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## APPROXIMATION BY HARMONIC POLYNOMIALS IN STAR-SHAPED DOMAINS AND EXPONENTIAL CONVERGENCE OF TREFFTZ $hp$ -dGFEM<sup>\*</sup>

RALF HIPTMAIR<sup>1</sup>, ANDREA MOIOLA<sup>2</sup>, ILARIA PERUGIA<sup>3</sup> AND CHRISTOPH SCHWAB<sup>4</sup>

**Abstract.** We study the approximation of harmonic functions by means of harmonic polynomials in two-dimensional, bounded, star-shaped domains. Assuming that the functions possess analytic extensions to a  $\delta$ -neighbourhood of the domain, we prove exponential convergence of the approximation error with respect to the degree of the approximating harmonic polynomial. All the constants appearing in the bounds are explicit and depend only on the shape-regularity of the domain and on  $\delta$ . We apply the obtained estimates to show exponential convergence with rate  $O(\exp(-b\sqrt{N}))$ ,  $N$  being the number of degrees of freedom and  $b > 0$ , of a  $hp$ -dGFEM discretisation of the Laplace equation based on piecewise harmonic polynomials. This result is an improvement over the classical rate  $O(\exp(-b\sqrt[3]{N}))$ , and is due to the use of *harmonic polynomial spaces*, as opposed to complete polynomial spaces.

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### INTRODUCTION

We fix a domain that meets the following requirements, see Figure 1.

**Assumption 1.** *The domain  $D \subset \mathbb{C}$  is open and satisfies<sup>1</sup>*

- i)  $\text{diam}(D) = 1$ ;*
- ii) there exists  $0 < \rho \leq 1/2$  such that  $B_\rho \subseteq D$ ;*
- iii) there exists  $0 < \rho_0 < \rho$  such that  $D$  is star-shaped with respect to  $B_{\rho_0}$ , i.e.,  $\forall w \in D$  and  $\forall v \in B_{\rho_0}$ , the straight segment with endpoints  $w$  and  $v$  is contained in  $\overline{D}$ .*

In this article we investigate the best approximation on  $D$  of a function  $f : D \rightarrow \mathbb{C}$  by means of (complex variable) polynomials. We obtain *exponential convergence* in the polynomial degree provided that  $f$  is holomorphic in an open neighbourhood of  $D$ . Our main approximation result from Section 3.2 reads as follows.<sup>2</sup>

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*Keywords and phrases:* Approximation by harmonic polynomials, exponential orders of convergence,  $hp$ -finite elements

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<sup>1</sup>We write  $B_r(w_0) := \{w \in \mathbb{C} : |w - w_0| < r\}$  and  $B_r := B_r(0)$ .

<sup>2</sup>Here and in the following, we denote the distance between a point  $w \in \mathbb{C}$  and a set  $\mathcal{D} \subset \mathbb{C}$  and the distance between two sets  $\mathcal{D}_1, \mathcal{D}_2 \subset \mathbb{C}$  by  $d(w, \mathcal{D}) := \inf_{w' \in \mathcal{D}} |w - w'|$  and  $d(\mathcal{D}_1, \mathcal{D}_2) := \inf_{w_1 \in \mathcal{D}_1, w_2 \in \mathcal{D}_2} |w_1 - w_2|$ .

**Theorem 2.** Fix  $0 < \delta \leq 1/2$  and define the inflated domain  $D_\delta = \{w \in \mathbb{C} : d(w, D) < \delta\}$ . There exist  $C, b > 0$  only depending on  $\rho, \rho_0$  and  $\delta$  such that, for any function  $f$  which is holomorphic and bounded in  $D_\delta$ , there is a sequence of polynomials  $\{q_p\}_{p \geq 1}$  of degree at most  $p$  such that

$$\|f - q_p\|_{L^\infty(D)} \leq C e^{-bp} \|f\|_{L^\infty(D_\delta)}.$$

In Section 3, the values of  $C$  and  $b$  will be made *fully explicit* in terms of  $\delta$  and the geometry of  $D$  (Theorem 18), and we will prove similar results for the derivatives of  $f$  (Corollary 20).

Our considerations follow the pioneering work of M. Melenk in [27, Chapter II] and [28], refining and completing his arguments. The linchpin is Hermite’s representation formula for the error of polynomial interpolation of holomorphic functions in complex domains, see Section 3.2. It is applied using, as integration contours, the level lines of the holomorphic mapping  $\varphi_D : \mathbb{C} \setminus \overline{B_1} \rightarrow \mathbb{C} \setminus \overline{D}$  provided by the Riemann mapping theorem. Thus we need rather precise information about the position of these level lines, and this information is gleaned in Section 2 by means of fairly intricate estimates. A result similar to Theorem 2 was stated in [27, Theorem 2.2.10]; the novelty of the present contribution lies in the *explicit expressions* for the constants  $C$  and  $b$  in terms of the parameters  $\delta, \rho$  and  $\rho_0$  only. The importance of having explicit dependence of the constants on the geometry, in the context of Trefftz methods, is described in Remark 30.

Our work was motivated by the desire to obtain convergence estimates for the *hp*-version of Trefftz-type discontinuous Galerkin finite element methods (dGFEM) for second-order scalar elliptic boundary value problems. For the Laplace equation  $\Delta u = 0$ , these methods rely on harmonic polynomials for the local approximation on the mesh cells. Thus, with  $D$  standing for a mesh cell (after the identification of  $\mathbb{R}^2$  with  $\mathbb{C}$  and, possibly, a similarity transformation), estimates like that of Theorem 2 become instrumental for showing exponential convergence of the discretisation error in terms of the dimensions of the trial spaces. This will be outlined in Section 4, in the case of (straight) triangular and quadrilateral meshes, building on the *hp*-dGFEM convergence theory of [40]. On geometrically graded meshes, this scheme features faster exponential convergence than standard methods: the energy norm of the error decays as  $\exp(-b\sqrt{N})$ ,  $N$  being the number of degrees of freedom and  $b > 0$ , as opposed to standard schemes which achieve only  $\exp(-b\sqrt[3]{N})$ . A dGFEM based on harmonic polynomials was already introduced in [24, 25]; only the convergence under mesh refinement was discussed there.

The results of the present paper can be extended to general second-order elliptic equations by means of the so-called Vekua theory [31, 36]. This technique provides continuous bijections between spaces of harmonic functions and spaces of solutions to the considered elliptic equation. In particular, the case of Helmholtz boundary value problems is relevant as several Trefftz-type numerical schemes have recently been proposed for their efficient approximation at medium and high wave numbers; see [8, 13, 14, 19, 20, 23, 29, 32], the references therein, and the review in [30, §1.2]. In this case, Vekua’s theory allows to translate approximation estimates for harmonic polynomials into similar bounds for circular waves and can be related to approximation results for plane waves. This is pursued in recent work [21].

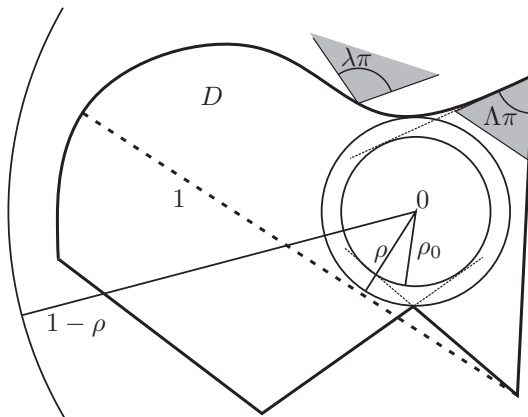
We close this introduction with some remarks on the geometry of the domain  $D$  in our approximation results. We refer to Figure 1 for an illustration of the notation in the following statements. By Assumption 1,  $D$  is bounded, simply connected,  $0 \in D$  and  $D \subseteq B_{1-\rho}$ . Moreover,  $D$  satisfies the following *uniform cone conditions*: there exist  $H_0 > 0$ , and  $\Lambda, \lambda \in (0, 1]$  satisfying

$$\min\{\Lambda, \lambda\} \geq \frac{2}{\pi} \arcsin \frac{\rho_0}{1-\rho}, \quad (1)$$

such that, for any  $w \in \partial D$ ,

- a) there exists a cone<sup>3</sup> with vertex  $w$ , opening angle  $\Lambda\pi$  and height  $H_0$  contained in  $\overline{D}$ ,
- b) there exists an infinite cone with vertex  $w$  and opening angle  $\lambda\pi$  contained in  $\mathbb{C} \setminus D$ .

<sup>3</sup>Following [27, Proposition 2.1.6] we call “cone” an isosceles triangle, “infinite cone” the sector of the plane delimited by two half lines with common origin, and “opening angle” the angle adjacent to the two sides with equal length of a cone, or to the two half lines of a infinite cone.

FIGURE 1. Geometry of domain  $D$ , see Assumption 1.

The proof can be found in Theorem A.1 of [22, Appendix A]. The uniform cone conditions imply that  $D$  is Lipschitz (see, e.g., [16, Theorem 1.2.2.2]).

**Remark 3.** If  $D$  is convex, we could choose  $\rho_0 = \rho$ . However, in order to avoid the discussion of special cases, we will always assume  $\rho_0 < \rho$ , obviously with no loss of generality.

We also notice that, in the convex case, the exterior cone condition holds with  $\lambda = 1$  (the cone is a half plane through  $w$  that does not intersect  $D$ ), while for the interior cone condition one always has  $\Lambda < 1$ .

**Remark 4.** We chose to consider star-shaped domains since, as mentioned above, the main application we have in mind involves the use of the Vekua operators, which are defined only under this assumption, see [31]. On the other hand, all the proofs in the present article would be hugely simplified if we assumed convex domains instead (e.g. compare the proof of bound (8) in the convex case in Section 2.2 and in the star-shaped one in the appendix), and it could be argued that convex elements suffice for applications to finite elements. However, we believe that the abstract approximation results we state may be of much wider interest than for Galerkin schemes only. Moreover, if a boundary value problem is to be solved in a piecewise-smooth, non-convex domain, it is not possible to partition it into convex elements. Finally, we envisage the use of the bounds proved here in the context of finite elements for more general elliptic PDEs, where elements might be analytically mapped to transform the original PDE into the Laplace equation; in this case convexity would not be preserved.

## 1. EXTERIOR CONFORMAL MAPPINGS

Let  $\mathcal{D} \subset \mathbb{C}$  be a non-empty, simply connected “generic” domain that is either compact or open and bounded. Set  $\mathcal{D}^c := (\mathbb{C} \cup \{\infty\}) \setminus \overline{\mathcal{D}}$  and  $B_1^c := (\mathbb{C} \cup \{\infty\}) \setminus \overline{B_1}$ . Owing to the Riemann mapping theorem, there exists a unique one-to-one conformal mapping  $\varphi_{\mathcal{D}} : B_1^c \rightarrow \mathcal{D}^c$  such that  $\varphi_{\mathcal{D}}(\infty) = \infty$  and  $\varphi'_{\mathcal{D}}(\infty)$  is real and positive. The proof can be found in [17, Corollary 5.10c] (where “regions” are non-empty, open, connected sets as defined in [17, §3.2]) or in [26, Vol. III, Theorems 1.2 and 1.3] after using the inversion across  $\partial B_1$ . If  $\partial \mathcal{D}$  is a Jordan curve, then  $\varphi_{\mathcal{D}}$  can be extended to a homeomorphism from  $\overline{B_1^c} \rightarrow \overline{\mathcal{D}^c}$ , i.e., it is bijective and continuous, with continuous inverse also on the boundary (see [33, §17.20] or [17, Theorem 5.10e]).

For every  $h > 0$ , we define the *level line* of  $\varphi_{\mathcal{D}}$  by

$$L_h[\overline{\mathcal{D}}] := \varphi_{\mathcal{D}}(\partial B_{1+h}). \quad (2)$$

Since  $\varphi_{\mathcal{D}}$  is a homeomorphism, the level line  $L_h[\overline{\mathcal{D}}]$  partitions  $\mathbb{C}$  into two connected components and we denote by  $\text{Int } L_h[\overline{\mathcal{D}}]$  the closure of the bounded connected component. Whenever  $\mathcal{D} = D$ , which satisfies Assumption 1, we set for brevity  $\varphi := \varphi_D$  and  $L_h := L_h[\overline{D}]$ .

In Section 16.5.II of [18] (eq. (16.5-7), page 374) and in [37, Sec. 4.4, p. 74]<sup>4</sup>, the value  $\varphi'_D(\infty)$  (which is real and positive by definition of  $\varphi_D$ ) is identified as the classic analytic *capacity* of  $\mathcal{D}$ .

If  $\mathcal{D}_1 \subset \mathcal{D}_2$ , then  $\varphi'_{\mathcal{D}_1}(\infty) \leq \varphi'_{\mathcal{D}_2}(\infty)$ . Besides, Theorem 16.6j of [18] asserts that  $\varphi'_{B_R(w)}(\infty) = R$  and, thus, for the domain  $D$ ,

$$\rho < \varphi'_D(\infty) < 1 - \rho. \quad (3)$$

Let  $P$  be a bounded *polygon* with counterclockwise ordered vertices  $\{w_k\}_{k=1}^N$  and corresponding internal angles  $\{\alpha_k\}_{k=1}^N$ . Then, using conformal inversion across  $\partial B_1$  and [11, Eq. 4.6], the conformal mapping  $\varphi_P$  is given by the Schwartz–Christoffel formula

$$\varphi_P(z) = A + C_{SC} \int^{1/z} \zeta^{-2} \prod_{k=1}^N \left(1 - \frac{\zeta}{z_k}\right)^{1-\alpha_k} d\zeta \quad |z| > 1, \quad (4)$$

where  $z_k = \varphi_P^{-1}(w_k)$ ,  $|z_k| = 1$ . We have  $\sum_{k=1}^N \alpha_k = N - 2$  (or  $\sum_{k=1}^N (1 - \alpha_k) = 2$ ); see also [26, Vol. III, eq. (9.10) page 331]. The constant  $A \in \mathbb{C}$  depends on translations of  $P$  and on the initial point in the integration; the constant  $C_{SC}$  is related to rotations/dilations and from [11, p. 53] we have

$$|C_{SC}| = \varphi'_P(\infty). \quad (5)$$

The complex derivative of the Schwarz–Christoffel mapping can easily be computed as

$$\varphi'_P(z) = -C_{SC} \prod_{k=1}^N \left(1 - \frac{1}{z z_k}\right)^{1-\alpha_k} = -C_{SC} \frac{1}{z^2} \prod_{k=1}^N (z - \bar{z}_k)^{1-\alpha_k}, \quad (6)$$

where in the last step we have used  $|z_k| = 1$  and  $\sum_k (1 - \alpha_k) = 2$ . When  $z$  approaches one of the  $z_k$ 's, then  $\varphi'_P(z)$  tends either to 0 or to  $\infty$ , depending on the sign of  $1 - \alpha_k$ .

Next, we recall the estimates of [27, Lemma 2.1.3] applied to our domain  $D$ .

**Lemma 5.** *Let  $\varphi$  be the conformal mapping from  $B_1^c$  onto  $D^c$ . Then, for  $0 < h' < h$*

$$d(L_h, L_{h'}) \geq \frac{\rho^2}{8\varphi'(\infty)} \frac{h'}{(1+h)^3} (h - h'), \quad |\varphi'(z)| \leq \frac{\varphi'(\infty)|z|}{|z| - 1} \quad |z| > 1.$$

*Proof.* We refer to [27, Appendix A.2] for the second bound, which is based on the “area formula” of [26, Vol. III, Th. 1.4], while here we report the proof of the first bound given in [27, Appendix A.2], taking into account our assumptions on  $D$ .

Fix  $0 < h' < h$ ;  $L_h, L_{h'}$  are compact, thus we can choose  $1 < |z_1| = 1 + h' < |z_2| = 1 + h$  such that  $d(L_h, L_{h'}) = |\varphi(z_2) - \varphi(z_1)|$ . Then,

$$\begin{aligned} h - h' &\leq |z_2 - z_1| = |\varphi^{-1}(\varphi(z_2)) - \varphi^{-1}(\varphi(z_1))| \\ &= \left| \int_{\varphi(z_1)}^{\varphi(z_2)} (\varphi^{-1})'(w) dw \right| \leq |\varphi(z_2) - \varphi(z_1)| \sup_{1+h' \leq |\varphi^{-1}(w)| \leq 1+h} |(\varphi^{-1})'(w)| \\ &= |\varphi(z_2) - \varphi(z_1)| \sup_{1+h' \leq |z| \leq 1+h} \left| \frac{1}{\varphi'(z)} \right| \\ &\stackrel{[27, p. 165]}{\leq} |\varphi(z_2) - \varphi(z_1)| \varphi'(\infty) \sup_{1+h' \leq |z| \leq 1+h} \frac{(1+|z|)^3}{(|z|-1)|\varphi(z)|^2} \end{aligned}$$

<sup>4</sup>Notice that, in both these references, the inverse conformal map  $\varphi_D^{-1}$  is used.

$$\leq d(L_h, L_{h'})\varphi'(\infty)\frac{(2+h)^3}{h'\rho^2} \leq d(L_h, L_{h'})\varphi'(\infty)\frac{8(1+h)^3}{h'\rho^2},$$

which gives the result. The bound we used from [27, p. 165] is a consequence of the ‘‘distortion theorem’’, see [26, Vol. III, Theorems 1.7 and 1.9].  $\square$

The following result is a direct consequence of Schwarz’s Lemma [17, Theorem 5.10b], i.e., of the fact that every holomorphic function  $f : B_1 \rightarrow B_1$  satisfies  $|f(z)| \leq |z| \forall z \in B_1$ , applied to the function  $z \mapsto 1/(\varphi_{\mathcal{D}_1}^{-1}(\varphi_{\mathcal{D}_2}(1/z)))$ .

**Lemma 6.** *Let  $\mathcal{D}_1 \subset \mathcal{D}_2$  be two bounded, simply connected, Lipschitz domains. Then,  $\text{Int } L_h[\mathcal{D}_1] \subset \text{Int } L_h[\mathcal{D}_2]$ , for all  $h > 0$ .*

## 2. DISTANCE ESTIMATES FOR LEVEL LINES OF $\varphi_D$

We need precise quantitative information of how far the level lines  $L_h$  move away from  $\partial D$  as  $h$  increases. It is provided by the following key result.

**Theorem 7.** *Let  $L_h$  be the  $h$ -level line of the conformal mapping of  $D$ . Define  $0 < \xi \leq 1$  as*

$$\xi := \begin{cases} \frac{2}{\pi} \arcsin \frac{\rho_0}{1-\rho} & \text{if } D \text{ is non convex,} \\ 1 & \text{if } D \text{ is convex.} \end{cases}$$

Then, provided that  $0 < h \leq 1$ , we have

$$\forall w \in \partial D, \forall w_h \in L_h, \quad |w - w_h| \geq C_I h^2, \quad (7)$$

$$\forall w_h \in L_h \exists w \in \partial D : \quad |w - w_h| \leq C_E h^\xi, \quad (8)$$

where we have set  $C_I := \frac{\rho}{4}$  and  $C_E := \frac{27}{\xi}$ .

**Remark 8.** *In the case of a convex polygonal domain  $D$ , (8) holds with  $C_E = 9$  instead of 27 and, for more general convex domains,  $C_E$  can be improved up to  $9 + c_0$ , with any  $c_0 > 0$ ; see Section 2.2 below.*

**Remark 9.** *The bounds (7) and (8) can be rewritten as*

$$d(L_h, \partial D) \geq C_I h^2, \quad d(w_h, \partial D) \leq C_E h^\xi \quad \forall w_h \in L_h.$$

A result in the spirit of Theorem 7 is proved in [27, Proposition 2.1.6]. There, an upper bound for  $d(w, L_h)$  with  $w \in \partial D$  is given, which is different from our estimate (8). Another difference is that the exponents of  $h$  in the bounds (7) and (8), as well as the constants  $C_I$  and  $C_E$ , are specified under our assumptions on  $D$ .

The proofs of the two bounds (7) and (8) are given in the following Section 2.1 and 2.2, respectively. On first reading these may be skipped.

### 2.1. Proof of the lower bound (7)

**Lemma 10.** *Let  $S \subset \mathbb{C}$  be the segment  $[-\rho, \rho]$ ,  $\rho > 0$ , on the real axis. Then  $d(\rho, L_h[S]) = \frac{\rho h^2}{2(1+h)}$  for all  $h > 0$ .*

*Proof.* For any  $\rho > 0$ , the Joukowski map [17, §5.1, page 294]  $J(z) = \frac{\rho}{2} \left( z + \frac{1}{z} \right)$  is the conformal mapping that maps  $B_1^c$  in the exterior of the segment  $S$ , with  $J(\partial B_1) = S$ ,  $J(\infty) = \infty$  and  $J'(\infty) = \rho/2$ . Its level lines are ellipses whose foci are the endpoints of  $S$ . For every  $h > 0$ ,

$$d(\rho, L_h[S]) = \min_{z \in \partial B_{1+h}} |\rho - J(z)| = \min_{\theta \in [-\pi, \pi]} \left| \rho - \frac{\rho}{2} \left( (1+h)e^{i\theta} + \frac{1}{(1+h)e^{i\theta}} \right) \right|$$

$$= \frac{\rho}{2(1+h)} \min_{\theta \in [-\pi, \pi]} \left| \frac{2(1+h)e^{i\theta} - (1+h)^2 e^{2i\theta} - 1}{e^{i\theta}} \right| = \frac{\rho}{2(1+h)} \min_{\theta \in [-\pi, \pi]} |(1+h)e^{i\theta} - 1|^2;$$

the minimum is  $h^2$  and it is achieved for  $\theta = 0$ ; the proof is complete.  $\square$

*Proof of (7).* The proof proceeds along the lines of [27, Proposition 2.1.6]. Since  $D$  is star-shaped with respect to the origin and  $B_\rho \subseteq D$ , then for any  $w \in \partial D$ , there exists a (closed) straight segment  $S_w$  with one endpoint at  $w$  and length  $2\rho$  such that  $S_w \subset \overline{D}$ . By Lemma 10 and Lemma 6, we have

$$\frac{\rho h^2}{2(1+h)} = d(w, L_h[S_w]) \leq d(w, L_h) \quad \forall w \in \partial D,$$

which implies (7) with  $C_I = \rho/4$ , since  $h \leq 1$ .  $\square$

**Remark 11.** In Proposition 2.1.6 of [27] a bound similar to (7) was established with a better power of  $h$ , i.e.,  $2 - \Lambda$  instead of 2. This was proved by comparing the level lines of  $D$  with those of a triangle, instead of comparing with those of a segment. We were not able to prove this result with a fully explicit constant  $C_I$ . On the other hand, exponent 2 is sufficient to establish exponential convergence for the approximations of holomorphic functions by complex polynomials.

## 2.2. Proof of the upper bound (8) for convex domains

In this section we consider the case of convex  $D$ , which already reveals the key ideas with moderate technical complexity. For the much more intricate case of general  $D$  with non convex boundary, we refer to Appendix A.

*Proof of (8) for convex domains.* We consider first the case when  $D$  is a convex polygon (with straight sides) with vertices  $\{w_k\}_{k=1}^N$  and corresponding internal angles  $\{\alpha_k\}_{k=1}^N$ ; set  $z_k = \varphi^{-1}(w_k) \in \partial B_1$ ,  $k = 1, \dots, N$ .

Fix  $w_h \in L_h$  and set  $z_h = \varphi^{-1}(w_h) \in \partial B_{1+h}$ ; thus  $z_h = (1+h)e^{i\theta}$ , for some  $\theta \in [-\pi, \pi]$ . Define  $z = e^{i\theta}$ ,  $w = \varphi(z)$ , and denote by  $S$  the straight segment of length  $h$  connecting  $z$  and  $z_h$ . From (6) and (5) we have

$$|w_h - w| = |\varphi(z_h) - \varphi(z)| \leq \int_S |\varphi'(y)| dy \leq \varphi'(\infty) \int_S \frac{1}{|y|^2} \prod_{k=1}^N |y - \bar{z}_k|^{1-\alpha_k} dy.$$

For any  $y \in S$ , we have  $|y - \bar{z}_k| \leq 2 + h$  and, due to the convexity of  $D$ ,  $1 - \alpha_k \geq 0$ ,  $k = 1, \dots, N$ . Then, recalling that  $\sum_{k=1}^N (1 - \alpha_k) = 2$ , we arrive at

$$\prod_{k=1}^N |y - \bar{z}_k|^{1-\alpha_k} \leq (2+h)^{\sum_{k=1}^N (1-\alpha_k)} = (2+h)^2.$$

Notice that this bound is independent of the number  $N$  of the vertices of  $P$ . Using  $|y| \geq 1$  and (3), since  $h \leq 1$ , we obtain

$$|w_h - w| \leq (1-\rho)(2+h)^2 \int_S 1 dy \leq (2+h)^2 h \leq 9h.$$

If a convex  $D$  has more general shape, we exploit the fact that, for any fixed  $\varepsilon > 0$ , we can find a convex polygon  $P_\varepsilon$  containing  $D$  such that, for all  $w \in \partial P_\varepsilon$ ,  $d(w, \partial D) < \varepsilon$ , [38, Theorem 3.1.6]; for  $\varepsilon$  small enough,  $P_\varepsilon \subset B_1$ , thus  $\varphi'_{P_\varepsilon}(\infty) \leq 1$ .

Fix  $w_h \in L_h = L_h[\overline{D}]$ . Let  $P_\varepsilon$  be an approximating polygon as before, with  $\varepsilon \leq \frac{1}{2}d(w_h, \partial D)$ . Then,  $w_h \in L_{h'}[\overline{P_\varepsilon}]$  with  $h' \leq h$ , as a consequence of Lemma 6. Let  $z_{h'} = \varphi_{P_\varepsilon}^{-1}(w_h) = (1+h')e^{i\theta}$ , and define  $z = e^{i\theta}$ . Then,

$$d(w_h, \partial D) \leq d(w_h, \varphi_{P_\varepsilon}(z)) + d(\varphi_{P_\varepsilon}(z), \partial D) = |\varphi_{P_\varepsilon}(z_{h'}) - \varphi_{P_\varepsilon}(z)| + d(\varphi_{P_\varepsilon}(z), \partial D)$$



$$\leq (2 + h')^2 h' + \varepsilon \leq (2 + h)^2 h + \frac{1}{2} d(w_h, \partial D),$$

which implies  $d(w_h, \partial D) \leq 2(2 + h)^2 h \leq 18h$ .  $\square$

### 3. INTERPOLATION ESTIMATES

In this section, we prove error estimates for the approximation of holomorphic functions by means of polynomials. We first state some auxiliary results.

#### 3.1. Auxiliary results

We define the “polar parametrisation”  $\Psi : \mathbb{C} \rightarrow \mathbb{C}$  such that

$$\Psi(B_1) = D, \quad \Psi(re^{i\theta}) = \psi(\theta)re^{i\theta}, \quad \psi : [-\pi, \pi] \rightarrow [\rho, 1 - \rho].$$

**Lemma 12.** *The function  $\psi : [-\pi, \pi] \rightarrow [\rho, 1 - \rho]$  is Lipschitz continuous with constant  $L_\psi$  satisfying*

$$L_\psi := \sup_{\theta \in [-\pi, \pi]} \psi'(\theta) \leq \frac{(1 - \rho)^2}{\rho_0}.$$

*Proof.* Assumption 1 guarantees that  $D$  is a Lipschitz domain, therefore by Rademacher’s theorem (see [12, §3.1.2]),  $\psi$  is differentiable almost everywhere and, for almost every point of  $\partial D$ , there exists a tangent line. Because of the star-shapedness requirement, no tangent line to  $\partial D$  can intersect the open ball  $B_{\rho_0}$ .

Therefore the steepest (in polar coordinates) possible tangent line at a point  $\psi(\theta)$  is tangent to  $\partial B_{\rho_0}$ . Since the angular derivative of a straight line is larger for points with larger moduli, we can bound  $\psi'(\theta)$  with the angular derivative at  $\theta = 0$  of one of the two straight lines through  $1 - \rho$  that are tangent to  $B_{\rho_0}$ .

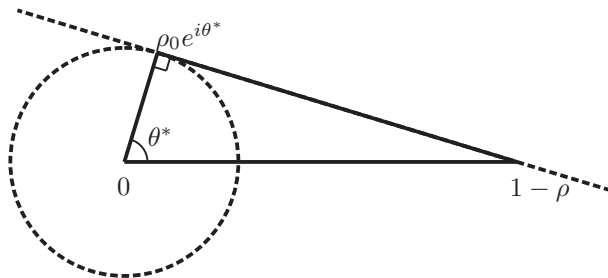


FIGURE 2. The extremal case in the proof of Lemma 12 and the angle  $\theta^*$ .

This line has polar representation  $r(\theta) = \rho_0 / \cos(\theta^* - \theta)$ , where  $\theta^* = \arccos \frac{\rho_0}{1 - \rho}$  (i.e.,  $\theta^*$  is the angle at 0 of the rectangular triangle of vertices 0,  $1 - \rho$  and the tangent point of the line to  $\partial B_{\rho_0}$ ; see Figure 2). Its “polar slope” in  $\theta = 0$  is given by

$$|r'(\theta)|_{\theta=0} = \left[ \frac{\rho_0 |\sin(\theta^* - \theta)|}{\cos^2(\theta^* - \theta)} \right]_{\theta=0} = \frac{\rho_0 |\sin \theta^*|}{\cos^2 \theta^*} = \frac{(1 - \rho)^2}{\rho_0} \left| \sin \arccos \frac{\rho_0}{1 - \rho} \right| \leq \frac{(1 - \rho)^2}{\rho_0}.$$

Then  $|\psi'(0)| \leq \frac{(1 - \rho)^2}{\rho_0}$  and the proof is complete.  $\square$

The inverse of  $\Psi$  is given by  $\Psi^{-1}(re^{i\theta}) = \frac{1}{\psi(\theta)}re^{i\theta}$  or, in Cartesian coordinates (after identifying  $\mathbb{C}$  with  $\mathbb{R}^2$ ),

$$\Psi^{-1}(r \cos \theta, r \sin \theta) = \left( \frac{r}{\psi(\theta)} \cos \theta, \frac{r}{\psi(\theta)} \sin \theta \right) =: (F_1, F_2). \quad (9)$$

Moreover,  $\Psi^{-1}$  is Lipschitz continuous, and an estimate for its Lipschitz constant is given in the next Lemma.

**Lemma 13.** *The function  $\Psi^{-1} : \mathbb{C} \rightarrow \mathbb{C}$  is Lipschitz continuous with constant  $L_{\Psi^{-1}}$  satisfying*

$$L_{\Psi^{-1}} := \sup_{w, v \in \mathbb{C}, w \neq v} \frac{|w - v|}{|\Psi(w) - \Psi(v)|} \leq \frac{2(2\rho + L_\psi)}{\rho^2},$$

with  $L_\psi$  as in Lemma 12.

*Proof.* Let  $D\Psi^{-1}$  be the Jacobian of  $\Psi^{-1}$ . Considering the representation (9) of  $\Psi^{-1}$ , we have

$$\begin{aligned} \frac{\partial F_1}{\partial x} &= \frac{x \cos \theta}{r \psi(\theta)} - \frac{y - \sin \theta \psi(\theta) + \cos \theta \psi'(\theta)}{r (\psi(\theta))^2}, & \frac{\partial F_2}{\partial x} &= \frac{x \sin \theta}{r \psi(\theta)} - \frac{y \cos \theta \psi(\theta) + \sin \theta \psi'(\theta)}{r (\psi(\theta))^2}, \\ \frac{\partial F_1}{\partial y} &= \frac{y \cos \theta}{r \psi(\theta)} + \frac{x - \sin \theta \psi(\theta) + \cos \theta \psi'(\theta)}{r (\psi(\theta))^2}, & \frac{\partial F_2}{\partial y} &= \frac{y \sin \theta}{r \psi(\theta)} + \frac{x \cos \theta \psi(\theta) + \sin \theta \psi'(\theta)}{r (\psi(\theta))^2}. \end{aligned}$$

Since  $|x|, |y| \leq r$  and  $0 < \rho < |\psi(\theta)| < 1$ , we can bound  $\|D\Psi^{-1}\|_{L^\infty(\mathbb{C})}$  (in the matrix  $\infty$ -norm) as

$$\|D\Psi^{-1}\|_{L^\infty(\mathbb{C})} \leq 2 \left( \frac{1}{\rho} + \frac{1}{\rho} + \frac{L_\psi}{\rho^2} \right) = \frac{2(2\rho + L_\psi)}{\rho^2}.$$

Since  $L_{\Psi^{-1}} = \|D\Psi^{-1}\|_{L^\infty(\mathbb{C})}$  the proof is complete.  $\square$

**Lemma 14.** *For every positive  $h$ , the following bound holds:*

$$|e^{i\theta} - (1+h)|^2 \geq \left( \frac{2}{\pi} \right)^2 (\theta^2 + h^2) =: C_B^2 (\theta^2 + h^2) \quad \forall \theta \in [-\pi, \pi].$$

*Proof.* Using  $1 - \cos \theta \geq \frac{2}{\pi^2} \theta^2$  for any  $\theta \in [-\pi, \pi]$ , we have

$$|e^{i\theta} - (1+h)|^2 = (1+h - \cos \theta)^2 + (\sin \theta)^2 = h^2 + 2(1 - \cos \theta)(h+1) \geq \frac{4}{\pi^2} (\theta^2 + h^2).$$

$\square$

Now, we provide a refined version of [27, Lemma 2.1.8].

**Lemma 15.** *If  $0 < h \leq C_I$  is such that  $L_h \subset B_{1+\rho}$  and  $w_0 \in L_h$ , then*

$$\int_{\partial D} \frac{1}{|w - w_0|} dw \leq C_D |\log h|, \quad \text{where} \quad C_D = 4\pi\sqrt{2}L_\psi L_{\Psi^{-1}},$$

with  $L_\psi$  and  $L_{\Psi^{-1}}$  as in Lemma 12 and Lemma 13, respectively.

*Proof.* Fix  $w_0 \in L_h$ , and assume, with no loss of generality, that  $w_0$  is on the positive real axis. Define  $d := w_0 - \psi(0)$  and notice that  $d(w_0, \partial D) \leq d \leq 1$ .

Setting  $w(\theta) := \Psi(e^{i\theta}) = \psi(\theta)e^{i\theta} \in \partial D$ , using Lemma 13, Lemma 14 and  $\psi(\theta) < 1$ , we obtain, for all  $\theta \in [-\pi, \pi]$ ,

$$|w(\theta) - w_0|^2 \geq L_{\Psi^{-1}}^{-2} |\Psi^{-1}(w(\theta)) - \Psi^{-1}(w_0)|^2 = L_{\Psi^{-1}}^{-2} |e^{i\theta} - w_0/\psi(0)|^2 \quad (10)$$

$$\begin{aligned}
&\geq L_{\Psi^{-1}}^{-2} C_B^2 \left[ \theta^2 + \left( \frac{w_0}{\psi(0)} - 1 \right)^2 \right] = L_{\Psi^{-1}}^{-2} C_B^2 \left[ \theta^2 + \left( \frac{w_0 - \psi(0)}{\psi(0)} \right)^2 \right] \\
&> L_{\Psi^{-1}}^{-2} C_B^2 [\theta^2 + (w_0 - \psi(0))^2] = \frac{4}{\pi^2} L_{\Psi^{-1}}^{-2} (\theta^2 + d^2) =: L_D^2 (\theta^2 + d^2).
\end{aligned}$$

Then,

$$\begin{aligned}
\int_{\partial D} \frac{1}{|w - w_0|} dw &= \int_{-\pi}^{\pi} \frac{1}{|w(\theta) - w_0|} |w'(\theta)| d\theta \stackrel{\text{Lem. 12}}{\leq} L_{\psi} \int_{-\pi}^{\pi} \frac{1}{|w(\theta) - w_0|} d\theta \\
&\stackrel{(10)}{\leq} L_{\psi} L_D^{-1} \int_{-\pi}^{\pi} \frac{1}{\sqrt{\theta^2 + d^2}} d\theta \leq 2\sqrt{2} L_{\psi} L_D^{-1} \int_0^{\pi} \frac{1}{\theta + d} d\theta \leq 2\sqrt{2} L_{\psi} L_D^{-1} (\log(\pi + d) - \log d).
\end{aligned}$$

Since

$$h \leq C_I = \frac{\rho}{4} \leq \frac{1}{8} < \frac{1}{\pi + 1} \leq \frac{1}{\pi + d} < 1,$$

we have  $\log(\pi + d) \leq |\log h|$  and

$$\begin{aligned}
\int_{\partial D} \frac{1}{|w - w_0|} dw &\leq 2\sqrt{2} L_{\psi} L_D^{-1} (|\log h| + |\log d|) \\
&\stackrel{d \geq d(w_0, \partial D)}{\leq} 2\sqrt{2} L_{\psi} L_D^{-1} (|\log h| + |\log d(w_0, \partial D)|) \\
&\stackrel{w_0 \in L_h, (7)}{\leq} 2\sqrt{2} L_{\psi} L_D^{-1} (|\log C_I| + 3|\log h|) \\
&\stackrel{h \leq C_I < 1}{\leq} 8\sqrt{2} L_{\psi} L_D^{-1} |\log h| = \underbrace{4\pi\sqrt{2} L_{\psi} L_{\Psi^{-1}}}_{=: C_D} |\log h|,
\end{aligned}$$

where we can use (7), because  $h \leq C_I < 1$ . □

**Remark 16.** Using Lemma 13, Lemma 12 and  $\rho_0 < \rho \leq 1/2$ , we have the bound

$$\begin{aligned}
C_D &= 4\pi\sqrt{2} L_{\psi} L_{\Psi^{-1}} \leq 4\pi\sqrt{2} L_{\psi} \frac{2(2\rho + L_{\psi})}{\rho^2} \leq 8\pi\sqrt{2} \frac{(1 - \rho)^2 (2\rho\rho_0 + (1 - \rho)^2)}{\rho_0^2 \rho^2} \\
&\leq \frac{8\pi\sqrt{2}}{\rho_0^2 \rho^2} [(1 - \rho)^2 (2\rho^2 + (1 - \rho)^2)] \leq \frac{8\pi\sqrt{2}}{\rho_0^2 \rho^2} [(1 - \rho)^2 (1 + \rho^2)] \leq \frac{8\pi\sqrt{2}}{\rho_0^2 \rho^2},
\end{aligned}$$

since  $(1 - \rho)^2 (1 + \rho^2) = 1 - 2\rho + 2\rho^2 - 2\rho^3 + \rho^4 < 1$ .

Define the sequence of complex polynomials  $\{\omega_p\}_{p \in \mathbb{N}}$  with

$$\omega_p(w) := \prod_{k=0}^{p-1} (w - \varphi(e^{2\pi ik/p})),$$

where  $\varphi$  is the exterior conformal mapping of  $D$ .

**Lemma 17.** [27, Lemma 2.2.9] Under the same hypothesis on  $h$  as in Lemma 15 we find

$$h^{C_D} |\varphi'(\infty)|^p (1 + h)^p \leq |\omega_p(w)| \leq h^{-C_D} |\varphi'(\infty)|^p (1 + h)^p \quad \forall w \in L_h, \forall p \in \mathbb{N},$$

where  $C_D$  is the constant in Lemma 15.

*Proof.* We refer to the proof of [27, Lemma 2.2.9]. The constant at the exponents of  $h$  is equal to  $C_D$  and the threshold on  $h$  is the one needed by Lemma 15. □

### 3.2. Main interpolation estimates

As in Theorem 2, for  $\delta > 0$ , define the inflated domain

$$D_\delta := \{w \in \mathbb{C} : d(w, D) < \delta\}. \quad (11)$$

Assume  $\ell > 0$ ; then Theorem 7 guarantees that, if  $0 < h < \frac{1}{\ell} \left(\frac{\delta}{C_E}\right)^{1/\xi}$ , then  $L_{\ell h} \subset D_\delta$ .

Our main approximation results is a refinement of [27, Theorem 2.2.10].

**Theorem 18.** *Fix  $0 < \delta \leq 1/2$ . Provided that*

$$0 < h < h^*(\delta) := \min \left\{ \frac{1}{3} \left(\frac{\delta}{C_E}\right)^{1/\xi}, \frac{\rho}{4} \right\}, \quad (12)$$

there exist  $C_{\text{appr}} > 0$  and  $\alpha > 0$  depending only on  $D$  through  $\rho$  and  $\rho_0$ , such that, for any  $f$  holomorphic in  $D_\delta$ , there is a sequence of polynomials  $\{q_p\}_{p \geq 1}$  of degree at most  $p$  such that

$$\|f - q_p\|_{L^\infty(\text{Int } L_h)} \leq C_{\text{appr}} h^{-\alpha} (1+h)^{-p} \|f\|_{L^\infty(\text{Int } L_{3h})},$$

where

$$C_{\text{appr}} \leq \frac{20(1-\rho)^2}{3\rho^2} \leq \frac{7}{\rho^2}, \quad \alpha \leq 3 + \frac{16\sqrt{2}\pi}{\rho_0^2 \rho^2} \leq \frac{72}{\rho_0^2 \rho^2}.$$

**Remark 19.** *Compared to [27, Theorem 2.2.10], this estimate features fully explicit bounds in terms of shape parameters of  $D$ . Moreover, no complete proof of Theorem 2.2.10 was given in [27], cf. Remark 9.*

*Proof of Theorem 18.* We choose  $q_p$  as the polynomial of degree  $p$  which interpolates  $f$  at the  $p+1$  points  $\varphi(e^{2\pi i k/(p+1)})$ ,  $k = 0, \dots, p$ . Since  $L_{3h} \subset D_\delta$ , using the Hermite interpolation error formula (see [27, p. 17] or [9, Theorem 3.6.1]), we have

$$\|f - q_p\|_{L^\infty(\text{Int } L_h)} = \sup_{w \in \text{Int } L_h} \left| \frac{1}{2\pi i} \int_{L_{3h}} \frac{\omega_p(w)f(t)}{\omega_p(t)(t-w)} dt \right| \leq \frac{\text{length}(L_{3h}) \sup_{w \in \text{Int } L_h} |\omega_p(w)| \|f\|_{L^\infty(\text{Int } L_{3h})}}{2\pi \inf_{t \in \text{Int } L_{3h}} |\omega_p(t)| d(L_h, L_{3h})}.$$

Since  $\varphi$  is a curve parametrisation  $\varphi : \partial B_{1+3h} \rightarrow L_{3h}$ , it satisfies  $\text{length}(L_{3h}) \leq 2\pi(1+3h) \sup_{|z|=1+3h} |\varphi'(z)|$ . This, together with the lower bound of  $d(L_h, L_{3h})$  and the upper bound of  $|\varphi'(z)|$  given in Lemma 5, and the bounds in Lemma 17, gives

$$\begin{aligned} \|f - q_p\|_{L^\infty(\text{Int } L_h)} &\leq \frac{8(1+3h)^5 \varphi'(\infty)^2}{6h^3 \rho^2} (3h^2)^{-C_D} \left(\frac{1+h}{1+3h}\right)^p \|f\|_{L^\infty(\text{Int } L_{3h})} \\ &\leq \frac{4\varphi'(\infty)^2}{3^{1+C_D} \rho^2} h^{-3-2C_D} \left(\frac{1+h}{1+3h}\right)^p (1+3h)^5 \|f\|_{L^\infty(\text{Int } L_{3h})} \\ &\leq \frac{20(1-\rho)^2}{3\rho^2} h^{-3-2C_D} \left(\frac{1}{1+h}\right)^p \|f\|_{L^\infty(\text{Int } L_{3h})}, \end{aligned}$$

where in the last step we have used  $3^{1+C_D} > 3$ ,  $|\varphi'(\infty)| < 1 - \rho$ ,  $\frac{1+h}{1+3h} \leq \frac{1}{1+h}$ , and  $(1+3h)^5 < 5$ , since  $h \leq \rho/4 \Rightarrow h < 1/8$ . The use of Lemma 17 (and thus of Lemma 15) is legitimate due to the hypothesis imposed on  $h$  and  $\delta$ . The result of the theorem follows from the bound of  $C_D$  derived in Remark 16.  $\square$

Obviously, Theorem 2 from the Introduction is an immediate consequence of Theorem 18: given  $0 < h < h^*$ , just define  $C := C_{\text{appr}}(h^*(\delta))^{-\alpha}$  and  $b := \log(1+h^*(\delta))$ .

The polynomials  $q_p$  defined in the proof of Theorem 18 as the complex interpolants of  $f$  in special points, *simultaneously* approximate the first  $p$  derivatives of  $f$  (denoted  $f^{(j)}$ ,  $j = 1, \dots, p$ ), as established by the following corollary.

**Corollary 20.** *Under the assumptions of Theorem 18, for any<sup>5</sup>  $j \in \mathbb{N}$ ,  $j \leq p$ , we have*

$$\left\| f^{(j)} - q_p^{(j)} \right\|_{L^\infty(D)} \leq C_{\text{appr}} \frac{j!}{(C_I h^2)^j} h^{-\alpha} (1+h)^{-p} \|f\|_{L^\infty(\text{Int } L_{3h})}.$$

*Proof.* We use Cauchy's inequalities [26, Vol. I, Theorem 14.7] for the interpolation error  $f^{(j)} - q_p^{(j)}$  to obtain a sharp bound on the complex derivatives of holomorphic functions:

$$\left\| f^{(j)} - q_p^{(j)} \right\|_{L^\infty(D)} \leq \frac{j!}{d(L_h, \partial D)^j} \|f - q_p\|_{L^\infty(\text{Int } L_h)};$$

the assertion of the corollary follows from the bound (7) and from Theorem 18.  $\square$

As a consequence of the previous results, we can gauge the approximation of real-valued harmonic functions by harmonic polynomials. To this purpose, setting  $z = x + iy$  we identify  $S \subseteq \mathbb{C}$  and  $\{(x, y) \in \mathbb{R}^2 \mid z = x + iy \in S\}$  and now regard  $f : D_\delta \rightarrow \mathbb{C}$  as a *real analytic* function of two real variables  $f = f(x, y)$ . We also adopt this perspective for the polynomials  $q_p$ , which have been defined in the proof of Theorem 18 as the complex interpolants of  $f$  in special points.

The statement of the following results makes use of the (standard)  $W^{j,\infty}(S)$ -seminorms,  $j \in \mathbb{N}$ , and of the weighted Sobolev  $W^{1,\infty}(S)$ -norm, for sufficiently smooth functions, and  $S \subseteq D_\delta$ :

$$|u|_{W^{j,\infty}(S)} := \sup_{\beta \in \mathbb{N}_0^2, |\beta|=j} \|D^\beta u\|_{L^\infty(S)}, \quad \|u\|_{W^{1,\infty}(S)} := \|u\|_{L^\infty(S)} + \text{diam}(D_\delta) \|\nabla u\|_{L^\infty(S)}.$$

**Theorem 21.** *Fix  $0 < \delta \leq 1/2$ , and let  $h$  satisfy (12). For any real, harmonic function  $u$  in the inflated domain  $D_\delta$  defined in (11), there is a sequence of harmonic polynomials  $\{Q_p\}_{p \geq 1}$  of degree at most  $p$  such that*

$$\begin{aligned} \|u - Q_p\|_{L^\infty(D)} &\leq C_{\text{appr}} h^{-\alpha} (1+h)^{-p} \|u\|_{W^{1,\infty}(\text{Int } L_{3h})}, \\ |u - Q_p|_{W^{j,\infty}(D)} &\leq C_{\text{appr}} \left( \frac{2j}{C_I h^2} \right)^j h^{-\alpha} (1+h)^{-p} \|u\|_{W^{1,\infty}(\text{Int } L_{3h})}, \\ \|u - Q_p\|_{L^2(D)} &\leq \sqrt{|D|} C_{\text{appr}} h^{-\alpha} (1+h)^{-p} \|u\|_{W^{1,\infty}(\text{Int } L_{3h})}, \\ |u - Q_p|_{H^j(D)} &\leq \sqrt{|D|(j+1)} C_{\text{appr}} \left( \frac{2j}{C_I h^2} \right)^j h^{-\alpha} (1+h)^{-p} \|u\|_{W^{1,\infty}(\text{Int } L_{3h})} \end{aligned}$$

for all  $j \in \mathbb{N}$ ,  $j \leq p$ , where  $|D| < 1$  is the Lebesgue measure of  $D$ , and the constants  $C_{\text{appr}}$  and  $\alpha$  are the same as in Theorem 18.

*Proof.* For any real, harmonic function  $u$  on a simply-connected domain  $\mathcal{D} \ni (x_0, y_0)$ , there exists a unique holomorphic function  $f$  on  $\mathcal{D}$ , with  $f(x_0 + iy_0) \in \mathbb{R}$ , such that  $u(x, y) = \text{Re } f(x + iy)$  [26, Vol. II, Theorem 5.2]. More precisely,  $f(z) = u(x, y) + iv(x, y)$ , with  $z = x + iy$  and  $v$  a real, harmonic function satisfying the Cauchy–Riemann equations  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ,  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ , and  $v(x_0, y_0) = 0$ . If  $\mathcal{D}$  is star-shaped with respect to  $(x_0, y_0)$ , and  $\|u\|_{L^\infty(\mathcal{D})}$ ,  $\|\nabla u\|_{L^\infty(\mathcal{D})}$  are bounded, it holds  $\|f\|_{L^\infty(\mathcal{D})} \leq \|u\|_{L^\infty(\mathcal{D})} + \text{diam}(\mathcal{D}) \|\nabla u\|_{L^\infty(\mathcal{D})}$ . Moreover, if  $f$  is a holomorphic function, then  $u(\text{Re } z, \text{Im } z) = \text{Re } f(z)$  is harmonic; thus, the real part of any complex polynomial is a harmonic polynomial. Obviously,  $\|u\|_{L^\infty(\mathcal{D})} \leq \|f\|_{L^\infty(\mathcal{D})}$  holds true.

With these considerations, defining  $Q_p := \text{Re } q_p$ , with  $q_p$  as in Theorem 18, the desired bound in  $L^\infty$ -norm is direct consequence of Theorem 18. Notice that  $\|u\|_{L^\infty(\text{Int } L_{3h})}$  and  $\|\nabla u\|_{L^\infty(\text{Int } L_{3h})}$  are bounded (and thus  $\|u\|_{W^{1,\infty}(\text{Int } L_{3h})} < +\infty$ ) because, by (12), the (closed) set  $\text{Int } L_{3h}$  is contained in  $D_\delta$ , the (open) domain of analyticity of  $u$ .

<sup>5</sup>We use the following standard notation:  $\mathbb{N} = \{1, 2, \dots\}$ ,  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ .

For the bounds in  $W^{j,\infty}$ -norms, the inclusion  $D \subset L_h$ , the interior estimates for the derivatives of harmonic functions in [15, Theorem 2.10], and the bound (7) give

$$\begin{aligned} |u - Q_p|_{W^{j,\infty}(D)} &= \sup_{\beta \in \mathbb{N}_0^2, |\beta|=j} \|D^\beta(u - Q_p)\|_{L^\infty(D)} \\ &\leq \left(\frac{2j}{d(L_h, \Omega)}\right)^j \|u - Q_p\|_{L^\infty(\text{Int } L_h)} \leq \left(\frac{2j}{C_I h^2}\right)^j \|u - Q_p\|_{L^\infty(\text{Int } L_h)}; \end{aligned}$$

again, Theorem 18 allows to conclude. Finally, the bounds in integral norms follow from

$$|u - Q_p|_{H^j(D)}^2 := \sum_{\beta \in \mathbb{N}_0^2, |\beta|=j} \int_D |D^\beta(u(x) - Q_p(x))|^2 dx \leq |D|(j+1) |u - Q_p|_{W^{j,\infty}(D)}^2$$

and the previous inequalities.  $\square$

From Theorem 2, with the same considerations as in the proof of Theorem 21, we obtain the following result.

**Corollary 22.** *Fix  $0 < \delta \leq 1/2$  and  $j \in \mathbb{N}_0$ . There exist  $C > 0$  and  $b > 0$ , depending only on  $\rho, \rho_0, \delta$  and  $j$ , such that, for any real-valued, harmonic function  $u$  which is bounded along with its first-order derivatives in the inflated domain  $D_\delta$  defined in (11), there is a sequence of harmonic polynomials  $\{Q_p\}_p$  of degree at most  $p$  such that*

$$|u - Q_p|_{W^{j,\infty}(D)} \leq C e^{-bp} \|u\|_{W^{1,\infty}(D_\delta)}, \quad |u - Q_p|_{H^j(D)} \leq C e^{-bp} \|u\|_{W^{1,\infty}(D_\delta)}.$$

**Remark 23.** *The boundedness of  $f, u$  and  $\nabla u$  in Theorem 2 and Corollary 22 is assumed only in order to write estimates with  $L^\infty$ -norms in the whole  $D_\delta$  on the right-hand side. Actually, the estimates hold true also with  $\|f\|_{L^\infty(\text{Int } L_{3h})}$  and  $\|u\|_{W^{1,\infty}(\text{Int } L_{3h})}$  respectively, on the right-hand side, for any  $0 < h < h^*$ , with no need of assuming boundedness of  $f, u$  and  $\nabla u$  in  $D_\delta$ .*

*The constants  $C$  and  $b$  in Theorem 2 and Corollary 22 depend on  $\delta$  only through  $h^*(\delta)$  defined in (12).*

**Remark 24.** *The interpolating polynomials  $q_p$  (and  $Q_p$ ) in Theorem 2, Theorem 18 and Corollary 20 (Theorem 21 and Corollary 22, respectively) interpolate exactly the function  $f$  ( $u$ , respectively) in at least  $p+1$  points lying on the boundary of  $D$ . The exact location of the points depend on the conformal map  $\varphi_D$ . This fact follows from the definition of  $q_p$  given in the proof of Theorem 18 and the relations  $u = \text{Re } f$  and  $Q_p = \text{Re } q_p$ .*

#### 4. APPLICATION: EXPONENTIAL CONVERGENCE OF TREFFTZ $hp$ -dGFEM

In this section, we outline how to apply the estimates of Corollary 22 and prove exponential convergence of a *Trefftz  $hp$ -dGFEM* for the mixed Laplace boundary value problem (BVP), i.e. a FEM with discontinuous, piecewise harmonic, polynomial basis functions on a geometrically graded mesh. We establish exponential convergence with rate  $O(\exp(-b\sqrt{N}))$ , for some  $b > 0$ , in terms of the overall number  $N$  of degrees of freedom. This result is an improvement over the classical rate  $O(\exp(-b\sqrt[3]{N}))$  shown for inhomogeneous problems in [2,4]; this improvement is due to the use of harmonic polynomials, instead of complete polynomials, in the trial spaces. Indeed, as it was already observed e.g. in [28, Page 38], the  $(2p+1)$ -dimensional space of harmonic polynomials of degree at most  $p$  enjoys the same approximation properties (when approximating harmonic functions) of the space of continuous polynomials of the same degree, which has higher dimension  $\frac{1}{2}(p+1)(p+2)$ ; this is the reason of the better asymptotic properties (both in  $h$  and in  $p$ ) of Trefftz methods compared to classic schemes, and is reflected in the  $hp$ -analysis performed here.

Since we rely on the  $hp$ -dGFEM theory from [39], we restrict ourselves to the case of (straight) polygonal domains and meshes comprising (straight) triangles or parallelograms. The extension to curvilinear domains and mesh elements would require to develop, for such elements, several tools as polynomial  $hp$ -inverse estimates,

scaling estimates of Sobolev seminorms, and approximation estimates for linear and bilinear polynomials near corners. This goes beyond the scope of this paper.

#### 4.1. The Laplace BVP

Without further explanation, we use the notation for the weighted Sobolev spaces ( $H_{\underline{\beta}}^{m,l}(\Omega)$ ) and the countably normed spaces ( $\mathcal{B}_{\underline{\beta}}^{\ell}(\Omega)$  and  $\mathcal{C}_{\underline{\beta}}^{\ell}(\Omega)$ ) from [2, §2], along with the analyticity and analytic continuation results given in [2–5].

Let  $\Omega \subset \mathbb{R}^2$  be a bounded, Lipschitz polygon with corners  $\mathbf{c}_{\nu}$ ,  $1 \leq \nu \leq n_a$ , whose boundary is partitioned into a Dirichlet and a Neumann boundary  $\Gamma^{[0]}$  and  $\Gamma^{[1]}$ , respectively, such that the interiors of  $\Gamma^{[0]}$  and  $\Gamma^{[1]}$  do not overlap and  $\overline{\Gamma^{[0]}} \cup \overline{\Gamma^{[1]}} = \partial\Omega$ . Moreover, we assume that  $\Gamma^{[0]}$  has positive 1-dimensional measure. Consider the following (well-posed) boundary value problem: given  $g^{[i]}$ ,  $i = 0, 1$ , find  $u \in H^1(\Omega)$  such that

$$\Delta u = 0 \quad \text{in } \Omega, \quad (13a)$$

$$\gamma_0 u|_{\Gamma^{[0]}} = g^{[0]} \quad \text{on } \Gamma^{[0]}, \quad \gamma_1 u|_{\Gamma^{[1]}} = g^{[1]} \quad \text{on } \Gamma^{[1]}. \quad (13b)$$

Here,  $\gamma_0$  and  $\gamma_1$  denote trace and normal derivative operators, respectively.

There exists a weight vector  $\underline{\beta} \in (0, 1)^{n_a}$  such that, if  $g^{[i]} \in \mathcal{B}_{\underline{\beta}}^{\frac{3}{2}-i}(\Gamma^{[i]})$ ,  $i = 0, 1$ , problem (13) admits a unique solution  $u$  which belongs to  $\mathcal{C}_{\underline{\beta}}^2(\Omega)$ , [2, Theorem 3.5]. Moreover, as in [2, page 841], it can be proved that there exist two constants  $C_u > 0$  and  $d_u \geq 1$  such that

$$|(D^{\alpha}u)(\mathbf{x}_0)| \leq C_u \left( \frac{d_u}{\Phi(\mathbf{x}_0)} \right)^k k! \quad \forall \mathbf{x}_0 \in \Omega, \quad \alpha \in \mathbb{N}_0^2, \quad |\alpha| = k \geq 1, \quad (14)$$

where  $\Phi(\mathbf{x}_0) := \prod_{\nu=1}^{n_a} \min\{1, |\mathbf{x}_0 - \mathbf{c}_{\nu}|\}$ , thus  $u$  admits a real analytic continuation to the set

$$\mathcal{N}(u) := \bigcup_{\mathbf{x}_0 \in \overline{\Omega} \setminus \bigcup_{\nu=1}^{n_a} \mathbf{c}_{\nu}} \left\{ \mathbf{x} \in \mathbb{R}^2 : |\mathbf{x} - \mathbf{x}_0| < \frac{\Phi(\mathbf{x}_0)}{2d_u} \right\} \subset \mathbb{R}^2. \quad (15)$$

#### 4.2. Trefftz $hp$ -dGFEM

We now formulate the  $hp$ -dGFEM discretisation of the BVP (13) on *geometric mesh families*  $\mathfrak{M}_{\sigma} = \{\mathcal{T}_{\sigma}^{\ell}\}_{\ell=1}^{\infty}$  in  $\Omega$ , with increasing number  $\ell$  of layers and geometric grading factor  $0 < \sigma < 1$ .

##### 4.2.1. Geometric meshes

Given  $\ell \in \mathbb{N}$ , the mesh  $\mathcal{T}_{\sigma}^{\ell}$  is a partition of the domain  $\Omega$  into open triangles or parallelograms  $\Omega_{ij}^{\ell}$  (such that  $\overline{\Omega} = \bigcup_{i,j} \overline{\Omega}_{ij}^{\ell}$  and  $\Omega_{ij}^{\ell} \cap \Omega_{i'j'}^{\ell} = \emptyset$  if  $(i, j) \neq (i', j')$ ). The elements are grouped in *layers*, denoted by  $\mathcal{L}_{\sigma,i}^{\ell}$ ,  $1 \leq i \leq \ell$ , such that

$$\mathcal{T}_{\sigma}^{\ell} = \bigcup_{i=1}^{\ell} \mathcal{L}_{\sigma,i}^{\ell} = \left\{ \Omega_{ij}^{\ell} : 1 \leq i \leq \ell, 1 \leq j \leq \widehat{\mathcal{J}}(i) \right\},$$

where  $\widehat{\mathcal{J}}(i) \geq 1$  is the number of the elements in  $i$ -th layer  $\mathcal{L}_{\sigma,i}^{\ell}$ . Given an element  $\Omega_{ij}^{\ell} \in \mathcal{T}_{\sigma}^{\ell}$ , the index  $i$  denotes the layer  $\mathcal{L}_{\sigma,i}^{\ell}$  belongs to, and  $j$  identifies it among the  $\widehat{\mathcal{J}}(i)$  elements belonging to the  $i$ -th layer. We say that  $\mathcal{T}_{\sigma}^{\ell}$  is a geometric mesh if it belongs to a family  $\mathfrak{M}_{\sigma} = \{\mathcal{T}_{\sigma}^{\ell}\}_{\ell=1}^{\infty}$  that satisfies the assumptions (GM1)–(GM4) listed below.

For every element  $\Omega_{ij}^{\ell}$ , we define the following parameters:  $h_{ij}^{\ell} := \text{diam}(\Omega_{ij}^{\ell})$ ;  $\rho_{ij}^{\ell}$  and  $\mathbf{x}_{ij}^{\ell}$  the radius and the centre, respectively, of the largest ball inscribed in  $\Omega_{ij}^{\ell}$ ; and  $r_{ij}^{\ell} := \min_{1 \leq \nu \leq n_a} d(\mathbf{c}_{\nu}, \Omega_{ij}^{\ell})$  its distance from the nearest corner of  $\Omega$ .

**Assumption 25.** The family  $\mathfrak{M}_\sigma = \{\mathcal{T}_\sigma^\ell\}_{\ell=1}^\infty$  satisfies the following conditions.

(GM1) The elements are uniformly shape-regular triangles/parallelograms:  $\exists 0 < \kappa_1 \leq 1/2$ , independent of  $\sigma$ ,  $\ell$ ,  $i$  and  $j$ , such that, for all  $\mathcal{T}_\sigma^\ell \in \mathfrak{M}_\sigma$  and  $\Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell$ ,

$$\rho_{ij}^\ell \geq \kappa_1 h_{ij}^\ell > 0.$$

(GM2) The distance  $r_{ij}^\ell$  between an element  $\Omega_{ij}^\ell$  and the nearest corner of  $\Omega$  depends geometrically on its layer index  $i$ :  $\exists 0 < \kappa_{2-} \leq \kappa_{2+} < \infty$ , independent of  $\sigma$ ,  $\ell$ ,  $i$  and  $j$ , such that for all  $\mathcal{T}_\sigma^\ell \in \mathfrak{M}_\sigma$  and  $\Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell$ , with  $1 \leq i < \ell$ ,

$$\kappa_{2-} \sigma^i \leq r_{ij}^\ell \leq \kappa_{2+} \sigma^i.$$

The  $\ell$ -th layer is the set of the elements abutting at domain corners (i.e.,  $r_{ij}^\ell = 0 \Leftrightarrow i = \ell$ ).

(GM3) The size of an element  $\Omega_{ij}^\ell$  depends geometrically on its layer index  $i$ :  $\exists 0 < \kappa_{3-} \leq \kappa_{3+} < \infty$ , independent of  $\sigma$ ,  $\ell$ ,  $i$  and  $j$ , such that for all  $\mathcal{T}_\sigma^\ell \in \mathfrak{M}_\sigma$  and  $\Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell$ ,

$$\kappa_{3-} \sigma^i \leq h_{ij}^\ell \leq \kappa_{3+} \sigma^i.$$

(GM4) For  $\ell \geq 2$ ,  $\mathcal{T}_\sigma^\ell$  is obtained from  $\mathcal{T}_\sigma^{\ell-1}$  by only refining the elements in the layer  $\mathcal{L}_{\sigma, \ell-1}^{\ell-1}$  adjacent to the domain corners, forming two new layers  $\mathcal{L}_{\sigma, \ell-1}^\ell$  and  $\mathcal{L}_{\sigma, \ell}^\ell$ . Equivalently, the elements of  $\mathcal{L}_{\sigma, i}^\ell$  are uniquely defined for all  $\ell \geq i + 1$ :

$$\mathcal{L}_{\sigma, i}^\ell = \mathcal{L}_{\sigma, i}^{\ell'} \quad \forall i \in \{1, 2, \dots, \min(\ell, \ell') - 1\}; \quad \mathcal{L}_{\sigma, \ell}^\ell = \bigcup_{i=\ell}^{\ell'} \mathcal{L}_{\sigma, i}^{\ell'} \quad \forall \ell' > \ell \geq 1. \quad (16)$$

Note that (GM2) and (GM3) imply that the diameter of an element  $\Omega_{ij}^\ell$  is proportional to its distance from the domain corners:

$$\frac{\kappa_{3-}}{\kappa_{2+}} r_{ij}^\ell \leq h_{ij}^\ell \leq \frac{\kappa_{3+}}{\kappa_{2-}} r_{ij}^\ell \quad 1 \leq i < \ell, \quad 1 \leq j \leq \widehat{J}(\ell). \quad (17)$$

Using (GM1) and (GM3), we can control the area  $|\Omega_{ij}^\ell|$  of each element: for all  $\Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell$ ,  $\ell \in \mathbb{N}$ ,

$$(h_{ij}^\ell)^2 \geq |\Omega_{ij}^\ell| \geq \left| B_{\rho_{ij}^\ell}(\mathbf{x}_{ij}^\ell) \right| = \pi(\rho_{ij}^\ell)^2 \geq \pi(\kappa_1 h_{ij}^\ell)^2 \geq \pi(\kappa_1 \kappa_{3-})^2 \sigma^{2i}.$$

Moreover, (GM2) and (GM3) imply

$$\bigcup_{j=1}^{\widehat{J}(i)} \Omega_{ij}^\ell \subseteq \bigcup_{\nu=1}^{n_a} \left( B_{\max_{1 \leq j \leq \widehat{J}(i)} (r_{ij}^\ell + h_{ij}^\ell)}(\mathbf{c}_\nu) \right) \subseteq \bigcup_{\nu=1}^{n_a} \left( B_{(\kappa_{2+} + \kappa_{3+})\sigma^i}(\mathbf{c}_\nu) \right) \quad 1 \leq i \leq \ell,$$

from which  $\left| \bigcup_{j=1}^{\widehat{J}(i)} \Omega_{ij}^\ell \right| \leq n_a \pi (\kappa_{2+} + \kappa_{3+})^2 \sigma^{2i}$ . Therefore, the number of elements per layer is uniformly bounded in  $i$ :

$$\widehat{J}(i) \leq \frac{\left| \bigcup_{j=1}^{\widehat{J}(i)} \Omega_{ij}^\ell \right|}{\min_{1 \leq j \leq \widehat{J}(i)} |\Omega_{ij}^\ell|} \leq n_a \left( \frac{\kappa_{2+} + \kappa_{3+}}{\kappa_1 \kappa_{3-}} \right)^2 =: J^*(\mathfrak{M}_\sigma) \quad 1 \leq i \leq \ell, \quad \ell \in \mathbb{N}. \quad (18)$$

#### 4.2.2. $hp$ -subspaces on $\mathfrak{M}_\sigma$

For a positive integer  $p$ , let  $\mathbb{P}_p(D)$  be the space of bivariate real polynomials of degree at most  $p$  on a domain  $D \subset \mathbb{R}^2$ . Define the spaces  $\mathcal{S}^p(\mathcal{T}_\sigma^\ell)$  of discontinuous, piecewise polynomial functions of degree at most  $p$  on  $\mathcal{T}_\sigma^\ell$ :

$$\mathcal{S}^p(\mathcal{T}_\sigma^\ell) := \{v \in L^2(\Omega) : v|_{\Omega_{ij}^\ell} \in \mathbb{P}_p(\Omega_{ij}^\ell) \text{ for every } \Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell\}, \quad (19)$$



and its subspace of discontinuous, piecewise harmonic polynomials (i.e., the Trefftz subspace):

$$\mathcal{S}^{p,\Delta}(\mathcal{T}_\sigma^\ell) := \{v \in \mathcal{S}^p(\mathcal{T}_\sigma^\ell) : \Delta(v|_{\Omega_{ij}^\ell}) = 0 \text{ for every } \Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell\}. \quad (20)$$

For the sake of simplicity, we confine ourselves to the case where the same polynomial degree is used in every element of the mesh; the results below can be extended to more sophisticated degree distributions. For example, in the elements adjacent to the domain corners, the use of linear polynomials on triangles and bilinear polynomials on parallelograms preserves the exponential convergence. Thus, if quadrilateral elements are used near the domain corners, we will choose  $p \geq 2$  in order to include bilinear functions in the trial space. Polynomial degrees linearly decreasing with the layer index will also give the same convergence.

**Lemma 26.** *If the family  $\mathfrak{M}_\sigma$  satisfies Assumption 25, for all  $p, \ell \geq 1$ , we have*

$$\dim(\mathcal{S}^p(\mathcal{T}_\sigma^\ell)) \leq J^*(\mathfrak{M}_\sigma) \frac{(p+1)(p+2)\ell}{2} = O(p^2\ell), \quad \dim(\mathcal{S}^{p,\Delta}(\mathcal{T}_\sigma^\ell)) \leq J^*(\mathfrak{M}_\sigma) (2p+1)\ell = O(p\ell),$$

where  $J^*(\mathfrak{M}_\sigma)$  is defined in (18) and is independent of  $\ell$  and  $p$ .

*Proof.* The mesh  $\mathcal{T}_\sigma^\ell$  has at most  $J^*(\mathfrak{M}_\sigma)$  elements in each layer  $\mathcal{L}_{\sigma,i}^\ell$ ,  $1 \leq i \leq \ell$ , thus at most  $\ell J^*(\mathfrak{M}_\sigma)$  elements in total. Thus,  $\dim(\mathbb{P}_p(\Omega_{ij}^\ell)) = (p+1)(p+2)/2$  and  $\dim(\mathbb{P}_p(\Omega_{ij}^\ell) \cap \{v : \Delta v = 0\}) = 2p+1$  imply the assertion.  $\square$

#### 4.2.3. $hp$ -dGFEM

We consider both the *symmetric interior penalty* (SIP) and the *non symmetric interior penalty* (NIP) methods introduced, respectively, in [10], [6], and in [7], [34] (see [1] for a survey of interior penalty and other dGFEM for elliptic problems).

For a given mesh  $\mathcal{T}_\sigma^\ell \in \mathfrak{M}_\sigma$  on  $\Omega$ , let  $V_p(\mathcal{T}_\sigma^\ell)$  be either of the subspaces defined in (19) and (20). For simplicity, we denote here by  $K$  a generic element of  $\mathcal{T}_\sigma^\ell$ , instead of using the more detailed notation  $\Omega_{ij}^\ell$ . Let  $\mathcal{E}_{\text{int}}$  be the set the interior edges of  $\mathcal{T}_\sigma^\ell$ , i.e., the intersections between two elements of  $\mathcal{T}_\sigma^\ell$  that have positive 1-dimensional measure; moreover, let  $\mathcal{E}_D$  be the set of the edges of  $\mathcal{T}_\sigma^\ell$  lying on  $\Gamma^{[0]}$ , and set  $\mathcal{E}_{\text{int},D} := \mathcal{E}_{\text{int}} \cup \mathcal{E}_D$ . For a piecewise smooth function  $v$ , we define jumps and averages across the edges  $e \in \mathcal{E}_{\text{int},D}$ :

$$\begin{aligned} \{\!\!\{v}\!\!\}_e &:= \frac{v|_K + v|_{K'}}{2}, & \llbracket v \rrbracket_e &:= v|_K \mathbf{n}_K + v|_{K'} \mathbf{n}_{K'} & e \in \overline{K} \cap \overline{K'}, \quad K, K' \in \mathcal{T}_\sigma^\ell, \\ \{\!\!\{v}\!\!\}_e &:= \llbracket v \rrbracket_e := v|_e & & & e \subseteq \Gamma^{[0]}, \end{aligned}$$

where  $\mathbf{n}_K$  is the outgoing unit normal on  $\partial K$ . We set  $h_K := \text{diam}(K)$  and define the meshwidth function  $h : \mathcal{E}_{\text{int},D} \rightarrow \mathbb{R}$  as  $h(\mathbf{x}) := \min\{h_K : \mathbf{x} \in K \in \mathcal{T}_\sigma^\ell\}$ .

For  $\theta \in \{1, -1\}$  and  $v, w \in V_p(\mathcal{T}_\sigma^\ell)$ , define the two bilinear forms and linear functionals

$$\begin{aligned} B_{\mathcal{T}}^\theta(v, w) &:= \sum_{K \in \mathcal{T}_\sigma^\ell} \int_K \nabla v \cdot \nabla w \, d\mathbf{x} + \sum_{e \in \mathcal{E}_{\text{int},D}} \int_e \left( -\{\!\!\{\nabla v\}\!\!\} \cdot \llbracket w \rrbracket + \theta \llbracket v \rrbracket \cdot \{\!\!\{\nabla w\}\!\!\} + \mathbf{a} \llbracket v \rrbracket \cdot \llbracket w \rrbracket \right) ds, \\ L_{\mathcal{T}}^\theta(w) &:= \int_{\Gamma^{[1]}} g^{[1]} w \, ds + \theta \int_{\Gamma^{[0]}} g^{[0]} \gamma^{[1]} w \, ds + \int_{\Gamma^{[0]}} \mathbf{a} g^{[0]} w \, ds. \end{aligned}$$

Here,  $\mathbf{a}$  is the discontinuity stabilisation function given by  $\mathbf{a}(\mathbf{x}) := \alpha p^2 / h(\mathbf{x})$ , where  $\alpha > 0$  is a parameter independent of  $h$  and  $p$ . Fixing  $\theta \in \{1, -1\}$ , the  $hp$ -dGFEM reads: find  $u_p^\theta \in V_p(\mathcal{T}_\sigma^\ell)$  such that

$$B_{\mathcal{T}}^\theta(u_p^\theta, v_p) = L_{\mathcal{T}}^\theta(v_p) \quad \forall v_p \in V_p(\mathcal{T}_\sigma^\ell). \quad (21)$$

The method defined in (21) is SIP, for  $\theta = -1$ , and NIP, for  $\theta = 1$ .

Integrating by parts the volume term in  $B_{\mathcal{T}}^{\theta}$ , and using the fact that the discrete functions are harmonic, one would obtain an “ultra weak formulation”, containing only skeleton and boundary integrals. For Helmholtz BVPs, a similar approach has been first adopted in [8]; the corresponding  $hp$ -version is analysed in [21].

We recall the following result from [39], where the mesh-dependent norm  $\|\cdot\|_{dG}$  is defined by

$$\|w\|_{dG}^2 := \sum_{K \in \mathcal{T}_{\sigma}^{\ell}} \|\nabla w\|_{L^2(K)}^2 + \sum_{e \in \mathcal{E}_{\text{int},D}} \|\sqrt{\mathbf{a}} \llbracket w \rrbracket\|_{L^2(e)}^2 \quad w \in V_p(\mathcal{T}_{\sigma}^{\ell}).$$

**Proposition 27.** [39, Theorem 2.3.7, Corollary 2.4.2] *Let  $\underline{\beta} \in (0,1)^{n_a}$  be such that the analytical solution  $u$  to (13) belongs to  $C_{\underline{\beta}}^2(\Omega)$ . If either  $\theta = 1$  and  $\alpha$  is positive, or  $\theta = -1$  and  $\alpha$  is sufficiently large, then the  $hp$ -dGFEM (21) admits a unique solution.*

Moreover, let  $\pi_{\mathcal{T}} : H_{\underline{\beta}}^{2,2}(\Omega) \rightarrow V_p(\mathcal{T}_{\sigma}^{\ell})$  be an arbitrary operator such that, for every element  $K \in \mathcal{T}_{\sigma}^{\ell}$ , there exist at least two zeros of  $\eta := u - \pi_{\mathcal{T}}u$  in  $\overline{K}$ . For  $\theta = \pm 1$  (with sufficiently large  $\alpha$ , if  $\theta = -1$ ), it holds

$$\|u - u_p^{\theta}\|_{dG}^2 \leq Cp^2 \left\{ \sum_{K \in \mathcal{T}_{\sigma}^{\ell}} |\eta|_{H^1(K)}^2 + \sum_{K \in \mathcal{T}_{\sigma}^{\ell} \setminus \mathcal{K}_{\sigma}^{\ell}} h_K^2 |\eta|_{H^2(K)}^2 + \sum_{K \in \mathcal{K}_{\sigma}^{\ell}} h_K^{2(1-\beta_{[K]})} |\eta|_{H_{\underline{\beta}}^{2,2}(K)}^2 \right\} \quad (22)$$

where  $C > 0$  is independent of  $\sigma$ ,  $\ell$  and  $p$ . Here,  $\mathcal{K}_{\sigma}^{\ell} := \mathcal{L}_{\sigma,\ell}^{\ell} \subseteq \mathcal{T}_{\sigma}^{\ell}$  designates the set of elements abutting at domain corners and, for any  $K \in \mathcal{K}_{\sigma}^{\ell}$ ,  $\beta_{[K]} := \sup\{\beta_{\nu} : \mathbf{c}_{\nu} \in \partial K\}$ .

### 4.3. Exponential convergence of $hp$ -dGFEM

We apply the approximation estimates proved in Section 3.2 to establish exponential convergence of the  $hp$ -dGFEM scheme. We begin with the following lemma, which puts in relation the domain of analyticity of  $u$  and the geometric mesh family  $\mathfrak{M}_{\sigma}$ .

**Lemma 28.** *Let  $\mathfrak{M}_{\sigma}$  be a family of geometric meshes  $\mathcal{T}_{\sigma}^{\ell}$  on  $\Omega$  satisfying Assumption 25, and let  $u$  be the solution of the BVP (13) on  $\Omega$ . Then, there exists  $\delta_* > 0$  depending on  $u$  (only through  $d_u$  in (14)),  $\sigma$  and  $\mathfrak{M}_{\sigma}$ , such that  $u$  is analytic in  $\Omega_{ij}^{\ell} + B_{\delta_* h_{ij}^{\ell}} = \{\mathbf{x} \in \mathbb{R}^2 : d(\mathbf{x}, \Omega_{ij}^{\ell}) < \delta_* h_{ij}^{\ell}\}$  for all  $\Omega_{ij}^{\ell} \in \mathcal{T}_{\sigma}^{\ell} \setminus \mathcal{K}_{\sigma}^{\ell}$ ,  $\mathcal{T}_{\sigma}^{\ell} \in \mathfrak{M}_{\sigma}$ .*

*Proof.* We define the domain parameter

$$E_{\Omega} := \min \left\{ 1, \min_{1 \leq \nu \neq \nu' \leq n_a} \frac{|\mathbf{c}_{\nu} - \mathbf{c}_{\nu'}|}{2} \right\},$$

which depends only on the position of the corners of  $\Omega$ , and consider an arbitrary element  $\Omega_{ij}^{\ell} \in \mathcal{T}_{\sigma}^{\ell} \setminus \mathcal{K}_{\sigma}^{\ell}$ .

First, we consider the case  $\Omega_{ij}^{\ell} \subseteq B_{E_{\Omega}}(\mathbf{c}_{\nu'})$  for some  $\nu' \in \{1, \dots, n_a\}$ . Fix  $\mathbf{x} \in \Omega_{ij}^{\ell}$ ; by the triangular inequality  $|\mathbf{x} - \mathbf{c}_{\nu}| \geq |\mathbf{c}_{\nu} - \mathbf{c}_{\nu'}| - |\mathbf{x} - \mathbf{c}_{\nu'}| \geq E_{\Omega}$ , for all  $\nu \neq \nu'$ . The definition of  $\Phi$  and the bound (17) give

$$\Phi(\mathbf{x}) = \prod_{\nu=1}^{n_a} \max\{1, |\mathbf{x} - \mathbf{c}_{\nu}|\} \geq |\mathbf{x} - \mathbf{c}_{\nu'}| E_{\Omega}^{n_a-1} \geq r_{ij}^{\ell} E_{\Omega}^{n_a-1} \geq h_{ij}^{\ell} \frac{\kappa_{2-}}{\kappa_{3+}} E_{\Omega}^{n_a-1} \quad \forall \mathbf{x} \in \Omega_{ij}^{\ell}.$$

This, together with the definition of the domain of analyticity  $\mathcal{N}(u)$  in (15) and of the parameter  $d_u$  in (14), implies that

$$\frac{d(\Omega_{ij}^{\ell}, \partial \mathcal{N}(u))}{h_{ij}^{\ell}} \geq \frac{\inf_{\mathbf{x} \in \Omega_{ij}^{\ell}} \Phi(\mathbf{x})}{2d_u h_{ij}^{\ell}} \geq \frac{\kappa_{2-} E_{\Omega}^{n_a-1}}{2d_u \kappa_{3+}} =: \delta_1.$$

Now consider the case when  $\Omega_{ij}^\ell \not\subset B_{E_\Omega}(\mathbf{c}_\nu)$  for any  $\nu \in \{1, \dots, n_a\}$ . Fix  $\mathbf{x} \in \Omega_{ij}^\ell$  such that  $|\mathbf{x} - \mathbf{c}_\nu| \geq E_\Omega$  for every  $\nu \in \{1, \dots, n_a\}$ . Thus, by (GM2) and (GM3),

$$E_\Omega \leq \inf_{1 \leq \nu \leq n_a} |\mathbf{x} - \mathbf{c}_\nu| \leq r_{ij}^\ell + h_{ij}^\ell \leq (\kappa_{2+} + \kappa_{3+})\sigma^i \quad \Rightarrow \quad i \leq \left\lceil \frac{\log \frac{E_\Omega}{\kappa_{2+} + \kappa_{3+}}}{\log \sigma} \right\rceil =: i^*,$$

i.e.,  $\Omega_{ij}^\ell$  belongs to one of the first  $i^*$  layers. The elements in first  $i^*$  layers are uniquely defined in all the meshes with at least  $i^* + 1$  layers, see (GM4). Thus we can define

$$\delta_2 := \min_{\substack{i, j, \ell \in \mathbb{N} \\ i \leq i^*, i < \ell, 1 \leq j \leq \mathcal{J}(i)}} \frac{d(\Omega_{ij}^\ell, \partial \mathcal{N}(u))}{h_{ij}^\ell},$$

which is positive since is the minimum of a finite number of positive values, although  $\ell$  can take any value in  $\mathbb{N}$ .

Therefore, if  $\delta_* := \min\{\delta_1, \delta_2\}$ , for any element  $\Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell \setminus \mathcal{K}_\sigma^\ell$ , for any  $\mathcal{T}_\sigma^\ell \in \mathfrak{M}_\sigma$ , the solution  $u$  is analytic in  $\Omega_{ij}^\ell + B_{\delta_* h_{ij}^\ell}$ . Note that  $\delta_*$  depends on  $u$  through  $d_u$ , on  $\sigma$  through  $i^*$ , but is independent of  $i, j$  and  $\ell$ .  $\square$

**Theorem 29.** *Consider the solution  $u \in \mathcal{C}_\beta^2(\Omega)$  of the Laplace mixed BVP (13) and its approximation  $u_p^\theta \in V_p(\mathcal{T}_\sigma^\ell) := \mathcal{S}^{p, \Delta}(\mathcal{T}_\sigma^\ell)$  computed with the Trefftz hp-dGFEM (21) (with  $\alpha > 0$ , if  $\theta = 1$ , or  $\alpha$  sufficiently large, if  $\theta = -1$ ) on a family  $\mathfrak{M}_\sigma$  of geometric meshes  $\mathcal{T}_\sigma^\ell$  satisfying Assumption 25. Assume uniform polynomial degree  $p = \ell$  and define  $N := \dim(V_p(\mathcal{T}_\sigma^\ell))$ . Then,  $u_p^\theta$  converges exponentially to  $u$ : there exist  $b, C > 0$  (depending on  $u, \Omega, \sigma$  and  $\mathfrak{M}_\sigma$ , but independent of  $p = \ell$ ) such that*

$$\|u - u_p^\theta\|_{dG} \leq C \exp(-b\sqrt{N}).$$

*Proof.* Since  $N = O(p\ell)$  by Lemma 26 and  $p = \ell$ , we have to prove  $\|u - u_p^\theta\|_{dG} \leq Ce^{-b\ell}$ . Thanks to Proposition 27, we only need to define an operator  $\pi_\mathcal{T} : H_{\beta}^{2,2}(\Omega) \rightarrow V_p(\mathcal{T}_\sigma^\ell)$  with suitable approximation and interpolation properties. We treat separately the elements  $\Omega_{ij}^\ell$  adjacent to a domain corner ( $\Omega_{ij}^\ell \in \mathcal{K}_\sigma^\ell$ ) and the remaining ones ( $\Omega_{ij}^\ell$  with  $1 \leq i \leq \ell - 1$ ).

In the elements  $\Omega_{ij}^\ell \in \mathcal{K}_\sigma^\ell$ , we define  $\pi_\mathcal{T}(u)$  as the (piecewise) linear or bilinear interpolant of  $u$  at the vertices of  $\Omega_{ij}^\ell$ , if  $\Omega_{ij}^\ell$  is a triangle or a parallelogram, respectively. Then,  $\pi_\mathcal{T}(u)$  is obviously harmonic. Using [35, Lemma 4.16, Lemma 4.25] (see also [39, Lemma 2.5.2]), and taking into account (GM3) with  $i = \ell$ , the contribution of the elements  $\Omega_{ij}^\ell \in \mathcal{K}_\sigma^\ell$  to the right-hand side of (22) has exponential order of convergence in  $p = \ell$  (for some  $b \geq (1 - \max_\nu \beta_\nu)(-\log \sigma)$ ).

Consider now the elements  $\Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell \setminus \mathcal{K}_\sigma^\ell$ . For any  $\Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell \setminus \mathcal{K}_\sigma^\ell$ , due to Lemma 28, the solution  $u$  is analytic in  $\Omega_{ij}^\ell + B_{\delta_* h_{ij}^\ell}$ , for some  $\delta_*$  independent of  $i, j$  and  $\ell$ . Define the corresponding scaled element  $D := \widehat{\Omega}_{ij}^\ell := \{\widehat{\mathbf{x}} := (\mathbf{x} - \mathbf{x}_{ij}^\ell)/h_{ij}^\ell \in \mathbb{R}^2 : \mathbf{x} \in \Omega_{ij}^\ell\}$  and the scaled solution  $\widehat{u}(\widehat{\mathbf{x}}) := u(h_{ij}^\ell \widehat{\mathbf{x}} + \mathbf{x}_{ij}^\ell)$ . The scaled element satisfies Assumption 1 with  $\rho = \rho_{ij}^\ell/h_{ij}^\ell \geq \kappa_1$  and for any  $0 < \rho_0 < \rho$ , due to (GM1) and the convexity of  $\Omega_{ij}^\ell$ . The domain of analyticity of  $\widehat{u}$  is dilated in the same way, therefore the hypothesis of Corollary 22 are verified with  $\delta = \delta_*$ . Thus, there exists a harmonic polynomial  $\widehat{Q}_p$  of degree at most  $p$  such that

$$\left| \widehat{u} - \widehat{Q}_p \right|_{H^m(\widehat{\Omega}_{ij}^\ell)} \leq Ce^{-bp} \quad m = 0, 1, 2,$$

for some constants  $C$  and  $b > 0$  depending only on  $\kappa_1, \delta_*$  (which, again, depends on  $\mathfrak{M}_\sigma, \sigma$  and  $u$ , through  $d_u$ ) and  $\|\widehat{u}\|_{W^{1,\infty}(\widehat{\mathcal{N}}(u))}$  (which, again, depends only on  $u$  and  $\Omega$ ). We scale  $\widehat{Q}_p$  back to  $\Omega_{ij}^\ell$  and define the local interpolant as

$$(\pi_\mathcal{T}u(\mathbf{x}))|_{\Omega_{ij}^\ell} := \widehat{Q}_p((\mathbf{x} - \mathbf{x}_{ij}^\ell)/h_{ij}^\ell).$$

Remark 24 guarantees that the interpolation is exact in at least  $p + 1$  points on the boundary of  $\Omega_{ij}^\ell$ . From the usual scaling of Sobolev seminorms  $|\cdot|_{H^k(\Omega_{ij}^\ell)} \leq C(h_{ij}^\ell)^{1-k} |\cdot|_{H^k(\widehat{\Omega}_{ij}^\ell)}$ , we obtain

$$\sum_{1 \leq i \leq \ell-1, 1 \leq j \leq \widehat{J}(i)} \left( |\eta|_{H^1(\Omega_{ij}^\ell)}^2 + (h_{ij}^\ell)^2 |\eta|_{H^2(\Omega_{ij}^\ell)}^2 \right) \leq C \ell e^{-b\ell},$$

with  $C$  and  $b$  depending only on  $u$ ,  $\sigma$ ,  $\Omega$  and  $\mathfrak{M}_\sigma$ . Here we used the fact that the number of elements in  $\mathcal{T}_\sigma^\ell$  is  $O(\ell)$ , as proved in Lemma 26.

The assertion is then obtained by combining the last bound with the one previously obtained for the elements incident to the corners, using  $\ell = O(\sqrt{N})$ , and noting that  $\pi_{\mathcal{T}}(u)$  interpolates  $u$  at least in two points per element, thus Proposition 27 applies, and the  $hp$ -dGFEM error is bounded by the approximation error.  $\square$

**Remark 30.** *In standard FEM convergence analysis, approximation estimates are derived only for few reference elements, which are then mapped to the “physical” mesh elements. For Trefftz schemes this is usually not possible: spaces made of harmonic functions (or harmonic polynomials) are not invariant under general affine mappings but only under similarity transformations, thus estimates that are uniform for every element shape must be proven, up to scaling and isometry only. This is one of the reasons for deriving the approximation estimates of Section 3.2; however, they hold in much more generality than what we used in the  $hp$ -dGFEM analysis (i.e., for star-shaped elements instead of triangles and parallelograms). The explicit dependence on the geometry, only through  $\rho$  and  $\rho_0$ , shows that these bounds are uniform for all the elements of a shape-regular family of meshes. The obstruction to extending the results of Theorem 29 to more general (e.g., curvilinear) geometries is not due to the new approximation estimates, but only to the limitations of the existing theory on quasi-optimality of dGFEM solutions.*

## ACKNOWLEDGMENT

The authors wish to thank D.P. Hewett for inspiring discussions concerning the proof in Appendix A.2.

## APPENDIX A. PROOF OF THE UPPER BOUND (8) FOR NON CONVEX DOMAINS

We consider first the case of polygonal domains (with straight sides) in Section A.1, then we extend the result to more general curvilinear domains in Section A.2. We recall that we are assuming  $0 < h \leq 1$ .

### A.1. Polygonal domains

Denote by  $\{\alpha_k^C \pi\}_{k=1}^{n_C}$  and  $\{\alpha_k^{NC} \pi\}_{k=1}^{n_{NC}}$  the convex and non convex internal angles, respectively, of  $D$ , by  $\{w_k^C\}_{k=1}^{n_C}$  and  $\{w_k^{NC}\}_{k=1}^{n_{NC}}$  the corresponding vertices and set

$$z_k^C = \varphi^{-1}(w_k^C) \quad \text{for } k = 1, \dots, n_C, \quad z_k^{NC} = \varphi^{-1}(w_k^{NC}) \quad \text{for } k = 1, \dots, n_{NC}.$$

The following relations hold (see the left plot of Figure 3 for the geometrical meaning of the parameters):

$$\begin{aligned} 0 \leq \alpha_k^C \leq 1, \quad 0 \leq \beta_k^C := 1 - \alpha_k^C \leq 1 & \quad k = 1, \dots, n_C, \\ 1 < \alpha_k^{NC} \leq 2, \quad -1 \leq \beta_k^{NC} := 1 - \alpha_k^{NC} < 0 & \quad k = 1, \dots, n_{NC}, \\ \sum_{k=1}^{n_C} \beta_k^C + \sum_{k=1}^{n_{NC}} \beta_k^{NC} = 2. \end{aligned}$$

Recalling the definition  $\xi = \frac{2}{\pi} \arcsin \frac{\rho_0}{1-\rho}$  for non convex  $D$ , from Assumption 1 and [22, Remark A.1] we have

$$\alpha_k^C \geq \xi, \quad \beta_k^C \leq 1 - \xi \quad k = 1, \dots, n_C, \quad 2 - \alpha_k^{NC} \geq \xi, \quad \beta_k^{NC} \geq \xi - 1 \quad k = 1, \dots, n_{NC}. \quad (23)$$

One of the crucial ideas of this proof is the fact that the sum of the  $\beta_k$ 's corresponding to an arbitrary set  $V$  of *consecutive* vertices of a polygon  $P \subset B_{1-\rho}$ , which is star-shaped with respect to  $B_{\rho_0}$ , satisfies the inequalities  $\xi - 1 \leq \sum_{k \in V} \beta_k \leq 1 - \xi$ . It will be therefore necessary to take into account the ordering of the vertices along the polygon.

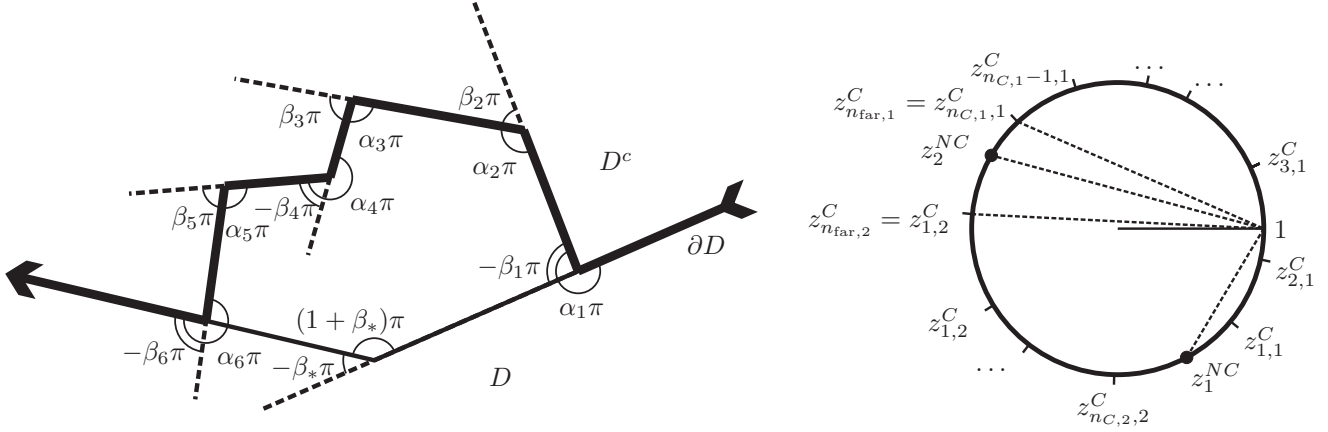


FIGURE 3. Left plot: the geometrical meaning of the parameters  $\alpha_k$ 's and  $\beta_k$ 's. The  $\alpha_k$ 's are all positive, while the  $\beta_k$ 's are positive only on convex corners:  $\beta_1, \beta_4, \beta_6 < 0 < \beta_2, \beta_3, \beta_5$ . The angle between the first and the last segment can be computed by summing over the  $\beta_k$ 's, i.e.,  $\beta_* := \sum_{k=1}^6 \beta_k$ . In this example  $\beta_*$  is negative since the corresponding internal angle is non convex.

Right plot: the location of the pre-vertices  $z_k$ 's in case *ii*) with two non consecutive non convex corners. The four dashed segments have lengths  $\max \left\{ |1 - \bar{z}_1^{NC}|; |1 - \bar{z}_{n_{far,1}}^C| \right\} \leq \min \left\{ |1 - \bar{z}_2^{NC}|; |1 - \bar{z}_{n_{far,2}}^C| \right\}$ .

As in Section 2.2, fix  $w_h \in L_h$  and set  $z_h = \varphi^{-1}(w_h) \in \partial B_{1+h}$ ; thus  $z_h = (1+h)e^{i\theta}$ , for some  $\theta \in [-\pi, \pi]$ . Define  $z = e^{i\theta}$ ,  $w = \varphi(z)$ , and denote by  $S$  the (straight) segment of length  $h$  connecting  $z$  and  $z_h$ . From (6) and (5) we have

$$\begin{aligned} |w_h - w| &= |\varphi(z_h) - \varphi(z)| \leq \int_S |\varphi'(y)| dy \leq \varphi'(\infty) \int_S \frac{1}{|y|^2} \prod_{k=1}^{n_C} |y - \bar{z}_k^C|^{1-\alpha_k^C} \prod_{k=1}^{n_{NC}} |y - \bar{z}_k^{NC}|^{1-\alpha_k^{NC}} dy \\ &\leq \int_S \prod_{k=1}^{n_C} |y - \bar{z}_k^C|^{\beta_k^C} \prod_{k=1}^{n_{NC}} |y - \bar{z}_k^{NC}|^{\beta_k^{NC}} dy =: T, \end{aligned} \quad (24)$$

since  $\varphi'(\infty) < 1$  and  $|y| \geq 1$ . Finally, for any  $y \in S$ ,

$$|y - \bar{z}_k^C| \leq 2 + h \quad k = 1, \dots, n_C, \quad |y - \bar{z}_k^{NC}| \leq 2 + h \quad k = 1, \dots, n_{NC}.$$

With no loss of generality, we consider  $\theta = 0$ , i.e.,  $z = 1$ ,  $z_h = 1 + h$  and  $S$  lies in the positive real axis.

We consider separately four situations.

*i*)  $D$  has only one non convex angle. In this case, the term  $T$  in (24) can be bounded by

$$T \leq (2+h)^{\sum_k \beta_k^C} \int_S |y - \bar{z}_1^{NC}|^{\beta_1^{NC}} dy \leq 27 \int_S |y - \bar{z}_1^{NC}|^{\beta_1^{NC}} dy,$$

since  $h \leq 1$  and  $\sum_k \beta_k^C \leq 3$ , due to  $\sum_k \beta_k^C + \beta_1^{NC} = 2$  and  $\beta_1^{NC} \geq -1$ .

Since  $\beta_1^{NC} < 0$  and  $|y - \bar{z}_1^{NC}| \geq |y - 1|$  for all  $y \in S$ , we have

$$T \leq 27 \int_S |y - 1|^{\beta_1^{NC}} dy = 27 \int_0^h s^{\beta_1^{NC}} ds = 27 \frac{h^{\beta_1^{NC}+1}}{\beta_1^{NC}+1} \leq \frac{27h\xi}{\xi},$$

because  $\beta_1^{NC} > -1$ ,  $h \leq 1$  and  $\beta_1^{NC} + 1 \geq \xi$ .

ii)  $D$  has only two non convex angles, and these angles are non consecutive. Assume  $|1 - \bar{z}_1^{NC}| \leq |1 - \bar{z}_2^{NC}|$ .

The points  $\bar{z}_1^{NC}$  and  $\bar{z}_2^{NC}$  separate the points in  $\{\bar{z}_k^C\}_{k=1}^{n_C}$  into two blocks,  $\{\bar{z}_{j,1}^C\}_{j=1}^{n_{C,1}}$  and  $\{\bar{z}_{j,2}^C\}_{j=1}^{n_{C,2}}$ . We set

$$n_{\text{far},1} = \arg \max_{j=1, \dots, n_{C,1}} |1 - \bar{z}_{j,1}^C|, \quad n_{\text{far},2} = \arg \max_{j=1, \dots, n_{C,2}} |1 - \bar{z}_{j,2}^C|,$$

and assume

$$|1 - \bar{z}_{n_{\text{far},1}}^C| \leq |1 - \bar{z}_{n_{\text{far},2}}^C|;$$

consequently, as can be inferred from the right plot in Figure 4,

$$|1 - \bar{z}_{n_{\text{far},1}}^C| \leq |1 - \bar{z}_2^{NC}|. \quad (25)$$

We have

$$\begin{aligned} T &= \int_S |y - \bar{z}_1^{NC}|^{\beta_1^{NC}} |y - \bar{z}_2^{NC}|^{\beta_2^{NC}} \prod_{j=1}^{n_{C,1}} |y - \bar{z}_{j,1}^C|^{\beta_{j,1}^C} \prod_{j=1}^{n_{C,2}} |y - \bar{z}_{j,2}^C|^{\beta_{j,2}^C} dy \\ &\leq \int_S |y - \bar{z}_1^{NC}|^{\beta_1^{NC}} |y - \bar{z}_2^{NC}|^{\beta_2^{NC}} |y - \bar{z}_{n_{\text{far},1}}^C|^{\sum_j \beta_{j,1}^C} |y - \bar{z}_{n_{\text{far},2}}^C|^{\sum_j \beta_{j,2}^C} dy \\ &\stackrel{(25), \beta_{j,1}^C \geq 0}{\leq} \int_S |y - \bar{z}_1^{NC}|^{\beta_1^{NC}} |y - \bar{z}_2^{NC}|^{\beta_2^{NC} + \sum_j \beta_{j,1}^C} |y - \bar{z}_{n_{\text{far},2}}^C|^{\sum_j \beta_{j,2}^C} dy. \end{aligned}$$

a) If  $\beta_2^{NC} + \sum_j \beta_{j,1}^C \geq 0$ ,

$$T \leq (2+h)^{2-\beta_1^{NC}} \int_S |y - \bar{z}_1^{NC}|^{\beta_1^{NC}} dy \leq 27 \frac{h^{\beta_1^{NC}+1}}{\beta_1^{NC}+1} \leq \frac{27h\xi}{\xi}.$$

b) If  $\beta_2^{NC} + \sum_j \beta_{j,1}^C < 0$ , we write

$$T \leq (2+h)^{\sum_j \beta_{j,2}^C} \int_S |y - \bar{z}_1^{NC}|^{\beta_1^{NC} + \beta_2^{NC} + \sum_j \beta_{j,1}^C} dy.$$

If we prove that

$$\beta^* := \beta_1^{NC} + \beta_2^{NC} + \sum_j \beta_{j,1}^C \geq \xi - 1, \quad (26)$$

then  $\sum_j \beta_{j,2}^C = 2 - \beta^* < 3$ , from which

$$T \leq 27 \int_S |y - \bar{z}_1^{NC}|^{\beta^*} dy \leq \frac{27h\xi}{\xi}. \quad (27)$$

In order to conclude, we only need to prove (26).

Consider the counterclockwise oriented part of  $\partial D$  formed by the consecutive (oriented) sides  $s_i$ ,  $i = 1, \dots, m := n_{C,1} + 3$ , abutting  $w_1^{NC}$ ,  $w_{j,1}^C$ ,  $j = 1, \dots, n_{C,1}$ , and  $w_2^{NC}$ . Let  $\ell_i$  be the oriented line containing  $s_i$ ,  $i = 1, \dots, m$ . Since  $D$  is star-shaped with respect to  $B_{\rho_0}$ , then  $B_{\rho_0}$  lies in the intersection of the half planes lying on the left of the  $\ell_i$ 's.

Let  $K$  be the infinite cone obtained by intersecting the right half planes generated by  $\ell_1$  and  $\ell_m$ . Its opening is  $(1 + \beta^*)\pi < \pi$ , with  $\beta^* < 0$  (cf. the left plots of Figures 3 and 4).

Define  $D' := D \setminus \overline{K}$ ;  $D'$  only has one non convex angle of internal amplitude  $(1 - \beta^*)\pi$ . The ball  $B_{\rho_0}$  lies on the left side of every edge of  $D'$ , thus this domain is star-shaped with respect to  $B_{\rho_0}$  and  $D' \subset B_{\rho_0}$ , by the bounds (23) we have  $1 + \beta^* \geq \xi$  (cf. the left plot of Figure 4). Therefore,  $\beta^* \geq \xi - 1 > -1$ , which concludes the argument.

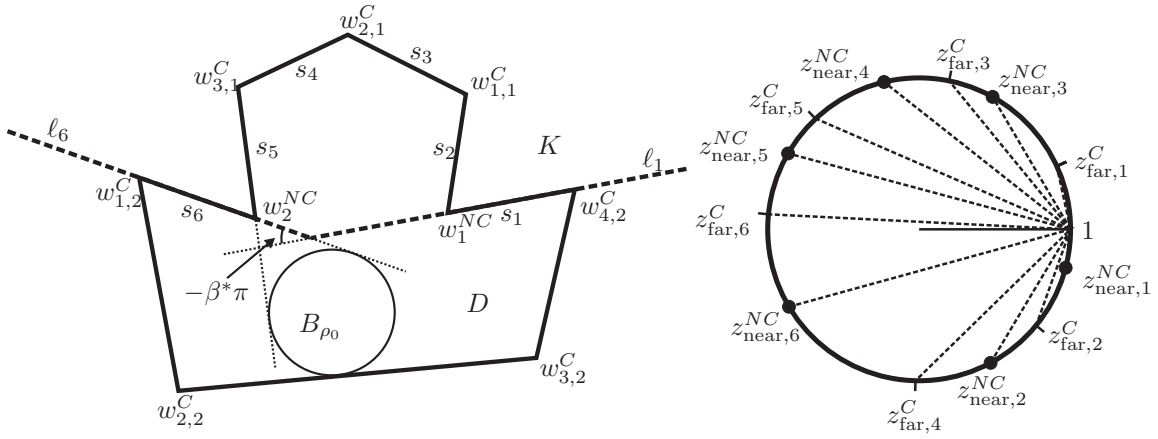


FIGURE 4. Left plot: an example of a domain  $D$  with two non consecutive, non convex corners ( $w_1^{NC}$  and  $w_2^{NC}$ ). In this case  $n_{C,1} = 3$ ,  $n_{C,2} = 4$  and  $m = 6$ . The cone  $K$  is delimited by the dashed lines  $\ell_1$  and  $\ell_6$ . The ball  $B_{\rho_0}$  lies on the left of every edge when  $\partial D$  is run counterclockwise. The set  $D' = D \setminus \overline{K}$  is star-shaped with respect to  $B_{\rho_0}$ .

Right plot: the pre-vertices  $z_k$  satisfy the ordering relation (29). Notice that  $z_{near,1}^{NC}$  and  $z_{far,n}^C$  (in the picture  $n = 6$ ) do not enter the relation. Therefore it is not relevant which one between  $z_{far,1}^C$  and  $z_{near,1}^{NC}$  is closest to 1. The number of pre-vertices lying in the upper and in the lower half of the complex plane does not affect the ordering of the distances.

iii)  $D$  has only two non convex angles, and these angles are consecutive. We have

$$T \leq (2 + h) \sum_j \beta_j^C \int_S |y - \bar{z}_1^{NC}|^{\beta_1^{NC} + \beta_2^{NC}} dy,$$

assuming again  $|1 - \bar{z}_1^{NC}| \leq |1 - \bar{z}_2^{NC}|$ . If we prove that

$$\beta^* := \beta_1^{NC} + \beta_2^{NC} \geq \xi - 1 > -1, \quad (28)$$

then  $\sum_j \beta_j^C = 2 - \beta^* < 3$ , from which we get again (27).

For the proof of (28), consider the part of  $\partial D$  formed by the  $m = 3$  consecutive sides abutting  $w_1^{NC}$  and  $w_2^{NC}$ ; the rest of the proof is identical to that of (26).

iv)  $D$  has more than two non convex angles. We generalise the argument of step ii). Assume that we have  $n$  blocks of consecutive convex angles, alternated by  $n$  blocks of consecutive convex angles. With a

similar notation as before, we can write

$$T \leq \int_S \prod_{i=1}^n \left[ \prod_{j=1}^{n_{NC,i}} |y - \bar{z}_{j,i}^{NC}|^{\beta_{j,i}^{NC}} \prod_{j=1}^{n_{C,i}} |y - \bar{z}_{j,i}^C|^{\beta_{j,i}^C} \right] dy.$$

Setting, for  $i = 1, \dots, n$ ,

$$n_{\text{far},i} = \arg \max_{j=1, \dots, n_{NC,i}} |1 - \bar{z}_{j,i}^C|, \quad n_{\text{near},i} = \arg \min_{j=1, \dots, n_{NC,i}} |1 - \bar{z}_{j,i}^{NC}|,$$

we can bound  $T$  as

$$T \leq \int_S \prod_{i=1}^n \left[ |y - \bar{z}_{\text{near},i}^{NC}|^{\sum_j \beta_{j,i}^{NC}} |y - \bar{z}_{\text{far},i}^C|^{\sum_j \beta_{j,i}^C} \right] dy =: \int_S P(y) dy.$$

We order the blocks in such a way that

$$|1 - \bar{z}_{\text{near},i}^{NC}| \leq |1 - \bar{z}_{\text{near},i+1}^{NC}| \quad \text{and} \quad |1 - \bar{z}_{\text{far},i}^C| \leq |1 - \bar{z}_{\text{far},i+1}^C| \quad i = 1, \dots, n-1;$$

consequently (see the left plot in Figure 4),

$$|1 - \bar{z}_{\text{far},i}^C| \leq |1 - \bar{z}_{\text{near},i+1}^{NC}| \quad i = 1, \dots, n-1. \quad (29)$$

Thus, we have

$$P(y) \leq |y - \bar{z}_{\text{near},1}^{NC}|^{\sum_j \beta_{j,1}^{NC}} \left[ \prod_{i=1}^{n-1} |y - \bar{z}_{\text{near},i+1}^{NC}|^{\sum_j \beta_{j,i}^C + \sum_j \beta_{j,i+1}^{NC}} \right] (2+h)^{\sum_j \beta_{j,n}^C}.$$

We consider the term with index  $n-1$  in the product and look at its exponent ( $\sum_j \beta_{j,n-1}^C + \sum_j \beta_{j,n}^{NC}$ );  
a) if it is  $\geq 0$ , we combine the term with index  $n-1$  with the following term (the last one) and obtain

$$P(y) \leq |y - \bar{z}_{\text{near},1}^{NC}|^{\sum_j \beta_{j,1}^{NC}} \left[ \prod_{i=1}^{n-2} |y - \bar{z}_{\text{near},i+1}^{NC}|^{\sum_j \beta_{j,i}^C + \sum_j \beta_{j,i+1}^{NC}} \right] (2+h)^{\sum_j \beta_{j,n-1}^C + \sum_j \beta_{j,n}^{NC} + \sum_j \beta_{j,n}^C};$$

b) if it is  $< 0$ , we combine the term with index  $n-1$  with the previous term (the one with index  $n-2$ ) and obtain

$$P(y) \leq |y - \bar{z}_{\text{near},1}^{NC}|^{\sum_j \beta_{j,1}^{NC}} \left[ \prod_{i=1}^{n-3} |y - \bar{z}_{\text{near},i+1}^{NC}|^{\sum_j \beta_{j,i}^C + \sum_j \beta_{j,i+1}^{NC}} \right] \\ \cdot |y - \bar{z}_{\text{near},n-1}^{NC}|^{\sum_j \beta_{j,n-2}^C + \sum_j \beta_{j,n-1}^{NC} + \sum_j \beta_{j,n-1}^C + \sum_j \beta_{j,n}^{NC}} (2+h)^{\sum_j \beta_{j,n}^C}.$$

Then, we proceed backward, considering the term with index  $i = n-2$  and, depending on whether its exponent is  $\geq 0$  or  $< 0$ , we combine it either with the following term or with the previous term the way we did before, and so on, until the term  $i = 1$ . We end up with three factors in the upper bound of  $P(y)$ : the first one is  $|y - \bar{z}_{\text{near},1}^{NC}|^{\sum_j \beta_{j,1}^{NC}}$ , the third one is  $(2+h)^B$ , with  $B > 0$ , and the second one is  $|y - \bar{z}_{\text{near},2}^{NC}|^A$ . If  $A \geq 0$ , we conclude as in step *ii*), case a), while if  $A < 0$ , in order to conclude as in step *ii*), case b), we need to prove that

$$\beta^* := \sum_j \beta_{j,1}^{NC} + A \geq \xi - 1 > -1. \quad (30)$$



Since the blocks of (convex and non convex) angles corresponding to the  $\beta$ 's entering the expression  $\beta_{j,1}^{NC} + A$  are consecutive, the proof of (30) can be carried out as the proof of (26).

The proof in the polygonal case is complete.

## A.2. Domains with non-polygonal boundaries

We begin with the following trigonometric lemma.

**Lemma 31.** *Let  $0 < R_1 < R_2 < +\infty$ , and fix two distinct straight lines  $\ell_1$  and  $\ell_2$  that are tangent to  $B_{R_1}$  in the two points  $y_1$  and  $y_2$  ( $y_1 \neq y_2$ ), respectively. We denote by  $x_1$  the intersection between the circle  $\partial B_{R_2}$  and the line  $\ell_1$  such that, in a counterclockwise orientation,  $x_1$ ,  $y_1$  and  $0$  appear with this ordering as vertices of a triangle. Symmetrically, we denote by  $x_2$  the intersection between the circle  $\partial B_{R_2}$  and the line  $\ell_2$  such that, in a clockwise orientation,  $x_2$ ,  $y_2$  and  $0$  appear with this ordering. We denote by  $C_\eta$  the infinite convex sector with opening  $\eta\pi$ ,  $0 \leq \eta \leq 1$ , defined by the two half lines generating at the origin and passing through  $x_1$  and  $x_2$ , respectively (see the left plot in Figure 5).*

*If  $\eta < \frac{2}{\pi} \arcsin \frac{R_1}{R_2}$ , then  $\ell_1$  and  $\ell_2$  intersect at a point  $w$  that lies in the interior of  $C_\eta$ . Moreover, if  $|w| > R_2$  and if we define  $\varepsilon := |w - x_1| = |w - x_2| > 0$ ,  $\eta$  is related to  $\varepsilon$ ,  $R_1$  and  $R_2$  by the following formula:*

$$0 < \eta(\varepsilon, R_1, R_2) = \frac{2}{\pi} \arccos \frac{R_2^2 + \varepsilon \sqrt{R_2^2 - R_1^2}}{R_2 \sqrt{\varepsilon^2 + R_2^2 + 2\varepsilon \sqrt{R_2^2 - R_1^2}}} < \frac{2}{\pi} \arcsin \frac{R_1}{R_2}. \quad (31)$$

*For  $\varepsilon > 0$ , the function  $\varepsilon \mapsto \eta(\varepsilon, R_1, R_2)$  is continuous and strictly increasing. For  $R_2 > R_1$ , the function  $R_2 \mapsto \eta(\varepsilon, R_1, R_2)$  is continuous and strictly decreasing.*

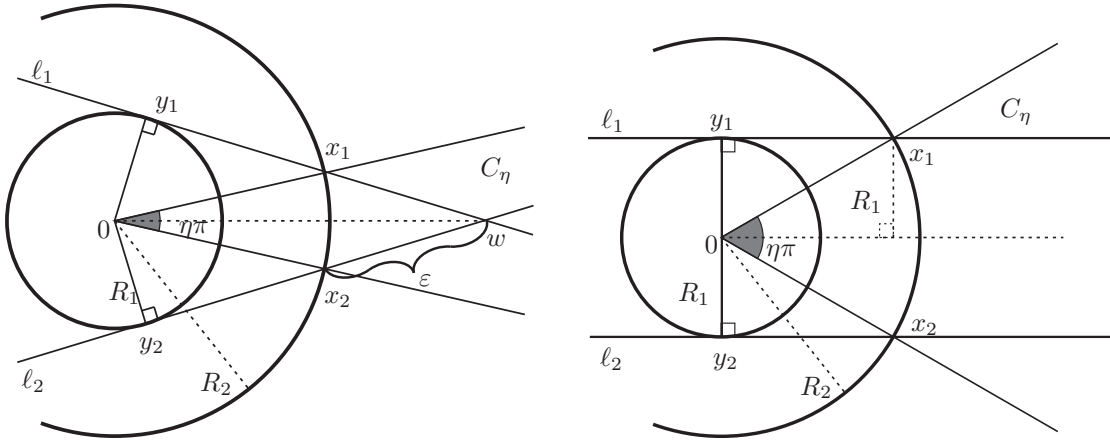


FIGURE 5. Left plot: the geometric configuration in Lemma 31. Right plot: the limit case  $\eta = \frac{2}{\pi} \arcsin \frac{R_1}{R_2}$  in the proof.

*Proof.* We consider the limit case  $\eta = \frac{2}{\pi} \arcsin \frac{R_1}{R_2} < 1$ . Then,  $R_2 \sin \frac{\eta\pi}{2} = R_1$  and, as depicted in the right plot of Figure 5, the lines  $\ell_1$  and  $\ell_2$  are parallel to each other. Therefore, whenever  $\eta$  is smaller than this threshold value,  $\ell_1$  and  $\ell_2$  will intersect on the central half line of  $C_\eta$ .

We apply Pythagoras's theorem twice: to the triangle of vertices  $x_1$ ,  $y_1$  and  $0$ , yielding  $|x_1 - y_1|^2 = R_2^2 - R_1^2$ , and then to the triangle of vertices  $w$ ,  $y_1$  and  $0$ , leading to

$$|w|^2 = R_1^2 + \left( \varepsilon + \sqrt{R_2^2 - R_1^2} \right)^2 = \varepsilon^2 + R_2^2 + 2\varepsilon \sqrt{R_2^2 - R_1^2}.$$

From the law of cosines applied to the triangle of vertices  $w$ ,  $x_1$  and  $0$ , we obtain  $2|w|R_2 \cos \frac{\eta\pi}{2} = |w|^2 + R_2^2 - \varepsilon^2$  from which the identity in the assertion follows.

The monotonicity in dependence of  $\varepsilon$  and  $R_2$  can be verified by computing the derivative of the expression in (31). The last inequality in the assertion follows from

$$\eta(\varepsilon, R_1, R_2) < \lim_{\varepsilon \rightarrow \infty} \eta(\varepsilon, R_1, R_2) = \frac{2}{\pi} \arccos \sqrt{1 - \frac{R_1^2}{R_2^2}} = \frac{2}{\pi} \arcsin \frac{R_1}{R_2}$$

which uses the monotonicity of  $\eta$  as a function of  $\varepsilon$ , and the identity  $\sin \arccos \sqrt{1 - t^2} = |t|$ .  $\square$

We can now complete the proof of the bound (8) in the general case. In order to do that, we will construct a polygon  $P_\varepsilon \supset D$ , star-shaped with respect to  $B_{\rho_0}$ . The maximal distance  $\sup_{w \in \partial P_\varepsilon} d(w, \partial D)$  will be made arbitrarily small, and the parameter  $\xi$  (defined in Theorem 7) relative to  $P_\varepsilon$  will converge to the one relative to  $D$ . Then, invoking the result of Section A.1 completes the proof in the case of non-polygonal domains.

Consider a domain  $D$  satisfying Assumption 1. Fix  $\varepsilon > 0$ . Define an integer  $N \in \mathbb{N}$  such that  $\eta_\varepsilon := \frac{2}{N} \leq \eta(\varepsilon, \rho_0, 1 - \rho)$ , where  $\eta(\cdot, \cdot, \cdot)$  was defined in formula (31). We select the points  $w_j \in \partial D$ ,  $j = 1, \dots, N$ , that have complex argument (namely, angular polar coordinate) equal to  $\theta_j \pi := j \eta_\varepsilon \pi$  for  $j = 1, \dots, N$ . (In this proof we assume that all the indices  $j$  are taken modulus  $N$ .)

Let  $\ell_j^+$  and  $\ell_j^-$  be the two tangent (straight) lines to  $B_{\rho_0}$  passing through  $w_j$  and such that, sitting in  $w_j$  and looking at  $B_{\rho_0}$ ,  $\ell_j^+$  is on the left and  $\ell_j^-$  is on the right (notice that the two lines do not coincide, since  $\rho_0 < \rho \leq |w_j|$ ), see the right plot in Figure 6.

Consider the two lines  $\ell_j^+$  and  $\ell_{j+1}^-$ . If  $|w_j| = |w_{j+1}|$ , then they satisfy the assumptions of Lemma 31 (since  $R_2 \mapsto \frac{2}{\pi} \arcsin \frac{\rho_0}{R_2}$  is monotonically decreasing, the definition of  $\eta_\varepsilon$  made above guarantees the needed bound for any value of  $\rho_0 \leq R_2 = |w_j| \leq 1 - \rho$ ). Thus they intersect at a point  $v_j$  such that  $|v_j| \geq |w_j| = |w_{j+1}|$  and whose complex argument satisfies  $\theta_j \pi \leq \arg(v_j) \leq \theta_{j+1} \pi$ . Moreover,  $|v_j - w_j| = |v_j - w_{j+1}| \leq \varepsilon$ , due to the monotonicity of the map  $\eta(\varepsilon, \rho_0, |w_j|) \mapsto \varepsilon$ .

On the other hand, if  $|w_j| < |w_{j+1}|$  (the opposite case is analogous), then  $v_j$  lies closer to  $w_{j+1}$  than in the previous case (see the left plot in Figure 6); therefore, in all the situations, we have

$$d(v_j, \partial D) \leq \min\{|v_j - w_j|, |v_j - w_{j+1}|\} \leq \varepsilon. \quad (32)$$

Notice that, given  $|w_{j+1}|$ ,  $\rho_0$  and  $\eta_\varepsilon$ , due to the star-shapedness assumption,  $|w_j|$  can not be arbitrarily small, namely it can not trespass the point denoted with  $z$  in the left plot in Figure 6.

Notice that every domain which is star-shaped with respect to  $B_{\rho_0}$  and such that its boundary contains the point  $w_j$  (e.g., the domain  $D$  satisfies these requests) can not cross the segments  $[v_{j-1}, w_j]$  and  $[w_j, v_j]$ .

Now we define the polygon  $P_\varepsilon$  with  $2N$  sides whose vertices are  $w_1, v_1, w_2, \dots, w_N, v_N$ . Every edge of  $P_\varepsilon$  is part of either  $\ell_j^+$  or  $\ell_j^-$ . The polygon  $P_\varepsilon$  satisfies the following conditions:

- i)  $P_\varepsilon$  is star-shaped with respect to  $B_{\rho_0}$ , since the continuation of each of its edges is tangent to  $B_{\rho_0}$  and (in a counterclockwise orientation of  $\partial P_\varepsilon$ ) leaves  $B_{\rho_0}$  on its left;
- ii)  $D \subseteq P_\varepsilon$ , as it contains every domain  $D'$  star-shaped with respect to  $B_{\rho_0}$  and satisfying  $\{w_j\}_{j=1}^N \subset \partial D'$ ;
- iii) for every  $w \in \partial P_\varepsilon$ ,  $d(w, \partial D) \leq \varepsilon$ ; in fact, since  $\{w_j\}_{j=1}^N \subset \partial D$ , the maximum distance from  $\partial D$  is achieved in one of the vertices  $v_j$  and this is controlled by the bound (32).

Then we can conclude as in the convex case. Fix  $w_h \in L_h = L_h[\overline{D}]$ . Choose  $0 < \varepsilon < \rho$  and define the polygon  $P_\varepsilon$  as above (so that  $P_\varepsilon \subset B_1$  and  $\varphi_{P_\varepsilon}'(\infty) \leq 1$ ). Then,  $w_h \in L_{h'}[\overline{P_\varepsilon}]$  with  $h' \leq h$ , as a consequence of Lemma 6. Let  $z_{h'} := \varphi_{P_\varepsilon}^{-1}(w_h) = (1 + h')e^{i\theta}$ , and define  $z := e^{i\theta}$ ,  $\xi_\varepsilon := \frac{2}{\pi} \arcsin \frac{\rho_0}{1 - \rho + \varepsilon} < \xi$ . Then, from Section A.1,

$$d(w_h, \partial D) \leq d(w_h, \varphi_{P_\varepsilon}(z)) + d(\varphi_{P_\varepsilon}(z), \partial D) = |\varphi_{P_\varepsilon}(z_{h'}) - \varphi_{P_\varepsilon}(z)| + d(\varphi_{P_\varepsilon}(z), \partial D) \leq 27\xi_\varepsilon^{-1}h^{\xi_\varepsilon} + \varepsilon.$$

Since this is true for every  $0 < \varepsilon < \rho$ , by taking the limit for  $\varepsilon \rightarrow 0$ , we get  $d(w_h, \partial D) \leq \frac{27}{\xi}h^\xi$  for all  $w_h \in L_h$ .

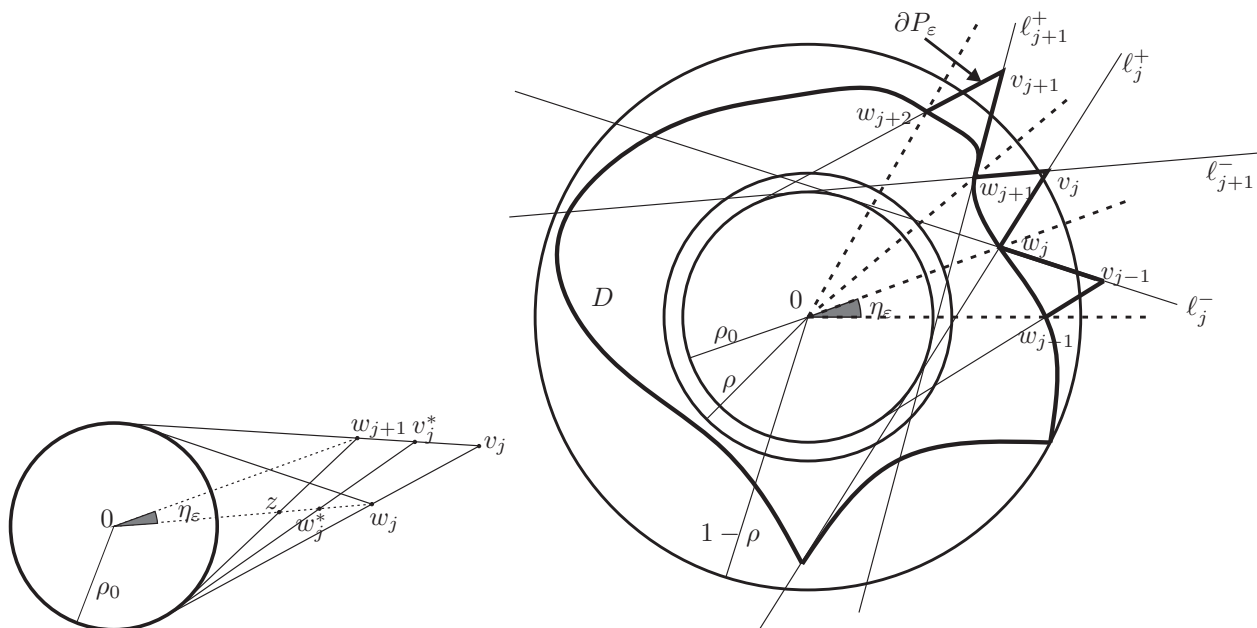


FIGURE 6. Left plot: the comparison of the cases  $|w_j| = |w_{j+1}|$  and  $|w_j^*| < |w_{j+1}|$  for a fixed  $w_{j+1}$ . In the second case, the constructed point  $v_j^*$  is closer to  $w_{j+1}$  than in the first case, namely,  $|v_j^* - w_{j+1}| < |v_j - w_{j+1}|$ .

Right plot: the construction of the star-shaped polygon  $P_\varepsilon$  enclosing the non-polygonal, non-convex domain  $D$ .

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