galilean-invariant quantity, being dependent on  $u_0$ , so we ought to choose  $u_0$  to give the lowest value of total energy. This choice is evidently  $u_0 = -\beta D_1 (1 + \epsilon) / F_2 (D_1 + D_2)$ , for which

$$\mathcal{E} = \frac{1}{2} \frac{D_1 \beta^2}{F_2^2} (1 + \epsilon)^2 \left( \frac{D_2}{D_1 + D_2} + \frac{F_1}{12} \right). \quad (4.21)$$

This total energy (4.21) is to be compared with the upper bound (4.20) on the eddy energy. Certainly for sufficiently small  $\epsilon$ , (4.20) will be much smaller; while for  $\epsilon = O(1)$ , the two expressions will be of comparable magnitude.

It is also of interest to compare (4.20) with the bound on the eddy energy that is derivable from Shepherd's (1988a) bound on the eddy potential enstrophy. The latter was obtained in the special case  $F_1 = F_2 = F$ ,  $D_1 = D_2 = 1$ , and is [cf. Shepherd 1988a, Eqs. (8.2) and (5.5)]

$$\frac{1}{\pi^2} \frac{\beta^2 \epsilon}{3}, \quad (4.22)$$

valid for  $\epsilon < 1$ . The ratio of the two in this case is

$$\frac{(4.20)}{(4.22)} = \frac{\pi^2}{2F} \left( 1 + \frac{12}{F} \epsilon \right) \rightarrow \frac{\pi^2}{2F}, \quad \text{as } \epsilon \rightarrow 0. \quad (4.23)$$

For instability to be possible at all,  $\pi^2 < 2F$ ; so the new bound based on pseudoenergy conservation is evidently superior to the old one. In the "wide jet" limit  $F \rightarrow \infty$  (where the deformation radius is much less than the channel width), the new bound becomes increasingly more powerful compared with (4.22).

## 5. Comparison with weakly nonlinear theory

When the supercriticality  $\epsilon$  is not too large, an explicit nonlinear theory for the time evolution of a baroclinic wave in the Phillips model can be constructed using a perturbation expansion based on the supercriticality as the small parameter. It is clearly of interest to compare the saturation amplitudes predicted by this weakly nonlinear theory with the rigorous bounds derived in the previous section. Because the nonlinear theory takes full account of the wave-mean interaction, it should be compatible with the rigorous bounds, so long as the perturbation expansion on which the theory is based remains valid.

The original theory for the Phillips model was worked out by Pedlosky (1970) based on single-wave equilibration. It turns out that the single-wave analysis breaks down at the point of minimum critical shear; the correct theory at that point was derived by Pedlosky (1982b), and later solved in an elegant analytical fashion by Warn and Gauthier (1989). As one "de-tunes" the system away from the resonance that occurs at minimum critical shear, either by increasing the supercriticality or by changing the zonal wavenumber, the single-wave theory becomes relevant (Pedlosky 1982a; Gauthier 1990). In Part I of this study (Shepherd 1988a, section 7), rigorous saturation bounds on the eddy potential enstrophy were seen to compare favorably with the amplitudes predicted by Pedlosky's 1970 theory. Here we compare the new energy bounds derived in the previous section with the weakly nonlinear theories of Pedlosky (1970) and Warn and Gauthier (1989).

The weakly nonlinear theories treat the case of equal layer depths, whence we take

$$D_1 = D_2 = 1, \quad F_1 = F_2 = F. \quad (5.1)$$

Now, consider the evolution of the most unstable wave, with total wavenumber  $\kappa$  satisfying  $\kappa^2 = \sqrt{2}F$ . Following Pedlosky (1970), we write the marginal wave as

$$\begin{aligned} \Phi'_1 &= \Delta^{1/2} \Re \{ A e^{i\theta} \sin(\pi y) \}, \\ \Phi'_2 &= \Delta^{1/2} \Re \{ A \tilde{\gamma} e^{i\theta} \sin(\pi y) \}, \end{aligned} \quad (5.2)$$

where  $\Delta \equiv \beta\epsilon/F$  is the absolute supercriticality,  $\theta$  is the phase,  $\tilde{\gamma} = \sqrt{2} - 1$  for this case, and  $A$  is an amplitude that varies on the slow (nonlinear) time scale. Note that there is no phase shift between  $\Phi'_1$  and  $\Phi'_2$  to leading order. The corresponding wave energy is then



$$E' = \frac{1}{8} \Delta[\kappa^2(1 + \tilde{\gamma}^2) + F(1 - \tilde{\gamma})^2]|A|^2 = \frac{1}{4} \beta \epsilon |A|^2. \tag{5.3}$$

Pedlosky (1970) predicts, for inviscid evolution, a maximum amplitude of

$$|A|_{\max}^2 = \frac{c_{oi}^2}{N} + \frac{c_{oi}^2}{N} \left( 1 + \frac{2N}{c_{oi}^2} A_0^2 \right)^{1/2} + A_0^2, \tag{5.4}$$

where  $A_0 = |A(t=0)|$ ; the general expressions for  $c_{oi}$  and  $N$  are given by Pedlosky (1970), but for  $\kappa^2 = \sqrt{2}F$  they take the values

$$c_{oi}^2 = \frac{\beta}{2(\sqrt{2} + 1)^2 F}, \quad N = \frac{(\sqrt{2} - 1)\pi^2}{4(\sqrt{2} + 1)}, \tag{5.5}$$

in which case

$$|A|_{\max}^2 = \frac{2\beta}{\pi^2 F} \left[ 1 + \frac{\pi^2 F A_0^2}{2\beta} + \left( 1 + \frac{\pi^2 F A_0^2}{\beta} \right)^{1/2} \right]. \tag{5.6}$$

To obtain the maximum wave energy we simply substitute (5.6) into (5.3).

Consider first the case where the initial wave amplitude is small, that is,

$$\mu \equiv \frac{\pi^2 F A_0^2}{2\beta} \ll 1. \tag{5.7}$$

Then, substituting (5.6) into (5.3) yields

$$E'_{\max} = \frac{\beta^2 \epsilon}{\pi^2 F} [1 + \mu + O(\mu^2)]. \tag{5.8}$$

We wish to compare (5.8) with the rigorous bound (4.16). From (5.3), it is evident that

$$E'_0 = \frac{1}{4} \beta \epsilon A_0^2, \tag{5.9}$$

while  $P'_2 = 0$  to leading order (see Pedlosky 1970), so we may set  $Z'_{0,2} = 0$ . This suggests that the smallest value of (4.16) is well approximated by taking  $\delta = \epsilon$ ; note that this would not be true for finite  $Z'_{0,2}$ . Taking  $\delta = \epsilon$ , then, and substituting (5.9), (4.16) implies the rigorous upper bound

$$E' \leq \frac{1}{6} \frac{\beta^2 \epsilon^2}{F^2} \left( 12 + \frac{F}{\epsilon} \right) + \frac{1}{4} \beta \epsilon A_0^2 = \frac{\beta^2 \epsilon}{6F} \left[ 1 + \frac{12\epsilon}{F} + \frac{3\mu}{\pi^2} \right]. \tag{5.10}$$

The ratio of the weakly nonlinear prediction to the rigorous bound is then given by

$$\frac{(5.8)}{(5.10)} \approx \frac{6}{\pi^2} \frac{1 + \mu}{1 + 12F^{-1}\epsilon + 3\pi^{-2}\mu} \rightarrow \frac{6}{\pi^2} \text{ as } \mu \rightarrow 0, \epsilon \rightarrow 0. \tag{5.11}$$

Thus, in the limit  $\mu \ll 1$  (infinitesimal initial wave amplitude) and  $\epsilon \ll 1$  (weak supercriticality), the rigorous upper bound on the eddy energy is  $\pi^2/6 \approx 1.65$  times the maximum eddy energy predicted by Pedlosky's (1970) single-wave theory. This ratio is independent of  $F$ ,  $\epsilon$ , or  $\beta$ , indicating that the rigorous bound has captured the essential scaling of the saturation amplitude of the instability. As the initial wave amplitude increases, the ratio (5.11) increases towards unity.

In the case  $\mu \gg 1$ , the maximum wave energy predicted by Pedlosky (1970) is, from (5.3) and (5.6),

$$E'_{\max} \approx \frac{1}{4} \beta \epsilon A_0^2 = E'_0,$$

while the rigorous bound (4.16) is also approximately given by  $E'_0$ ; thus both theories give the same value. The fact that  $E'$  is bounded by  $E'_0$  is not trivial in the sense that the total amount of energy contained in the initial flow, which is given by  $E'_0$  plus (4.21), may be much larger than  $E'_0$  for small  $E'_0$ .

We now compare the bounds with the results of Warn and Gauthier (1989). They use the form (5.2) with  $\Delta^{1/2}$  replaced by  $\epsilon$  (not to be confused with  $\epsilon$  used above), for which the corresponding wave energy is then

$$E' = \frac{1}{4} F \epsilon^2 |A|^2. \tag{5.12}$$

In the inviscid case, Warn and Gauthier obtain an explicit solution for the wave amplitude as a function of time. When the initial wave amplitude is small (the case  $\mu \ll 1$  considered above), they show that the wave equilibrates with amplitude given by

$$|A|_{\text{eq}}^2 = \frac{1}{3} \frac{\beta \Delta}{F \epsilon^2} = \frac{1}{3} \frac{\beta^2 \epsilon}{F^2 \epsilon^2}, \tag{5.13}$$

whence

$$E'_{\text{eq}} = \frac{\beta^2 \epsilon}{12F} \quad (\epsilon \ll 1, \mu \ll 1). \tag{5.14}$$

This is one-half the rigorous bound (5.10) in the same limit, and has the same dependence on  $\beta$ ,  $\epsilon$ , and  $F$ . Of course, (5.14) is the *equilibrated* wave energy, not its maximum value. Although Warn and Gauthier (1989) do not provide an analytical expression for  $|A|_{\max}$ , based on their Fig. 3 one can estimate

$$\frac{|A|_{\max}^2}{|A|_{\text{eq}}^2} \approx \frac{1.3}{\pi^2/12} \approx \frac{3}{2},$$

whence

$$E'_{\max} \approx \frac{\beta^2 \epsilon}{8F}. \tag{5.15}$$

[This estimate is also consistent with Fig. 2 of Gauthier (1990).] The maximum wave energy (5.15) is larger than the single-wave prediction (5.8) by a factor of

$\pi^2/8 \approx 1.3$ , which is consistent with the early numerical findings of Boville (1981), and it comes within a factor of  $6/8 = 0.75$  of the rigorous upper bound (5.10).

Although Warn and Gauthier (1989) treat the case of finite  $\mu$ , they did not present a simple expression for the maximum amplitude (or numerical results), which could be compared with (5.10) in this case.

In summary, then, the rigorous upper bound on the wave energy derived in section 4 captures the essential scaling (in terms of  $\beta$ ,  $\epsilon$ , and  $F$ ) of the maximum wave energy predicted by the weakly nonlinear theories of Pedlosky (1970) and Warn and Gauthier (1989) and is, moreover, in remarkable quantitative agreement with those predictions.

## 6. Comparison with baroclinic adjustment estimates

A popular approach to estimating the amplitude of an unstable wave is to assume that it will grow until it renders the zonal flow neutral to (normal mode) instability: in the present context this is usually referred to as the *baroclinic adjustment hypothesis* (Stone 1978; Lindzen and Farrell 1980). Of course, there are many difficulties with this sort of “quasilinear” argument, not the least being that it is flatly contradicted by weakly nonlinear theory: according to Pedlosky (1970), the zonal flow becomes *subcritical* (not just neutral) by an amount equal to the initial supercriticality before the wave stops growing. Numerical evidence from fully developed nonlinear simulations is also not supportive of the baroclinic adjustment hypothesis (e.g., Salmon 1980; Vallis 1988).

Nevertheless, the concept of baroclinic adjustment provides a heuristic and easily calculated estimate of wave amplitude, and it is instructive to work out its implications in the present context. The hypothesis is exceedingly simple to implement: the zonal flow is presumed to adjust from its initial unstable configuration (4.1) to a neutral configuration. One has merely to be careful about the choice of the reference frame, as this will affect the estimate of the energy released in the adjustment process. A little thought suggests that the correct estimate is the minimum one, which is obtained on supposing opposite flows in the two layers: thus, the zonal flow is presumed to adjust from an initial configuration

$$-\frac{d\bar{\Phi}_1^{(0)}}{dy} = \frac{\beta}{2F}(1 + \epsilon), \quad -\frac{d\bar{\Phi}_2^{(0)}}{dy} = -\frac{\beta}{2F}(1 + \epsilon), \quad (6.1)$$

to a final (neutral) configuration

$$-\frac{d\bar{\Phi}_1^{(f)}}{dy} = \frac{\beta}{2F}, \quad -\frac{d\bar{\Phi}_2^{(f)}}{dy} = -\frac{\beta}{2F}. \quad (6.2)$$

The energy released thereby can be shown to be given by (4.21) less the same expression with  $\epsilon = 0$  [under the condition (5.1)]; namely,

$$\begin{aligned} E'_{\text{est}} &= \frac{1}{2} \frac{\beta^2}{F^2} (2\epsilon + \epsilon^2) \left( \frac{1}{2} + \frac{F}{12} \right) \\ &= \frac{\beta^2 \epsilon}{12F} \left( 1 + \frac{6}{F} \right) \left( 1 + \frac{\epsilon}{2} \right). \end{aligned} \quad (6.3)$$

Since certainly  $F > 6$  for instability to be possible at all, it is clear that the estimate (6.3) is consistent with the rigorous upper bound (5.10). For  $F \gg 6$  and  $\epsilon \ll 1$ , (6.3) is equal to Warn and Gauthier's (1989) equilibrated energy (5.14). This is not at all surprising: in the regime where (5.14) is valid, the flow evolves to a state where the potential vorticity is homogenized in a coarse-grain sense (see section 6 of Warn and Gauthier 1989), which upon inversion yields the zonal flow (6.2).<sup>1</sup> This also explains why the baroclinic adjustment hypothesis generally provides an underestimate of the maximum amplitude, since during the equilibration process the flow initially “overmixes” the potential vorticity, producing virtually a reversal of the potential vorticity gradient in the lower layer (see Warn and Gauthier 1989, Fig. 2b).

## 7. Bounds for general profiles

In this section an upper bound on the eddy energy is worked out for a general class of unstable westerly jets. The point is not so much to produce quantitatively accurate bounds—experience (cf. Shepherd 1988a,b) suggests this is usually best done on a case-by-case basis—but rather just to demonstrate that the method is fully general, and to obtain the scaling dependencies of the bounds.

The initial condition is therefore supposed to consist of a zonal flow of the form

$$-\frac{d\bar{\Phi}_1^{(0)}}{dy} = \frac{\beta}{F_2}(1 + \epsilon)g(y) + u_0, \quad -\frac{d\bar{\Phi}_2^{(0)}}{dy} = u_0, \quad (7.1)$$

plus a nonzonal (eddy) perturbation  $\{\Phi_i^{(0)}, P_i^{(0)}\}$ . Apart from the constant  $u_0$ , and the allowance for different layer depths, this is the same class of unstable flows considered by Shepherd (1988a, section 4) in obtaining bounds on the eddy potential enstrophy. As in Shepherd, we take  $g(y)$  positive with a maximum of unity (to give a westerly jet), and presume that  $g''(y) < F_2/(1 + \epsilon)$  so that the initial upper-layer zonal flow is barotropically stable (in the sense that  $d\bar{P}_1^{(0)}/dy$  is single signed). The parameter  $u_0$  is arbitrary since the problem is Galilean invariant, so we keep it free for the time being. The potential vorticity gradients associated with (7.1) are given by

<sup>1</sup> It is relevant in this respect that although the equilibrated zonal flow generally differs from the neutral configuration (6.2), the difference vanishes in the limit  $F \rightarrow \infty$  (see Gauthier 1990, appendix A).

$$\begin{aligned} \frac{d\bar{P}_1^{(0)}}{dy} &= \beta + \frac{\beta F_1}{F_2} (1 + \epsilon)g(y) - \frac{\beta}{F_2} (1 + \epsilon)g''(y), \\ \frac{d\bar{P}_2^{(0)}}{dy} &= \beta - \beta(1 + \epsilon)g(y). \end{aligned} \quad (7.2)$$

By hypothesis,  $d\bar{P}_1^{(0)}/dy > 0$ ; and evidently  $d\bar{P}_2^{(0)}/dy < 0$  for at least some  $y$  when  $\epsilon > 0$ . Since the upper-layer velocity is always larger (more westerly) than the lower-layer velocity, it follows that the stability criterion (3.10) can never be satisfied for the initial flow (7.1), (7.2) when  $\epsilon > 0$ . We, therefore, restrict attention to such cases, referring to them as supercritical, even though they may not actually be subject to an instability in every case.

Now introduce a subcritical (stable) basic flow,

$$U_1 = \frac{\beta}{F_2} (1 - \delta)g(y) + u_0, \quad U_2 = u_0, \quad (7.3)$$

where  $\delta$  is a free parameter, with corresponding potential-vorticity gradients

$$\begin{aligned} \frac{dQ_1}{dy} &= \beta + \beta \frac{F_1}{F_2} (1 - \delta)g(y) - \frac{\beta}{F_2} (1 - \delta)g''(y), \\ \frac{dQ_2}{dy} &= \beta - \beta(1 - \delta)g(y). \end{aligned} \quad (7.4)$$

For this basic flow, we have

$$\begin{aligned} \Psi_1'(Q_1) &= -\frac{U_1}{Q_1'(y)} \\ &= \frac{-\beta(1 - \delta)g(y) - u_0 F_2}{\beta[F_2 + F_1(1 - \delta)g(y) - (1 - \delta)g''(y)]}, \end{aligned} \quad (7.5)$$

$$\Psi_2'(Q_2) = -\frac{U_2}{Q_2'(y)} = \frac{-u_0}{\beta[1 - (1 - \delta)g(y)]}. \quad (7.6)$$

Unlike the case with (4.4), these expressions are functions of  $y$ . It is evident that  $dQ_i/dy > 0$  for both layers (for  $\delta > 0$ ), so in order to satisfy the stability criterion (3.9) it is clearly sufficient if

$$\begin{aligned} \mathcal{A}(0) &\leq \int_0^1 \frac{1}{2} \left\{ D_1 \frac{\beta^2}{F_2^2} (\epsilon + \delta)^2 [g(y)]^2 + D_1 F_1 \frac{\beta^2}{F_2^2} (\epsilon + \delta)^2 [G(y)]^2 + \Psi_1'(Q_1) |_{\max} D_1 \frac{\beta^2}{F_2^2} (\epsilon + \delta)^2 \right. \\ &\quad \times [g'(y) - F_1 G(y)]^2 + \Psi_2'(Q_2) |_{\max} D_2 \frac{\beta^2}{F_2^2} (\epsilon + \delta)^2 [F_2 G(y)]^2 \left. \right\} dy \\ &\quad + E'_0 + \Psi_1'(Q_1) |_{\max} Z'_{0,1} + \Psi_2'(Q_2) |_{\max} Z'_{0,2}, \end{aligned} \quad (7.11)$$

where  $E'_0$  and  $Z'_{0,i}$  are given by (4.11) and (4.12). We now must determine (or at least bound) the factors

$$-u_0 \geq \frac{\beta}{F_2} (1 - \delta) \quad \text{and} \quad -u_0 \geq 0. \quad (7.7)$$

Thus, we may consider all  $u_0$  and  $\delta$  consistent with (7.7), in order to evaluate the right-hand side of (3.17) and obtain an upper bound on the eddy energy. Note that (7.7) suggests we keep  $\delta \leq 1$ .

It is clear that an explicit evaluation of  $\mathcal{A}(0)$ , as was done for the case of the Phillips model (section 4), is not possible for general  $g(y)$ . Instead, we must make do with the fact that

$$\int_0^{q_i} [\Psi_i(Q_i + \tilde{q}) - \Psi_i(Q_i)] d\tilde{q} \leq \frac{1}{2} \Psi_i'(Q_i) |_{\max} q_i^2, \quad (7.8)$$

which is the counterpart to (3.12). From (7.8), (3.8), and (3.11), it follows that

$$\begin{aligned} \mathcal{A}(0) &\leq \int_0^1 \frac{1}{2} \{ D_1 |\overline{\nabla \psi_1^{(0)}}|^2 + D_2 |\overline{\nabla \psi_2^{(0)}}|^2 \\ &\quad + D_1 F_1 \overline{(\psi_1^{(0)} - \psi_2^{(0)})^2} + \Psi_1'(Q_1) |_{\max} D_1 \overline{(q_1^{(0)})^2} \\ &\quad + \Psi_2'(Q_2) |_{\max} D_2 \overline{(q_2^{(0)})^2} \} dy, \end{aligned} \quad (7.9)$$

where the initial disturbance  $\{\psi_i^{(0)}, q_i^{(0)}\}$  is given by

$$\psi_1^{(0)} = \Phi_1^{(0)} - \Psi_1 = -\frac{\beta}{F_2} (\epsilon + \delta)G(y) + \Phi_1^{(0)}, \quad (7.10a)$$

$$\psi_2^{(0)} = \Phi_2^{(0)} - \Psi_2 = \Phi_2^{(0)}, \quad (7.10b)$$

$$\begin{aligned} q_1^{(0)} &= P_1^{(0)} - Q_1 = -\frac{\beta}{F_2} (\epsilon + \delta)g'(y) \\ &\quad + \frac{\beta F_1}{F_2} (\epsilon + \delta)G(y) + P_1^{(0)}, \end{aligned} \quad (7.10c)$$

$$q_2^{(0)} = P_2^{(0)} - Q_2 = -\beta(\epsilon + \delta)G(y) + P_2^{(0)}; \quad (7.10d)$$

here  $G(y) = \int g(y)dy + \lambda$ , with  $\lambda$  a (as yet arbitrary) constant of integration.

Substituting (7.10) into (7.9), and separating zonal-mean and eddy contributions, yields

$\Psi_i'(Q_i) |_{\max}$ . Using the hypothesized properties of  $g(y)$ , note from (7.5) that

$\Psi'_1(Q_1)$

$$\begin{aligned} &< \frac{-\beta(1-\delta)g(y) - u_0F_2}{\beta[F_2 + F_1(1-\delta)g(y) - F_2(1-\delta)(1+\epsilon)^{-1}]} \\ &< \frac{-\beta(1-\delta)g(y) - u_0F_2}{\beta[F_2\delta + F_1(1-\delta)g(y)]} < \frac{-u_0}{\beta\delta} \end{aligned} \quad (7.12)$$

[it does not seem possible to do better than this without knowledge of the minimum value of  $g(y)$ ]; while likewise, from (7.6)

$$\Psi'_2(Q_2) < \frac{-u_0}{\beta\delta}. \quad (7.13)$$

One may therefore take  $\Psi'_i(Q_i)|_{\max} < -u_0/\beta\delta$  [recall this is positive, by condition (7.7)] in (7.11), which gives

$$\begin{aligned} \mathcal{A}(0) \leq \frac{\beta^2(\epsilon + \delta)^2}{F_2^2} \left[ \Lambda_1 + \left( \frac{-u_0}{\beta\delta} \right) F_2 \Lambda_2 \right] \\ + E'_0 + \left( \frac{-u_0}{\beta\delta} \right) Z'_0, \end{aligned} \quad (7.14)$$

[after using (2.3)], where  $Z'_0 \equiv Z'_{0,1} + Z'_{0,2}$  and

$$\Lambda_1 = \Lambda_1(\lambda) \equiv \frac{D_1}{2} \int_0^1 [g^2 + F_1 G^2] dy,$$

$$\Lambda_2 = \Lambda_2(\lambda)$$

$$\equiv \frac{D_1 F_1}{2 F_2} \int_0^1 \left[ \frac{1}{F_1} g'^2 - 2Gg' + (F_1 + F_2)G^2 \right] dy.$$

It seems clear that the right-hand side of (7.14) will be minimized by making  $-u_0$  as small as possible consistent with (7.7), namely by taking

$$-u_0 = \frac{\beta}{F_2} (1 - \delta). \quad (7.15)$$

It then follows from (7.6) that  $\Psi'_i(Q_i)|_{\min} = 0$ , so the rigorous upper bound on the eddy energy—the right-hand side of (3.17)—is simply given by (7.14) above, with the substitution (7.15): that is,

$$\frac{\beta^2(\epsilon + \delta)^2}{F_2^2} \left[ \Lambda_1 + \frac{(1-\delta)}{\delta} \Lambda_2 \right] + E'_0 + \frac{(1-\delta)}{F_2\delta} Z'_0. \quad (7.16)$$

The expression (7.16) is the counterpart to (4.16) for the general class of unstable profiles (7.1), (7.2), and it evidently bears a fair degree of resemblance. Apart from the obvious dependence on  $\delta$ , (7.16) also depends on the free parameter  $\lambda$ . Without more information on  $g(y)$ , it is not possible to determine the optimal choice of  $\lambda$ . For fixed  $\lambda$ , however, the choice of  $\delta$  that yields the minimum value of (7.16) will be one of the roots of the cubic equation

$$\begin{aligned} 2(\epsilon + \delta)[\Lambda_1\delta^2 + \delta(1-\delta)\Lambda_2] \\ - (\epsilon + \delta)^2\Lambda_2 - \frac{F_2}{\beta^2} Z'_0 = 0. \end{aligned} \quad (7.17)$$

In the special case of infinitesimal initial eddy amplitudes,  $Z'_0 \rightarrow 0$ , and the cubic (7.17) factors to yield

$$(\epsilon + \delta)(2(\Lambda_1 - \Lambda_2)\delta^2 + \Lambda_2\delta - \Lambda_2\epsilon) = 0, \quad (7.18)$$

with roots

$$\begin{aligned} \delta = -\epsilon, \quad \delta = \frac{1}{4(\Lambda_1 - \Lambda_2)} \\ \times (-\Lambda_2 \pm \sqrt{\Lambda_2^2 + 8(\Lambda_1 - \Lambda_2)\Lambda_2\epsilon}). \end{aligned} \quad (7.19)$$

The root in (7.19) corresponding to the local minimum of (7.16) is the one corresponding to the positive square root in the second choice. For  $\epsilon \ll 1$ , this root can be seen to be well approximated by  $\delta \approx \epsilon$ . Since we are only interested in rough bounds anyway, we therefore take  $\delta = \epsilon$  across the full range of  $\epsilon$ , whereupon (7.16) takes the form

$$\frac{4\beta^2}{F_2^2} (\Lambda_2 + (\Lambda_1 - \Lambda_2)\epsilon)\epsilon \quad (7.20)$$

(valid in the limit  $E'_0 \rightarrow 0, Z'_0 \rightarrow 0$ ). The expression (7.20) represents a rigorous upper bound on the eddy energy for all  $\epsilon \leq 1$  (since we have the restriction  $\delta \leq 1$ ). Note in particular that the bound goes to zero like  $\epsilon$  as the supercriticality  $\epsilon$  goes to zero.

It is of interest to compare the rigorous bound (7.20) with the total amount of energy in the system. In the limit  $E'_0 \rightarrow 0, Z'_0 \rightarrow 0$  the total energy at  $t = 0$  is given by

$$\begin{aligned} \mathcal{E} = \mathcal{E}[\bar{P}_i^{(0)}, \Sigma_i^j] = \int_0^1 \frac{1}{2} \left\{ D_1 \left[ \frac{\beta}{F_2} (1 + \epsilon)g + u_0 \right]^2 + D_2 u_0^2 + D_1 F_1 \frac{\beta^2}{F_2^2} (1 + \epsilon)^2 G^2 \right\} dy \\ = \frac{1}{2} \left( (D_1 + D_2) u_0^2 + 2D_1 \frac{\beta}{F_2} u_0 (1 + \epsilon) \left[ \int_0^1 g dy \right] \right. \\ \left. + D_1 \frac{\beta^2}{F_2^2} (1 + \epsilon)^2 \left[ \int_0^1 g^2 dy \right] + D_1 F_1 \frac{\beta^2}{F_2^2} (1 + \epsilon)^2 \left[ \int_0^1 G^2 dy \right] \right). \end{aligned}$$

As in section 4, we choose  $u_0$  to give the lowest value of total energy. This choice is evidently

$$u_0 = - \frac{\beta D_1 (1 + \epsilon) \left[ \int_0^1 g dy \right]}{F_2 (D_1 + D_2)},$$

for which

$$\mathcal{E} = \frac{1}{2} \frac{D_1 \beta^2}{F_2^2} (1 + \epsilon)^2 \left( \left[ \int_0^1 g^2 dy \right] - \left( \frac{D_1}{D_1 + D_2} \right) \left[ \int_0^1 g dy \right]^2 + F_1 \left[ \int_0^1 G^2 dy \right] \right). \quad (7.21)$$

For sufficiently small  $\epsilon$ , the rigorous upper bound (7.20) is much smaller than the total energy (7.21) and is therefore providing a nontrivial constraint on the dynamics. For  $\epsilon = O(1)$ , however, the two expressions will be of comparable magnitude.

We may also compare (7.20) with the bound on the eddy energy that is derivable from Shepherd's (1988a) bound on the eddy potential enstrophy for the initial flow (7.1), (7.2) in the special case  $F_1 = F_2 = F$ ,  $D_1 = D_2 = 1$ . From Eqs. (8.2) and (4.17) of Shepherd (1988a), this bound is

$$\begin{aligned} & \frac{1}{\pi^2} 4\beta^2 \Lambda (2 + \gamma - (1 + \gamma)\epsilon) \epsilon \\ &= \frac{1}{\pi^2} \frac{4\beta^2 \Lambda_2 \epsilon}{F} (2 + \gamma - (1 + \gamma)\epsilon), \quad (7.22) \end{aligned}$$

valid for  $\epsilon \leq (5 + 4\gamma)^{-1}$ , where  $\gamma \equiv F^{-1} \max \{-g''(y)\}$ . [It should be pointed out that the footnote on p. 2017 of Shepherd (1988a) is incorrect (A. A. White, personal communication): no such assumption is required in order to obtain Eq. (4.5a).] Note that  $\Lambda_2$  in the present paper is identical (for the case of equal layer depths) to  $F$  times  $\Lambda$  in the earlier paper. The ratio of the two bounds in this case is

$$\begin{aligned} \frac{(7.20)}{(7.22)} &= \frac{\pi^2}{F} \frac{\Lambda_2 + (\Lambda_1 - \Lambda_2)\epsilon}{\Lambda_2 (2 + \gamma - (1 + \gamma)\epsilon)} \\ &\rightarrow \frac{\pi^2}{2F} \frac{1}{1 + (\gamma/2)} \quad \text{as } \epsilon \rightarrow 0. \quad (7.23) \end{aligned}$$

As with the Phillips model (section 4), the new bound is evidently generally superior to the old one, becoming increasingly so in the wide-jet limit  $F \rightarrow \infty$ . When  $F \rightarrow \infty$  or the horizontal curvature  $g''(y)$  is small,  $\gamma$  becomes negligible, and the ratios (7.23) and (4.23) are seen to be identical (in the limit  $\epsilon \rightarrow 0$ ).

### 8. Discussion

In this paper, rigorous upper bounds have been derived on the nonlinear saturation of baroclinic instability in the two-layer model. The bounds apply to the eddy energy, and are obtained by appealing to the conservation of disturbance pseudoenergy (McIntyre and Shepherd 1987) relative to some stable basic flow. This stable basic flow has no intrinsic physical significance, but is simply a mathematical device: by considering a family of basic flows, and minimizing the upper bound over this family, an optimal (least) bound is obtained.

The present work is an extension of Part I of this study (Shepherd 1988a), which employed pseudomomentum conservation to derive upper bounds on the eddy potential enstrophy for the same system. Since energy and potential enstrophy do not provide equivalent norms, it is of intrinsic interest to obtain bounds on both quantities. For a bounded domain (like the channel considered here), a potential enstrophy bound leads immediately to an energy bound, after using a Poincaré inequality [Shepherd 1988a, Eq. (8.2)], but it is shown here that the energy bounds derived in this manner are much weaker (less constraining) than those obtainable from a direct appeal to pseudoenergy conservation, especially so in the wide-jet limit  $F \rightarrow \infty$ .

Bounds on the energy have been worked out for the special case of the Phillips (1954) model of baroclinic instability (section 4). When the initial eddy amplitude is infinitesimal, and in the special case of equal layer depths, the bound takes the form

$$\begin{aligned} & \int_0^1 \frac{1}{2} \{ |\nabla \Phi'_1|^2 + |\nabla \Phi'_2|^2 \\ & + F(\Phi'_1 - \Phi'_2)^2 \} dy \leq \frac{\beta^2}{6F} \left( 1 + \frac{12}{F} \epsilon \right) \epsilon \quad (8.1) \end{aligned}$$

[a combination of (3.17) and (4.20) with (5.1)], where  $\epsilon = (U - U_{crit})/U_{crit}$  is the (relative) supercriticality. This bound captures the essential dynamical scalings (i.e., the dependence on  $\epsilon$ ,  $\beta$ , and  $F$ ) of the saturation amplitudes predicted by weakly nonlinear theory, as well as exhibiting remarkable quantitative agreement with those predictions (section 5), and is also consistent with heuristic baroclinic adjustment estimates (section 6). For  $\epsilon \ll 1$ , the bound (8.1) is significant in the sense that it is much smaller than the total amount of energy in the system (which provides an obvious bound on the eddy energy). In fact, (8.1) is smaller than the total energy [given by (4.21) in the case (5.1)] for all  $\epsilon$  satisfying

$$\left( 2 - \frac{12}{F} \right) \epsilon - \left( 1 - \frac{42}{F} \right) \epsilon^2 < \left( 1 + \frac{6}{F} \right),$$

hence, in practice for  $\epsilon \leq O(1)$ .

It is instructive to write the bounds (8.1) and (4.21)

in terms of the initial vertical shear,  $U \equiv \beta(1 + \epsilon)/F$ . This gives the combined bound

$$\int_0^1 \frac{1}{2} \{ |\nabla \Phi'_1|^2 + |\nabla \Phi'_2|^2 + F(\Phi'_1 - \Phi'_2)^2 \} dy \leq \begin{cases} \frac{\beta}{6F} \left[ 1 + \frac{12}{\beta F} (UF - \beta) \right] (UF - \beta) \\ \frac{U^2 F}{24} \left( 1 + \frac{6}{F} \right). \end{cases} \quad (8.2)$$

The upper of the two bounds in (8.2) corresponds to (8.1) and applies for  $1 < UF/\beta \leq 2$ , roughly; whereas the lower of the two bounds corresponds to (4.21) and represents the total amount of energy in the system: it applies for  $UF/\beta \geq 2$ , roughly. It is clear that in the limit  $\beta \rightarrow 0$ , the lower of the two bounds always applies, and the nonlinear stability constraints are therefore giving no useful information.

In terms of dimensional variables, the bound (8.1) takes the form

$$(\text{eddy energy}) \leq \frac{\beta_0^2 N^2 D^3 L^3}{6f_0^2} \left( 1 + \frac{12N^2 D^2}{f_0^2 L^2} \epsilon \right), \quad (8.3)$$

where  $\beta_0$  and  $f_0$  are the dimensional  $\beta$  and  $f$  parameters,  $D$  is the depth of each layer,  $L$  is the channel width, and  $N^2 = g'/D$  where  $g'$  is the reduced gravity. The interesting thing to note about (8.3) is that it scales like  $L^3$ . The bound can thus be expected to be an overestimate of actual eddy amplitudes in the wide-jet limit, where the eddy statistics are independent of  $L$  and the spatially integrated eddy energy should therefore scale like  $L$  (Haidvogel and Held 1980).

It has been pointed out to the author (P. H. Stone, personal communication) that the rigorous bounds on the eddy potential enstrophy derived previously for the Charney problem of baroclinic instability in a semi-infinite, continuously stratified fluid (Shepherd 1989) are consistent with conventional energy equipartition arguments: to wit, upon taking the zonal length scale of the eddies to be the deformation radius, and their meridional length scale to be the jet scale, one can show that the saturation bounds, if attained, allow the eddy kinetic energy to be of the same order of magnitude as the mean (zonal) available potential energy. However, a little thought quickly shows that this will not be the case with the two-layer model. As the supercriticality  $\epsilon$  approaches zero, the bound on the eddy energy likewise approaches zero; yet there remains plenty of available potential energy in the zonal flow.

The theory in this paper has been derived under the assumption of conservative (inviscid, unforced) flow. Perhaps surprisingly, it turns out that the results go through for a certain kind of forced-dissipative prob-

lem, namely, where the potential vorticity is relaxed back to the initial unstable state. The governing equation (2.1) is then replaced by

$$\frac{D_i P_i}{Dt} = -r(P_i - \bar{P}_i^{(0)}), \quad (8.4)$$

where  $r$  is the dissipation coefficient. This sort of system was considered by Pedlosky (1982b), for example. The pseudoenergy conservation law (3.7) is of course no longer valid, but although one cannot show that  $d\mathcal{A}/dt < 0$  in general, it is nevertheless possible to establish that

$$\mathcal{A}(t) \leq \mathcal{A}(0), \quad (8.5)$$

from which the principal inequality (3.17) follows directly. The proof of (8.5) for the system (8.4) is given for the case of two-dimensional (barotropic) flow in Shepherd (1988b, section 4.2), where  $P = \nabla^2 \Phi + f + \beta y$  rather than (2.2). It is a straightforward exercise to verify that it goes through in the present case.

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## APPENDIX

### Derivation of (3.8) from (3.5)

From the definition (2.6), it follows that

$$\begin{aligned} \mathcal{E}[P_i, \Sigma'_i] - \mathcal{E}[Q_i, \Gamma'_i] &= \int_0^1 \left\{ \frac{1}{2} D_1 |\nabla \psi_1|^2 + D_1 \nabla \Psi_1 \cdot \nabla \psi_1 + \frac{1}{2} D_2 |\nabla \psi_2|^2 \right. \\ &\quad \left. + D_2 \nabla \Psi_2 \cdot \nabla \psi_2 + \frac{1}{2} D_1 F_1 (\psi_1 - \psi_2)^2 \right. \\ &\quad \left. + D_1 F_1 (\Psi_1 - \Psi_2) (\psi_1 - \psi_2) \right\} dy \\ &= \int_0^1 \frac{1}{2} \{ D_1 |\nabla \psi_1|^2 + D_2 |\nabla \psi_2|^2 \\ &\quad + D_1 F_1 (\psi_1 - \psi_2)^2 \} dy + \left[ D_1 \Psi_1 \frac{\partial \bar{\psi}_1}{\partial y} \right]_{y=0}^{y=1} \\ &\quad + \left[ D_2 \Psi_2 \frac{\partial \bar{\psi}_2}{\partial y} \right]_{y=0}^{y=1} - \int_0^1 \{ D_1 \Psi_1 \nabla^2 \psi_1 + D_2 \Psi_2 \nabla^2 \psi_2 \\ &\quad + D_1 \Psi_1 F_1 (\psi_2 - \psi_1) + D_2 \Psi_2 F_2 (\psi_1 - \psi_2) \} dy, \end{aligned} \quad (A1)$$

where (2.3) has been used. From the definition (2.7) together with (3.6), it follows that

$$\begin{aligned}
& \mathcal{E}[P_i, \Sigma_i^j] - \mathcal{E}[Q_i, \Gamma_i^j] \\
&= \int_0^1 \left\{ D_1 \int_{Q_1}^{\overline{P_1}} \Psi_1(\tilde{q}) d\tilde{q} + D_2 \int_{Q_2}^{\overline{P_2}} \Psi_2(\tilde{q}) d\tilde{q} \right\} dy \\
&\quad - D_1 \Psi_1|_{y=0} \gamma_1^0 - D_1 \Psi_1|_{y=1} \gamma_1^1 \\
&\quad\quad\quad - D_2 \Psi_2|_{y=0} \gamma_2^0 - D_2 \Psi_2|_{y=1} \gamma_2^1 \\
&= \int_0^1 \left\{ D_1 \int_0^{Q_1} \Psi_1(Q_1 + \tilde{q}) d\tilde{q} \right. \\
&\quad\quad\quad \left. + D_2 \int_0^{Q_2} \Psi_2(Q_2 + \tilde{q}) d\tilde{q} \right\} dy \\
&\quad - \left[ D_1 \Psi_1 \frac{\partial \bar{\Psi}_1}{\partial y} \right]_{y=0}^{y=1} - \left[ D_2 \Psi_2 \frac{\partial \bar{\Psi}_2}{\partial y} \right]_{y=0}^{y=1}, \quad (\text{A2})
\end{aligned}$$

where (3.4b,c) have been used. We now use (3.4a) to note that the integrand in the second  $y$  integral of (A1) may be rewritten as

$$\begin{aligned}
& D_1 \Psi_1 q_1 + D_2 \Psi_2 q_2 \\
&= D_1 \int_0^{Q_1} \Psi_1(Q_1) d\tilde{q} + D_2 \int_0^{Q_2} \Psi_2(Q_2) d\tilde{q}. \quad (\text{A3})
\end{aligned}$$

Substituting (A3) in the aforementioned integral in (A1), and combining (A1) with (A2) as in (3.5), then yields (3.8).

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