Strategies for Constrained Optimisation

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STRATEGIES FOR CONSTRAINED OPTIMISATION

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ABSTRACT

The latest 6-man chess endgame results confirm that there are many deep forced mates beyond the 50-move rule. Players with potential wins near this limit naturally want to avoid a claim for a draw: optimal play to current metrics does not guarantee feasible wins or maximise the chances of winning against fallible opposition. A new metric and further strategies are defined which support players' aspirations and improve their prospects of securing wins in the context of a *k*-move rule.

1. INTRODUCTION

Endgame tables (EGTs) have to date not acknowledged the FIDE 50-move rule of Article 9.3. It is irrelevant for all but 8 of the 3- to 5-man endgames. Further, EGT authors share an interest with chess composers in the absolute capabilities of the chessmen. They have reasonably not given priority to FIDE's flexible rule which has indeed changed five times (see below) and whose detail has been difficult to implement.

However, recent progress on 6-man endgames (Nalimov, Wirth and Haworth, 1999; Hyatt, 2000; Karrer, 2000; Tamplin, 2000; Thompson, 2000) has renewed interest in having endgame data which serves both the practical player and the theoretician. The deeper maximum-depth wins imply that the 50-move rule will become a more frequent consideration. Let a won position be termed a *k-win* if *k* is the least integer for which optimal play would not risk a draw claim under a *k-move rule.* About half of the 6-man endgames computed to date feature *k*-wins with $k > 50$. Currently, practical players may be said to have two objectives:

- to win positions which are *k*-wins for $k \leq 50$ without risking a 50-move draw claim, and
- to maximise the probability of winning a *k*-win position for $k > 50$.

These objectives are addressed here. Section 2 questions the appropriateness of the rule given a demonstrably effective aggressor. Section 3 introduces a number of metrics that define varieties of *optimal play*. Section 4 shows the value of the now disused metric *Depth to Zeroing move*[2] (DTZ). Section 5 defines the new metric *Depth by the Rule* (DTR) and describes algorithms for generating DTR data. Section 6 demonstrates the failings of a naive strategy for using DTR and defines further strategies using DTR and DTZ data.

2. HISTORY OF THE RULE

Ruy López suggested a 50-move limit in Article 17 of his Chess Code of 1561, perhaps in the interests of his fellow coffee-house professionals who played for wagers. The 1883 London Tournament's rules, the basis of FIDE's rules today, were the first to state that a P-push or capture would zero the count.

In 1974, FIDE first enabled the 50-move rule to be varied. They did so with 100-move clauses, in 1978 for KNNKP (Troitzkiĭ, 1906-1910, 1934), in 1982 for KRP(a2)KbBP(a3) following the Timman-Velimirović game (Van den Herik, Herschberg and Nakad, 1987), and in 1984 for KRBKR (59) (Croskill, 1864; Nunn, 1994). They did not meet all the requirements defined by Roycroft (1984) at the first opportunity.

By 1988, computer results, albeit single-sourced, were plentiful (Thompson, 1986) and endgame-specific limits were suggested. However, FIDE adopted a simpler stance, replacing the 100-move clauses by a 75-move allowance for just the six endgames KBBKN (Roycroft, 1983), KNNKP, KQKBB, KQKNN, KQP(x7)KQ and KRBKR (Kažíc, 1989; Mednis, 1989). KRPKBP with blocked Pawns ceased to be an exception.

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[2] A *zeroing move* is defined as one which zeroes the move count by FIDE Article 9.3, i.e., a pawn push, capture or mate. A *phase* of play is defined as a sequence of moves starting just after and ending with a zeroing move.

Following Stiller's (1991) discovery that KRBKNN's maximum depth is 223, FIDE gave up the chase and restored the 50-move limit for all endgames in 1992 (Herschberg and Van den Herik, 1993). KRNKNN then took the record phase length to 243 (Stiller, 1996) and this could well be extended by 7-man pawnless endgames. More details of some games and studies associated with the 50-move rule are in Appendix B.

Clearly a balance has to be struck between the extremes of denying players attainable wins and requiring the opposition to be eternally vigilent in a drawn position. Today, the main concerns are social ones for the welfare of defenders and tournament directors who wish to run their events to a schedule (Levy and Newborn, 1991): however, it is clear that these need not apply to computer-assisted play. There is an argument for waiving the 50-move rule where a player can demonstrably achieve a theoretical win. An EGT is currently the only way to establish theoretical position values and benchmark the aggressor's effectiveness. Certainly, computers with EGTs can play won or drawn positions quickly, even if they initially assume a fallible opponent and take time to choose between equi-optimal moves (Levy, 1991). Other means of winning effectively may be created in the future. To deny players the opportunity of achieving complex wins foreshortens the domain of chess itself. It prevents us from seeing immaculate play building on the smallest of advantages and exploring the deep space of the endgame where humans may never go without the vehicle of perfect information.

3. METRICS FOR OPTIMALITY

Table 1 refines a previously published version (Nalimov *et al.*, 1999) and contains a systematic notation to describe the various optimisation goals and related concepts. It provides a way of referring to and comparing different metrics, position depths, endgame tables, maximal depths, types of optimality and minimax strategies. Each strategy selects a subset of *equi-optimal* moves: one strategy may win where another draws. The actual line of play is determined by both sides' respective strategies and their ultimate choice of equi-optimal moves.

Goal	**GZ**	**GC**	**GM**	**GR**
Goal Description	Zero move-count in maximin no. of moves	'Conversion', i.e., mate, capture or P-conversion in maximin no. of moves	Mate in maximin no. of moves	Mate within *k*-move rule ... with maximin *k*
Reference player	PZ	PC	PM	PR
Metric	DTZ	DTC	DTM	DTR
Position depth	dz, dz_i	dc, dc_i	dm, dm_i	dr, dr_i
Endgame Table	EZ	EC	EM	ER
Maximal depth	mxZ	mxC	mxM	mxR
Type of optimality	Z-optimal	C-optimal	M-optimal	R-optimal
Minimax Strategy ... move-subset chosen	SZ	SC	SM	SR
Example EG tables by:				
Thompson (1986)	KxPKx, x = Q, R	5-man	3- & 4-man	3-man
Tamplin (2000); Thompson (2000)	none	5- & 6-man	3- & 4-man	3-man
Stiller (1989, 1991, 1992, 1996)	none	5- & 6-man	none	none
Hyatt (2000); Nalimov (2000)	none	none	3- to 6-man	3-man
Wirth (1999)	none	KPPKP, KQQKQQ	3- & 4-man	3-man

Table 1: Endgame goals and associated concepts.

For pawnless endgames, DTC ≡ DTZ and SC ≡ SZ. The notation allows for more comprehensive goals. Let the *nested strategy* $SX_1X_2...X_n$ be defined as subsetting the available moves with strategies SX_1, SX_2, ... , SX_n in turn. A *line X-Y* is an optimal line of play where White is reference player PX using strategy SX and Black is reference player PY using strategy SY. Appendix A shows Black, then White, having to choose between C- and M-optimal play as they approach the events of force conversion and mate.

For a specific *k*-win position P, a strategy SY is said to *(k-)succeed on P* if each move chosen by SY avoids the risk of a *k*-move draw claim. If not, SY *(k-)fails* on *P* and SY risks a draw claim on any positions from which Y-optimal play can arrive at *P*. Let σ denote any move-subsetting strategy. If SY succeeds on *P*, SYσ succeeds on *P*. However, as position Q-NN2 of Table 2 demonstrates, if SYσ succeeds on *P*, SY may still fail. Let *SA* ≥ *SB* denote that if strategy SA fails, strategy SB fails; SYσ ≥ SY. Let SA > SB denote that SA ≥ SB and that SA sometimes succeeds where SB fails.

Key	Position	stm	Val.	DTZ	DTC	DTM	DTR	Notes
				ply	ply	ply	ply	
Maximal Positions								
mxNN-P1	6N1/8/7p/8/8/8/3N1k2/7K	w	1-0	?	228	229	?	(Dekker, 1990). max DTM pos. ... 24. ... h4 {NN-P2}
mxQ-NN	7Q/8/8/4n3/2k5/8/3K3n	b	1-0	126	126	144	126	(Nunn, 1994, p. 307). ... 14. ... Kc6 {Q-NN1}
mxQP-Q1	8/q7/P6k/Q7/8/8/8/6K1	w	1-0	141	213	235	141	(Thompson, 1986, p. 138). max DTZ wtm KQPKQ win
mxRN-NN	6N1/5KR1/2n5/8/8/8/2n5/1k6	w	1-0	485	485	523	485	(Thompson, 2000). max DTC/DTM wtm KRNKNN win
Positions to test strategies								
NN-P2	8/8/4k3/8/7p/6KN/6N1/8	w	1-0	?	180	181	?	*h3* needed before move 75. ... 62. ... Ke1 {NN-P3}
NN-P3	8/8/8/8/7p/1N1K4/7N/4k3	w	1-0	≤ 24	104	105	?	SZ succeeds; SC, SM fail with 63. Nd2 {NN-P4}
NN-P4	8/8/8/8/7p/3K4/3N3N/4k3	b	'1-0'	35	103	104	?	63. .. Kf2 forces h3 to at least m80 and a draw claim
QP-Q2	8/8/P7/6k1/3q4/8/4Q1K1/8	w	1-0	99	171	193	99	(Thompson, 1986, p. 138 after 21. ... Qd4)
QP-Q3	K7/7k/P7/8/6q1/2Q5/8/8	w	1-0	1	17	41	?	(Thompson, 1986, p. 138 after 70. ... Kh7)
QBB-N1	6Q1/8/8/1n6/8/7K/7B/k6B	w	1-0	2	2	7	5	PM-PM: 1. Be5+ Nd4 2. Bxd4 Kb1 3. Qb3+ Kc1 4. Be3#.
QBB-N2	8/8/8/1n6/8/7K/k6B/7B	w	1-0	103	103	127	103	{QBB-N1} PC-PC: 1. Qa2+ Kxa2 {QBB-N2}
Q-NN1	8/8/2k5/8/4n3/3Q2n1/8/4K3	w	1-0	99	99	117	99	PMC-PCM: 1. Qd4 ... to 37. ... Ka4 {Q-NN2}
Q-NN2	8/8/1n1K4/n1Q5/k7/8/8/8	w	1-0	25	25	41	25	SC, SMC succeed with Qg1; SM fails with Qc3/Qf2/Qg1
Miscellaneous								
NN-P5	7k/5K2/8/4N1N1/8/8/7p/8	b	1-0	1	1	2	1	The defender is forced to effect the conversion

Table 2: Illustrative chess positions.

4. ENDGAMES DEEPER THAN 50 MOVES

Below is a list of known endgames with *k*-wins for *k* > 50 (Thompson, 1986, 2000; Van den Herik *et al.*, 1987; Stiller, 1989, 1991, 1992, 1996; Dekker, Van den Herik and Herschberg, 1990; Wirth, 1999; Hyatt, 2000; Lincke, 2000; Nalimov, 2000; Tamplin, 2000). Obtrusive forces are in italics: mxR is in brackets.

5 man KBBKN (66), KBNKN (77), KNNKP (70+), KQKBB (71), KQKNN (63), KQRKQ (60), KRBKR (59), KQPKQ with wP on a6 (71), a7 (69), b3 (51), b6 (61), b7 (55), c3 (53), d3 (54), d4 (64), d6 (58).

6 man *KBBBKR* (69), KBBNKR (68), *KNNNKB* (92), *KNNNKN* (86), KQBKRR (85), *KQKBBB* (51+), KQKBBN (63+), KQNKRR (153), KQNNKQ (72), KQRKQB (73), KQRKQN (71), KQRKQR (92), KRBKBB (83), KRBKBN (98), KRBKNN (223), KRBNKQ (99), KRNKBB (140), KRNKBN (190), KRNKNN (243), KRP(a2)KbBP(a3) (54+), KRRBKQ (82), KRRKRB (54), KRRKRN (73), KRRNKQ (101), *KRRRKQ* (65).

Zeroing move, Conversion and Mate are increasingly distant goals. While the corresponding minimax strategies SZ, SC and SM are highly correlated, one strategy can preclude another. A focus on the longer-term objectives can extend the first phase beyond 50 moves but equally, an exclusive focus on the first phase can overextend a subsequent phase. In practice, players today have a choice only of tables providing DTC data (Thompson, 1986; ChessBase, 2000) or DTM data (Hyatt, 2000; Nalimov, 2000); no DTZ data is easily available for P-endgames. Table 2, which collates all positions cited in this paper, gives examples of blind adherence to strategies SZ, SC or SM missing 50-wins, starting with three first-phase failures.

The KQKNN position Q-NN1 (Tamplin, 2000) leads with MC-optimal play to position Q-NN2 on move 38. With just 13 moves left and all required for conversion, strategy SM selects 38. Qc3, Qf2 and Qg1 of which only Qg1 is C-optimal: SM therefore fails. Strategy SMC succeeds by narrowing the choice to Qg1.

The maximal KNNKP position mxNN-P1 (Dekker, Van den Herik and Herschberg, 1990) leads by MC-CM play to NN-P2 after 24. ... h4. White must now force *h3* by move 74. However, at position NN-P3, the SC and SM strategies dictate 63. Nd2 leading to position NN-P4. This allows 63. ... Kf2, postponing *h3* until move 80. Strategies SC and SM therefore fail but Dekker *et al.* imply that strategy SZ forces *h3* in time to win NN-P3.

The KQPKQ position QP-Q3 is the result of 49 moves of Z-optimal play from QP-Q2 (Thompson, 1986, p. 138 after 21. ... Qd4) but could equally well have been the result of 49 moves of C-optimal play from another position. Strategy SZ succeeds just in time with 50. a7 but SC and SM fail by dictating 50. Kb8.

The KQBBKN position QBB-N1 shows that strategy SZ, far from being a panacea, also fails. It misses the four-move mate, sacrifices the Queen unnecessarily and leaves a second phase of 52 moves. Perhaps one should never resign against a computer. Line *f* of Appendix A features a more benign knight sacrifice.

The positions above show SM, SC and SZ failing individually. However, with *mleft* denoting the number of moves left in the current phase, the following non-minimax strategies optimise against longer-term goals but safeguard the length of the current phase. They are defined in terms of the subset of moves they return:

SZ' $\equiv$ {move to P(dz_i) | $dz_i \leq mleft - 1$}
Sσ* $\equiv$ **if** $dz > mleft$ **then** SZ **else** SZ'σ .. e.g., SC*, SM* and SMC* $\equiv$ (SMC)*. Sσ* $\geq$ Sσ but SZσ* $\equiv$ SZσ.
SA_1 $\equiv$ **if** $dm \leq mleft$ **then** SM **else if** $dc \leq mleft$ **then** SC **else** SZCM.

SM* and SA_1 succeed on the positions above but it is conjectured that they fail to win some winnable positions and that a metric recognising a *k*-move rule explicitly is needed. For example, KBBKNN has mxM = 106 moves and mxZ = 38 moves (Stiller, 1996); 24% of wtm positions are wins and 65% of these have $dm > 50$. It converts to KBBKN which has $dz > 50$ for some 11.16% of White wins. Let the KBBKN wins for White be partitioned into sets W ($dz \leq 50$) and D ($dz > 50$). Let three subsets of KBBKNN wins be defined as follows:

A $\equiv$ {P | P is a Wh. win: Wh. cannot mate in KBBKNN but can force P to $P_w \in$ W in $dz_w \leq 50$ moves}
B $\equiv$ {P | P is a Wh. win: Wh. cannot mate in KBBKNN but can force P to $P_d \in$ D in $dz_d \leq 50$ moves}
AB $\equiv$ {P | P $\in A \cap B$, $dm > 50$ and $dz_d \leq dz_w$ implying $dz = dz_d \leq 50$}, see Figure 1.

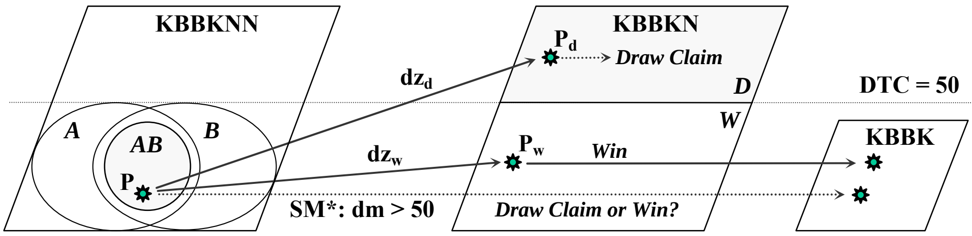

Figure 1: Winning a 'difficult' position P in KBBKNN.

For any P $\in$ AB, $SA_1 \equiv$ SC $\equiv$ SC* because $dm > 50$: this strategy fails because unconstrained C-optimal play could arrive at position P_d. Neither is SM* constrained to avoid P_d. However, P can be won in two phases of $\leq$ 50 moves by conversion to a position P_w in W and then to mate in KBBK. The positions in set AB require a more circuitous route to secure the win, i.e., constrained optimisation that recognises the 50-move rule.

The same scenario may occur before the long phases of KQPKQ, corresponding to pawn positions a6, a7, b3, b6, b7, c3, d3, d4 and d6. The position sets equivalent to A, B and AB above are also more easily computed.

Let G(*dr*, *bm*) be the goal of ending a phase by converting to a position with DTR = *dr* in *bm* moves or less. Figure 2 illustrates a scenario in which various goals may be achieved, some requiring more moves than others. The initial default goal is $G_1(50, 50)$. However, an initial DTR of 30 implies that $G_2(30, 30)$ is achievable. Further goals $G_3(25, 43)$ and $G_4(22, 56)$ are also achievable although the last is beyond the 50-move limit.

As already observed, strategy SZ may not win under the minimal *k*-move rule possible. Equally, conversions to lower DTR values than the phase's initial *dr* may be achievable by extending the current phase beyond *dr* moves but not beyond the *k*-move limit. Section 6 returns to this scenario and considers how White, traditionally pursuing a win, can narrow down its choice of moves while also managing a risk, which is in fact present, that the latest G(*dr*, *bm*) goal may be missed.

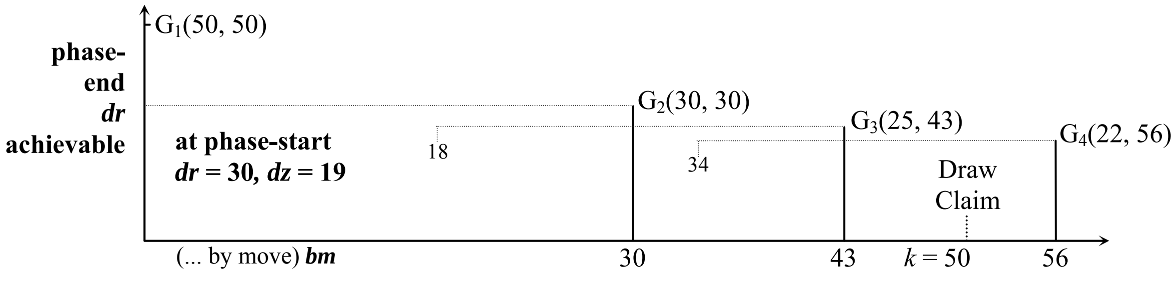

Figure 2: Phase-ending goals $G_i(dr, bm)$.

5. THE DTR METRIC AND ENDGAME TABLE

The 50-move rule generalises to the *k-move rule* and suggests metric DTR as follows:

a position's *Depth by the Rule*, DTR, is the least k for which the position can be won without the risk of a draw claim under a k-move rule.

It immediately follows that $dz \le dc \le dm$, $dz \le dr \le dm$, and that a position can be moved to the next phase in at most a further dr moves, leaving a position with DTR $\le dr$. Further, a position's dr satisfies:

- dr = max [dz, min (dr of won successors)] for side-to-move, *stm*, wins
- dr = max [dz, max (dr of lost successors)] for stm losses

The data for metric DTR is stored in endgame table ER. ER ≡ EM for K*x*K where x = Q, R, (B, N,) BB or BN. When computing ER for an endgame, it is assumed that table EZ already exists and that ER has already been computed for subsequent phases of play following a pawn-push or capture.

5.1 Algorithm AL1: generating table ER from table EZ

The formulae above suggest a straightforward, if relatively inefficient, algorithm AL1 for generating an EGT table ER from the table EZ. The figure *50* does not feature and the table ER can be used under any k-move rule.

{initialise} *ChangeFlag* ← True; **for** i = 1 **to** *index_range* **do** ER[i] ← EZ[i] **end_do**;
  max_next_dr = max(mxR of a subgame of this endgame) {$dr \ge max\_next\_dr \Rightarrow dr = dz$};
{cycle} **while** *ChangeFlag* = True **do** *ChangeFlag* ← False;
  for i = 1 **to** *index_range* **do**
    if $dr < max\_next\_dr \wedge$ ER[i] ≠ Draw ∧ ER[i] ≠ Broken[3] **then**
      if position(i) is an stm win **then** *dr2* = max(EZ[i], min(ER[j] of won successors)) **end_if**;
      if position(i) is an stm loss **then** *dr2* = max(EZ[i], max(ER[j] of successors)) **end_if**;
      if *dr2* ≠ ER[*i*] **then** ER[i] ← *dr2*; *ChangeFlag* ← True **end_if**;
    end_if ;
  end_do; **end_do** {end: ER is now the definitive ER table with ER[i] ≡ dr_i}

Note that it is sometimes necessary to adjust an original dz value more than once. For example, the position QBB-N1 would start with $dr = dz = 2$ plies, would then be given dr = 103 plies and finally dr = 5 plies. The same is true in the Edwards/Nalimov DTM algorithm. For QBB-N1, dm = 105 plies and later dm = 7 plies.

With $dr > k$, the ER table can be used by the infallible attacker to minimise dr and give a fallible opponent an opportunity to lower dr. Conversely, if $dr \le k$, an infallible defender can maximise dr and give a fallible opponent an opportunity to raise dr.

5.2 Algorithm AL2: generating table ER by modified retro-method

The algorithm AL2 for constructing the endgame table ER is based on the established retro-search algorithm (Thompson, 1986, 1996; Nalimov *et al.*, 1999) used in the past to create EZ and EC tables to the DTZ and DTC metrics respectively. It is now more convenient to think of depth in plies and assume a *2k*-plies rule. The modified algorithm introduces two constraints. First, it considers only conversions to i-plies-wins with $i \le 2k$ and its retro-search only reaches back to positions with $dz \le 2k$ plies.

The following definitions are used:

C_0 = {subgame positions P | the stm is mated}
C_i = {subgame positions P | P is an h-plies win for $h \le i$, i.e., no phase has more than h plies}
M = {endgame positions P | the stm is mated}: $X_{i,0}$ = M.
W_i = {endgame positions P | the stm, winning, can mate or move to a won P' ∈ C_i in one ply}
L_i = {endgame positions P | the stm, losing, must move to a lost P' ∈ C_i in one ply}
  see, for example, position NN-P5 or QBB-N1 after 1. Qa2+. $X_{i,1} = W_i \cup L_i$.
$X_{i,j}$ = {endgame positions P | stm can force or must allow conversion to a P in C_i, or mate, in $\le j$ plies}

[3] A table entry is marked *broken* if it corresponds to a clearly illegal position, an unwanted position or no position.

Imagine that the algorithm is divided up into phases and that the *i*th phase finds those positions in, say, KBBKNN which can be won under an *i*-plies rule. Each phase starts by identifying the mates in KBBKNN. Then it identifies those *boundary positions* which can be or must be converted in one ply to mate or to a value-preserving subgame position, also winnable under an *i*-plies rule. The phase then computes just *i-1* cycles of retro-search, each one identifying positions one ply deeper into KBBKNN.

Given a set X of positions in an endgame, let the functions $\mathbf{W}(X)$, $\mathbf{L}(X)$, $\mathbf{\Pi}(X)$ and $\mathbf{\Sigma}(X)$ be defined as:

$\mathbf{W}(X)$ = {positions P | the winner, to move, can move to some won P' $\in$ X}: $\mathbf{W}(X \cup Y) = \mathbf{W}(X) \cup \mathbf{W}(Y)$
$\mathbf{L}(X)$ = {positions P | the loser, to move, must move to some (lost) P' $\in$ X}: $\mathbf{L}(X \cup Y) = \mathbf{L}(X) \cup \mathbf{L}(Y)$
$\mathbf{\Pi}(X) = \mathbf{W}(X) \cup \mathbf{L}(X)$:
$\mathbf{\Pi}(X \cup Y) = \mathbf{W}(X \cup Y) \cup \mathbf{L}(X \cup Y) = \mathbf{W}(X) \cup \mathbf{W}(Y) \cup \mathbf{L}(X) \cup \mathbf{L}(Y) = \mathbf{\Pi}(X) \cup \mathbf{\Pi}(Y)$
$\mathbf{\Sigma}(X) = X \cup \mathbf{\Pi}(X)$: $\mathbf{\Sigma}(X \cup Y) = X \cup Y \cup \mathbf{\Pi}(X \cup Y) = X \cup Y \cup \mathbf{\Pi}(X) \cup \mathbf{\Pi}(Y) = \mathbf{\Sigma}(X) \cup \mathbf{\Sigma}(Y)$

It follows that $X_{i,\,j+1} = \mathbf{\Sigma}(X_{i,\,j})$, $X_{i,\,2j} = \mathbf{\Sigma}^{2j-1}(X_{i,\,1})$ and that the set of *h*-plies wins for $h \leq i$ is the set $X_{i,\,i}$. The set of *i*-plies wins is $X_{i,\,i} - X_{i-1,\,i-1}$ and the set of *k*-wins is $X_{2k,\,2k} - X_{2k-2,\,2k-2}$.

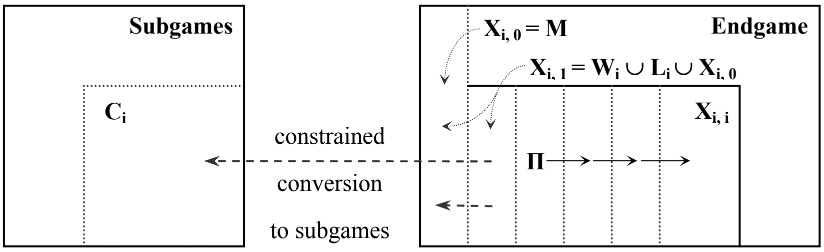

Figure 3: Algorithm AL2 for computing endgame table ER , phase *i*.

It is obvious that each phase recomputes much of what has been computed before. Further, the function $\mathbf{\Pi}(X)$ makes random access to data, particularly expensive when confirming that the loser is forced to move to some position in X. There is however an opportunity to exploit previous data to increase efficiency, reducing the use of $\mathbf{\Pi}$ at the expense only of some sequential access to and manipulation of interim sets of results.

5.3 Using known subsets of $X_{i,\,j}$

Let $X_{i,\,j} \equiv \varnothing$ for i < 0 and j < 0. $X_{i,\,0}$ = M. Note that $X_{i,\,j}$ is not defined for $j > i$.
Let $Y_{i,\,j} = X_{i,\,j} - X_{i,\,j-1}$ and $Z_{i,\,j} = Y_{i,\,j} - Y_{i-1,\,j}$. Then $Y_{i,\,j} \subseteq Y_{i-1,\,j} \cup Z_{i,\,j}$ and $X_{i,\,j} = X_{i,\,j-1} \cup Y_{i,\,j} = X_{i,\,j-1} \cup Y_{i-1,\,j} \cup Z_{i,\,j}$.
Note that, because a position may be in $Y_{i-1,\,j} \cap X_{i,\,j-1}$, it is possible that $Y_{i-1,\,j} - Y_{i,\,j} \neq \varnothing$.

The computation of $X_{i,\,j}$ involves only the computation of $\mathbf{\Pi}(Z_{i,\,j-1})$ for $j < i$ and $\mathbf{\Pi}(Y_{i,\,j-1})$ for $j = i$, see Figure 4:

$X_{i,\,j} = \mathbf{\Sigma}(X_{i,\,j-1}) = \mathbf{\Sigma}(X_{i,\,j-2} \cup Y_{i,\,j-1}) = \mathbf{\Sigma}(X_{i,\,j-2}) \cup \mathbf{\Sigma}(Y_{i,\,j-1}) = X_{i,\,j-1} \cup \mathbf{\Pi}(Y_{i,\,j-1})$, i.e., $X_{i,\,i} = X_{i,\,i-1} \cup \mathbf{\Pi}(Y_{i,\,i-1})$.
$X_{i,\,j} = \mathbf{\Sigma}(X_{i,\,j-1}) = \mathbf{\Sigma}(X_{i,\,j-2} \cup Y_{i,\,j-1}) = \mathbf{\Sigma}(X_{i,\,j-2} \cup Y_{i-1,\,j-1} \cup Z_{i,\,j-1})$
$= \mathbf{\Sigma}(X_{i,\,j-2} \cup X_{i-1,\,j-2} \cup Y_{i-1,\,j-1} \cup Z_{i,\,j-1})$ because $X_{i-1,\,j-2} \subseteq X_{i,\,j-2}$
$= \mathbf{\Sigma}(X_{i,\,j-2} \cup X_{i-1,\,j-1} \cup Z_{i,\,j-1}) = \mathbf{\Sigma}(X_{i,\,j-2}) \cup \mathbf{\Sigma}(X_{i-1,\,j-1}) \cup \mathbf{\Sigma}(Z_{i,\,j-1}) = X_{i,\,j-1} \cup X_{i-1,\,j} \cup \mathbf{\Pi}(Z_{i,\,j-1})$.
$Y_{i,\,j} = [X_{i-1,\,j} \cup \mathbf{\Sigma}(Z_{i,\,j-1})] - X_{i,\,j-1} = [X_{i-1,\,j-1} \cup Y_{i-1,\,j} \cup \mathbf{\Sigma}(Z_{i,\,j-1})] - X_{i,\,j-1} = [Y_{i-1,\,j} \cup \mathbf{\Sigma}(Z_{i,\,j-1})] - X_{i,\,j-1}$.
$Z_{i,\,j} = Y_{i,\,j} - Y_{i-1,\,j} = \mathbf{\Sigma}(Z_{i,\,j-1}) - X_{i,\,j-1} - Y_{i-1,\,j}$.

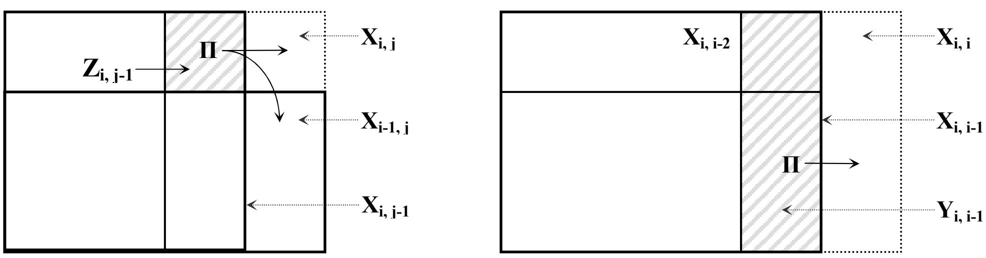

Figure 4: Subsetting $X_{i,\,j}$ to minimise use of function $\mathbf{\Pi}$.

6. USES OF THE DTR DATA

Let us suppose, as is usual, that White is pursuing a win under a *k*-move rule against a possibly fallible player. For convenience, moves will be numbered from *1* in the current phase. The following notation is used:

k	indicates the k-move rule in force: currently, FIDE has set $k = 50$ for all endgames
mplayed	the *history factor*, the number of white moves played in this phase
mleft	the number of white moves left before the risk of a draw claim; $mleft = k - mplayed$
$P(dr, dz)$	a wtm position P with depths dr in metric DTR and dz in metric DTZ
CP	the set $\{P_i(dr_i, dz_i)\}$ of btm successors of position P
CQ	the subset $\{P_i \in CP \mid dz_i \leq mleft - 1\}$: $\varnothing \subseteq CQ \subseteq CP$.
$G_j(dr_j, bm_j)$	the *jth goal*, to conclude the phase by conversion with DTR dr_j on or before move bm_j
gi	index of the last goal defined
CR	the subset $\{P_i \in CQ \mid dr_i \leq dr_{gi} \wedge mplayed + 1 + dz_i \leq bm_{gi}\}$: $\varnothing \subseteq CR \subseteq CQ \subseteq CP$.

As any strategy Sσ can be transformed into a strategy Sσ* considering only those moves that safeguard the first phase, it is assumed below that $dz \leq mleft$.

6.1 The minimax strategy SR

The equivalent of the strategies SM, SC and SZ in Table 1 is SR which selects {move to $P_i(dr_i, dz_i) \mid dr_i \leq dr_j$} as the set of options. The two ways in which SR and SR* fail suggest the definition of a range of strategies which leave a wider choice of moves available. It is assumed that the opponent is playing to maximise DTR.

In the example of Figure 2 with a 50-move rule, the default goal $G_1(50, 50)$ is immediately superceded by $G_2(30, 30)$ before White's first move. With 18 moves played, the position $P(25, dz)$ implies a potential goal $G_3(25, 43)$. This, if reached, will also safeguard the win at the expense of more moves in the current phase. Strategy SR always aims for the lowest *dr* and therefore, in effect, adopts goal $G_3(25, 43)$.

As position QBB-N1 of Table 1 shows, albeit with the entirely hypothetical targets of $dr = 3$ and $dz = 1$, White may have to choose between its *dr* and *dz* targets in position $P(dr, dz)$. It cannot necessarily achieve both and in the event of conflict, will safeguard the current phase in an Sσ* strategy. Figure 5 shows that after 28 moves, a *feasible* move, apparently compatible with the *dr* target and *dz* constraint, is in fact a wrong choice. It leads to conversion positions with $dr \leq 25$ but these are not only beyond move 43 but beyond move 50. The *dr* target would therefore have to be abandoned with unpredictable consequences. This demonstrates that there is a risk which is not present with the DTM, DTC and DTZ metrics: the aggressor may stray off the winning line.

After 33 moves, in position $P(22, dz)$, SR again adopts the implied goal $G_4(22, 56)$. This, in the worst case, is not achievable until after Black claims a 50-move draw. The risk of failing to win in this way is easily avoided by strategies which do not adopt goals with $bm > 50$.

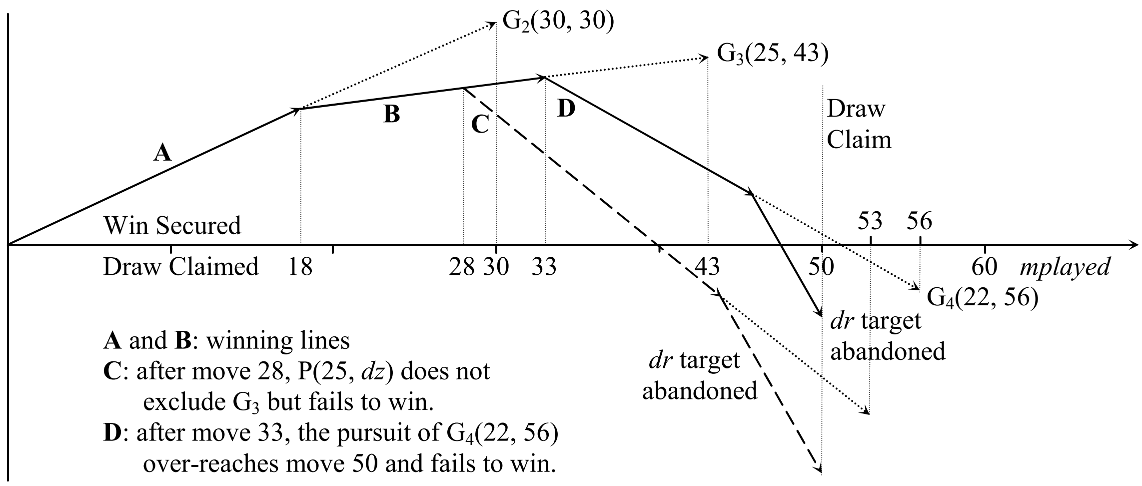

Figure 5: Winning, incorrect and over-reaching lines of play.

The following measures are suggested to lower the residual risk of a draw claim when using SR*:

- avoid subsetting the moves offered by SR*, e.g., by minimum *dz*
- instead, search forward a number of plies continuing to subset options with SR*
- if the lowest visible *dr* in unattainable within *k* moves, relax the *dr* goal.

Given the failings of SR*, a range of strategies SR_a is defined, featuring constraints on the *dr* attempted.

6.2 The SR_a Strategies

Four strategies are defined differing in the criteria applied to potential goals before they are adopted as actual goals. All strategies adopt the default goal $G_1(k, k)$ in the context of a *k* -move rule, even though it might not be achievable against infallible play. SR* above is equivalent to SR_4* below. In summary:

- SR_1 focuses only on goal $G_1(k, k)$, in effect, providing a fixed filter on the move options
- SR_2, given goal $G_i(dr_i, bm_i)$, adopts $G_j(dr_j, bm_j)$ provided $dr_j < dr_i$ and $bm_j \le bm_i$: $SR_2 \ge SR_1$
- SR_3, given goal $G_i(dr_i, bm_i)$, adopts $G_j(dr_j, bm_j)$ provided $dr_j < dr_i$ and $bm_j \le k$: $SR_3 \ge SR_1$
- SR_4, given goal $G_i(dr_i, bm_i)$, adopts $G_j(dr_j, bm_j)$ provided $dr_j < dr_i$.

Where SR_2 and SR_3 adopt a new goal, they confirm that the aggressor has a winning line and the effect of any previous, suboptimal choices of move may be ignored. Returning to the example of Figure 2, SR_1 uses only goal G_1, SR_2 uses G_1 and G_2, SR_3 uses G_1-G_3 and SR_4 uses G_1-G_4.

6.3 An algorithm for the SR_a

The algorithm, written for the attacker, returns a subset *CM* of moves. To guard against DTR > *k*, *CM* is first defined to be the same subset which strategy SR will return, i.e., those moves with minimal *dr*.

{initialise: $a \equiv dr\_focus$ is assumed set} $high\_value = 10^9$;
  if $mplayed = 0$ **then** $G_1 \leftarrow G_1(k, k)$; $gi \leftarrow 1$ **end_if**;
{step 1: in case $dr > k$}
  $CM \leftarrow \{P_i \mid P_i$ is selected by strategy SR$\}$
{step 2: re-adopt a previous goal if this exists and the current goal is clearly unattainable}
  while $CR = \varnothing \wedge gi > 1$ **do** $gi \leftarrow gi - 1$;
{step 3: if possible, set a stronger goal from those implied by the current move options}
  if $a \ne 1 \wedge CR \ne \varnothing$ **then**
    $drmin = \min\{dr_i \mid P_i(dr_i, dz_i) \in CR\}$;
    if $mplayed + 1 + drmin \le bm\_limit(a)$ **then** *add_new_goal*(G, *gi*) **end_if**
  end_if;
{step 4: if possible, subset to P_i not excluding the current goal}
  if $CR \ne \varnothing$ **then** $CM \leftarrow CR$;

where *bm_limit*(*a*) ≡ **begin if** a = 1, 2, 3 or 4 **then** k, bm_{gi}, k or *high_value* respectively **end_if**
and *add_new_goal*(*G*, *gi*, ...) ≡ **begin** gi $\leftarrow gi + 1$; $G_{gi} = G_{gi}(drmin, drmin + mplayed + 1)$ **end**

6.4 Examples of SR_a in use

SR_1 succeeds where SZ fails on position QBB-N1 of Table 1 and positions P ∈ AB in Figure 1. Strategy SA_1 of section 4 can be strengthened to:

SA_2 ≡ **if** $dm \le mleft$ **then** SM **else if** $dc \le mleft$ **then** SR_aC **else** SR_aZCM*

The example of Figure 6 shows that goals G_1-G_5 have been logged starting with the default goal $G_1(50, 50)$. After Black's first move, the goal can be improved to '45 by move 45' and later to '44/43 by 45' and '41 by 44'. With White about to play its 17th move in the phase, three scenarios labeled *a*, *b* and *c* are portrayed. For each, a single move to a position $P_\alpha(dr, dz)$ is indicated, implying a potential goal which may or may not be adopted. SR_2 will adopt goal $G_a(23, 40)$ but not G_b or G_c as these may only be achievable after the current goal horizon of move 44. SR_3 will adopt G_a and $G_b(29, 46)$ as the latter is theoretically achievable before 50 moves elapse. SR_4 will adopt any of G_a-G_c, possibly over-reaching with goal G_c and being forced back to a previous goal or to strategy SZ.

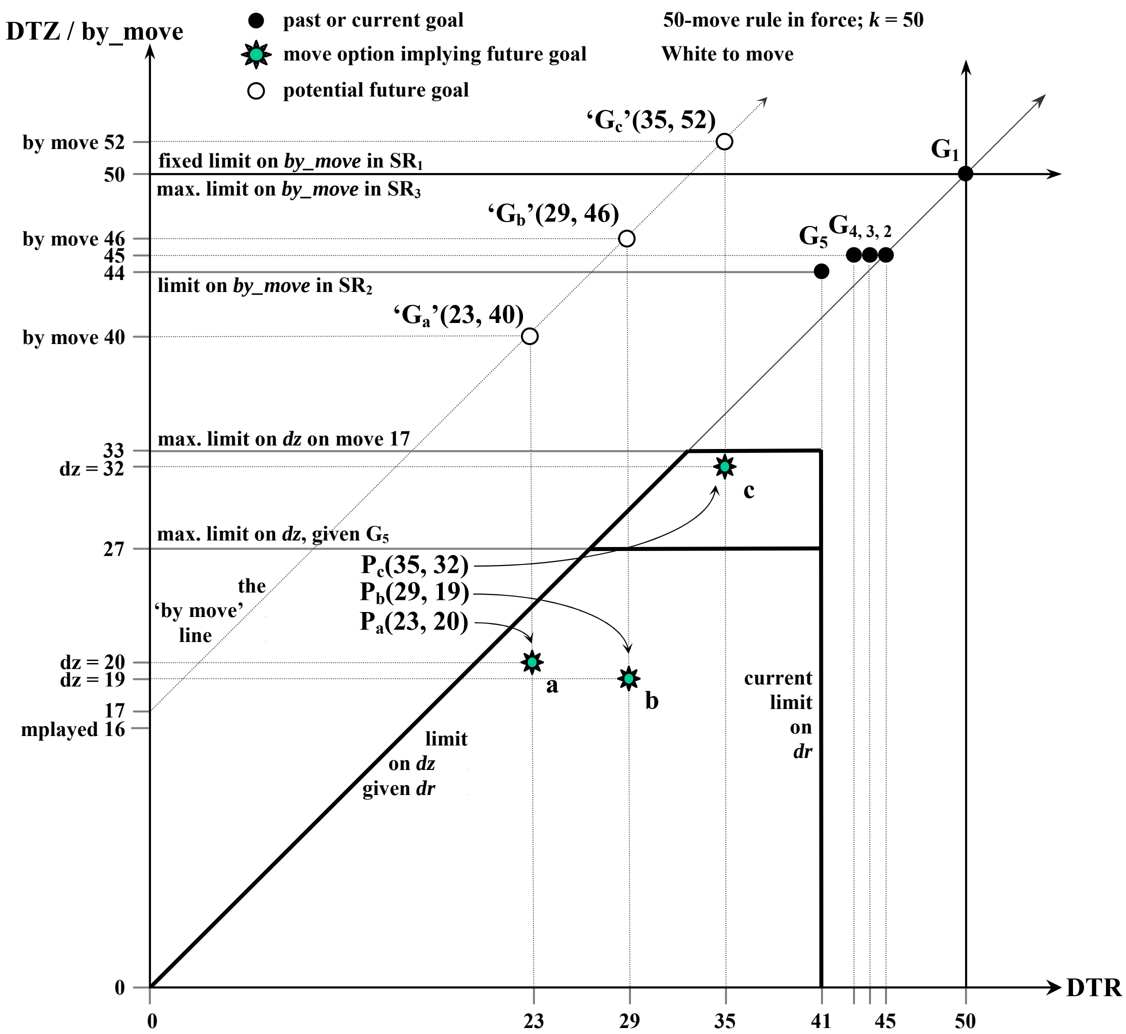

Figure 6: Example game phase with various destination positions $P_\alpha(dr, dz)$ and implied goals $G_\alpha(dr, bm)$.

7. SUMMARY

The basic minimax strategies SM and SC currently in use can fail in the context of the 50-move rule by allowing unnecessary draws. Even strategy SZ can fail by minimising the length of the current phase of play at the cost of an over-long subsequent phase. More complex *nested strategies* such as SMC, SCM and SZCM improve the minimax approach. The non-minimax strategies such as SZ' and SMCZ* also require no more than the relatively easy production of the endgame tables EZ to the DTZ metric for P-endgames.

However, the general *k*-move rule suggests a new metric DTR. It is conjectured that strategies using metrics which do not recognise this rule explicitly will eventually fail. The endgames KQPKQ and KBBKNN may well harbour positions which demonstrate this. The endgame table ER may be generated by various methods, the most complex approach striving for greater efficiency.

Both the naive minimax strategy SR and its stronger derivative SR* can fail in two ways. Therefore, a range of four strategies SR_a has been defined, using both DTR and DTZ information to provide a wider choice of moves. The SR_a can be used iteratively in a focussed conventional search. In the absence of empirical data, the author believes that the repeated use of SR_3* in a search will be more effective than strategy SR_3ZCM*.

Experiments to verify the power of the SR_a* strategies require actual EZ and ER tables. The author therefore invites others in this field to produce such tables. The prime candidates are the deep 5-man endgames and those 6-man endgames which can precede them, as listed in Section 4.

8. ACKNOWLEDGMENTS

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APPENDIX A: TYPES OF OPTIMAL PLAY

These optimal lines proceed from the maximal KRNKNN position mxRN-NN of Table 1. A published line (Stiller, 1996) is, up to and including move 205w, both C-C optimal as intended and M-M optimal. First Black and then White must choose between C-optimal and M-optimal moves, their choices eventually defining four different types of optimal line. Different equi-optimal choices would of course produce different specific lines.

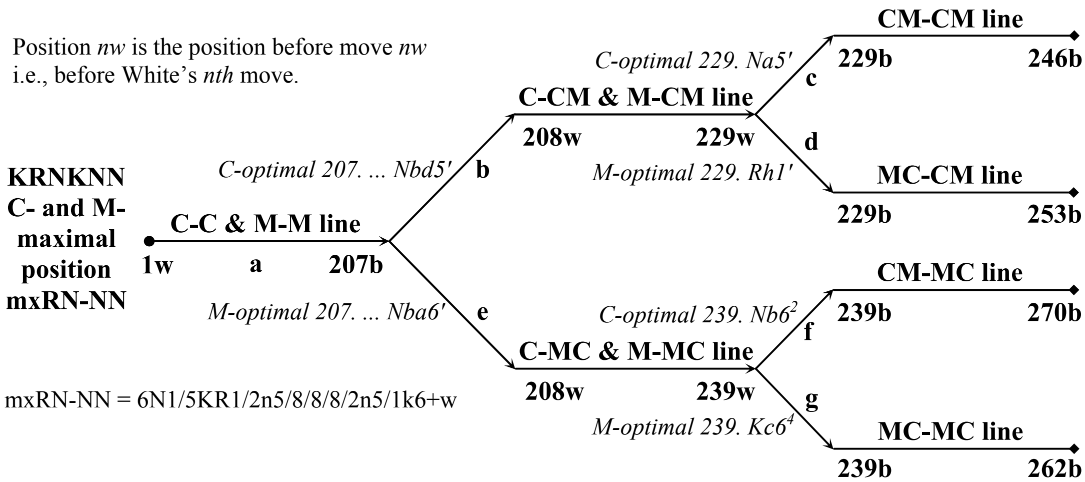

Figure 7: Both sides eventually choose between C- and M-optimal play

The following notation is used to indicate various properties of the moves:

> ' *only X-optimal move, given strategy SX*; " *only value-preserving move*; ° *only legal move*; [..., ...] *equi-optimal moves*; ^{n} *one of* n *equi-optimal moves*; *-n a move suboptimal by* n *moves*; ^{v} value-changing move.

a) {C-C and M-M line} 1. Ke6" Nb4' 2. Ke5" Nd3' 3. Ke4" Nf2' 4. Kf3" ..... 205. Re5' Nec7' 206. Nd6' Nb4' 207. Re4' {and now Black must choose between C- and M-optimality: 5k2/2n5/3N4/6K1/1n2R3/8/8/8+207b}.

b) {C-CM and M-CM line} 207... Nbd5' 208. Re1' Kg7 [Nb4, Nb6, Nc3] 209. Kf5' Kf8' 210. Ke5' Nb4' 211. Rf1+' Ke7' 212. Rf7+' Kd8° 213. Nb7+' Kc8' 214. Nc5' Nb5' 215. Rg7 [Rh7] Kd8' 216. Rb7' Nc6+' 217. Ke6' Kc8' 218. Rh7' Nb4' 219. Na4' Na6' 220. Kd5' Nbc7+' 221. Kd6' Ne8+' 222. Ke7' Nec7' 223. Rh6' Nb8' 224. Nb6+' Kb7° 225. Nc4' Nc6+' 226. Kd7 [Kd6] Nb8+' 227. Kd6' Nba6' 228. Rh7' Kc8' {and now White must choose between C- and M-optimality: 2k5/2n4R/n2K4/8/2N5/8/8/8+229w}.

c) {CM-CM line} 229. Na5' Kd8' 230. Nc6+' Kc8' 231. Ne7+' Kd8' 232. Nd5' Ne8+' 233. Kc6' Nb8+' 234. Kb5' Nd6+' 235. Kc5' Nc8' 236. Rh8+' Kd7° 237. Nf6+' Kc7' 238. Rh7+' Kd8' 239. Rb7' Na6+' 240. Kc6' Ne7+' 241. Kb6' Nb4' 242. Rd7+' Kc8° 243. Rxe7' **{KRNKN}** Nc6 [Nd5] 244. Kxc6' Kb8' 245. Kb6' Kc8^2 246. Re8#' 1-0.

d) {MC-CM line} 229. Rh1' Kd8' 230. Rh8+' Ne8+° 231. Ke6' Nc7+' 232. Kf7' Kd7' 233. Rh6" Nd5' 234. Ne5+" Kd8' 235. Rh2' Ndc7' 236. Rh8' Kc8' 237. Rh6' Kd8' 238. Nc4' Kc8' 239. Rg6' Kb8' 240. Ke7' Kc8' 241. Ne3' Kb8' 242. Kd8' Kb7^2 243. Kd7 [Nc4] Kb8^3 244. Rb6+' Ka7' 245. Rb1' Ka6^2 246. Nc4' Ka7' 247. Rb2' Ka6' 248. Kc6' Ka7' 249. Nb6' Kb8' 250. Nd7+' Ka7' 251. Rb4^2 Ka8^2 252. Ra4+' Na6° 253. Rxa6#' **{KRNKN}** 1-0.

e) {C-MC and M-MC line} 207. ... Nba6' 208. Kf5' Nc5' 209. Re2' N5e6' 210. Ke5' Nd8' 211. Rf2+' Ke7' 212. Nf5+' Ke8' 213. Re2' Nb5' 214. Kd5+' Kf7' 215. Rf2' Kg6' 216. Nh4+' Kh5' 217. Ng2' Nc3+' 218. Kd4' Nb5+' 219. Ke5' Kg6' 220. Rf6+' Kg7' 221. Nf4' Nf7+' 222. Ke6" Nd4+' 223. Ke7" Ne5' 224. Nh5+' Kh7' 225. Ra6' Kg8' 226. Kd6' Nf7+' 227. Kd5' Nb5' 228. Rc6' Kf8' 229. Rc5' Na7 [Nbd6] 230. Ra5' Nc8' 231. Ke6' Nd8+' 232. Kd7" Nd6' 233. Ra1' N8b7' 234. Kc7' Ke8 [Ke7, Kf7] 235. Nf4' Ke7 [Kf7] 236. Re1+' Kf7 [Kf6] 237. Nd5' Kg6' 238. Rf1' Kg5' {8/1nK5/3n4/3N2k1/8/8/8/5R2+239w ... and now White must choose between C- and M-optimality}.

f) {CM-MC line} 239. Nb6 [Ne3] Kg4' 240. Rc1' Kf5^3 241. Nc4' Ne4' 242. Kxb7' **{KRNKN}** Nc5+' 243. Kc6' Nd3' 244. Ra1' Ke4 [Nf4] 245. Ra4' Nf4' 246. Kd6 [Nd2+] Kd3 [Ne2] 247. Ke5 Ng6+' 248. Kf5' Nh4+' 249. Kg4' Ng6' {*dm* = 18} 250. Ra7' Kxc4' **{KRKN**, *dm* = 20} 251. Kf5" Nh4+ 252. Ke4" {and mate, m270} 1-0.

g) {MC-MC line} 239. Kc6^4 Kg6' 240. Nc7' Kg7' 241. Ne6+' Kg6' 242. Kc7' Kh6' 243. Rg1' Kh5' 244. Kc6 Kh6' 245. Ng7 [Nf4] Kh7' 246. Nh5' Kh6' 247. Nf6' Nf7 [Nf5] 248. Kxb7' **{KRNKN}** Ng5' 249. Kc7 [Kc6] Nf3' 250. Rg8' Ne5' 251. Kd6' Ng6' 252. Ke6' Kg5' 253. Nd5' Kh5' 254. Kf6' Kg4' 255. Rxg6+ **{KRNK}** Kf3' 256. Ke5^2 Ke2^2 257. Ke4' Kd2' 258. Rg2+' Kd1' 259. Nb4^4 Kc1^2 260. Kd3' Kb1' 261. Kc3' Ka1 262. Rg1#' 1-0.

APPENDIX B: SOME MONSTERS OF THE DEEP

This appendix notes some games (ChessLab, 2000) and positions associated with the 50-move rule.

'KBBKN' (Roycroft, 1986) b7/b1K5/8/3P4/3k4/8/8/8+w. 1. d6' Kc5" 2. d7' Bb6+" 3. Kc8' Kc6" 4. d8=N+' {*dc* = 57 moves} Kb5" 5. Nb7' Ba7" 6. Kc7' Ka6' 7. Nd6' and White can get to the Kling-Horwitz position.

'KBBKN'. Pinter-Bronstein (1977, ECO B14, ½-½), 8/2b5/8/3b2k1/8/4K3/8/4N3+68w {*dc* = 54m, *dm* = 66m}. The 44-move win from move 70 would have just beaten a 50-move draw claim. However, Pinter was allowed to set up Kling-Horwitz positions on moves 71, 90 and 112 in the b2, g2 and b7 corners respectively and could have taken more moves doing so (Roycroft, 1988). A draw was agreed on move 117.

KBNNKR. Karpov-Kasparov (1991, ECO E97, ½-½), {63. Kxh4} 3r4/8/2B2k2/8/5N1K/3N4/8/8/+63b {=}. In 51 moves, Kasparov never allowed a win (Stiller, 1991b) and set up a stalemate finish with a Rook sacrifice.

KQNKQ. Ljubojevic-Hjartarson (1991, ECO A22, ½-½), {70. Qxg5+} 6k1/3q4/8/6Q1/6N1/7K/8/8+70b: {=}. Contrary to Nefkens (1991), Black's defences slip on move 88 but White misses the win on the next move. On move 117, Black sets up a mate for White which is then promoted to just two moves beyond the draw claim.

KQPKQ. Wegner-Johnsen (1991, ECO D30, ½-½), {126. … a2} 8/8/7K/3q4/k7/8/p7/1Q6+127w. The game entered KQPKQ with move 53w. Although *dc* = 13m and *dm* = 28m, a '75-move' draw resulted on move 201.

KRBKR (Croskill, 1864). k7/6R1/2K5/8/2B5/8/8/7r+w (RB-R1) and 1k6/8/2K5/8/2B3R1/8/7r/8+w (RB-R2). Croskill incorrectly claimed *dc* = 57 for RB-R1 (*dc* = 49) but then arrived at RB-R2 for which he gave the correct *dc* = 51 and an almost correct line: "a high point of 19th century endgame analysis" (Nunn, 1994, p. 232).

KRBKR. Deep Thought - Fishbein (1988, ECO C69, 1-0), {58. Rxc4} k7/1r6/8/8/2R2B2/8/5K2/8+58b {=}. Black's defence slips after 12 moves. This leaves a 13-move win and resignation follows on move 81.

KRBKR. Nikolic-Arsovic (1989, ECO E95, ½-½), {167. Bxd5} 8/8/8/1r1B3R/3K1k2/8/8/8+167b {=}. This is the longest game on record. White misses wins on moves 201, 238, 239, 241, 244 and 255 with depths *dc* = 4, 9, 3, 3, 9 and 20 respectively ...

4k3/5R2/3K4/3B48/8/8/3r4+201w: 201. **Rg7**^{v} (*201. Rf2*^{3} *Rd4 202. Ra2 Kf8 203. Rg2" Ke8*^{7} *204. Rg8#'*).
4r3/8/8/k2B4/3K4/8/1R6/8+239w: 239. **Rb7**^{v} (*239. Kc5" Rc8+ [Re3] 240. Bc6" Ka6*^{6} *241. Ra2#'*).

KRP(a2)KbBP(a3). Timman-Velimirović (1979, ECO D30, 1-0), 8/8/4k3/2R5/7b/p2K4/P7/8+64b. Won on move 103, this game brought about the 100-move allowance for this ending (Van den Herik *et al*., 1987).

POST-PUBLICATION NOTE

The iteration formulae for DTR defined in Section 5, and algorithms AL1 and Al2, are incorrect.

The correction was published in:

Haworth, G.McC. (2001). Depth by The Rule, *ICGA Journal*, Vol. 24, No. 3, p. 160.

Examples of positions where SC^-, SM^- and SZ^- all fail to defend a win have been published in:

Tamplin, J.A. and Haworth, G.McC. (2003). Chess Endgames: Data and Strategy. *Advances in Computer Games 10*, Graz, Austria (eds. H.J.van den Herik, H. Iida and E.A. Heinz), pp. 81-96. Kluwer Academic Publishers, Norwell, MA. ISBN 1-4020-7709-2.

Bourzutschky, M.S., Tamplin, J.A. and Haworth, G.McC. (2004). Chess Endgames: Data and Strategy, 2. *Journal of Theoretical Computer Science* (to appear).