STRATEGIES FOR CONSTRAINED OPTIMISATION

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ABSTRACT
The latest 6-man chess endgame results confirm that there are many deep forced mates beyond the 50-move rule. Players with potential wins near this limit naturally want to avoid a claim for a draw: optimal play to current metrics does not guarantee feasible wins or maximise the chances of winning against fallible opposition. A new metric and further strategies are defined which support players’ aspirations and improve their prospects of securing wins in the context of a k-move rule.

1. INTRODUCTION
Endgame tables (EGTs) have to date not acknowledged the FIDE 50-move rule of Article 9.3. It is irrelevant for all but 8 of the 3- to 5-man endgames. Further, EGT authors share an interest with chess composers in the absolute capabilities of the chessmen. They have reasonably not given priority to FIDE’s flexible rule which has indeed changed five times (see below) and whose detail has been difficult to implement.

However, recent progress on 6-man endgames (Nalimov, Wirth and Haworth, 1999; Hyatt, 2000; Karrer, 2000; Tamplin, 2000; Thompson, 2000) has renewed interest in having endgame data which serves both the practical player and the theoretician. The deeper maximum-depth wins imply that the 50-move rule will become a more frequent consideration. Let a won position be termed a k-win if k is the least integer for which optimal play would not risk a draw claim under a k-move rule. About half of the 6-man endgames computed to date feature k-wins with k > 50. Currently, practical players may be said to have two objectives:

• to win positions which are k-wins for k ≤ 50 without risking a 50-move draw claim, and
• to maximise the probability of winning a k-win position for k > 50.

These objectives are addressed here. Section 2 questions the appropriateness of the rule given a demonstrably effective aggressor. Section 3 introduces a number of metrics that define varieties of optimal play. Section 4 shows the value of the now disused metric Depth to Zeroing move

2 (DTZ). Section 5 defines the new metric Depth by the Rule (DTR) and describes algorithms for generating DTR data. Section 6 demonstrates the failings of a naive strategy for using DTR and defines further strategies using DTR and DTZ data.

2. HISTORY OF THE RULE
Ruy López suggested a 50-move limit in Article 17 of his Chess Code of 1561, perhaps in the interests of his fellow coffee-house professionals who played for wagers. The 1883 London Tournament’s rules, the basis of FIDE’s rules today, were the first to state that a P-push or capture would zero the count.

In 1974, FIDE first enabled the 50-move rule to be varied. They did so with 100-move clauses, in 1978 for KNNKP (Troitzkii, 1906-1910, 1934), in 1982 for KRP(a2)KbBP(a3) following the Timman-Velimirović game (Van den Herik, Herschberg and Nakad, 1987), and in 1984 for KRBKR (59) (Croshikill, 1864; Nunn, 1994). They did not meet all the requirements defined by Roycroft (1984) at the first opportunity.

By 1988, computer results, albeit single-sourced, were plentiful (Thompson, 1986) and endgame-specific limits were suggested. However, FIDE adopted a simpler stance, replacing the 100-move clauses by a 75-move allowance for just the six endgames KBBKN (Roycroft, 1983), KNNKP, KQKBB, KQKNN, KQP(x7)KQ and KRBKR (Kajić, 1989; Mednis, 1989). KRPKBP with blocked Pawns ceased to be an exception.

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2 A zeroing move is defined as one which zeroes the move count by FIDE Article 9.3, i.e., a pawn push, capture or mate. A phase of play is defined as a sequence of moves starting just after and ending with a zeroing move.
Following Stiller’s (1991) discovery that KRBKNN’s maximum depth is 223, FIDE gave up the chase and restored the 50-move limit for all endgames in 1992 (Herschberg and Van den Herik, 1993). KRNKNN then took the record phase length to 243 (Stiller, 1996) and this could well be extended by 7-man pawnless endgames. More details of some games and studies associated with the 50-move rule are in Appendix B.

Clearly a balance has to be struck between the extremes of denying players attainable wins and requiring the opposition to be eternally vigilant in a drawn position. Today, the main concerns are social ones for the welfare of defenders and tournament directors who wish to run their events to a schedule (Levy and Newborn, 1991): however, it is clear that these need not apply to computer-assisted play. There is an argument for waiving the 50-move rule where a player can demonstrably achieve a theoretical win. An EGT is currently the only way to establish theoretical position values and benchmark the aggressor’s effectiveness. Certainly, computers with EGTs can play won or drawn positions quickly, even if they initially assume a fallible opponent and take time to choose between equi-optimal moves (Levy, 1991). Other means of winning effectively may be created in the future. To deny players the opportunity of achieving complex wins foreshortens the domain of chess itself. It prevents us from seeing immaculate play building on the smallest of advantages and exploring the deep space of the endgame where humans may never go without the vehicle of perfect information.

3. METRICS FOR OPTIMALITY

Table 1 refines a previously published version (Nalimov et al., 1999) and contains a systematic notation to describe the various optimisation goals and related concepts. It provides a way of referring to and comparing different metrics, position depths, endgame tables, maximal depths, types of optimality and minimax strategies. Each strategy selects a subset of equi-optimal moves: one strategy may win where another draws. The actual line of play is determined by both sides’ respective strategies and their ultimate choice of equi-optimal moves.

<table>
<thead>
<tr>
<th>Goal Description</th>
<th>Goal (GZ)</th>
<th>Goal (GC)</th>
<th>Goal (GM)</th>
<th>Goal (GR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero move-count</td>
<td>Zero</td>
<td>Conversion</td>
<td>Mate</td>
<td>Mate within k-move rule</td>
</tr>
<tr>
<td>in maximin no. of moves</td>
<td>GZ</td>
<td>in maximin no. of moves</td>
<td>in maximin no. of moves</td>
<td>with maximin k</td>
</tr>
<tr>
<td>Reference player</td>
<td>PZ</td>
<td>PC</td>
<td>PM</td>
<td>PR</td>
</tr>
<tr>
<td>Metric</td>
<td>DZ</td>
<td>DTC</td>
<td>DTM</td>
<td>DTR</td>
</tr>
<tr>
<td>Position depth</td>
<td>dz, dz1</td>
<td>dc, dc1</td>
<td>dm, dm1</td>
<td>dr, dr1</td>
</tr>
<tr>
<td>Endgame Table</td>
<td>EZ</td>
<td>EC</td>
<td>EM</td>
<td>ER</td>
</tr>
<tr>
<td>Maximal depth</td>
<td>mxZ</td>
<td>mxC</td>
<td>mxM</td>
<td>mxR</td>
</tr>
<tr>
<td>Type of optimality</td>
<td>Z-optimal</td>
<td>C-optimal</td>
<td>M-optimal</td>
<td>R-optimal</td>
</tr>
<tr>
<td>Minimax Strategy</td>
<td>... move-subset chosen</td>
<td>SZ</td>
<td>SC</td>
<td>SM</td>
</tr>
</tbody>
</table>

Table 1: Endgame goals and associated concepts.

For pawnless endgames, DTC = DTZ and SC = SZ. The notation allows for more comprehensive goals. Let the nested strategy $SX_1X_2...X_n$ be defined as subsetting the available moves with strategies $SX_1$, $SX_2$, ..., $SX_n$ in turn. A line $X-Y$ is an optimal line of play where White is reference player PX using strategy SX and Black is reference player PY using strategy SY. Appendix A shows Black, then White, having to choose between C- and M-optimal play as they approach the events of force conversion and mate.

For a specific $k$-win position $P$, a strategy $SY$ is said to (k-)succeed on $P$ if each move chosen by $SY$ avoids the risk of a $k$-move draw claim. If not, $SY$ (k-)fails on $P$ and $SY$ risks a draw claim on any positions from which $Y$-optimal play can arrive at $P$. Let $\sigma$ denote any move-subsetting strategy. If $SY$ succeeds on $P$, $SY\sigma$ succeeds on $P$. However, as position Q-NN2 of Table 2 demonstrates, if $SY\sigma$ succeeds on $P$, $SY\sigma$ may still fail. Let $SA \geq SB$ denote that if strategy $SA$ fails, strategy $SB$ fails; $SY\sigma \geq SY$. Let $SA > SB$ denote that $SA \geq SB$ and that $SA$ sometimes succeeds where $SB$ fails.
Table 2: Illustrative chess positions.

4. ENDGAMES DEEPER THAN 50 MOVES


5 man KBBKN (66), KBNKN (77), KNNKP (70+), KQKBB (71), KRBBK (59), KQPKQ with wP on a6 (71), a7 (69), b3 (51), b6 (61), b7 (55), c3 (53), d3 (54), d4 (64), d6 (58).

6 man KBBBKR (69), KBBNKR (68), KNNNKB (92), KNNKN (86), KQBBKR (85), KQBBBN (63+), KQKBN (153), KQKNK (72), KQKQR (73), KQKQB (73), KQKQQ (71), KQKBQ (92), KRBBK (83), KRBBN (98), KRBNK (223), KRBNKQ (99), KRBBKN (140), KRBBKN (243), KRP(a2)KbBP(a3) (54+), KRBKQ (82), KRRKB (54), KRRKN (73), KRRNK (101), KRRRQ (65).

Zeroing move, Conversion and Mate are increasingly distant goals. While the corresponding minimax strategies SZ, SC and SM are highly correlated, one strategy can preclude another. A focus on the longer-term objectives can extend the first phase beyond 50 moves but equally, an exclusive focus on the first phase can overextend a subsequent phase. In practice, players today have a choice only of tables providing DTC data (Thompson, 1986; ChessBase, 2000) or DTM data (Hyatt, 2000; Nalimov, 2000); no DTZ data is easily available for $P$-endgames. Table 2, which collates all positions cited in this paper, gives examples of blind adherence to strategies SZ, SC or SM missing 50-wins, starting with three first-phase failures.

The KQKNN position Q-NN1 (Tamplin, 2000) leads with MC-optimal play to position Q-NN2 on move 38. With just 13 moves left and all required for conversion, strategy SM selects 38. Qc3, Qf2 and Qg1 of which only Qg1 is C-optimal: SM therefore fails. Strategy SMC succeeds by narrowing the choice to Qg1.

The maximal KNNKP position mxNN-P1 (Dekker, Van den Herik and Herschberg, 1990) leads by MC-CM play to position NN-P2 after 24. ... h4. White must now force $h3$ by move 74. However, at position NN-P3, the SC and SM strategies dictate 63. Nd2 leading to position NN-P4. This allows 63. ... Kf2, postponing $h3$ until move 80. Strategies SC and SM therefore fail but Dekker et al. imply that strategy SZ forces $h3$ in time to win NN-P3.

The KQPKQ position QP-Q3 is the result of 49 moves of Z-optimal play from QP-Q2 (Thompson, 1986, p. 138 after 21. ... Qd4) but could equally well have been the result of 49 moves of C-optimal play from another position. Strategy SZ succeeds just in time with 50. a7 but SC and SM fail by dictating 50. Kb8.

The KQBBKN position QBB-N1 shows that strategy SZ, far from being a panacea, also fails. It misses the four-move mate, sacrifices the Queen unnecessarily and leaves a second phase of 52 moves. Perhaps one should never resign against a computer. Line f of Appendix A features a more benign knight sacrifice.
The positions above show SM, SC and SZ failing individually. However, with \( m_{\text{left}} \) denoting the number of moves left in the current phase, the following non-minimax strategies optimise against longer-term goals but safeguard the length of the current phase. They are defined in terms of the subset of moves they return:

\[
SZ' \equiv \{ \text{move to } P(d_z) \mid d_z \leq m_{\text{left}} - 1 \}
\]

\[
S_{\sigma}^* \equiv \begin{cases} \text{if } d_z > m_{\text{left}} \text{ then } SZ' & \text{else } \text{if } d_c \leq m_{\text{left}} \text{ then } SZ \end{cases}
\]

\[
\sigma \geq \sigma_0 \text{ but } SZ_{\sigma}^* = SZ_0.
\]

SM* and SA1 succeed on the positions above but it is conjectured that they fail to win some winnable positions and that a metric recognising a \( k \)-move rule explicitly is needed. For example, KBBKNN has \( m_x M = 106 \) moves and \( m_x Z = 38 \) moves (Stiller, 1996); 24% of wtm positions are wins and 65% of these have \( dm > 50 \). It converts to KBBK which has \( d_z > 50 \) for some 11.16% of White wins. Let the KBBK wins for White be partitioned into sets \( W(d_z \leq 50) \) and \( D(d_z > 50) \). Let three subsets of KBBKNN wins be defined as follows:

\[
A \equiv \{ P \mid P \text{ is a Wh. win: Wh. cannot mate in KBBKNN but can force } P \rightarrow P_w \in W \text{ in } d_{zw} \leq 50 \text{ moves} \}
\]

\[
B \equiv \{ P \mid P \text{ is a Wh. win: Wh. cannot mate in KBBKNN but can force } P \rightarrow P_d \in D \text{ in } d_{zd} \leq 50 \text{ moves} \}
\]

\[
AB \equiv \{ P \mid P \in A \cap B, \text{ dm > 50 and } d_{zd} \leq d_{zw} \text{ implying } d_z = d_{zd} \leq 50 \}, \text{ see Figure 1.}
\]

For any \( P \in AB, \) SA1 = SC = SC* because \( dm > 50 \): this strategy fails because unconstrained C-optimal play could arrive at position \( P_d \). Neither is SM* constrained to avoid \( P_d \). However, \( P \) can be won in two phases of \( \leq 50 \) moves by conversion to a position \( P_w \) in \( W \) and then to mate in KBBK. The positions in set AB require a more circuitous route to secure the win, i.e., constrained optimisation that recognises the 50-move rule.

The same scenario may occur before the long phases of KQPKQ, corresponding to pawn positions a6, a7, b3, b6, b7, c3, d3, d4 and d6. The position sets equivalent to A, B and AB above are also more easily computed.

Let \( G(dr, bm) \) be the goal of ending a phase by converting to a position with DTR = \( dr \) in \( bm \) moves or less. Figure 2 illustrates a scenario in which various goals may be achieved, some requiring more moves than others. The initial default goal is \( G_1(50, 50) \). However, an initial DTR of 30 implies that \( G_2(30, 30) \) is achievable. Further goals \( G_3(25, 43) \) and \( G_4(22, 56) \) are also achievable although the last is beyond the 50-move limit.

As already observed, strategy SZ may not win under the minimal \( k \)-move rule possible. Equally, conversions to lower DTR values than the phase’s initial \( dr \) may be achievable by extending the current phase beyond \( dr \) moves but not beyond the \( k \)-move limit. Section 6 returns to this scenario and considers how White, traditionally pursuing a win, can narrow down its choice of moves while also managing a risk, which is in fact present, that the latest \( G(dr, bm) \) goal may be missed.

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**Figure 1:** Winning a ‘difficult’ position \( P \) in KBBKNN.

**Figure 2:** Phase-ending goals \( G(dr, bm) \).
5. THE DTR METRIC AND ENDGAME TABLE

The 50-move rule generalises to the \(k\)-move rule and suggests metric DTR as follows:

a position’s Depth by the Rule, DTR, is the least \(k\) for which the position can be won without the risk of a draw claim under a \(k\)-move rule.

It immediately follows that \(dz \leq dc \leq dm\), \(dz \leq dr \leq dm\), and that a position can be moved to the next phase in at most a further \(dr\) moves, leaving a position with DTR \(\leq dr\). Further, a position’s \(dr\) satisfies:

- \(dr = \max\{dz, \min\{dr\ \text{of won successors}\}\}\) for side-to-move, \(stm\), wins
- \(dr = \max\{dz, \max\{dr\ \text{of lost successors}\}\}\) for \(stm\) losses

The data for metric DTR is stored in endgame table \(ER\). \(ER \equiv EM\) for \(K \times K\) where \(x = Q, R, (B, N,) BB\) or \(BN\).

When computing \(ER\) for an endgame, it is assumed that table \(EZ\) already exists and that \(ER\) has already been computed for subsequent phases of play following a pawn-push or capture.

5.1 Algorithm AL1: generating table \(ER\) from table \(EZ\)

The formulae above suggest a straightforward, if relatively inefficient, algorithm AL1 for generating an EGT table \(ER\) from the table \(EZ\). The figure 50 does not feature and the table \(ER\) can be used under any \(k\)-move rule.

```plaintext
{initialise} ChangeFlag ← True; for \(i = 1\) to index_range do ER\[i\] ← EZ\[i\] end_do;
max_next_dr = max(mxR of a subgame of this endgame) \{\(dr \geq max\_next\_dr \Rightarrow dr = dz\}\};
{cycle} while ChangeFlag = True do ChangeFlag ← False;
for \(i = 1\) to index_range do
    if \(dr < max\_next\_dr \land ER\[i\] \neq Draw \land ER\[i\] \neq Broken\} then
        if position\(i\) is an \(stm\) win then \(dr2 = \max\{EZ\[i\], \min\{ER\[j\] of won successors\}\}\) end_if;
        if position\(i\) is an \(stm\) loss then \(dr2 = \max\{EZ\[i\], \max\{ER\[j\] of successors\}\}\) end_if;
        if \(dr2 \neq ER\[i\]\} then \(ER\[i\] ← \(dr2\); ChangeFlag ← True end_if;
    end_if;
end_if;
end_do; end_do {end: \(ER\) is now the definitive \(ER\) table with \(ER\[i\] = dr\)}
```

Note that it is sometimes necessary to adjust an original \(dz\) value more than once. For example, the position QBB-N1 would start with \(dr = dz = 2\) plies, would then be given \(dr = 103\) plies and finally \(dr = 5\) plies. The same is true in the Edwards/Nalimov DTM algorithm. For QBB-N1, \(dm = 105\) plies and later \(dm = 7\) plies.

With \(dr > k\), the \(ER\) table can be used by the infallible attacker to minimise \(dr\) and give a fallible opponent an opportunity to lower \(dr\). Conversely, if \(dr \leq k\), an infallible defender can maximise \(dr\) and give a fallible opponent an opportunity to raise \(dr\).

5.2 Algorithm AL2: generating table \(ER\) by modified retro-method

The algorithm AL2 for constructing the endgame table \(ER\) is based on the established retro-search algorithm (Thompson, 1986, 1996; Nalimov et al., 1999) used in the past to create \(EZ\) and \(EC\) tables to the DTZ and DTC metrics respectively. It is now more convenient to think of depth in plies and assume a \(2k\)-plies rule. The modified algorithm introduces two constraints. First, it considers only conversions to \(i\)-plies-wins with \(i \leq 2k\) and its retro-search only reaches back to positions with \(dz \leq 2k\) plies.

The following definitions are used:

- \(C_0 = \{\text{subgame positions } P | \text{ the } stm \text{ is mated}\}\)
- \(C_i = \{\text{subgame positions } P | P \text{ is an } h\text{-plies win for } h \leq i, \text{i.e., no phase has more than } h\text{ plies}\}\)
- \(M = \{\text{endgame positions } P | \text{ the } stm \text{ is mated}\}: X_{i,0} = M.\)
- \(W_i = \{\text{endgame positions } P | \text{ the } stm, \text{ winning, can mate or move to a won } P^* \in C_i \text{ in one ply}\}\)
- \(L_i = \{\text{endgame positions } P | \text{ the } stm, \text{ losing, must move to a lost } P^* \in C_i \text{ in one ply}\}\)
  see, for example, position NN-P5 or QB-BN1 after 1. Qa2+. \(X_{i,1} = W_i \cup L_i.\)
- \(X_{i,j} = \{\text{endgame positions } P | \text{ stm can force or must allow conversion to a } P \in C_i, \text{ or mate, in } \leq j \text{ plies}\}\)

\(^3\) A table entry is marked broken if it corresponds to a clearly illegal position, an unwanted position or no position.
Imagine that the algorithm is divided up into phases and that the $i$th phase finds those positions in, say, KBBKNN which can be won under an $i$-plies rule. Each phase starts by identifying the mates in KBBKNN. Then it identifies those boundary positions which can be or must be converted in one ply to mate or to a value-preserving subgame position, also winnable under an $i$-plies rule. The phase then computes just $i-1$ cycles of retro-search, each one identifying positions one ply deeper into KBBKNN.

Given a set $X$ of positions in an endgame, let the functions $W(X)$, $L(X)$, $\Pi(X)$ and $\Sigma(X)$ be defined as:

- $W(X) = \{\text{positions } P \mid \text{the winner, to move, can move to some won } P' \in X\}$.
- $L(X) = \{\text{positions } P \mid \text{the loser, to move, must move to some (lost) } P' \in X\}$.
- $\Pi(X) = W(X) \cup L(X)$.
- $\Sigma(X) = X \cup \Pi(X)$.

It follows that $X_{i,j+1} = \Sigma(X_{i,j})$, $X_{i,2j} = \Sigma^{2j-1}(X_{i,1})$ and that the set of $h$-plies wins for $h \leq i$ is the set $X_{i,h}$. The set of $i$-plies wins is $X_{i,i} - X_{i-1,i-1}$ and the set of $k$-wins is $X_{2k,2k} - X_{2k-2,2k-2}$.

![Figure 3: Algorithm AL2 for computing endgame table $ER$, phase $i$.](image)

It is obvious that each phase recomputes much of what has been computed before. Further, the function $\Pi(X)$ makes random access to data, particularly expensive when confirming that the loser is forced to move to some position in $X$. There is however an opportunity to exploit previous data to increase efficiency, reducing the use of $\Pi$ at the expense only of some sequential access to and manipulation of interim sets of results.

### 5.3 Using known subsets of $X_{i,j}$

Let $X_{i,j} = \emptyset$ for $i < 0$ and $j < 0$. Let $X_{i,0} = M$. Note that $X_{i,j}$ is not defined for $j > i$.

Let $Y_{i,j} = X_{i,j} - X_{i,j+1}$ and $Z_{i,j} = Y_{i,j} - Y_{i,j+1}$. Then $Y_{i,j} \subseteq Y_{i+1,j} \cup Z_{i,j}$ and $X_{i,j} = X_{i,j+1} \cup Y_{i,j} = X_{i,j+1} \cup Y_{i+1,j} \cup Z_{i,j}$.

Note that, because a position may be in $Y_{i+1,j} \cap X_{i,j+1}$, it is possible that $Y_{i+1,j} - Y_{i,j} \neq \emptyset$.

The computation of $X_{i,j}$ involves only the computation of $\Pi(Z_{i,j+1})$ for $j < i$ and $\Pi(Y_{i+1,j})$ for $j = i$, see Figure 4:

- $X_{i,j} = \Sigma(X_{i,j+1}) = \Sigma(X_{i,j+2} \cup Y_{i,j+1}) = \Sigma(X_{i,j+2} \cup \Sigma(Y_{i,j+1})) = X_{i,j+1} \cup \Pi(Y_{i,j+1})$.
- $X_{i,j} = \Sigma(X_{i,j+1}) = \Sigma(X_{i,j+2} \cup Y_{i,j+1}) = \Sigma(X_{i,j+2} \cup Y_{i+1,j} \cup Z_{i,j+1})$.
- $Y_{i,j} = [X_{i,j+1} \cup \Sigma(Z_{i,j+1})] - X_{i,j+1} = [X_{i+1,j+1} \cup Y_{i+1,j} \cup \Sigma(Z_{i,j+1})] - X_{i,j+1} = [Y_{i+1,j} \cup \Sigma(Z_{i,j+1})] - X_{i,j+1} = [Y_{i+1,j} \cup \Sigma(Z_{i,j+1})] - X_{i,j+1}$.
- $Z_{i,j} = Y_{i,j} - Y_{i+1,j} = \Sigma(Z_{i,j+1}) - X_{i,j+1} - Y_{i,j+1}$.

![Figure 4: Subsetting $X_{i,j}$ to minimise use of function $\Pi$.](image)
6. USES OF THE DTR DATA

Let us suppose, as is usual, that White is pursuing a win under a \( k \)-move rule against a possibly fallible player. For convenience, moves will be numbered from 1 in the current phase. The following notation is used:

- \( k \) indicates the \( k \)-move rule in force: currently, FIDE has set \( k = 50 \) for all endgames
- \( m_{\text{played}} \) the history factor, the number of white moves played in this phase
- \( m_{\text{left}} \) the number of white moves left before the risk of a draw claim; \( m_{\text{left}} = k - m_{\text{played}} \)
- \( P(dr, dz) \) a wtm position \( P \) with depths \( dr \) in metric DTR and \( dz \) in metric DTZ
- \( CP \) the set \( \{ P(dr_1, dz_1) \} \) of btm successors of position \( P \)
- \( CQ \) the subset \( \{ P_i \in CP \mid dz_i \leq m_{\text{left}} - 1 \} \)
- \( G_j(dr_j, bm_j) \) the \( j \)th goal, to conclude the phase by conversion with DTR \( dr_j \) on or before move \( bm_j \)
- \( g_i \) index of the last goal defined
- \( CR \) the subset \( \{ P_i \in CQ \mid dr_i \leq dr_{gi} \wedge m_{\text{played}} + 1 + dz_i \leq bm_{gi} \} \)

As any strategy \( S_\sigma \) can be transformed into a strategy \( S_\sigma^* \) considering only those moves that safeguard the first phase, it is assumed below that \( dz \leq m_{\text{left}} \).

6.1 The minimax strategy SR

The equivalent of the strategies SM, SC and SZ in Table 1 is SR which selects \{move to \( P_i(dr_1, dz_1) \mid dr_1 \leq dr_j \}\} as the set of options. The two ways in which SR and SR* fail suggest the definition of a range of strategies which leave a wider choice of moves available. It is assumed that the opponent is playing to maximise DTR.

In the example of Figure 2 with a 50-move rule, the default goal \( G_1(50, 50) \) is immediately superceded by \( G_2(30, 30) \) before White’s first move. With 18 moves played, the position \( P(25, dz) \) implies a potential goal \( G_3(25, 43) \). This, if reached, will also safeguard the win at the expense of more moves in the current phase. Strategy SR always aims for the lowest \( dr \) and therefore, in effect, adopts goal \( G_3(25, 43) \).

As position QBB-N1 of Table 1 shows, albeit with the entirely hypothetical targets of \( dr = 3 \) and \( dz = 1 \), White may have to choose between its \( dr \) and \( dz \) targets in position \( P(dr, dz) \). It cannot necessarily achieve both and in the event of conflict, will safeguard the current phase in an \( S_\sigma^* \) strategy. Figure 5 shows that after 28 moves, a feasible move, apparently compatible with the \( dr \) target and \( dz \) constraint, is in fact a wrong choice. It leads to conversion positions with \( dr \leq 25 \) but these are not only beyond move 43 but beyond move 50. The \( dr \) target would therefore have to be abandoned with unpredictable consequences. This demonstrates that there is a risk which is not present with the DTM, DTC and DTZ metrics: the aggressor may stray off the winning line.

After 33 moves, in position \( P(22, dz) \), SR again adopts the implied goal \( G_4(22, 56) \). This, in the worst case, is not achievable until after Black claims a 50-move draw. The risk of failing to win in this way is easily avoided by strategies which do not adopt goals with \( bm > 50 \).

![Figure 5: Winning, incorrect and over-reaching lines of play.](image-url)
The following measures are suggested to lower the residual risk of a draw claim when using SR*:

- avoid subsetting the moves offered by SR*, e.g., by minimum \( dz \)
- instead, search forward a number of plies continuing to subset options with SR*
- if the lowest visible \( dr \) in unattainable within \( k \) moves, relax the \( dr \) goal.

Given the failings of SR*, a range of strategies SR\( a \) is defined, featuring constraints on the \( dr \) attempted.

### 6.2 The SR\( a \) Strategies

Four strategies are defined differing in the criteria applied to potential goals before they are adopted as actual goals. All strategies adopt the default goal \( G_i(k, k) \) in the context of a \( k \)-move rule, even though it might not be achievable against infallible play. SR* above is equivalent to SR\( 4 \) below. In summary:

- SR\( 1 \) focuses only on goal \( G_i(k, k) \), in effect, providing a fixed filter on the move options
- SR\( 2 \), given goal \( G_i(dr, bm) \), adopts \( G_i(dr, bm_j) \) provided \( dr_j < dr \) and \( bm_j \leq bm \); \( SR_2 \geq SR_1 \)
- SR\( 3 \), given goal \( G_i(dr, bm) \), adopts \( G_i(dr, bm_j) \) provided \( dr_j < dr \) and \( bm_j \leq k \); \( SR_3 \geq SR_1 \)
- SR\( 4 \), given goal \( G_i(dr, bm) \), adopts \( G_i(dr, bm_j) \) provided \( dr_j < dr \).

Where SR\( 2 \) and SR\( 3 \) adopt a new goal, they confirm that the aggressor has a winning line and the effect of any previous, suboptimal choices of move may be ignored. Returning to the example of Figure 2, SR\( 1 \) uses only goal \( G_1 \), SR\( 2 \) uses \( G_1 \) and \( G_2 \), SR\( 3 \) uses \( G_1-G_3 \) and SR\( 4 \) uses \( G_1-G_4 \).

### 6.3 An algorithm for the SR\( a \)

The algorithm, written for the attacker, returns a subset \( CM \) of moves. To guard against \( DTR > k \), \( CM \) is first defined to be the same subset which strategy SR\( a \) will return, i.e., those moves with minimal \( dr \).

```plaintext
{ initialise: \( a = dr_focus \) is assumed set} high_value = 10^9;
{ step 1: in case \( dr > k \)}
\( CM \leftarrow \{ P_i | P_i \) is selected by strategy SR\}
{ step 2: re-adopt a previous goal if this exists and the current goal is clearly unattainable}
\( while \ CR = \emptyset \land gi > 1 \) do \( gi \leftarrow gi - 1 \);
{ step 3: if possible, set a stronger goal from those implied by the current move options}
\( if a \neq 1 \land CR \neq \emptyset \)
\( drmin = \min \{ dr_j | P_i(dr, dz_j) \in CR \};
\( if mplayed + 1 + drmin \leq bm_limit(a) \) then \( add_new_goal(G, gi) \) end_if
{ end if;
{ step 4: if possible, subset to } \( P_i \) not excluding the current goal}
\( if CR \neq \emptyset \) then \( CM \leftarrow CR\);
{ add_new_goal(G, gi, ...) = begin gi \leftarrow gi + 1; G_{gi} = G_{gi}(drmin, drmin + mplayed + 1) \) end
```

### 6.4 Examples of SR\( a \) in use

SR\( 1 \) succeeds where SZ fails on position QBB-N1 of Table 1 and positions P ∈ AB in Figure 1. Strategy SA\( 1 \) of section 4 can be strengthened to:

\( SA_2 = if \) \( dm \leq mleft \) then \( SM \) else if \( dc \leq mleft \) then \( SR_2 \) \( C \) else \( SR_2 \) \( ZCM^{*} \)

The example of Figure 6 shows that goals \( G_c-G_e \) have been logged starting with the default goal \( G_1(50, 50) \). After Black’s first move, the goal can be improved to ‘45 by move 45’ and later to ‘44/43 by 45’ and ‘41 by 44’. With White about to play its 17th move in the phase, three scenarios labeled a, b and c are portrayed. For each, a single move to a position \( P_{a}(dr, dz) \) is indicated, implying a potential goal which may or may not be adopted. SR\( 2 \) will adopt goal \( G_1(23, 40) \) but not \( G_0 \) or \( G_2 \) as these may only be achievable after the current goal horizon of move 44. SR\( 3 \) will adopt \( G_1 \) and \( G_3(29, 46) \) as the latter is theoretically achievable before 50 moves elapse. SR\( 4 \) will adopt any of \( G_c-G_e \), possibly over-reaching with goal \( G_f \) and being forced back to a previous goal or to strategy SZ.
Figure 6: Example game phase with various destination positions \( P_\alpha(dr, dz) \) and implied goals \( G_\alpha(dr, bm) \).

7. SUMMARY

The basic minimax strategies SM and SC currently in use can fail in the context of the 50-move rule by allowing unnecessary draws. Even strategy SZ can fail by minimising the length of the current phase of play at the cost of an over-long subsequent phase. More complex nested strategies such as SMC, SCM and SZCM improve the minimax approach. The non-minimax strategies such as SZ' and SMZ* also require no more than the relatively easy production of the endgame tables EZ to the DTZ metric for P-endgames.

However, the general \( k \)-move rule suggests a new metric DTR. It is conjectured that strategies using metrics which do not recognise this rule explicitly will eventually fail. The endgames KQPKQ and KBBKNN may well harbour positions which demonstrate this. The endgame table ER may be generated by various methods, the most complex approach striving for greater efficiency.

Both the naive minimax strategy SR and its stronger derivative SR* can fail in two ways. Therefore, a range of four strategies SRa has been defined, using both DTR and DTZ information to provide a wider choice of moves. The SRa can be used iteratively in a focused conventional search. In the absence of empirical data, the author believes that the repeated use of SR3* in a search will be more effective than strategy SR3ZCM*.

Experiments to verify the power of the SRa* strategies require actual EZ and ER tables. The author therefore invites others in this field to produce such tables. The prime candidates are the deep 5-man endgames and those 6-man endgames which can precede them, as listed in Section 4.
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9. REFERENCES

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APPENDIX A: TYPES OF OPTIMAL PLAY

These optimal lines proceed from the maximal KRNNKN position mxRN-NN of Table 1. A published line (Stiller, 1996) is, up to and including move 205w, both C-C optimal as intended and M-M optimal. First Black and then White must choose between C-optimal and M-optimal moves, their choices eventually defining four different types of optimal line. Different equi-optimal choices would of course produce different specific lines.

The following notation is used to indicate various properties of the moves:

- only X-optimal move, given strategy SX; " only value-preserving move; ° only legal move; [...] equi-optimal moves; n one of n equi-optimal moves; -n a move suboptimal by n moves; v value-changing move.


**APPENDIX B: SOME MONSTERS OF THE DEEP**

This appendix notes some games (ChessLab, 2000) and positions associated with the 50-move rule.


‘KBBKN’. Pinter-Bronstein (1977, ECO B14, ½-½), 8/2b5/8/3b2k1/8/4K3/8/4N3/8+w {dc = 54m, dm = 66m}. The 44-move win from move 70 would have just beaten a 50-move draw claim. However, Pinter was allowed to set up Kling-Horwitz positions on moves 71, 90 and 112 in the b2, g2 and b7 corners respectively and could have taken more moves doing so (Roycroft, 1988). A draw was agreed on move 117.

KBNNKR. Karpov-Kasparov (1991, ECO E97, ½-½), {63. Kxh4} 3r4/8/2B2k2/8/5K1/3N4/8/8+63b {=.} In 51 moves, Kasparov never allowed a win (Stiller, 1991b) and set up a stalemate finish with a Rook sacrifice.

KQNKQ. Ljubojevic-Hjartarson (1991, ECO A22, ½-½), {70. Qxg5+} 6k1/3q4/8/6Q1/6N1/7K/8/8+70b: {.} Contrary to Nefkens (1991), Black’s defences slip on move 88 but White misses the win on the next move. On move 117, Black sets up a mate for White which is then promoted to just two moves beyond the draw claim.

KQPKQ. Wegner-Johnsen (1991, ECO D30, ½-½), {126. … a2} 8/8/7K/3q4/k7/8/p7/1Q6+127w. The game entered KQPKQ with move 53w. Although dc = 13m and dm = 28m, a ‘75-move’ draw resulted on move 201.


KRP(a2)KbBP(a3). Timman-Velimirović (1979, ECO D30, 1-0), 8/8/4k3/2R5/7b/p2K4/P7/8+64b. Won on move 103, this game brought about the 100-move allowance for this ending (Van den Herik et al., 1987).
POST-PUBLICATION NOTE

The iteration formulae for DTR defined in Section 5, and algorithms AL1 and AL2, are incorrect.

The correction was published in:

Examples of positions where SC', SM' and SZ' all fail to defend a win have been published in: