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A general method for finding extremal states of Hamiltonian dynamical systems, with applications to perfect fluids

By THEODORE G. SHEPHERD
Department of Physics, University of Toronto, Toronto M5S 1A7, Canada

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In addition to the Hamiltonian functional itself, non-canonical Hamiltonian dynamical systems generally possess integral invariants known as 'Casimir functionals'. In the case of the Euler equations for a perfect fluid, the Casimir functionals correspond to the vortex topology, whose invariance derives from the particle-relabelling symmetry of the underlying Lagrangian equations of motion. In a recent paper, Vallis, Carnevale & Young (1989) have presented algorithms for finding steady states of the Euler equations that represent extrema of energy subject to given vortex topology, and are therefore stable. The purpose of this note is to point out a very general method for modifying *any* Hamiltonian dynamical system into an algorithm that is analogous to those of Vallis *et al.* in that it will systematically increase or decrease the energy of the system while preserving all of the Casimir invariants. By incorporating momentum into the extremization procedure, the algorithm is able to find steadily-translating as well as steady stable states. The method is applied to a variety of perfect-fluid systems, including Euler flow as well as compressible and incompressible stratified flow.

1. Introduction

In a recent paper, Vallis, Carnevale & Young (1989) have presented algorithms for finding nonlinearly stable steady solutions of various inviscid fluid systems, including the two- and three-dimensional Euler equations. These algorithms are modifications to the inviscid governing equations that systematically increase or decrease the energy of the flow while preserving the vortex topology. Whenever the energy has a non-zero lower bound or a finite upper bound under such 'isovortical' evolution, the modified equations will generally approach a non-trivial steady state which, by construction, will be an extremum of energy subject to the given vortex topology. Such a steady state is therefore a stable solution of the original inviscid equations (Kelvin 1887; Arnol'd 1965, 1966); it is this fact that makes the method of Vallis *et al.* of considerable potential importance.

In this note, a very general method is described for modifying *any* Hamiltonian dynamical system into an algorithm that is analogous to those of Vallis *et al.* in that it will systematically increase or decrease the energy of the system while preserving all of the 'Casimir invariants'. (In the systems studied by Vallis *et al.* the Casimir invariants are just the invariants characterizing the vortex topology.) By incorporating momentum into the extremization procedure, the algorithm is able to find steadily-translating as well as steady stable states. The method is presented in §2, and then applied in §3 to the following fluid-dynamical systems: (i) two-dimensional Euler flow; (ii) three-dimensional Euler flow; (iii) baroclinic quasi-geostrophic flow over topography; (iv) two-dimensional stratified Boussinesq flow; (v) rotating homogeneous shallow-water flow; and (vi) three-dimensional rotating, stratified, compressible flow of an ideal gas (also known as the meteorological primitive equations). The results are discussed in §4.

2. The method

Consider a general continuous Hamiltonian dynamical system, whose governing equations may be written in the symplectic form

$$u_t = \mathbf{J}\frac{\delta \mathscr{H}}{\delta u}. \tag{2.1}$$

Here the dependent variable u is a function of time t and of position $\boldsymbol{x}$ within some domain D (for a discrete system u would just be a function of t); u_t is the partial derivative of u with respect to t; $\mathscr{H}(u)$ is the Hamiltonian functional, usually just the total energy of the system; $\delta\mathscr{H}/\delta u$ is the functional or variational derivative of $\mathscr{H}$, defined by

$$\delta\mathscr{H} \equiv \mathscr{H}(u+\delta u) - \mathscr{H}(u) = \left(\frac{\delta\mathscr{H}}{\delta u}, \delta u\right) + O(\delta u^2), \tag{2.2}$$

for admissible but otherwise arbitrary variations δu, where $(\cdot,\cdot)$ is the relevant inner product for the function space $\{u\}$; and $\mathbf{J}$ is a skew-symmetric transformation from $\{u\}$ to $\{u\}$, satisfying

$$(u, \mathbf{J}v) = -(\mathbf{J}u, v), \tag{2.3}$$

as well as the Jacobi condition. For further mathematical discussion one may refer to Benjamin (1984), Olver (1986, chapter 7) or Salmon (1988*a*). Various specific examples are considered in §3; there u is generally a vector function, $\mathbf{J}$ is a matrix operator, and $(\cdot,\cdot)$ is just the spatial integral over the domain D of the dot product of the vectors.

The system (2.1) possesses at least three sorts of integral invariants. The first is simply the Hamiltonian functional itself, since

$$\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}t} = \left(\frac{\delta\mathscr{H}}{\delta u}, u_t\right) = \left(\frac{\delta\mathscr{H}}{\delta u}, \mathbf{J}\frac{\delta\mathscr{H}}{\delta u}\right) = 0 \tag{2.4}$$

by the skew-symmetry of $\mathbf{J}$. The second are the momentum or impulse invariants, which are related by Noether's theorem to the spatial (translational) symmetries of the Hamiltonian; for example, if $\mathscr{H}$ is invariant under translations in x then the associated momentum functional $\mathscr{M}$, defined (to within a Casimir) by

$$-u_x = \mathbf{J}\frac{\delta\mathscr{M}}{\delta u}, \tag{2.5}$$

is conserved by the dynamics (2.1) since

$$\frac{\mathrm{d}\mathscr{M}}{\mathrm{d}t} = \left(\frac{\delta\mathscr{M}}{\delta u}, u_t\right) = \left(\frac{\delta\mathscr{M}}{\delta u}, \mathbf{J}\frac{\delta\mathscr{H}}{\delta u}\right) = -\left(\mathbf{J}\frac{\delta\mathscr{M}}{\delta u}, \frac{\delta\mathscr{H}}{\delta u}\right) = \left(u_x, \frac{\delta\mathscr{H}}{\delta u}\right) = 0 \tag{2.6}$$

(see e.g. Morrison 1982 or Benjamin 1984). The third sort of invariant consists of the Casimir functionals $\mathscr{C}(u)$, associated with the kernel of the operator $\mathbf{J}$, which are the solutions of

$$\mathbf{J}\frac{\delta \mathscr{C}}{\delta u} = 0; \tag{2.7}$$

their invariance follows from

$$\frac{\mathrm{d}\mathscr{C}}{\mathrm{d}t} = \left(\frac{\delta \mathscr{C}}{\delta u}, u_t\right) = \left(\frac{\delta \mathscr{C}}{\delta u}, \mathbf{J}\frac{\delta \mathscr{H}}{\delta u}\right) = -\left(\mathbf{J}\frac{\delta \mathscr{C}}{\delta u}, \frac{\delta \mathscr{H}}{\delta u}\right) = 0. \tag{2.8}$$

For fluid systems the Casimir invariants include the 'topological invariants', examples being helicity as well as the families of invariants corresponding to materially-conserved quantities (e.g. entropy, potential vorticity); they also include the mass of the system, where appropriate. The existence of non-trivial solutions to (2.7) depends on the Hamiltonian system being non-canonical (see e.g. Littlejohn 1982), something which is generally the case for Eulerian representations of perfect fluids.

At this point it may be noted for completeness that the Hamiltonian system (2.1) may be alternatively represented in bracket notation as

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = [\mathscr{F}, \mathscr{H}], \tag{2.9}$$

where $\mathscr{F}$ is any functional of u whose functional derivative is well-defined, and $[\cdot, \cdot]$ is the (generally non-canonical) Poisson bracket defined by

$$[\mathscr{F}, \mathscr{G}] \equiv \left(\frac{\delta \mathscr{F}}{\delta u}, \mathbf{J}\frac{\delta \mathscr{G}}{\delta u}\right) \tag{2.10}$$

(see e.g. Morrison 1982; Littlejohn 1982). In this notation a Casimir functional is any functional $\mathscr{C}$ that satisfies

$$[\mathscr{C}, \mathscr{G}] = 0 \tag{2.11}$$

for all admissible functionals $\mathscr{G}$; that this condition is equivalent to (2.7) follows directly from (2.10) and (2.3).

Now, given the system (2.1), consider a new system defined by

$$u_t = \mathbf{J}\frac{\delta \mathscr{H}}{\delta u} + \mathbf{J}\alpha\mathbf{J}\frac{\delta \mathscr{H}}{\delta u}, \tag{2.12}$$

where α is a symmetric transformation with $(u, \alpha u)$ of definite sign for all u. In the examples of §3, α is taken to be a constant, single-signed diagonal matrix (whose entries generally have differing dimensions), but more general forms are clearly possible. (The iterated Laplacian forms discussed by Vallis *et al.* represent examples of this.) Actually the first term on the right-hand side of (2.12) may be dispensed with altogether without affecting the essential properties of (2.12), but the full form is more general and so is retained here.

The Casimir invariants of (2.1) are also invariants of (2.12), for if $\mathscr{C}$ satisfies (2.7) then under (2.12) one has

$$\frac{\mathrm{d}\mathscr{C}}{\mathrm{d}t} = \left(\frac{\delta \mathscr{C}}{\delta u}, u_t\right) = \left(\frac{\delta \mathscr{C}}{\delta u}, \mathbf{J}\frac{\delta \mathscr{H}}{\delta u} + \mathbf{J}\alpha\mathbf{J}\frac{\delta \mathscr{H}}{\delta u}\right) = -\left(\mathbf{J}\frac{\delta \mathscr{C}}{\delta u}, \frac{\delta \mathscr{H}}{\delta u} + \alpha\mathbf{J}\frac{\delta \mathscr{H}}{\delta u}\right) = 0. \tag{2.13}$$

On the other hand, the Hamiltonian functional $\mathscr{H}$ is no longer invariant, but will monotonically increase or decrease according to the sign of α:

$$\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}t} = \left(\frac{\delta\mathscr{H}}{\delta u}, u_t\right) = \left(\frac{\delta\mathscr{H}}{\delta u}, \mathbf{J}\frac{\delta\mathscr{H}}{\delta u} + \mathbf{J}\alpha\mathbf{J}\frac{\delta\mathscr{H}}{\delta u}\right) = -\left(\mathbf{J}\frac{\delta\mathscr{H}}{\delta u}, \alpha\mathbf{J}\frac{\delta\mathscr{H}}{\delta u}\right). \tag{2.14}$$

The right-hand side of (2.14) is of definite sign, and is non-zero unless

$$\mathbf{J}\frac{\delta\mathscr{H}}{\delta u} = 0; \tag{2.15}$$

hence the steady solutions of (2.12) are also steady solutions of the original system (2.1). So to recapitulate: the modified dynamical system (2.12) will monotonically increase or decrease the energy of the system while preserving all the Casimir invariants. If a finite upper bound or non-zero lower bound on the energy exists (subject to the constraints imposed by the Casimir invariants), then the process will generally converge to a state satisfying (2.15), which will be a steady solution of (2.1). (For topological reasons it may only approach it without attaining it – for example, in the case of two-dimensional Euler flow the connectedness of the vorticity distribution must be maintained – but this distinction will be moot in practice; see Carnevale & Vallis 1990.) Any steady solution of (2.1) must be at least a conditional extremum (critical point) of $\mathscr{H}$ under Casimir-preserving variations (cf. Arnol'd 1965). However, the steady states reached by the system (2.12) will, by construction, generally be true extrema, and will thus represent stable steady solutions of the Hamiltonian system (2.1) (but see the caveat at the end of §4).

If the system (2.1) is invariant under translations in x, as nearly all the systems to be considered in §3 are, then one may expect the existence of steadily-translating solutions of the form $u = U(x-ct)$ – the other spatial variables being implicit – where c is the translation velocity in the x-direction. Such solutions satisfy $U_t + cU_x = 0$, whence

$$\mathbf{J}\left(\frac{\delta\mathscr{H}}{\delta u} - c\frac{\delta\mathscr{M}}{\delta u}\right)\bigg|_{u=U} = 0. \tag{2.16}$$

To generalize the algorithm (2.12) to find steadily-translating states, it is enough to replace (2.12) by

$$u_t = \mathbf{J}\frac{\delta}{\delta u}(\mathscr{H} - c\mathscr{M}) + \mathbf{J}\alpha\mathbf{J}\frac{\delta}{\delta u}(\mathscr{H} - c\mathscr{M}), \tag{2.17}$$

with c specified. Under the evolution (2.17), one then has

$$\frac{\mathrm{d}\mathscr{C}}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{H} - c\mathscr{M}) = -\left(\mathbf{J}\frac{\delta}{\delta u}(\mathscr{H} - c\mathscr{M}), \alpha\mathbf{J}\frac{\delta}{\delta u}(\mathscr{H} - c\mathscr{M})\right), \tag{2.18a, b}$$

which are analogous to (2.13), (2.14). So the functional $\mathscr{H} - c\mathscr{M}$ will monotonically increase or decrease under (2.17); the process will only stop if the system converges to a steadily translating state, satisfying (2.16). As with steady solutions, steadily translating solutions of (2.1) are necessarily conditional extrema of $\mathscr{H} - c\mathscr{M}$ for Casimir-preserving variations (cf. Benjamin 1984); but any such states reached by the modified dynamics (2.17) will, by construction, generally be true extrema, and will thus be stable. By treating c as a Lagrange multiplier, such states may also be regarded as extrema of $\mathscr{H}$ for fixed $\mathscr{M}$ and $\mathscr{C}$.

It is obvious that in systems with rotational symmetry, the same method may be employed to find stable rotating states, with the angular momentum taking the place of $\mathscr{M}$.

3. Examples

In the examples presented below, the more general form (2.17) of the modified dynamics is used; if one wishes to find stable steady states, then it suffices to set $c = 0$ in all the formulae.

3.1. *Two-dimensional Euler flow*

The governing equation is commonly written in the form

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} \equiv \frac{\partial \omega}{\partial t} + \partial(\psi, \omega) = 0, \tag{3.1}$$

where $\omega(x, y, t) = \nabla^2\psi$ is the vorticity, $\psi(x, y, t)$ the stream function, and $\partial(a, b) \equiv a_x b_y - a_y b_x$ the two-dimensional Jacobian operator. Under appropriate boundary conditions, ψ is defined uniquely (at least to within a constant) in terms of the vorticity ω, so ω may be taken as the sole dependent variable and the equation of motion written in the Hamiltonian form (2.1), with the identification

$$u = \omega, \quad \mathbf{J} = -\partial(\omega, \cdot), \quad \mathscr{H}(\omega) = -\frac{1}{2}\iint \psi\omega \,\mathrm{d}x\,\mathrm{d}y \tag{3.2}$$

(see e.g. Benjamin 1984, §5.1, but note the different sign convention with respect to ψ; McIntyre & Shepherd 1987, §7). Under the definitions (3.2), and noting that

$$\frac{\delta\mathscr{H}}{\delta\omega} = -\psi, \quad \frac{\delta\mathscr{M}}{\delta\omega} = y, \tag{3.3}$$

so that

$$\frac{\delta}{\delta\omega}(\mathscr{H} - c\mathscr{M}) = -(\psi + cy) \equiv -\Psi, \tag{3.4}$$

the modified dynamics (2.17) takes the form

$$\omega_t + \partial(\Psi + \alpha\partial(\Psi, \omega), \omega) = \omega_t + \partial(\tilde{\psi}, \omega) = 0, \tag{3.5}$$

with $\tilde{\psi}$ defined thereby. With $c = 0$ this reduces to the scheme (4.4) of Vallis *et al.* (with $n = 0$). Here α is a single constant with dimensions of (time) $\times$ (length)^2. The Casimirs in this case are spatial integrals of any function of vorticity ω, and it is obvious from (3.5) that they remain invariant: the vorticity is still advected in the modified dynamics, just by $\tilde{\psi}$ rather than by ψ. Multiplication of (3.5) by Ψ and integration by parts yields the energy-momentum equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{H} - c\mathscr{M}) = -\alpha \iint (\partial(\Psi, \omega))^2 \,\mathrm{d}x\,\mathrm{d}y, \tag{3.6}$$

in agreement with (2.18*b*).

3.2. *Three-dimensional Euler flow*

The three-dimensional generalization of (3.1) is

$$\boldsymbol{\omega}_t = \nabla \times (\boldsymbol{v} \times \boldsymbol{\omega}), \quad \nabla \cdot \boldsymbol{v} = 0, \tag{3.7a, b}$$

where $\boldsymbol{\omega}(x,y,z,t) = \nabla \times \boldsymbol{v}$ is the vorticity vector and $\boldsymbol{v}(x,y,z,t)$ the velocity. Again, by virtue of (3.7 *b*), $\boldsymbol{v}$ may be determined from $\boldsymbol{\omega}$ under suitable boundary conditions, and the system takes the Hamiltonian form (2.1) with

$$u = \boldsymbol{\omega}, \quad \mathsf{J} = -\nabla \times (\boldsymbol{\omega} \times \nabla \times (\cdot)), \quad \mathscr{H}(\boldsymbol{\omega}) = -\frac{1}{2}\iiint \boldsymbol{\psi}\cdot\boldsymbol{\omega}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{3.8}$$

where $\nabla^2\boldsymbol{\psi} = \boldsymbol{\omega}$ and $\nabla\cdot\boldsymbol{\psi} = 0$ (see e.g. Benjamin 1984, §5.1, again noting the different sign convention). The x-component of momentum (or impulse), $\mathscr{M}$, satisfies

$$\frac{\delta\mathscr{M}}{\delta\boldsymbol{\omega}} = -\tfrac{1}{2}\boldsymbol{x}\times\hat{\boldsymbol{x}}, \quad \nabla\times\frac{\delta\mathscr{M}}{\delta\boldsymbol{\omega}} = \hat{\boldsymbol{x}}, \tag{3.9}$$

where $\hat{\boldsymbol{x}}$ is the unit vector in the x-direction, whence the modified dynamics (2.17) takes the form

$$\boldsymbol{\omega}_t = \nabla \times (\tilde{\boldsymbol{v}} \times \boldsymbol{\omega}), \tag{3.10a}$$

with

$$\tilde{\boldsymbol{v}} \equiv \boldsymbol{V} + \alpha\nabla\times\nabla\times(\boldsymbol{V}\times\boldsymbol{\omega}), \quad \boldsymbol{V} \equiv \boldsymbol{v} - c\hat{\boldsymbol{x}}. \tag{3.10b}$$

(Here a single constant α is sufficient because the three dependent variables ω_i all have the same dimension). This is equivalent to the scheme (2.2), (2.3 *b*) of Vallis *et al.* for the case $c = 0$. It is evident from (3.10 *a*) that the dynamics still retains a 'pseudo-advective' character, so the vortex topology is preserved, and that the helicity $\iiint \boldsymbol{v}\cdot\boldsymbol{\omega}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, which characterizes the vortex topology (Moffatt 1969) and is the only non-trivial Casimir, is likewise conserved. (The circulation around a closed material curve, while invariant, is not a Casimir of (3.7) because it cannot be expressed as a functional of the Eulerian representation of the flow.) The energy-momentum equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{H} - c\mathscr{M}) = -\alpha\iiint |\nabla\times(\boldsymbol{V}\times\boldsymbol{\omega})|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{3.11}$$

as expected from (2.18 *b*).

3.3. *Baroclinic quasi-geostrophic flow over topography*

The governing equations may be written in the form

$$\frac{\mathrm{D}q}{\mathrm{D}t} \equiv \frac{\partial q}{\partial t} + \partial(\psi, q) = 0 \quad \text{for } 0 < z < 1, \tag{3.12a}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}(\psi_z + Sh) = 0 \quad \text{on } z = 0, \tag{3.12b}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\psi_z = 0 \quad \text{on } z = 1, \tag{3.12c}$$

for fluid confined in the layer $0 \leqslant z \leqslant 1$, where

$$q = \psi_{xx} + \psi_{yy} + \frac{1}{\rho_s}\left(\frac{\rho_s}{S}\psi_z\right)_z + \beta y \tag{3.13}$$

is the potential vorticity, ψ the stream function, $\rho_s(z)$ and $S(z)$ prescribed vertical profiles of density and static stability, βy the linear approximation to the Coriolis parameter, and $\partial(a,b) \equiv a_x b_y - a_y b_x$. The effects of smooth topography at the lower boundary are represented by $h(x, y)$. The upper boundary condition (3.12*c*) may also be replaced by a radiation condition at $z = \infty$, in which case it plays no role in what follows.

Apart from the boundary conditions (3.12*b*, *c*), the system is entirely analogous to (3.1) with ω replaced by q. With specified circulation on lateral boundaries, it may be cast in Hamiltonian form (see Holm 1986) with the identification

$$u = [q, \lambda_0, \lambda_1]^{\mathrm{T}}, \tag{3.14a}$$

where $\lambda_0 \equiv \rho_s S^{-1}(\psi_z + Sh)|_{z=0}$ and $\lambda_1 \equiv \rho_s S^{-1}\psi_z|_{z=1}$, together with

$$\mathbf{J} = \begin{pmatrix} -\rho_s^{-1}\,\partial(q, \cdot) & 0 & 0 \\ 0 & -\partial(\lambda_0, \cdot) & 0 \\ 0 & 0 & \partial(\lambda_1, \cdot) \end{pmatrix}, \tag{3.14b}$$

$$\mathscr{H}(q, \lambda_0, \lambda_1) = \frac{1}{2}\iiint \rho_s(|\nabla\psi|^2 + S^{-1}\psi_z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{3.14c}$$

where $\nabla \equiv \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right)$, noting that

$$\frac{\delta\mathscr{H}}{\delta q} = -\rho_s\psi, \quad \frac{\delta\mathscr{H}}{\delta\lambda_0} = -\psi|_{z=0}, \quad \frac{\delta\mathscr{H}}{\delta\lambda_1} = \psi|_{z=1}. \tag{3.15}$$

For x-dependent topography h, the x-symmetry of the system is broken and $\mathscr{M}$ is no longer an invariant of (3.12). It follows that steadily-translating solutions cannot exist, and one is led to consider the modified dynamics (2.12) rather than (2.17). This leads to the system

$$q_t + \partial(\psi + \alpha\rho_s^{-1}\,\partial(\psi, q), q) = q_t + \partial(\tilde{\psi}, q) = 0, \tag{3.16a}$$

$$\lambda_{i,t} + \partial(\psi + (-1)^i\alpha_i\,\partial(\psi, \lambda_i), \lambda_i) = \lambda_{i,t} + \partial(\tilde{\psi}_i, \lambda_i) = 0 \quad (i = 0, 1), \tag{3.16b}$$

for arbitrary constants α, α_0 and α_1, with the 'pseudo-advecting' stream functions $\tilde{\psi}$, $\tilde{\psi}_0$ and $\tilde{\psi}_1$ defined thereby. The scheme (4.5) of Vallis *et al.* is a special case of this with $\alpha_0 = 0 = \alpha_1$. The Casmirs in this case are of the form

$$\iiint \rho_s C(q)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \iint C_0(\lambda_0)\,\mathrm{d}x\,\mathrm{d}y|_{z=0} + \iint C_1(\lambda_1)\,\mathrm{d}x\,\mathrm{d}y|_{z=1} \tag{3.17}$$

for arbitrary functions C, C_0 and C_1, and the energy equation corresponding to (3.16) is the obvious generalization of (3.6) with $\Psi = \psi$.

In the case of flat (or even x-symmetric) topography, the momentum invariant $\mathscr{M}$ (which includes boundary terms) may be incorporated as in (2.17), and the resulting modified dynamics is just (3.16) with ψ replaced by $\Psi \equiv \psi + cy$.

3.4. *Two-dimensional stratified Boussinesq flow*

The equations of motion for Boussinesq flow in the (x, z)-plane are conventionally written

$$\rho\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\boldsymbol{v} = -\nabla p - g\rho\hat{\boldsymbol{z}}, \tag{3.18a}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\rho = 0, \quad \nabla\cdot\boldsymbol{v} = 0, \tag{3.18b, c}$$

where ρ is the density, $\boldsymbol{v}$ the velocity, p the pressure, g the constant of gravitational acceleration, and $\hat{z}$ the unit vertical vector. The non-divergent property (3.18c) may be used to reduce the system to the more compact form

$$\rho_t = \partial(\rho, \psi), \tag{3.19a}$$

$$\sigma_t = \partial(\sigma, \psi) - \partial(\rho, gz - \tfrac{1}{2}|\nabla\psi|^2), \tag{3.19b}$$

(see Benjamin 1984, §5.3), where $\sigma \equiv \nabla \cdot (\rho \nabla \psi)$ is a vorticity-like variable, ψ the stream function defined by $\boldsymbol{v} = \hat{y} \times \nabla\psi$, and $\partial(a, b) \equiv a_z b_x - a_x b_z$ the Jacobian operator. Benjamin (1984, 1986) discusses the fact that this system may be written in the Hamiltonian form (2.1) with

$$u = [\rho, \sigma]^{\mathrm{T}}, \tag{3.20a}$$

$$\mathbf{J} = \begin{pmatrix} 0 & -\partial(\rho, \cdot) \\ -\partial(\rho, \cdot) & -\partial(\sigma, \cdot) \end{pmatrix}, \tag{3.20b}$$

$$\mathscr{H}(\rho, \sigma) = \iint \{\tfrac{1}{2}\rho|\nabla\psi|^2 + \rho g z\} \,\mathrm{d}x\,\mathrm{d}z, \tag{3.20c}$$

noting that

$$\frac{\delta\mathscr{H}}{\delta\rho} = gz - \tfrac{1}{2}|\nabla\psi|^2, \quad \frac{\delta\mathscr{H}}{\delta\sigma} = -\psi. \tag{3.21}$$

The matrix $\mathbf{J}\alpha\mathbf{J}$ then takes the form

$$\mathbf{J}\alpha\mathbf{J} = \begin{pmatrix} \alpha_2 \partial(\rho, \partial(\rho, \cdot)) & \alpha_2 \partial(\rho, \partial(\sigma, \cdot)) \\ \alpha_2 \partial(\sigma, \partial(\rho, \cdot)) & \alpha_2 \partial(\sigma, \partial(\sigma, \cdot)) + \alpha_1 \partial(\rho, \partial(\rho, \cdot)) \end{pmatrix},$$

while the x-component of momentum (or impulse) is given by $\mathscr{M} = -\iint z\sigma \,\mathrm{d}x\,\mathrm{d}z$, and satisfies

$$\frac{\delta\mathscr{M}}{\delta\rho} = 0, \quad \frac{\delta\mathscr{M}}{\delta\sigma} = -z. \tag{3.22}$$

It follows that the modified dynamics (2.17) in this case becomes

$$\rho_t = \partial(\rho, \tilde{\psi}), \tag{3.23a}$$

$$\sigma_t = \partial(\sigma, \tilde{\psi}) - \partial(\rho, gz - \tfrac{1}{2}|\nabla\psi|^2 + \alpha_1 \partial(\rho, \Psi)), \tag{3.23b}$$

where

$$\tilde{\psi} \equiv \Psi + \alpha_2(\partial(\rho, gz - \tfrac{1}{2}|\nabla\psi|^2) + \partial(\Psi, \sigma)), \quad \Psi \equiv \psi - cz. \tag{3.23c}$$

So the extremization process can proceed by an appropriate pseudo-advection (the case $\alpha_1 = 0, \alpha_2 \neq 0$), as in the three previous examples, but here it can also proceed by a different mechanism when $\alpha_1 \neq 0$. The Casimirs associated with (3.20) are the spatial integrals

$$\iint \sigma \,\mathrm{d}x\,\mathrm{d}z, \quad \iint C(\rho) \,\mathrm{d}x\,\mathrm{d}z, \quad \iint \sigma C(\rho) \,\mathrm{d}x\,\mathrm{d}z, \tag{3.24}$$

where C is an arbitrary function; it will be obvious from (3.23) that the first two remain invariant, as of course they must do by construction, and perhaps slightly less obvious that the third does so as well. It may be verified that the energy-momentum equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{H} - c\mathscr{M}) = -\iint \{\alpha_1(\partial(\rho, \Psi))^2 + \alpha_2(\partial(\sigma, \Psi) - \partial(\rho, gz - \tfrac{1}{2}|\nabla\psi|^2))^2\} \,\mathrm{d}x\,\mathrm{d}z, \tag{3.25}$$

as expected.

An important special case of the system (3.18) is that obtained under a strong form of the Boussinesq approximation in which ρ is replaced by a constant reference density ρ_0 on the left-hand side of (3.18*a*). In that case $\sigma = \rho_0 \nabla^2 \psi$ and is essentially just the flow vorticity, and ρ is replaced by ρ_0 in the kinetic energy component of the Hamiltonian (3.20*c*). As a result, the terms involving $|\nabla\psi|^2$ drop out in (3.19*b*), (3.21), (3.23*b*, *c*), and (3.25). Otherwise everything remains the same. In particular, the symmetry properties of the two systems and their Casimirs are identical.

3.5. *Shallow-water equations*

The equations of motion for a shallow homogeneous fluid, in a coordinate system rotating at constant angular velocity $\frac{1}{2}f$ about the vertical, are

$$\boldsymbol{v}_t + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} + f\hat{\boldsymbol{z}}\times\boldsymbol{v} = -g\boldsymbol{\nabla}h, \tag{3.26a}$$

$$h_t + \boldsymbol{\nabla}\cdot(h\boldsymbol{v}) = 0, \tag{3.26b}$$

where $\boldsymbol{v}(x,y,t)$ is the (horizontal) velocity, $h(x,y,t)$ the fluid depth, g the gravitational acceleration, $\hat{\boldsymbol{z}}$ the unit vertical vector, and $\boldsymbol{\nabla} \equiv ((\partial/\partial x),(\partial/\partial y)) \equiv (\partial_x, \partial_y)$. The system is Hamiltonian with dependent variable $u = [\boldsymbol{v}, h]^{\mathrm{T}}$, Hamiltonian functional

$$\mathscr{H}(\boldsymbol{v},h) = \frac{1}{2}\iint (h|\boldsymbol{v}|^2 + gh^2)\,\mathrm{d}x\,\mathrm{d}y, \tag{3.27}$$

and Poisson bracket

$$[\mathscr{F},\mathscr{G}] = \iint\left\{q\hat{\boldsymbol{z}}\cdot\left(\frac{\delta\mathscr{F}}{\delta\boldsymbol{v}}\times\frac{\delta\mathscr{G}}{\delta\boldsymbol{v}}\right)+\left(\boldsymbol{\nabla}\cdot\frac{\delta\mathscr{F}}{\delta\boldsymbol{v}}\right)\frac{\delta\mathscr{G}}{\delta h}-\frac{\delta\mathscr{F}}{\delta h}\left(\boldsymbol{\nabla}\cdot\frac{\delta\mathscr{G}}{\delta\boldsymbol{v}}\right)\right\}\mathrm{d}x\,\mathrm{d}y, \tag{3.28}$$

(see e.g. Salmon 1988*b*), where $q \equiv (f+\hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times\boldsymbol{v})/h$ is the potential vorticity. It follows immediately from (3.27) that

$$\frac{\delta\mathscr{H}}{\delta\boldsymbol{v}} = h\boldsymbol{v}, \quad \frac{\delta\mathscr{H}}{\delta h} = \tfrac{1}{2}|\boldsymbol{v}|^2 + gh. \tag{3.29}$$

After an integration by parts it may then be seen that the operator $\mathbf{J}$ corresponding to (3.28) is

$$\mathbf{J} = \begin{pmatrix} 0 & q & -\partial_x \\ -q & 0 & -\partial_y \\ -\partial_x & -\partial_y & 0 \end{pmatrix}. \tag{3.30}$$

The matrix $\mathbf{J}\alpha\mathbf{J}$ may thus be written as

$$\mathbf{J}\alpha\mathbf{J} = \begin{pmatrix} -\alpha_2 q^2 + \alpha_3\,\partial_{xx} & \alpha_3\,\partial_{xy} & -\alpha_2\, q\,\partial_y \\ \alpha_3\,\partial_{xy} & -\alpha_1 q^2 + \alpha_3\,\partial_{yy} & \alpha_1\, q\,\partial_x \\ \alpha_2\,\partial_y(q(\cdot)) & -\alpha_1\,\partial_x(q(\cdot)) & \alpha_1\,\partial_{xx} + \alpha_2\,\partial_{yy} \end{pmatrix},$$

but one may simplify matters by taking $\alpha_1 = \alpha_2$ since the two constants have the same dimensions. The x-component of momentum is given by $\mathscr{M} = \iint h(u-fy)\,\mathrm{d}x\,\mathrm{d}y$, and satisfies

$$\frac{\delta\mathscr{M}}{\delta u} = h, \quad \frac{\delta\mathscr{M}}{\delta v} = 0, \quad \frac{\delta\mathscr{M}}{\delta h} = u - fy, \tag{3.31}$$

and the modified dynamics (2.17) thus takes the form

$$\boldsymbol{v}_t+(\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}+f\hat{\boldsymbol{z}}\times\boldsymbol{v} = -g\boldsymbol{\nabla}h-\alpha_2\,q^2h\boldsymbol{V}+\alpha_2\,q\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}(\tfrac{1}{2}|\boldsymbol{V}|^2+gh+cfy)+\alpha_3\,\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot(h\boldsymbol{V})), \tag{3.32a}$$

$$h_t+\boldsymbol{\nabla}\cdot(h\boldsymbol{V}) = \alpha_2\,\nabla^2(\tfrac{1}{2}|\boldsymbol{V}|^2+gh+cfy)-\alpha_2\,\hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times(qh\boldsymbol{V}), \tag{3.32b}$$

where $\boldsymbol{V}\equiv\boldsymbol{v}-c\hat{\boldsymbol{x}}$. The scheme (4.8), (4.9) of Vallis *et al.* is actually a special case of this for $c=0$ with $\alpha_2=0$ (noting that then $\boldsymbol{\nabla}\cdot(h\boldsymbol{V})=\boldsymbol{\nabla}\cdot(h\boldsymbol{v})=-h_t$).

The Casimirs for the shallow-water equations are spatial integrals of the form

$$\iint hC(q)\,\mathrm{d}x\,\mathrm{d}y, \tag{3.33}$$

where C is an arbitrary function. By considering the potential-vorticity equation corresponding to (3.32), namely

$$q_t+\boldsymbol{V}\cdot\boldsymbol{\nabla}q = \alpha_2\left\{\hat{\boldsymbol{z}}\cdot\boldsymbol{V}\times\boldsymbol{\nabla}(\tfrac{1}{2}q^2)+\frac{1}{h}\boldsymbol{\nabla}q\cdot\boldsymbol{\nabla}(\tfrac{1}{2}|\boldsymbol{V}|^2+gh+cfy)\right\}, \tag{3.34}$$

it may be verified that the Casimirs (3.33) remain invariant under the modified dynamics, including the mass of the system (which is a special case of (3.33) with $C\equiv 1$). The energy-momentum equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{H}-c\mathscr{M}) = -\iint\{\alpha_2|\boldsymbol{\nabla}(\tfrac{1}{2}|\boldsymbol{V}|^2+gh+cfy)+qh\hat{\boldsymbol{z}}\times\boldsymbol{V}|^2+\alpha_3(\boldsymbol{\nabla}\cdot(h\boldsymbol{V}))^2\}\,\mathrm{d}x\,\mathrm{d}y; \tag{3.35}$$

to see that (3.35) vanishes if and only if the flow is a steadily-translating solution of (3.26), with translation velocity $c\hat{\boldsymbol{x}}$, it is helpful to note that (3.26*a*) may be re-written in the form

$$\boldsymbol{v}_t = \boldsymbol{v}\times(f\hat{\boldsymbol{z}}+\boldsymbol{\nabla}\times\boldsymbol{v})-\boldsymbol{\nabla}(\tfrac{1}{2}|\boldsymbol{v}|^2)-g\boldsymbol{\nabla}h. \tag{3.26a$'$}$$

3.6. *Meteorological primitive equations*

The final example to be considered is that of three-dimensional, compressible, rotating, adiabatic flow of an ideal gas, stratified under gravity, and governed by what is known in meteorology as the non-hydrostatic primitive equations. The equations consist of those for momentum,

$$\boldsymbol{v}_t+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}+f\hat{\boldsymbol{z}}\times\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p-g\hat{\boldsymbol{z}}, \tag{3.36a}$$

with notation as in §§3.4 and 3.5, for mass,

$$\rho_t+\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0, \tag{3.36b}$$

and for entropy η,

$$\eta_t+\boldsymbol{v}\cdot\boldsymbol{\nabla}\eta = 0. \tag{3.36c}$$

The system is completed by the ideal gas law

$$p = \rho RT, \tag{3.36d}$$

with R the gas constant and T the temperature, together with the fact that

$$\eta = c_{\mathrm{p}}\log\theta = c_{\mathrm{p}}\log[T(p/p_0)^{-\kappa}], \tag{3.36e}$$

where θ is the potential temperature, p_0 a (constant) reference pressure, c_{p} the specific heat at constant pressure, and $\kappa\equiv R/c_{\mathrm{p}}$. If (3.36*d, e*) were replaced by some other equation of state then the only change would be in the definition of the internal energy U (see below).

The system (3.36) is Hamiltonian in the variables $u = [\boldsymbol{v}, \rho, \eta]^{\mathrm{T}}$ (see e.g. Morrison 1982), with Hamiltonian functional

$$\mathscr{H} = \iiint \{\tfrac{1}{2}\rho|\boldsymbol{v}|^2 + U(\rho, \eta) + \rho g z\}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z, \tag{3.37}$$

where U is the internal energy, and

$$\mathbf{J} = \begin{pmatrix} 0 & \frac{1}{\rho}\omega_3 & -\frac{1}{\rho}\omega_2 & -\partial_x & \frac{1}{\rho}\eta_x \\ -\frac{1}{\rho}\omega_3 & 0 & \frac{1}{\rho}\omega_1 & -\partial_y & \frac{1}{\rho}\eta_y \\ \frac{1}{\rho}\omega_2 & -\frac{1}{\rho}\omega_1 & 0 & -\partial_z & \frac{1}{\rho}\eta_z \\ -\partial_x & -\partial_y & -\partial_z & 0 & 0 \\ -\frac{1}{\rho}\eta_x & -\frac{1}{\rho}\eta_y & -\frac{1}{\rho}\eta_z & 0 & 0 \end{pmatrix}, \tag{3.38}$$

where the absolute vorticity $(\omega_1, \omega_2, \omega_3) \equiv \boldsymbol{\omega} = f\hat{\boldsymbol{z}} + \boldsymbol{\nabla} \times \boldsymbol{v}$. It may be verified that

$$\frac{\delta \mathscr{H}}{\delta \boldsymbol{v}} = \rho \boldsymbol{v}, \quad \frac{\delta \mathscr{H}}{\delta \eta} = \frac{\partial U}{\partial \eta} = \rho T, \tag{3.39a}$$

$$\frac{\delta \mathscr{H}}{\delta \rho} = \tfrac{1}{2}|\boldsymbol{v}|^2 + gz + \frac{\partial U}{\partial \rho} = \tfrac{1}{2}|\boldsymbol{v}|^2 + gz + c_{\mathrm{p}} T, \tag{3.39b}$$

noting that $U = \rho c_{\mathrm{v}} T$ for an ideal gas (c_{v} being the specific heat at constant volume). The Casimirs for the system (3.36) are of the form

$$\iiint \rho C(\eta, q)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z, \tag{3.40}$$

where C is an arbitrary function and

$$q \equiv \frac{\boldsymbol{\omega} \cdot \boldsymbol{\nabla}\eta}{\rho} \tag{3.41}$$

the potential vorticity, noting that q satisfies

$$q_t + \boldsymbol{v} \cdot \boldsymbol{\nabla} q = 0, \tag{3.42}$$

and is thus (together with the entropy η) a material invariant. The x-component of momentum is given by $\mathscr{M} = \iiint \rho(u - fy)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$, and satisfies

$$\frac{\delta \mathscr{M}}{\delta u} = \rho, \quad \frac{\delta \mathscr{M}}{\delta v} = 0 = \frac{\delta \mathscr{M}}{\delta w}, \quad \frac{\delta \mathscr{M}}{\delta \rho} = u - fy, \quad \frac{\delta \mathscr{M}}{\delta \eta} = 0. \tag{3.43}$$

The matrix $\mathbf{J}\alpha\mathbf{J}$ is much too large to write out; its components are given in the Appendix. But introducing three different α (α_1 corresponding to $\boldsymbol{v}$, α_2 to ρ, and α_3 to η), the modified dynamical equations may be written

$$\begin{aligned} \boldsymbol{v}_t + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{v} + f\hat{\boldsymbol{z}} \times \boldsymbol{v} = {} & -\frac{1}{\rho}\boldsymbol{\nabla} p - g\hat{\boldsymbol{z}} + \alpha_2 \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot (\rho \boldsymbol{V})) - \frac{\alpha_3}{\rho}\boldsymbol{\nabla}\eta(\boldsymbol{\nabla}\eta \cdot \boldsymbol{V}) \\ & + \frac{\alpha_1}{\rho}\{\boldsymbol{\omega}(\boldsymbol{\omega} \cdot \boldsymbol{V}) - |\boldsymbol{\omega}|^2 \boldsymbol{V} + \boldsymbol{\omega} \times \boldsymbol{\nabla}(\tfrac{1}{2}|\boldsymbol{V}|^2 + c_{\mathrm{p}} T + gz + cfy) - T\boldsymbol{\omega} \times \boldsymbol{\nabla}\eta\}, \end{aligned} \tag{3.44a}$$

$$\rho_t + \nabla\cdot(\rho V) = \alpha_1\{V\cdot(\nabla\times\omega) - \omega\cdot(\nabla\times V) - \nabla\cdot(T\nabla\eta) + \nabla^2(\tfrac{1}{2}|V|^2 + c_p T + gz + cfy)\}, \tag{3.44b}$$

$$\eta_t + V\cdot\nabla\eta = -\frac{\alpha_1}{\rho}\{|\nabla\eta|^2 T - V\cdot(\nabla\eta\times\omega) + \nabla\eta\cdot\nabla(\tfrac{1}{2}|V|^2 + c_p T + gz + cfy)\}, \tag{3.44c}$$

where $V \equiv v - c\hat{x}$, together with (3.36*d, e*). It may be checked that the energy-momentum equation under (3.44) becomes

$$\frac{d}{dt}(\mathscr{H} - c\mathscr{M}) = -\iiint \{\alpha_1|\omega\times V + \nabla(\tfrac{1}{2}|V|^2 + c_p T + gz + cfy) - T\nabla\eta|^2 + \alpha_2(\nabla\cdot\rho V)^2 + \alpha_3(V\cdot\nabla\eta)^2\}\,dx\,dy\,dz, \tag{3.45}$$

which is indeed of definite sign for α_i of the same sign. That (3.45) vanishes if and only if the flow is a steadily-translating solution of (3.36*a–c*), with translation velocity $c\hat{x}$, is evident once one notes that (3.36*a*) may be re-written in the form

$$v_t = v\times\omega - \nabla(\tfrac{1}{2}|v|^2) - c_p\nabla T + T\nabla\eta - g\nabla z. \tag{3.36a$'$}$$

But to verify that the Casimirs (3.40) remain invariant under the modified dynamics (3.44) – as they must do by construction – is now a particularly tedious operation, except for the case of the total mass (a special case of (3.40) with $C \equiv 1$) which is evident at once from (3.44*b*).

4. Discussion

The foregoing has shown how the simple algorithm (2.12) can be used to turn any Hamiltonian dynamical system into one with monotonically increasing or decreasing energy, but with all its Casimir invariants preserved. Whenever the energy has a finite upper bound or a non-zero lower bound under such Casimir-preserving evolution, the 'modified dynamics' represented by (2.12) should generally approach a non-trivial steady state. Such steady states are also steady states of the original system (2.1), and by construction correspond to energy extrema (subject to the constraints imposed by the Casimir invariants). By incorporating momentum into the extremization procedure, as in (2.17), the steady states of the modified system correspond to steadily-translating solutions of the original system, and are extrema of a linear combination of the energy and momentum, viz. $\mathscr{H} - c\mathscr{M}$. Stability of these solutions is implied in both cases. It is obvious that the same approach, using a combination of energy and angular momentum, would similarly identify stable rotating states (details are left to the reader).

The method has been applied in §3 to a variety of perfect-fluid systems; in each case it may be seen explicitly from the resulting energy-momentum equation that under the modified dynamics $\mathscr{H} - c\mathscr{M}$ changes monotonically in time (depending on the sign of the symmetric transformation α), and that it is constant if and only if the flow is a steadily translating solution (with translation velocity $c\hat{x}$) of the original Hamiltonian dynamics. While in some cases the modified dynamics turns out to be a special kind of 'pseudo-advection' (meaning that the materially conserved quantities are still advected around the domain, just not by the true velocity), it is generally not so.

The method presented here provides a generalization of the algorithms given in Vallis *et al.* (1989). As discussed extensively there, these extremization algorithms provide a way to explore the structure of the phase space of the Hamiltonian dynamics, and in particular to seek out stable steady (as well as steadily-translating and rotating) states. In this respect it is important to note that one might hope by this approach to find stable states that are only local extrema of the relevant functional, and are thus not provably stable by Arnol'd's (1966) nonlinear stability theorems or their various generalizations (e.g. Holm *et al.* 1985); such theorems typically identify only global extrema (see McIntyre & Shepherd 1987, §6; Carnevale & Vallis 1990, §2). This expectation is indeed realized in some numerical experiments by Carnevale & Vallis (private communication) for two-dimensional flow over topography. The presence of nonlinearly stable states, whether global or only local extrema, may be expected to constrain the evolution of nearby states (cf. Carnevale & Frederiksen 1987; Shepherd 1987, 1988), provided that a suitable disturbance norm can be defined.

Insofar as the extremization in question is constrained by the constancy of the Casimir invariants, it is clear that the power of the method depends on the richness of the Casimir structure. A similar situation exists with regard to Arnol'd's hydrodynamical stability theorems; the more Casimirs that exist, the greater the chance of obtaining useful stability criteria (cf. Holm *et al.* 1985). In the case of irrotational water waves, for example, the system can be written in Hamiltonian form but is in fact canonical (see e.g. Benjamin 1984, §6), and therefore possesses no Casimir invariants; hence the present method would seem to be of no use in that context.

A final remark concerns the extent to which the modified dynamics can actually be expected to converge on a true energy or energy-momentum extremum. For example, suppose that the system started at the bottom of a 'potential valley' of $\mathscr{H}-c\mathscr{M}$ (namely a minimum in all directions, save one along which $\mathscr{H}-c\mathscr{M}$ was constant). Then the modified dynamics with positive α could not evolve at all, for to evolve $\mathscr{H}-c\mathscr{M}$ would have to decrease (which it could not do under the circumstances envisaged), and yet the system would not be at a true minimum. (Actually the modified dynamics could not even move away from a saddle point, as is evident from (2.14), but this is of no practical consequence since the point would be unstable.) Lest this situation with the potential valley seem pathological, it should be remarked that it will generally exist in the presence of a spatial symmetry, for any steady or steadily translating solution not sharing that symmetry; the neutral direction (the bottom of the valley) would then correspond to Casimir-preserving translations of the state in the direction of the symmetry. An example is a circular vortex in two-dimensional Euler flow; translation of the vortex in x or y generates a two-parameter family of steady flows having the same values of all the Casimirs. In practice, though, such solutions may nevertheless be stable provided that translations can be ruled out in the unmodified dynamics by consideration of the relevant momentum or impulse invariants (cf. Carnevale & Shepherd 1990). Even when the translations cannot be so ruled out, this apparent deficiency in the modified dynamics may actually prove to be an advantage, insofar as one may find states that are stable modulo translations (and thus would not be found by a straightforward application of Arnol'd's (1966) theorems).

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Appendix

The entries in the matrix $\mathbf{J}\alpha\mathbf{J}$ for the modified version of the meteorological primitive equations, discussed in §3.6, are given below.

$$(\mathbf{J}\alpha\mathbf{J})_{11} = -\frac{\alpha_1}{\rho^2}(\omega_2^2+\omega_3^2)+\alpha_2\,\partial_{xx}-\frac{\alpha_3}{\rho^2}\eta_x^2,$$

$$(\mathbf{J}\alpha\mathbf{J})_{22} = -\frac{\alpha_1}{\rho^2}(\omega_1^2+\omega_3^2)+\alpha_2\,\partial_{yy}-\frac{\alpha_3}{\rho^2}\eta_y^2,$$

$$(\mathbf{J}\alpha\mathbf{J})_{33} = -\frac{\alpha_1}{\rho^2}(\omega_1^2+\omega_2^2)+\alpha_2\,\partial_{zz}-\frac{\alpha_3}{\rho^2}\eta_z^2,$$

$$(\mathbf{J}\alpha\mathbf{J})_{44} = \alpha_1\nabla^2, \quad (\mathbf{J}\alpha\mathbf{J})_{55} = -\frac{\alpha_1}{\rho^2}(\eta_x^2+\eta_y^2+\eta_z^2),$$

$$(\mathbf{J}\alpha\mathbf{J})_{12} = \frac{\alpha_1}{\rho^2}\omega_1\,\omega_2+\alpha_2\,\partial_{xy}-\frac{\alpha_3}{\rho^2}\eta_x\,\eta_y = (\mathbf{J}\alpha\mathbf{J})_{21},$$

$$(\mathbf{J}\alpha\mathbf{J})_{13} = \frac{\alpha_1}{\rho^2}\omega_1\,\omega_3+\alpha_2\,\partial_{xz}-\frac{\alpha_3}{\rho^2}\eta_x\,\eta_z = (\mathbf{J}\alpha\mathbf{J})_{31},$$

$$(\mathbf{J}\alpha\mathbf{J})_{23} = \frac{\alpha_1}{\rho^2}\omega_2\,\omega_3+\alpha_2\,\partial_{yz}-\frac{\alpha_3}{\rho^2}\eta_y\,\eta_z = (\mathbf{J}\alpha\mathbf{J})_{32},$$

$$(\mathbf{J}\alpha\mathbf{J})_{14} = \frac{\alpha_1}{\rho}(\omega_2\,\partial_z-\omega_3\,\partial_y), \quad (\mathbf{J}\alpha\mathbf{J})_{41} = \alpha_1\left(\partial_y\left(\frac{\omega_3}{\rho}(\cdot)\right)-\partial_z\left(\frac{\omega_2}{\rho}(\cdot)\right)\right),$$

$$(\mathbf{J}\alpha\mathbf{J})_{24} = \frac{\alpha_1}{\rho}(\omega_3\,\partial_x-\omega_1\,\partial_z), \quad (\mathbf{J}\alpha\mathbf{J})_{42} = \alpha_1\left(\partial_z\left(\frac{\omega_1}{\rho}(\cdot)\right)-\partial_x\left(\frac{\omega_3}{\rho}(\cdot)\right)\right),$$

$$(\mathbf{J}\alpha\mathbf{J})_{34} = \frac{\alpha_1}{\rho}(\omega_1\,\partial_y-\omega_2\,\partial_x), \quad (\mathbf{J}\alpha\mathbf{J})_{43} = \alpha_1\left(\partial_x\left(\frac{\omega_2}{\rho}(\cdot)\right)-\partial_y\left(\frac{\omega_1}{\rho}(\cdot)\right)\right),$$

$$(\mathbf{J}\alpha\mathbf{J})_{15} = \frac{\alpha_1}{\rho^2}(\omega_3\,\eta_y-\omega_2\,\eta_z) = (\mathbf{J}\alpha\mathbf{J})_{51},$$

$$(\mathbf{J}\alpha\mathbf{J})_{25} = \frac{\alpha_1}{\rho^2}(\omega_1\,\eta_z-\omega_3\,\eta_x) = (\mathbf{J}\alpha\mathbf{J})_{52},$$

$$(\mathbf{J}\alpha\mathbf{J})_{35} = \frac{\alpha_1}{\rho^2}(\omega_2\,\eta_x-\omega_1\,\eta_y) = (\mathbf{J}\alpha\mathbf{J})_{53},$$

$$(\mathbf{J}\alpha\mathbf{J})_{45} = -\alpha_1\,\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\nabla}\eta}{\rho}(\cdot)\right), \quad (\mathbf{J}\alpha\mathbf{J})_{54} = \frac{\alpha_1}{\rho}\boldsymbol{\nabla}\eta\cdot\boldsymbol{\nabla}(\cdot).$$

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