On small bases for which 1 has countably many expansions

Article

Accepted Version

CentAUR

On small bases for which 1 has countably many expansions

Yuru Zou[a], Lijin Wang[a], Jian Lu[a,*], Simon Baker[b]

[a] College of Mathematics and Computational Science, Shenzhen University, Shenzhen 518060, China

[b] School of Mathematics, The University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom

Abstract

Let $q \in (1, 2)$. A q-expansion of a number x in $[0, \frac{1}{q-1}]$ is a sequence $(\delta_i)_{i=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ satisfying

$$x = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}.$$

Let $\mathcal{B}_{\aleph_0}$ denote the set of q for which there exists x with a countable number of q-expansions, and let $\mathcal{B}_{1,\aleph_0}$ denote the set of q for which 1 has a countable number of q-expansions. In [21] it was shown that $\min \mathcal{B}_{\aleph_0} = \min \mathcal{B}_{1,\aleph_0} = \frac{1+\sqrt{5}}{2}$, and in [1] it was shown that $\mathcal{B}_{\aleph_0} \cap (\frac{1+\sqrt{5}}{2}, q_1] = \{q_1\}$, where $q_1 (\approx 1.64541)$ is the positive root of $x^6 - x^4 - x^3 - 2x^2 - x - 1 = 0$. In this paper we show that the second smallest point of $\mathcal{B}_{1,\aleph_0}$ is $q_3 (\approx 1.68042)$, the positive root of $x^5 - x^4 - x^3 - x + 1 = 0$. Enroute to proving this result we show that $\mathcal{B}_{\aleph_0} \cap (q_1, q_3] = \{q_2, q_3\}$, where $q_2 (\approx 1.65462)$ is the positive root of $x^6 - 2x^4 - x^3 - 1 = 0$.

Key Words: beta-expansion, non-integer base, countable expansions

AMS Subject Classifications: 11A63, 37A45

1 Introduction

Let $q \in (1, 2)$ and $I_q := [0, \frac{1}{q-1}]$. For each $x \in I_q$ there exists a sequence $(\delta_i)_{i=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ such that

$$x = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}.$$

The sequence $(\delta_i)_{i=1}^{\infty}$ is called a q-expansion for x. Without confusion, we simplify $(\delta_i)_{i=1}^{\infty}$ as (δ_i). It is straightforward to show that a real number x has a q-expansion if and only if $x \in I_q$.

We now introduce some notation. The so-called *coding map* is defined to be $\Pi : \{0, 1\}^{\mathbb{N}} \to I_q$ where

$$\Pi((\delta_i)) = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}. \tag{1}$$

---

*Corresponding author. E-mail:jianlu1979@163.com

Throughout we let $(\varepsilon_1...\varepsilon_n)^k$ denote the k fold concatenation of $(\varepsilon_1...\varepsilon_n) \in \{0,1\}^n$, and similarly let $(\varepsilon_1...\varepsilon_n)^\infty$ denote the infinite concatenations of $(\varepsilon_1...\varepsilon_n)$. Given $x \in I_q$, let $\Sigma_q(x)$ denote the set of all q-expansions of x, that is

$$\Sigma_q(x) = \left\{ (\delta_i) \in \{0,1\}^{\mathbb{N}} : \Pi((\delta_i)) = x \right\}.$$

The cardinality of the set $\Sigma_q(x)$ plays an important role in the investigation of representations of real numbers in non-integer bases. It was shown in [11] that if $q \in (1, \frac{1+\sqrt{5}}{2})$ then for each $x \in (0, \frac{1}{q-1})$ there are $2^{\aleph_0}$ different q-expansions. Sidorov showed in [18, 19] that if $q \in (1,2)$ then Lebesgue almost every $x \in (0, \frac{1}{q-1})$ has $2^{\aleph_0}$ different q-expansions. Points belonging to I_q with a unique q-expansion were investigated in [7, 15] for $q \in (\frac{1+\sqrt{5}}{2}, 2)$. Some results concerning $x \in I_q$ having a fixed number of q-expansions were established in [1, 3, 8, 9, 10, 20, 22].

Let $m \in \mathbb{N} \cup \{\aleph_0\}$ and define

$$\mathcal{B}_m := \left\{ q \in \left[\frac{1+\sqrt{5}}{2}, 2 \right) : \text{ there exits } x \in I_q \text{ satisfying } \#\Sigma_q(x) = m \right\}.$$

Here and hereafter $\#A$ denotes the cardinality of a set A. The following results are known to hold:

- $\min \mathcal{B}_{\aleph_0} = \frac{1+\sqrt{5}}{2}$ [10].

- $\min \mathcal{B}_2 = \check{q} \approx 1.71064$ (the positive root of $x^4 - 2x^2 - x - 1 = 0$) [20].

- $\min \mathcal{B}_k = q_f \approx 1.75488, k \geq 3$ (the positive root of $x^3 - 2x^2 - 1 = 0$) [3].

- $\mathcal{B}_2 \cap (\check{q}, q_f] = \{q_f\}$ [3].

- The smallest element of $\mathcal{B}_{\aleph_0}$ strictly greater than $\frac{1+\sqrt{5}}{2}$ is $q_1 (\approx 1.64541)$ (the positive root of $x^6 - x^4 - x^3 - 2x^2 - x - 1 = 0$) [1].

Understanding the q-expansions of 1 is a classical problem, see [15, 16, 17] and the references therein. The motivation of this paper is to provide a clearer understanding of what values $\#\Sigma_q(x)$ can take. Let

$$\mathcal{B}_{1,m} := \left\{ q \in \left[\frac{1+\sqrt{5}}{2}, 2 \right) : \#\Sigma_q(1) = m \right\}.$$

It was shown in [16] that $\min \mathcal{B}_{1,1} \approx 1.78723$ (the Komornik-Loreti constant). In [10] it was proved that $\min \mathcal{B}_{1,\aleph_0} = \frac{1+\sqrt{5}}{2}$. For any $n \in (\mathbb{N} \setminus \{1\}) \cup \{\aleph_0\}$, Erdös and Joó [12, 13] constructed a continuum of real numbers $q \in [q_0, 2)$ ($q_0 > 1.99803$) for which the number 1 has precisely n q-expansions.

Motivated by the results listed above, a natural question arises: what is the second smallest point of $\mathcal{B}_{1,\aleph_0}$? In this paper we will answer this question.

Throughout this paper we let q_1, q_2, q_3 be as follows: $q_1 \approx 1.64541$, $q_2 \approx 1.65462$ and $q_3 \approx 1.68042$, which are the positive roots of $x^6 - x^4 - x^3 - 2x^2 - x - 1 = 0$, $x^6 - 2x^4 - x^3 - 1 = 0$ and $x^5 - x^4 - x^3 - x + 1 = 0$, respectively. Our main result is the following.

Theorem 1.1. *The smallest element of $\mathcal{B}_{1,\aleph_0}$ strictly greater than $\frac{1+\sqrt{5}}{2}$ is q_3.*

Enroute to proving this result we show the following.

Theorem 1.2. $\mathcal{B}_{\aleph_0} \cap (\frac{1+\sqrt{5}}{2}, q_3] = \{q_1, q_2, q_3\}$.

We remark that $\mathcal{B}_{\aleph_0} \cap (\frac{1+\sqrt{5}}{2}, q_1] = \{q_1\}$ is known [1]. The following corollary is an immediate consequence of Theorem 1.1, $\min \mathcal{B}_2 \approx 1.71064$ [21], $\min \mathcal{B}_k \approx 1.75488$ for $k \geq 3$ [3], $\mathcal{B}_2 \cap (\check{q}, q_f] = \{q_f\}$ [3], and $\min \mathcal{B}_{1,1} \approx 1.78723$ [16].

Corollary 1.3. *If* $q \in (1, q_3) \setminus \{\frac{1+\sqrt{5}}{2}\}$. *Then* 1 *has* $2^{\aleph_0}$ *different q-expansions.*

This paper is arranged as follows. Some definitions and results from [1] will be recalled in Section 2. Some results from this paper will be extended to our setting. In Section 3 we prove Theorem 1.2. The final section is devoted to the proof of Theorem 1.1.

2   Preliminaries

In this section we shall recall some definitions and results from [1]. An interpretation of q-expansions from the perspective of dynamical systems was given in [1], see also [4, 5, 6]. Let $T_{q,0}(x) = qx$ if $x \in [0, \frac{1}{q^2-q}]$, and let $T_{q,1}(x) = qx - 1$ if $x \in [\frac{1}{q}, \frac{1}{q-1}]$. We see that if $x \in L_q := [0, \frac{1}{q})$ or $x \in R_q := (\frac{1}{q^2-q}, \frac{1}{q-1}]$ then only one $T_{q,i}$ can be applied. However when $x \in S_q := [\frac{1}{q}, \frac{1}{q^2-q}]$, which is usually referred to as the switch region, we have a choice between $T_{q,0}$ and $T_{q,1}$. An element of $\bigcup_{n=0}^{\infty} \{T_{q,0}, T_{q,1}\}^n$ is denoted by $\mathbf{a}$, here $\{T_{q,0}, T_{q,1}\}^0$ denotes the identity map. Moreover, if $\mathbf{a} = a_1...a_n$ we shall use $\mathbf{a}(x)$ to denote $a_n(\cdots(a_1(x))\cdots)$. Given $x \in I_q$ we call a finite sequence of transformations $\mathbf{a} = a_1...a_n$ *minimal* for x if $\mathbf{a}(x) \in S_q$ and $\mathbf{a}|i(x) \notin S_q$ for all $i < n$. Here $\mathbf{a}|i = a_1...a_i$. We call $\mathbf{a}(x)$ a *branching point* of x if $\mathbf{a}(x) \in S_q$.

Define
$$\Omega_q(x) := \{(a_i)_{i=1}^{\infty} \in \{T_{q,0}, T_{q,1}\}^{\mathbb{N}} : a_n(\cdots(a_1(x))\cdots) \in I_q \text{ for all } n \in \mathbb{N}\}.$$

The set $\Omega_q(x)$ is significant because $\#\Sigma_q(x) = \#\Omega_q(x)$, where our bijection is given by mapping (δ_i) to (T_{q,δ_i}), see [2].

Construction of the branching tree The branching tree was constructed in [1] to study $\mathcal{B}_{\aleph_0}$. We now provide details of its construction. Suppose $x \in I_q$ and $\Omega_q(x)$ (or $\Sigma_q(x)$) is infinite. There exists a unique minimal $\mathbf{a} \in \bigcup_{n=0}^{\infty} \{T_{q,0}, T_{q,1}\}^n$ such that $\mathbf{a}(x) \in S_q$. Then there are two possibilities.

Case 1: There exists a unique $i \in \{0, 1\}$ such that $\Omega_q(T_{q,i}(\mathbf{a}(x)))$ is finite and $\Omega_q(T_{q,1-i}(\mathbf{a}(x)))$ is infinite. In this case, we draw a horizontal line of finite length that then bifurcates with an upper and lower branch. The lower branch corresponds to $T_{q,i}(\mathbf{a}(x))$ and stops bifurcating, the upper branch corresponds to $T_{q,1-i}(\mathbf{a}(x))$ and goes on bifurcating.

Case 2: Both $\Omega_q(T_{q,0}(\mathbf{a}(x)))$ and $\Omega_q(T_{q,1}(\mathbf{a}(x)))$ are infinite. In this case, we draw a horizontal line of finite length that then bifurcates with an upper and lower branch. The lower branch corresponds to $T_{q,0}(\mathbf{a}(x))$, the upper branch corresponds to $T_{q,1}(\mathbf{a}(x))$. Both of them go on bifurcating.

3

If $\Omega_q(T_{q,i}(\mathbf{a}(x)))$ is infinite, as in Case 1 or Case 2, then there exists a unique minimal $\mathbf{a}' \in \bigcup_{n=0}^{\infty}\{T_{q,0}, T_{q,1}\}^n$ such that point $\mathbf{a}'(T_{q,i}(\mathbf{a}(x)))$ goes to Case 1 or Case 2 again. This procedure continues indefinitely. The infinite tree we construct by repeating this process is known as the infinite branching tree corresponding to x. Fig.1 illustrates the bifurcating procedure.

Figure 1: The flow chart of the bifurcating procedure corresponding to $x \in I_q$

A point x for which $\Sigma_q(x)$ is infinite is said to be a q *null infinite point* if for each branching point of x, $\mathbf{a}(x)$ never goes to Case 2. It is easy to check that if x is a q *null infinite point* then $\#\Sigma_q(x) = \aleph_0$. The q null infinite points have a critical role in the proofs of the main results of [1].

Lemma A [1, Proposition 2.7] $q \in \mathcal{B}_{\aleph_0}$ *if and only if I_q contains a q null infinite point*

Given $\min \mathcal{B}_2 = \check{q} \approx 1.71064$, $\min \mathcal{B}_k = q_f \approx 1.75488$ for $k \geq 3$, and $\mathcal{B}_2 \cap (\check{q}, q_f] = \{q_f\}$ we have that if $q \in (\frac{1+\sqrt{5}}{2}, q_f) \setminus \{\check{q}\}$ and x is a q null infinite point, then for each branching point of x, $\mathbf{a}(x)$, we have $\#\Omega_q(T_{q,i}(\mathbf{a}(x))) = \aleph_0$ and $\#\Omega_q(T_{q,1-i}(\mathbf{a}(x))) = 1$. Which implies the following inclusion

$$\left\{\mathbf{a}(x) : \mathbf{a}(x) \in S_q, \mathbf{a} \in \bigcup_{n=0}^{\infty}\{T_{q,0}, T_{q,1}\}^n\right\} \subseteq \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap S_q. \tag{2}$$

Here U_q denotes the set of $x \in I_q$ having a unique q-expansion.

Unfortunately, it is difficult to deal with the set S_q. However, by some deductions we can restrict ourselves to a smaller set $J_q := [\frac{q+q^2}{q^4-1}, \frac{1+q^3}{q^4-1}] \subseteq S_q$. $J_q = S_q$ if and only if $q = q_f$.

Lemma B [1, Lemma 3.1] *Let $q \in [\frac{1+\sqrt{5}}{2}, q_f)$. Suppose $x \in I_q$ satisfies $\#\Sigma_q(x) > 1$, then there exists a finite sequence of transformations $\mathbf{a}$ such that $\mathbf{a}(x) \in J_q$. Here $q_f(\approx 1.75488)$ is the positive root of $x^3 - 2x^2 - 1 = 0$.*

Proposition 2.2 is devoted to characterizing the set $\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$ when $q \in [q_1, q_3]$. To prove this proposition we need Lemma C and Lemma 2.1.

4

Lemma C [14, Theorem 2] *Let* $q \in (\frac{1+\sqrt{5}}{2}, q_f)$. *Then*

$$U_q = \left\{ \Pi(0^k(10)^\infty),\ \Pi(1^k(10)^\infty),\ 0,\ \frac{1}{q-1} \right\}.$$

Where $k \geq 0$.

Set

$$\mathbf{y}_j = \Pi(01^j(10)^\infty) \ \text{ and } \ \mathbf{z}_j = \Pi(10^j(01)^\infty)$$

for $j \geq 1$. Here and hereafter we let $(S_q \setminus J_q)_L = [\frac{1}{q}, \frac{q+q^2}{q^4-1})$ and $(S_q \setminus J_q)_R = (\frac{1+q^3}{q^4-1}, \frac{1}{q^2-q}]$.

Lemma 2.1. *Let* $q \in [q_1, q_3]$. *Then the following hold:*

(i) $\mathbf{y}_j \in J_q$ *if and only if* $\mathbf{z}_j \in J_q$, *and* $\mathbf{y}_j \in (S_q \setminus J_q)_R$ *if and only if* $\mathbf{z}_j \in (S_q \setminus J_q)_L$.

(ii) $T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) \in J_q$ *if and only if* $T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) \in J_q$, *and* $T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) \in (S_q \setminus J_q)_R$ *if and only if* $T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) \in (S_q \setminus J_q)_L$;

(iii) $T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) = \mathbf{y}_k$ *if and only if* $T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) = \mathbf{z}_k$, *and* $T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) = \mathbf{z}_k$ *if and only if* $T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) = \mathbf{y}_k$. *Here* $j \geq 1$ *and* $k \geq 1$.

Proof. Direct computation shows the following equations hold.

$$\mathbf{y}_j - \frac{1+q^3}{q^4-1} = -\mathbf{z}_j + \frac{q+q^2}{q^4-1}, \quad \mathbf{y}_j - \frac{q+q^2}{q^4-1} = \frac{1+q^3}{q^4-1} - \mathbf{z}_j, \quad \frac{1}{q^2-q} - \mathbf{y}_j = \mathbf{z}_j - \frac{1}{q}, \tag{3}$$

$$T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) - \frac{1+q^3}{q^4-1} = -T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) + \frac{q+q^2}{q^4-1}, \tag{4}$$

$$T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) - \frac{q+q^2}{q^4-1} = \frac{1+q^3}{q^4-1} - T_{q,1}^m(T_{q,0}(\mathbf{z}_j)), \quad \frac{1}{q^2-q} - T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) = T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) - \frac{1}{q}. \tag{5}$$

(i) is implied by equation (3), and (ii) is implied by equations (4) and (5).

Simplifying $T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) = \mathbf{y}_k$ and $T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) = \mathbf{z}_k$, we see that they are both equivalent to

$$(1 + 2q^{-1} - q - q^{-1-j} - q^{-2-m} - q^{-1-m} + q^{-2-k-m})(q^2-1)^{-1} = 0.$$

Similarly, simplifying $T_{q,0}^m(T_{q,1}(\mathbf{y}_j)) = \mathbf{z}_k$ and $T_{q,1}^m(T_{q,0}(\mathbf{z}_j)) = \mathbf{y}_k$, we see that they are both equivalent to

$$(-1 - 2q^{-1} + q + q^{-1-j} - q^{-2-m} + q^{-2-k-m} + q^{-m})(q^2-1)^{-1} = 0.$$

Thus we obtain (iii). $\qquad\square$

Proposition 2.2. *Let* $q \in [q_1, q_3]$. *Then we have*

$$\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q = \{\mathbf{y}_j, \mathbf{z}_j, 1 \leq j \leq 3\} = \{\Pi(01^j(10)^\infty), \Pi(10^j(01)^\infty), 1 \leq j \leq 3\},$$

$$\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap (S_q \setminus J_q)_R = \{\mathbf{y}_j, j \geq 4,\ 1/(q^2-q)\} = \{\Pi(01^j(10)^\infty), j \geq 4,\ 1/(q^2-q)\},$$

$$\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap (S_q \setminus J_q)_L = \{\mathbf{z}_j, j \geq 4,\ 1/q\} = \{\Pi(10^j(01)^\infty), j \geq 4,\ 1/q\}.$$

Proof. It follows from Lemma C that

$$\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) = \left\{ \Pi(0^j(10)^\infty), \Pi(01^{j-1}(10)^\infty), \Pi(10^{j-1}(10)^\infty), \Pi(1^j(10)^\infty), 0, \frac{1}{q^2-q}, \frac{1}{q}, \frac{1}{q-1} \right\}.$$

Here $j \geq 1$. Then by some straightforward computation we have

$$\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap S_q = \left\{ \mathbf{y}_j, \ \mathbf{z}_j, \ j \geq 1, \frac{1}{q^2-q}, \frac{1}{q} \right\}. \tag{6}$$

Next, we prove that

$$\mathbf{y}_j - \frac{1+q^3}{q^4-1} \begin{cases} \leq 0 & \text{when } j < 4 \\ > 0 & \text{when } j \geq 4 \end{cases} \tag{7}$$

when $q \in [q_1, q_3]$. It suffices to show that

$$q^{-1} + q + q^2 - q^3 - q^{-1-j} - q^{1-j} \begin{cases} \leq 0 & \text{when } j < 4 \\ > 0 & \text{when } j \geq 4. \end{cases} \tag{8}$$

By noting that

$$\mathbf{y}_j - \frac{1+q^3}{q^4-1} = (q^{-1} + q + q^2 - q^3 - q^{-1-j} - q^{1-j})(q^4-1)^{-1},$$

we see that (8) is equivalent to

$$\ln \frac{(q^2+1)}{-q^4+q^3+q^2+1} (\ln q)^{-1} \begin{cases} \geq j & \text{when } j < 4 \\ < j & \text{when } j \geq 4. \end{cases} \tag{9}$$

The inequalities in (9) are true when $q \in [q_1, q_3]$, since $\ln \frac{(q^2+1)}{-q^4+q^3+q^2+1}(\ln q)^{-1}$ is strictly increasing on the interval $[q_1, q_4]$, and $\ln \frac{(q_4^2+1)}{-q_4^4+q_4^3+q_4^2+1}(\ln q_4)^{-1} = 4$ and $\ln \frac{(q_1^2+1)}{-q_1^4+q_1^3+q_1^2+1}(\ln q_1)^{-1} = 3$. Here $q_4 (\approx 1.69784)$ is the positive root of $x^7 - x^5 - x^4 - 2x^3 - 2x^2 - x - 1 = 0$. A direct computation shows that $\mathbf{y}_j \in [\frac{q+q^2}{q^4-1}, \frac{1}{q^2-q}]$ for all $j \geq 1$. This statement combined with (7) shows that $\mathbf{y}_j \in J_q$ if $1 \leq j \leq 3$ and $\mathbf{y}_j \in (S_q \setminus J_q)_R$ if $j \geq 4$. Using (i) of Lemma 2.1, we also have $\mathbf{z}_j \in J_q$ if $1 \leq j \leq 3$ and $\mathbf{z}_j \in (S_q \setminus J_q)_L$ if $j \geq 4$. Our proof now follows from (6). $\qquad \square$

3 Proof of Theorem 1.2

In this section we shall give a algorithm to find all elements of the set $\mathcal{B}_{\aleph_0} \cap [q_1, q_3]$. Recall that $\min \mathcal{B}_2 = \check{q} \approx 1.71064$.

Theorem 3.1. *Let $q \in [q_1, q_f) \setminus \{\check{q}\}$ and suppose $\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$ is a finite set. Then $q \in \mathcal{B}_{\aleph_0}$ if and only if there exists $w \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$ satisfying the following two properties.*
(i) There exist a finite sequence of transformations $\mathbf{b} \in \bigcup_{n=0}^{\infty} \{T_{q,0}, T_{q,1}\}^n$ such that

$$\mathbf{b}(w) = w.$$

(*ii*) *Let* **b** *be as above. Define*

$$B(w) := \{\mathbf{b}|i : \mathbf{b}|i(w) \in S_q, 1 \leq i \leq |\mathbf{b}|\}.$$

Then $B(w) \subset \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap S_q$. *Here* $|\mathbf{b}|$ *denotes the length of* **b**.

Proof. We begin with the rightwards implication. Suppose $q \in \mathcal{B}_{\aleph_0}$. By Lemma A, there exists $x \in (0, \frac{1}{q-1})$ such that x is a q null infinite point. Furthermore, we may assume $x \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$ by Lemma B and equation (2). Repeatedly applying Lemma B and equation (2), there exist $\mathbf{a}^1, ..., \mathbf{a}^m \in \bigcup_{n=0}^{\infty} \{T_{q,0}, T_{q1}\}^n$ satisfying

$$\mathbf{a}^i(\mathbf{a}^{i-1}(...(\mathbf{a}^1(x))...)) \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$$

for each $1 \leq i \leq m$. Note that the set $\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$ is finite. Therefore by the pigeonhole principle for m sufficiently large, there must exist $i, j \in \mathbb{N}$ with $1 \leq i < j \leq m$ such that

$$\mathbf{a}^j(\mathbf{a}^{j-1}(...(\mathbf{a}^1(x))...)) = \mathbf{a}^i(\mathbf{a}^{i-1}(...(\mathbf{a}^1(x))...)). \tag{10}$$

Set $\mathbf{b} = \mathbf{a}^{i+1} \cdots \mathbf{a}^j$, $\hat{\mathbf{a}} = \mathbf{a}^1 \cdots \mathbf{a}^i$ and suppose $\hat{\mathbf{a}}(x) = w$, here $w \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$. Then equation (10) implies that

$$\mathbf{b}(w) = w.$$

Since x is a q null infinite point so are all its branching points. Thus we have the second property.

To complete our if and only if it suffices to remark that (*i*) and (*ii*) imply that w is a q null infinite point. So $q \in \mathcal{B}_{\aleph_0}$. □

Now we search for all points belonging to $\mathcal{B}_{\aleph_0} \cap [q_1, q_3]$ by applying Theorem 3.1. Suppose x is a q null infinite point and $x \in J_q$. First we point out that

$$\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q = \{\mathbf{y}_j = \Pi(01^j(10)^\infty), \mathbf{z}_j = \Pi(10^j(01)^\infty), 1 \leq j \leq 3\}$$

when $q \in [q_1, q_3]$ by Proposition 2.2. By Theorem 3.1, we only need to consider the behavior of elements of $\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$ under maps belonging to $\bigcup_{n=0}^{\infty} \{T_{q,0}, T_{q,1}\}^n$. Without loss of generality we only need to consider the points $\mathbf{y}_j$, for $1 \leq j \leq 3$. We establish the following lemma.

Lemma 3.2. *(i)If* $q \in [q_1, q_3]$ *and* $\mathbf{y}_1$ *is a q null infinite point, then*

$$T_{q,1}(\mathbf{y}_1) \in L_q, \;\; T_{q,0}^j(T_{q,1}(\mathbf{y}_1)) \in L_q, j = 1, 2, \;\; T_{q,0}^3(T_{q,1}(\mathbf{y}_1)) \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q \tag{11}$$

(ii) If $q \in [q_1, q')$ *then* $\mathbf{y}_2$ *is not a q null infinite point. If* $q \in [q', q_3]$ *and* $\mathbf{y}_2$ *is a q null infinite point, then*

$$T_{q,1}(\mathbf{y}_2) \in L_q, \;\; T_{q,0}(T_{q,1}(\mathbf{y}_2)) \in L_q, \;\; T_{q,0}^2(T_{q,1}(\mathbf{y}_2)) \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q. \tag{12}$$

Here $q'(\approx 1.66184)$ is the positive root of $x^5 - x^3 - x^2 - 2x - 2 = 0$.

(iii)If $q = q_1$ and $\mathbf{y}_3$ is a q null infinite point, then

$$T_{q,1}(\mathbf{y}_3) \in L_q, \ \ T_{q,0}(T_{q,1}(\mathbf{y}_3)) \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q. \tag{13}$$

If $q \in (q_1, q_3]$ and $\mathbf{y}_3$ is a q null infinite point, then q must be q_3 or $q''(\approx 1.67365)$, the positive root of $x^5 - 2x^4 + x^3 - x^2 + x - 1 = 0$.

Proof. Direct computation yields (i). We prove (ii) now. A simple computation yields

$$T_{q,1}(\mathbf{y}_2) \in L_q, \ \ T_{q,0}(T_{q,1}(\mathbf{y}_2)) \in L_q, \ \ T_{q,0}^2(T_{q,1}(\mathbf{y}_2)) \in \begin{cases} \left[\frac{q+q^2}{q^4-1}, \frac{1+q^3}{q^4-1}\right] & \text{if } q \in [q', q_3] \\ \left(\frac{1+q^3}{q^4-1}, \frac{1}{q^2-q}\right] & \text{if } q \in [q_1, q'). \end{cases}$$

Here $q'(\approx 1.66184)$ is the positive root of $x^5 - x^3 - x^2 - 2x - 2 = 0$. Thus we obtain (12). We now assume $\mathbf{y}_2$ is q null infinite for some $q \in [q_1, q')$ and derive a contradiction. If $\mathbf{y}_2$ is a q null infinite point for some $q \in [q_1, q')$ then $T_{q,0}^2(T_{q,1}(\mathbf{y}_2)) \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap (S_q \setminus J_q)_R$ and it can be shown that

$$T_{q,0}(T_{q,1}(T_{q,0}^2(T_{q,1}(\mathbf{y}_2)))) \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q.$$

Therefore by Proposition 2.2 there exists $s \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap (S_q \setminus J_q)_R = \{\mathbf{y}_k, k \geq 4, \ 1/(q^2 - q)\}$ and $u \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$ such that

$$T_{q,0}^2(T_{q,1}(\mathbf{y}_2)) = s \ \text{ and } \ T_{q,0}(T_{q,1}(T_{q,0}^2(T_{q,1}(\mathbf{y}_2)))) = u. \tag{14}$$

We will show that this is not possible. That is $\mathbf{y}_2$ is not a q null infinite point when $q \in [q_1, q')$. In fact,

$$T_{q,0}^2(T_{q,1}(\mathbf{y}_2)) = s$$

means that

$$q^5 - q^4 - 2q^3 + 2q + 1 = q^{-k}.$$

That is

$$k = \frac{-\ln(q^5 - q^4 - 2q^3 + 2q + 1)}{\ln q}.$$

The function $-\ln(q^5 - q^4 - 2q^3 + 2q + 1)(\ln q)^{-1}$ is strictly decreasing on the interval $[q_1, q')$. Table 1 therefore implies that the only possible value of k that may occur within the interval $[q_1, q')$ is $k = 4$. It is not possible that $s = 1/(q^2 - q)$, since $\lim_{k \to \infty} \mathbf{y}_k = 1/(q^2 - q)$ and $-\ln(q^5 - q^4 - 2q^3 + 2q + 1)(\ln q)^{-1}$ is monotonic. Table 1 includes the values of q for which $T_{q,0}(T_{q,1}(T_{q,0}^2(T_{q,1}(\mathbf{y}_2)))) \in \bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \cap J_q$. Inspecting Table 1 shows that there are no values of q for which both equations in (14) hold. Therefore we may conclude (ii).

8

Table 1: the values of q for $\mathbf{y}_2$

	k	q	Polynomials
$T_{q,0}^2(T_{q,1}(\mathbf{y}_2)) = \mathbf{y}_k$	4	1.65027	$x^4 - x^3 - x^2 - x + 1$
	5	1.63923	$x^6 - 2x^4 - 2x^3 + x + 1$
$T_{q,0}(T_{q,1}(T_{q,0}^2(T_{q,1}(\mathbf{y}_2)))) = \mathbf{y}_k$	1	1.65637	$x^5 - 2x^3 - x^2 - x - 1$
	2	1.64308	$x^5 - x^4 - x^3 - 1$
	3	1.63420	$x^7 - 2x^5 - x^4 - x^2 - x - 1$
$T_{q,0}(T_{q,1}(T_{q,0}^2(T_{q,1}(\mathbf{y}_2)))) = \mathbf{z}_k$	1	1.64114	$x^2 - x - 1$
	2	1.65363	$x^5 - 2x^3 - x^2 + 1$
	3	1.66065	$x^5 - x^4 - x^3 + 1$

It remains to prove (iii). By direct computation, we have

$$T_{q,1}(\mathbf{y}_3) \in L_q, \quad T_{q,0}(T_{q,1}(\mathbf{y}_3)) \in \begin{cases} \left[\frac{q+q^2}{q^4-1}, \frac{1+q^3}{q^4-1}\right] & \text{if } q = q_1 \\ \left[\frac{1}{q}, \frac{q+q^2}{q^4-1}\right) & \text{if } q \in (q_1, q_3] \end{cases}$$

and we obtain (13). Furthermore if $\mathbf{y}_3$ is a q null infinite point for some $q \in (q_1, q_3]$ then it is straightforward to show that

$$T_{q,1}(T_{q,0}(T_{q,0}(T_{q,1}(\mathbf{y}_3)))) \in \bigcup_{i=0}^1 T_{q,i}^{-1}(U_q) \cap J_q.$$

Moreover, there must exist $s \in \bigcup_{i=0}^1 T_{q,i}^{-1}(U_q) \cap (S_q \setminus J_q)_L = \{\mathbf{z}_k, k \geq 4, \ 1/q\}$ and $u \in \bigcup_{i=0}^1 T_{q,i}^{-1}(U_q) \cap J_q$ such that

$$T_{q,0}(T_{q,1}(\mathbf{y}_3)) = s \quad \text{and} \quad T_{q,1}(T_{q,0}(T_{q,0}(T_{q,1}(\mathbf{y}_3)))) = u. \tag{15}$$

The equation $T_{q,0}(T_{q,1}(\mathbf{y}_3)) = s$ means that

$$k = \frac{-\ln(1 - q^{-1} + q^2 + q^3 - q^4)}{\ln q}.$$

The function $-\ln(1 - q^{-1} + q^2 + q^3 - q^4)(\ln q)^{-1}$ is strictly increasing on the interval (q_1, q_3) and

$$\lim_{q \to q_3} -\ln(1 - q^{-1} + q^2 + q^3 - q^4)(\ln q)^{-1} = +\infty. \tag{16}$$

Table 2 records the first few solutions of $-\ln(1 - q^{-1} + q^2 + q^3 - q^4)(\ln q)^{-1} = k$. It is easy to show that the case where $s = 1/q$ is only possible when $q = q_3$. In Table 2, we also list the q's for which $T_{q,1}(T_{q,0}(T_{q,0}(T_{q,1}(\mathbf{y}_3)))) = u$ holds. By inspecting Table 2 and using the fact $-\ln(1 - q^{-1} + q^2 + q^3 - q^4)(\ln q)^{-1}$ is increasing with q, we see that the only values of q for which both equations in (15) hold simultaneously are $q \approx 1.67365$ and when $q = q_3$.

$\square$

Table 3 lists the values of q for which equations (11-13, 15) hold true independently. In fact, it follows from Lemma 3.2 and the symmetric property of $\mathbf{y}_k$ and $\mathbf{z}_k$ shown in Lemma 2.1 that equations in (11-13, 15) give all the possible values of q such that $q \in \mathcal{B}_{\aleph_0} \cap [q_1, q_3]$.

Table 2: the values of q for $\mathbf{y}_3$

	k	q	Polynomials
$T_{q,0}(T_{q,1}(\mathbf{y}_3)) = \mathbf{z}_k$	4	1.66041	$x^6 - x^5 - x^3 - x^2 - 1$
	5	1.66883	$x^8 - x^6 - x^5 - 2x^4 - x^3 - x^2 - x - 1$
	6	1.67365	$x^5 - 2x^4 + x^3 - x^2 + x - 1$
	7	1.67644	$x^{10} - x^8 - x^7 - 2x^6 - x^5 - x^4 - x^3 - x^2 - x - 1$
$T_{q,0}(T_{q,1}(\mathbf{y}_3)) = q^{-1}$		1.68042	$x^5 - x^4 - x^3 - x + 1$
$T_{q,1}(T_{q,0}(T_{q,0}(T_{q,1}(\mathbf{y}_3)))) = \mathbf{y}_k$	1	1.68042	$x^5 - x^4 - x^3 - x + 1$
	2	1.65963	$x^7 - 2x^5 - x^4 - x^3 + x + 1$
	3	1.64541	$x^6 - x^4 - x^3 - 2x^2 - x - 1$
$T_{q,1}(T_{q,0}(T_{q,0}(T_{q,1}(\mathbf{y}_3)))) = \mathbf{z}_k$	1	1.65462	$x^6 - 2x^4 - x^3 - 1$
	2	1.67365	$x^5 - 2x^4 + x^3 - x^2 + x - 1$
	3	1.68400	$x^8 - 2x^6 - x^5 - x^2 - x - 1$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. By applying Theorem 3.1 and Lemma 3.2, we can find all points belonging to $\mathcal{B}_{\aleph_0} \cap [q_1, q_3]$ in Table 3. One can see from Table 3 that

$$T_{q,0}(T_{q,1}(\mathbf{y}_3)) = \mathbf{z}_3, T_{q,1}(T_{q,0}(\mathbf{z}_3)) = \mathbf{y}_3, T_{q,1}(T_{q,0}(T_{q,0}(T_{q,1}(\mathbf{y}_3)))) = \mathbf{y}_3,$$

when $q_1(\approx 1.64541)$, which is the positive root of $x^6 - x^4 - x^3 - 2x^2 - x - 1 = 0$,

$$T_{q,0}^3(T_{q,1}(\mathbf{y}_1)) = \mathbf{z}_1, T_{q,1}^3(T_{q,0}(\mathbf{z}_1)) = \mathbf{y}_1, T_{q,0}^3(T_{q,1}^4(T_{q,0}(\mathbf{z}_1))) = \mathbf{z}_1,$$

when $q_2(\approx 1.65462)$, which is the positive root of $x^6 - 2x^4 - x^3 - 1 = 0$, and

$$T_{q,0}^3(T_{q,1}(\mathbf{y}_1)) = \mathbf{y}_1,$$

when $q_3(\approx 1.68042)$, which is the positive root of $x^5 - x^4 - x^3 - x + 1 = 0$. So, the conditions in Theorem 3.1 are satisfied when $q = q_1, q_2, q_3$, respectively. That is $q_j \in \mathcal{B}_{\aleph_0}$ for each $1 \leq j \leq 3$.

Finally, it is easy to check that $q \notin \mathcal{B}_{\aleph_0}$ if q takes the values listed in Table 3, except when $q = q_j, 1 \leq j \leq 3$. For example, we have

$$T_{q,0}^3(T_{q,1}(\mathbf{y}_1)) = \mathbf{y}_2$$

when $q(\approx 1.65963)$ is the positive root of $x^5 - x^4 - x^3 - x + 1 = 0$. However, there exists no point in $\bigcup_{i=0}^{1} T_{q,i}^{-1}(U_q) \bigcap J_q$ such that condition (i) of Theorem 3.1 holds for this value of q. The other values of q are dealt with similarly. $\square$

4 Proof of Theorem 1.1

To prove Theorem 1.1 it suffices to prove 1 has $\aleph_0$ q-expansions when $q = q_3$ and 1 a continuum of q-expansions when $q = q_1$ and $q = q_2$. We only prove the case of $q = q_1$, the case of $q = q_2$ can be verified in a similar way.

Table 3: the values of q for $\mathbf{y}_k$ and $\mathbf{z}_k, 1 \leq k \leq 3$

	k	q	Polynomials
$T_{q,0}^3(T_{q,1}(\mathbf{y}_1)) = \mathbf{y}_k$ (or $T_{q,1}^3(T_{q,0}(\mathbf{z}_1)) = \mathbf{z}_k$)	1	1.68042	$x^5 - x^4 - x^3 - x + 1$
	2	1.65963	$x^7 - 2x^5 - x^4 - x^3 + x + 1$
	3	1.64541	$x^6 - x^4 - x^3 - 2x^2 - x - 1$
$T_{q,0}^3(T_{q,1}(\mathbf{y}_1)) = \mathbf{z}_k$ (or $T_{q,1}^3(T_{q,0}(\mathbf{z}_1)) = \mathbf{y}_k$)	1	1.65462	$x^6 - 2x^4 - x^3 - 1$
	2	1.67365	$x^5 - 2x^4 + x^3 - x^2 + x - 1$
	3	1.68400	$x^8 - 2x^6 - x^5 - x^2 - x - 1$
$T_{q,0}^2(T_{q,1}(\mathbf{y}_2)) = \mathbf{y}_k$ (or $T_{q,1}^2(T_{q,0}(\mathbf{z}_2)) = \mathbf{z}_k$)	1	1.72208	$x^4 - x^3 - x^2 - x + 1$
	2	1.68929	$x^6 - 2x^4 - 2x^3 + x + 1$
	3	1.6663	$x^6 - x^5 - x^4 - x^3 + x^2 + 1$
$T_{q,0}^2(T_{q,1}(\mathbf{y}_2)) = \mathbf{z}_k$ (or $T_{q,1}^2(T_{q,0}(\mathbf{z}_2)) = \mathbf{y}_k$)	1	1.67602	$x^5 - 2x^3 - x^2 - 1$
	2	1.7049	$x^5 - x^4 - x^3 - 1$
	3	1.72004	$x^7 - 2x^5 - x^4 - x^2 - x - 1$
$T_{q,0}(T_{q,1}(\mathbf{y}_3)) = \mathbf{z}_k$ (or $T_{q,1}(T_{q,0}(\mathbf{z}_3)) = \mathbf{y}_k$)	3	1.64541	$x^6 - x^4 - x^3 - 2x^2 - x - 1$
$T_{q,0}(T_{q,1}(\mathbf{y}_3)) = \mathbf{z}_6, \ T_{q,1}(T_{q,0}(\mathbf{z}_6)) = \mathbf{z}_2$ (or $T_{q,1}(T_{q,0}(\mathbf{z}_3)) = \mathbf{y}_6, \ T_{q,0}(T_{q,1}(\mathbf{y}_6)) = \mathbf{y}_2$)		1.67365	$x^5 - 2x^4 + x^3 - x^2 + x - 1$
$T_{q,0}(T_{q,1}(\mathbf{y}_3)) = 1/q, \ T_{q,1}(T_{q,0}(1/q)) = \mathbf{y}_1$ (or $T_{q,1}(T_{q,0}(\mathbf{z}_3)) = 1/(q^2 - q)$, $T_{q,0}(T_{q,1}(1/(q^2 - q))) = \mathbf{z}_1$)		1.68042	$x^5 - x^4 - x^3 - x + 1$

Theorem 4.1. *If $q_3(\approx 1.68042)$ is the positive root of $x^5 - x^4 - x^3 - x + 1 = 0$. Then 1 is a q_3 null infinite point and therefore has $\aleph_0$ q_3-expansions.*

Proof. It is straightforward to show that for all $k \geq 0$

$$(T_{q_3,0}^3 \circ T_{q_3,1})^k(T_{q_3,1}(1)) \in S_{q_3}.$$

Moreover these are the only $\mathbf{a} \in \bigcup_{n=0}^{\infty}\{T_{q_3,0}, T_{q_3,1}\}^n$ such that $\mathbf{a}(1) \in S_{q_3}$. This is sufficient to imply that 1 is a q_3 null infinite point . Namely that for each $k \geq 0$ we have

$$T_{q_3,0}\Big((T_{q_3,0}^3 \circ T_{q_3,1})^k(T_{q_3,1}(1))\Big) = \Pi(1(10)^\infty).$$

Therefore

$$\Sigma_{q_3}(1) = \left\{1(10^3)^k 01(10)^\infty, 1(10^3)^\infty\right\}.$$

$\square$

Figure 2 demonstrates the construction of $\Sigma_{q_3}(1)$.

Theorem 4.2. *Let $q_1(\approx 1.64541)$ be the positive solution of the equation $x^6 - x^4 - x^3 - 2x^2 - x - 1 = 0$. Then 1 has $2^{\aleph_0}$ q_1-expansions.*

Proof. We proceed via a proof by contradiction. We assume that 1 has $\aleph_0$ q_1-expansions and obtain a contradiction. It is a simple calculation to show that

$$w := (T_{q_1,0}^5 \circ T_{q_1,1}^2)(1) \in S_{q_1}.$$

11

Figure 2: The construction of $\Sigma_{q_3}(1)$

Therefore w has either $\aleph_0$ q_1-expansions or $2^{\aleph_0}$ q_1-expansions. Since we have assumed 1 has $\aleph_0$ q_1-expansions, w must also have $\aleph_0$ q_1-expansions. Therefore w can be mapped to a q_1 null infinite point, and by Lemma C it can be mapped to a point with a periodic q_1-expansion. The above implies that 1 has a q_1 expansion that begins $(1100000(\delta_i)_{i=1}^\infty)$ where $(\delta_i)_{i=1}^\infty$ is eventually periodic. This is obviously equivalent to

$$1 = \frac{1}{q_1} + \frac{1}{q_1^2} + \frac{1}{q_1^7} \sum_{i=1}^\infty \frac{\delta_i}{q_1^i}. \tag{17}$$

Since $(\delta_i)_{i=1}^\infty$ is eventually periodic we may use properties of geometric series to deduce that (17) is equivalent to

$$1 = \frac{1}{q_1} + \frac{1}{q_1^2} + \frac{f(q_1)}{g(q_1)} \tag{18}$$

where $f(x), g(x) \in \mathbb{Z}[x]$. Equation (18) is just an algebraic relation and so must also be satisfied by the conjugates of q_1, that is the other roots of $x^6 - x^4 - x^3 - 2x^2 - x - 1 = 0$. We now show that this cannot be the case for a particular choice of conjugate, namely $q_1^* \approx -1.20458$. Equation (18) is equivalent to equation (17), so (17) must also hold with q_1 replaced by q_1^*. We observe the following

$$1 = \frac{1}{q_1^*} + \frac{1}{(q_1^*)^2} + \frac{1}{(q_1^*)^7} \sum_{i=1}^\infty \frac{\delta_i}{(q_1^*)^i}$$
$$\leq \frac{1}{q_1^*} + \frac{1}{(q_1^*)^2} + \frac{1}{(q_1^*)^7} \frac{q_1^*}{(q_1^*)^2 - 1}$$
$$< 1.$$

Where the final strict inequality follows from a simple calculation. Thus we have our desired contradiction. $\qquad \square$

The proof that $q_2 \notin \mathcal{B}_{1,\aleph_0}$ is done analogously. In this case we similarly use a conjugate of q_2, namely the number $q_2^* \approx -1.26493$.

Acknowledgment

References

[1] S. Baker, On small bases which admit countably many expansions. *J. Number Theory* **147** (2015), 515-532.

[2] S. Baker, Generalized golden ratios over integer alphabets. *Integers* **14** (2014), Paper No. A15.

[3] S. Baker and N. Sidorov, Expansions in non-integer bases: lower order revisited. *Integers* **14** (2014), Paper No. A57.

[4] K. Dajani and M. de Vries, Measures of maximal entropy for random β-expansions. J. *Eur. Math. Soc.* **7**(2005), 51-68.

[5] K. Dajani and M. de Vries, Invariant densities for random β-expansions. *J. Eur. Math. Soc.* **9**(2007), 157-176.

[6] K. Dajani and C. Kraaikamp, Random β-expansions. *Ergodic Theory and Dynam. Sys.* **23**(2003), 461-479.

[7] Z. Daróczy and I. Kátai, Univoque sequences. *Publ. Math. Debrecen* **42**(1993), 397-407.

[8] Z. Daróczy and I. Kátai, On the structure of univoque numbers. *Publ. Math. Debrecen* **46**(1995), 385-408.

[9] M. de Vries and V. Komornik, Unique expansions of real numbers. *Adv. Math.* **221**(2009), 390-427.

[10] P. Erdős, M. Horváth and I. Joó, On the uniqueness of the expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$. *Acta Math. Hungar.* **58**(1991), 333-342.

[11] P. Erdős, I. Joó and V. Komornik, Characterization of the unique expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$ and related problems. *Bull. Soc. Math. Fr.* **118**(1990), 377-390.

[12] P. Erdös and I. Joó, On the number of expansions $1 = \sum q^{-n_i}$, II. *Ann. Univ. Sci. Budapest. Eötvös Sect.* **36**(1993), 229-233.

[13] P. Erdös and I. Joó, On the number of expansions $1 = \sum q^{-n_i}$. *Ann. Univ. Sci. Budapest. Eötvös Sect.* **35**(1992), 129-132.

[14] P. Erdős and V. Komornik, Developments in non-integer bases. *Acta Math. Hungar.* **79**(1998), 57-83.

[15] P. Glendinning and N. Sidorov, Unique representations of real numbers in non-integer bases. *Math. Res. Letters* **8**(2001), 535-543.

[16] V. Komornik and P. Loreti, Unique developments in non-integer bases. *Amer. Math. Monthly* **105**(1998), 636-639.

[17] W. Parry, On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hung.* **11** (1960), 401-416.

[18] N. Sidorov, Almost every number has a continuum of β-expansions. *Amer. Math. Monthly* **110**(2003), 838-842.

[19] N. Sidorov, Ergodic-theoretic properties of certain Bernoulli convolutions. *Acta Math. Hungar.* **101**(2003), 345-355.

[20] N. Sidorov, Expansions in non-integer bases: lower, middle and top orders. *J. Number Theorem* **129**(2009), 741-754.

[21] N. Sidorov and A. Vershik, Ergodic properties of the Erdős measure, the entropy of the golden shift, and related problems. *Monatsh. Math.* **126**(1998), 215-261.

[22] Y. R. Zou, J. Lu and W. X. Li, Unique expansion of points of a class of self-similar sets with overlaps. *Mathmatika* **58**(2012), 371-88.