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A Double Threshold GARCH Model for the French Franc / Deutschmark Exchange Rate

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Abstract

This paper combines and generalises a number of recent time series models of daily exchange rate series by using a SETAR model which also allows the variance equation of a GARCH specification for the error terms to be drawn from more than one regime. An application of the model to the French Franc / Deutschmark exchange rate demonstrates that out-of-sample forecasts for the exchange rate volatility are also improved when the restriction that the data it is drawn from a single regime is removed. This result highlights the importance of considering both types of regime shift (i.e. thresholds in variance as well as in mean) when analysing financial time series.

Keywords: threshold models, GARCH, non-linear, exchange rates, time series analysis

I. Introduction

It is now a fairly widely accepted stylised fact that financial time series exhibit strong signs of nonlinearity. Hsieh (1991) and Scheinkman and LeBaron (1989), for example, have shown very strong evidence of nonlinearity in stock returns using an application of the BDS (see Brock et al, 1996) test. Nonlinearities have also been observed in foreign exchange series (Hsieh, 1989 & 1993 and Brooks, 1996 are examples for US Dollar and Sterling base currencies respectively). Given these signs of nonlinearities, researchers have proceeded to fit non-linear time series to their financial data. By far the most popular specification has been the GARCH model due to Bollerslev (1986) or one of its variants. Under a GARCH model, the conditional dependence enters through the variance of the process rather than its mean. However, a number of recent studies (for example Hsieh, 1991 or Brooks, 1996) have found evidence that the GARCH model is unable to capture all of the observed nonlinearity in the series.

An alternative and entirely different set of non-linear models are the self-exciting threshold autoregressive (SETAR) family of models associated primarily with Tong (Tong and Lim, 1980, Tong, 1983, 1990; Chan and Tong, 1986; Tsay, 1989). This is a simple relaxation of a linear autoregressive model which allows a locally linear approximation over a number of states, so that globally the model is non-linear. These models have been less widely used than conditional variance models in the arena of economics and finance, although they are becoming more popular. The models are clearly of importance when the data may be drawn from one autoregressive model in one regime, but an entirely different autoregresssive model in another. Yadav *et al*. (1994) for example, have used the model to demonstrate that the futures “basis” (the futures price minus the corresponding spot price) for a futures contract could fluctuate within prescribed boundaries in a predictable manner without triggering a reaction from traders due to the existence of transactions costs. Thus if the absolute value of the basis is large enough (i.e. the basis is very large and positive or very large and negative), then agents in the market will trade until an approximate equilibrium is restored, but if the absolute value of the basis is small, then the transactions costs would imply that it is not worth market participants taking advantage of the minor mis-pricing. The upshot of this is that we might expect thresholds to exist, and model specification to be different between the regimes.[1]

Another useful application of a SETAR model in economics or finance is in modelling movements in exchange rates which are subject to intervention by central banks. Chappell *et al*. (1996), for example, apply a threshold model to the French Franc / Deutschmark exchange rate which is required to reside between two prescribed bands as part of the European Exchange Rate Mechanism (ERM). If the currency moves close to one of the bands, then the French and German central banks are obliged to intervene in the markets, buying the weaker currency and selling the stronger in an attempt to shift the exchange rate back closer to the central parity.

Until very recently, applied researchers had to choose between these two specifications: a GARCH model, or a threshold model. The former has the ability to capture many important stylised facts of financial and economic time series, such as unconditional leptokurtosis and volatility clustering, but Lamoureux and Lastrapes (1990) showed that structural breaks in the variance could lead to GARCH conditional variance equations which are spuriously highly persistent. The latter can accommodate these structural changes or regime shifts, but cannot generate volatility pooling or leverage effects. Both types of model are clearly important for time series modelling, and this has led a small number of recent studies to consider a combination of the models in an attempt to incorporate the important facets of both. Li and Li (1996), for example, propose a double threshold ARCH (DTARCH) model and apply it to Hong Kong stock index returns. They find the model useful in capturing important asymmetries in both the returns and their volatilities, which would not be found under one of the component models alone. Dueker (1997) uses a Markov switching GARCH model for modelling and predicting stock index

[1] In this case, we might expect the existence of two thresholds, and the basis to follow a random walk outside of the thresholds, with possibly some predictable slower-adjusting behaviour in between.

volatilities and finds that the switching model which allows the extent of the conditional heteroscedasticity to vary between regimes produces superior forecasts to a model which forces the volatility data generating process to be only one regime.

The purpose of this paper is to use the data series and SETAR specifications used in Chappell *et al*. (1996), and to extend the model so that the conditional variance is also allowed to vary depending on the regime. Then work of Li and Li (1996) is also extended to use a GARCH rather than ARCH specification for the conditional variance since the former is more widely used and represents a more parsimonious way to represent persistent volatility series. This study also extends previous research by considering the out-of-sample forecasting performance of the models for the variance of the exchange rate series. Modelling and forecasting stock market volatility has been the subject of much recent empirical and theoretical investigation by academics and practitioners alike. There are a number of motivations for this line of inquiry. First, volatility, as measured by the standard deviation of returns, is often used as a crude measure of the total risk of financial assets. Second, the volatility of stock market prices enters directly into the Black-Scholes formula for deriving the prices of traded options. Although in the past historical measures of volatility have been used as estimates of future volatility, there is a growing body of evidence that suggests that the use of volatility predicted from more sophisticated time series models will lead to more accurate option valuations (see, for example, Akgiray, 1989; or Chu and Freund, 1996).

The remainder of this paper is as follows. Section two presents a brief description of the data used in this study and describes the models which are specified. Section two also explains how forecasts are derived from the models and how both models and forecasts are evaluated. Finally, section three presents the results while section four concludes.

2. The Data and Methodology

2A The Data

The models in this study are those used in Chappell *et al.* (1996) estimated using 500 daily observations on the French Franc / German mark, spanning the period 1 May 1990 to 30 March 1992, obtained from Datastream International. Following Chappell *et al*., the models are estimated in the natural logarithms of the levels rather than the first differences, even though the null hypothesis that the former are unit root non-stationary cannot be rejected. This step is valid since, as they argue, it is the levels that will follow a threshold model and are required to lie within prescribed bands, and not the first differences. Thus a SETAR model in first differences would have little meaning[2]. Again, so that results can be compared with the Chappell *et al*. study, the first 450 observations (from 1 May 1990 until 20 January 1991) are used for in-sample model estimation, while the remaining 50 observations are retained entirely for out-of-sample forecasting comparison.

2B. The Double Threshold SETAR-GARCH Model and Estimation Issues

A general threshold autoregressive model may be expressed as

$$x_t = \sum_{j=1}^{J} I_t^{(j)} (\phi_0^{(j)} + \sum_{i=1}^{p_j} \phi_i^{(j)} x_{t-i} + \varepsilon_t), r_{j-1} \le z_{t-d} < r_j \qquad (1)$$

where $I_t^{(j)}$ is an indicator function for the j^{th} regime taking the value one if the underlying variable is in state j and zero otherwise. z_{t-d} is an observed variable determining the switching point and $\varepsilon_t^{(j)}$ is a zero-mean independently and identically distributed error process. If the regime changes are driven by own lags of the underlying variable, x_t, (i.e. $z_{t-d} = x_{t-d}$), then the model is a self-exciting TAR (SETAR). Estimation of the model parameters (ϕ_i, r_j, d, p_j) is considerably more difficult than for a standard autoregressive process, since in general they cannot be determined simultaneously, and the values chosen for one parameter are likely to influence estimates of the others. Tong (1983, 1990) suggests a complex non-parametric lag regression procedure to

[2] As Chappell *et al*. also point out, since the exchange rate is forced to always remain within the upper and lower bounds prescribed by the ERM, then it must be stationary.

estimate the values of the thresholds (r_j) and the delay parameter (d). Estimation of the autoregressive coefficients can then be achieved using nonlinear least squares (NLS), and the order of the piece-wise linear components (p_j) by Akaike's (as Tong, 1990, suggests), or some other information criterion.

This model has been extended to allow the conditional variance to be time varying and to be drawn from different regimes by Li and Li (1996). Their specification can be further generalised to allow the conditional variance to depend upon its own previous realisations (i.e. a double threshold GARCH model). The double threshold GARCH, or SETAR-GARCH model can then be written

$$x_t = \sum_{j=1}^{J} I_t^{(j)} (\phi_0^{(j)} + \sum_{i=1}^{p_j} \phi_i^{(j)} x_{t-i} + \varepsilon_t), r_{j-1} \leq x_{t-d} < r_j , \varepsilon_t \sim \mathrm{N}(0,h_t) \qquad (2)$$

$$h_t = \sum_{j=1}^{J} I_t^{(j)} (\alpha_0^{(j)} + \alpha_1^{(j)} \varepsilon_{t-1}^2 + \alpha_2^{(j)} h_{t-1})$$

Ideally, it may be preferable to endogenously estimate the values of the threshold(s) as part of the NLS [3] optimisation procedure, but this is not feasible. The underlying functional relationship between the variables of interest is discontinuous in the thresholds. Thus the procedure adopted here is as follows. The delay parameter, d, is set to one on theoretical grounds in common with Kräger and Kugler (1993): in the context of a financial market such as that analysed here, it is most likely that the previous day's return would be the one to determine the current state, rather than the return two days, two weeks or two months ago.

As a consequence of the exchange rate being required to reside between two boundaries, we might expect there to be two thresholds. Models are estimated in this study with one and two thresholds (i.e. J = 1 or 2 above), and with the number of lags of the dependent variable chosen using AIC, as in Chappell *et al*. As they note, however, the estimated values of the thresholds are too close together to represent the actual boundaries, and thus the models with only one threshold might be preferred. In any case, the pressure for central bank intervention was always to raise the Franc and reduce the value of the Mark rather than the other way around. When one allows the variances as well as the means to have 0,1, or 2 thresholds, then there is a possible total of 9 model combinations (e.g. no thresholds in mean, and one in variance, two thresholds in mean and two in variance etc.). All possible combinations are estimated but only those models with two thresholds in mean and two in variance, or one in mean and one in variance are shown here due to space constraints[4]. Following Chappell *et al*., the values of the upper and lower thresholds are 5.819 and 5.831 respectively, with the latter forming the threshold in the models where there is only one. The models are estimated by maximum likelihood using the BFGS algorithm, with White (1980) corrected standard errors.

2C: The Construction of Forecasts

For the purpose of forecast construction, the models are estimated using a rolling window of length 450 observations. A one-step-ahead forecast for the mean and the variance is then calculated, and the sample rolled forward to add another observation. The model is then re-calculated and another one-step-ahead forecast is computed until finally, there are 50 such forecasts which can then be compared with the actual values of the series for the exchange rate variance. The "true" variance of the series against which the forecasts are evaluated is the square of the return minus its mean value (i.e. r_t - $\bar{r}$) during each time period. This is the measure used in many recent papers which have used various GARCH and other models to forecast volatility (see, for example, Chan, Christie and Schultz, 1995; Day and Lewis, 1992; West and Cho, 1995). Three error measures are calculated for the volatility forecasts: mean squared error, mean absolute

[3] Subba Rao and Gabr (1980) use maximum likelihood estimation rather than NLS. This is used here since GARCH models are easily estimated within the quasi-maximum likelihood framework.
[4] A table containing the parameters of all estimated models is available upon request from the author.

error, and proportion of over-predictions (see Brooks, 1998, for a detailed discussion of these metrics).

3. Results

The estimated values of the models with two thresholds in the mean and two in the variance are given in table 1. The most important feature is clearly the difference in the behaviour of the series in each regime. When x_{t-1} is above the upper threshold, the exchange rate is much slower in adjusting to shocks; in the other two regimes, the behaviour closely approximates a random walk. The values above the upper threshold probably correspond to times when the central bank has intervened most heavily to pull the exchange rate back towards its central parity. The behaviour of the conditional variance also varies considerably between regimes; shocks to the conditional variance are more persistent when x_{t-1} is below the lower threshold, and least persistent above the upper threshold.

However, since the estimated values of the thresholds are rather close, the procedure is repeated allowing for only one threshold, for the reasons highlighted above. These results are presented in table 2. The pattern is very similar for that when two thresholds are used; that is, the conditional variance shows less persistence when x_{t-1} is above the threshold than when x_{t-1} is below it. Again, adjustment in the mean equation is much slower above the threshold.

It appears, then, that it is appropriate to model the conditional variance as switching between a number of regimes, as well as the conditional mean. Another method of evaluating this proposition is to compare the forecasting performance of models where the behaviour of the conditional variance is allowed to change, with models where it is fixed. Table 3 shows the results of applying the forecasting procedure suggested above to the SETAR-TGARCH models with all combinations of 0, 1, or 2 thresholds in the mean and in the variance. The forecasts of a pure GARCH-(1,1) model (i.e. with no thresholds in the variance and just a constant in the mean) and of a model which simply extrapolates the long-term mean of the volatility are also shown for comparison with the threshold models.

The results are very mixed, and seem to show that certain models with thresholds in the variances are able to beat the GARCH and long-term mean models, but also some of the models perform considerably worse than these benchmarks. The mean squared forecast error for the best threshold model is less than half the size of that of the GARCH model (although the MSE's of all the models are very small). The models with thresholds in mean but not in variance give worse forecasts of the conditional variance than models with no threshold in mean.

There is also some disagreement between the forecast evaluation criteria as to which is the best model. The model with the lowest mean absolute error is the most profligate one with two thresholds in both the mean and the variance; yet this was the model which had one of the strongest tendencies to over-predict.

4. Conclusions

This paper has extended a recently proposed application of the SETAR model to the French Franc / Deutschmark exchange rate to the case where the conditional variance is also allowed to be drawn from one of several regimes. It is observed that the behaviour of the conditional variance is quite different between the regimes, and that models which allow for different regimes can provide superior volatility forecasts compared to those which do not. This result has implications for risk management tools and option pricing models where volatility forecasts are an essential ingredient. Whether threshold-GARCH models can lead to improvements in the accuracies of these models when employed in a financial context is an important question for future research.

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Table 1: Estimated Coefficients for SETAR-TGARCH Model with 2 Thresholds

Model:

$$x_t = \begin{cases} \phi_0^{(1)} + \phi_1^{(1)} x_{t-1} + \varepsilon_t^{(1)} & if \quad x_{t-1} < 5.819 \\ \phi_0^{(2)} + \phi_1^{(2)} x_{t-1} + \varepsilon_t^{(2)} & if \quad 5.819 \leq x_{t-1} < 5.831 \\ \phi_0^{(3)} + \phi_1^{(3)} x_{t-1} + \phi_1^{(3)} x_{t-1} + \phi_2^{(3)} x_{t-2} + \phi_3^{(3)} x_{t-3} + \varepsilon_t^{(3)} & if \quad x_{t-1} \geq 5.831 \end{cases}$$

$$\varepsilon_t \sim \mathrm{N}(0,h_t) \qquad h_t = \begin{cases} \alpha_0^{(1)} + \alpha_1^{(1)} h_{t-1} + \alpha_2^{(1)} \varepsilon_t^2 & if \quad x_{t-1} < 5.819 \\ \alpha_0^{(2)} + \alpha_1^{(2)} h_{t-1} + \alpha_2^{(2)} \varepsilon_t^2 & if \quad 5.819 \leq x_{t-1} < 5.831 \\ \alpha_0^{(3)} + \alpha_1^{(3)} h_{t-1} + \alpha_2^{(3)} \varepsilon_t^2 & if \quad x_{t-1} \geq 5.831 \end{cases}$$

Parameter	Below lower threshold, i.e. $x_{t-1} < 5.819$	Between thresholds, i.e. $5.819 \leq x_{t-1} < 5.831$	Above upper threshold, i.e. $x_{t-1} \geq 5.831$
ϕ_0	0.0504	0.0964	0.0403
	(0.0054)**	(0.0964)	(0.3129)
ϕ_1	0.9946	0.9809	0.5038
	(0.0009)**	(0.0166)**	(0.0537)**
ϕ_2	-	-	0.3301
			(0.0537)**
ϕ_3	-	-	0.1568
			(0.0537)**
α_0	0.0001	0.0001	0.0002
	(0.0018)	(0.0001)	(0.0002)
α_1	1.0411	0.7431	0.6025
	(0.3938)**	(1.500)	(1.9907)
α_2	0.1994	0.1194	0.1618
	(0.0973)**	(0.0417)**	(0.1438)

Notes: Heteroscedasticity consistent standard errors in parentheses. * and ** denote significance at the 5% and 1% levels respectively.

Table 2: Estimated Coefficients for SETAR-TGARCH Model with 1 Threshold in Mean and One In Variance

Model: $x_t = \begin{cases} \phi_0^{(1)} + \phi_1^{(1)} x_{t-1} + \varepsilon_t^{(1)} & if \quad x_{t-1} < 5.831 \\ \phi_0^{(3)} + \phi_1^{(3)} x_{t-1} + \phi_1^{(3)} x_{t-1} + \phi_2^{(3)} x_{t-2} + \phi_3^{(3)} x_{t-3} + \varepsilon_t^{(3)} & if \quad x_{t-1} \geq 5.831 \end{cases}$

$\varepsilon_t \sim \mathrm{N}(0,h_t)$ $\quad h_t = \begin{cases} \alpha_0^{(1)} + \alpha_1^{(1)} h_{t-1} + \alpha_2^{(1)} \varepsilon_t^2 & if \quad x_{t-1} < 5.831 \\ \alpha_0^{(2)} + \alpha_1^{(2)} h_{t-1} + \alpha_2^{(2)} \varepsilon_t^2 & if \quad 5.819 < x_{t-1} \geq 5.831 \end{cases}$

Parameter	Below threshold, i.e. $x_{t-1} < 5.831$	Above threshold, i.e. $x_{t-1} \geq 5.831$
ϕ_0	0.0445	0.0616
	(0.0016)**	(0.0048)**
ϕ_1	0.9948	0.4948
	(0.0003)**	(0.0008)**
ϕ_2	-	0.2678
		(0.0008)**
ϕ_3	-	0.2279
		(0.0008)**
α_0	0.0001	0.0001
	(000006)	(0.0001)
α_1	0.5483	0.4783
	(0.1954)**	(0.6168)
α_2	0.0954	0.0955
	(0.0301)**	(0.0624)

Notes: Heteroscedasticity consistent standard errors in parentheses. * and ** denote significance at the 5% and 1% levels respectively.

Table 3: Volatility Forecasts Using Threshold Models

Number of thresholds in mean / variance	Mean Squared error ($X\ 10^{-12}$)	Mean Absolute Error ($X\ 10^{-5}$)	Proportion of over-predictions (%)
0 / 1	55.2	1.87	0.57
0 / 2	8.92	1.79*	0.62
1 / 0	526	18.0	0.59
1 / 1	88.3	5.70	0.52*
1 / 2	57.9	7.32	0.61
2 / 0	247	12.0	0.56
2 / 1	55.2	1.87	0.55
2 / 2	5.21*	3.40	0.68
GARCH (1,1)	13.3	2.73	0.52
Long term mean	18.3	3.03	0.42

Note: * denotes the "best" model order according to each separate criterion.