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Generalized convective quasi-equilibrium principle

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ABSTRACT

A generalization of Arakawa and Schubert's convective quasi-equilibrium principle is presented for a closure formulation of mass-flux convection parameterization. The original principle is based on the budget of the cloud work function. This principle is generalized by considering the budget for a vertical integral of an arbitrary convection-related quantity. The closure formulation includes Arakawa and Schubert's quasi-equilibrium, as well as both CAPE and moisture closures as special cases. The formulation also includes new possibilities for considering vertical integrals that are dependent on convective-scale variables, such as the moisture within convection.

The generalized convective quasi-equilibrium is defined by a balance between large-scale forcing and convective response for a given vertically-integrated quantity. The latter takes the form of a convolution of a kernel matrix and a mass-flux spectrum, as in the original convective quasi-equilibrium. The kernel reduces to a scalar when either a bulk formulation is adopted, or only large-scale variables are considered within the vertical integral. Various physical implications of the generalized closure are discussed. These include the possibility that precipitation might be considered as a potentially-significant contribution to the large-scale forcing. Two dicta are proposed as guiding physical principles for the specifying a suitable vertically-integrated quantity.

1. Introduction

Quasi-equilibrium is often considered an important guiding principle for understanding the role of moist convection in the large-scale atmospheric circulations (*cf.*, Emanuel et al., 1994). However, it is often forgotten that the concept of convective quasi-equilibrium was originally introduced by Arakawa and Schubert (1974: hereafter AS) in a rather specific and technical manner. In spite of the great influence of this concept, the original specific formulation is strangely not much investigated in the literature (*cf.*, Yano and Plant, 2012a). The goal of the present paper is to present its direct generalization.

Arakawa and Schubert's original quasi-equilibrium principle is specifically introduced as a closure condition for mass-flux convection parameterization. Thus, the present paper pursues its generalization also in the context of parameterization closure. The importance of subgrid-scale parameterization and the challenges that we still face cannot be overemphasized (*cf.*, McFarlane, 2011). The closure problem remains one of the major difficulties, for which many hypotheses have been proposed but without any clear consensus (*cf.*, Yano et al., 2013, 2014).

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For this goal, a general formulation for the convection parameterization closure is developed. Some existing closure hypotheses are also examined and compared on a equal footing in the light of the developed general formulation. Although the general formulation itself does not provide an ultimate answer to the question of the closure choice, a well-defined and consistent perspective on the possible formulations is definitely a step forward. A well-defined physical basis for closure must be capable of being incorporated within a suitably general framework, and this statement in itself may help to narrow down the acceptable possibilities.

General conceptual reflections on the convective closure problem are provided by Arakawa and Chen (1987), and Arakawa (1993), in which they propose to categorize the closures into the four types. The present paper develops a general closure formulation in mathematical terms, focusing on the type IV in their categorizations.

We take the mass-flux parameterization structure as the basis for considering this general closure formulation, also because the majority of current operational parameterizations follow this approach. For a presentation and discussion of the whole structure of the mass-flux convection parameterization, we refer to Yano (2014a). In terms of the mass-flux parameterization, the closure refers more specifically to the problem of defining the value of the convective mass-flux at the convection base.

This problem arises in the following manner. In the process of computing the tendency of resolved-scale variables due to convection, a mass-flux convection parameterization needs to determine both the mass flux and the values of convective-scale variables. A spectrum of the mass flux may be considered, M_i, with i an index for a convection type. Alternatively, a single bulk mass flux, M, may be preferred. Once this quantity is defined, all of the convective-scale variables, here designated by φ_i, are then calculated from:

$$\frac{\partial}{\partial z} M_i \varphi_i = E_i \overline{\varphi} - D_i \varphi_i + \rho \sigma_i F_i. \tag{1.1}$$

Here the bar designates the grid-box average, ρ is the air density, σ_i is the fractional area occupied by the ith convection type and F_i is the forcing on φ averaged over the ith convection type. z is the height coordinate.

The procedure for solving Eq. (1.1) is relatively straightforward once the mass flux, M_i, is known, so long as the variable is conserved (*i.e.*, $F_i = 0$). For issues concerning non-conservative processes, we refer to Donner (1993). The issues of prescribing the entrainment and detrainment rates, E_i and D_i, are reviewed by de Rooy et al. (2013).

Thus, the main problem in mass-flux convection parameterization reduces to that of defining the mass flux, M_i. The usual practice is to consider it under a separation of variables into a vertical dependence and temporal dependence:

$$M_i = \eta_i(z) M_{B,i}(t). \tag{1.2}$$

Here, $\eta_i(z)$ is a "normalized" vertical profile and $M_{B,i}(t)$ is a time-dependent amplitude of convection. The problems of defining them are usually called "the cloud model" and "the closure" respectively, and the latter is the topic of the present paper. $M_{B,i}(t)$ is usually defined as a value at the convection base, but mostly for a historical reason (*cf.*, Yano, 2011).

The cloud model is usually formulated in terms of prescribed fractional rates, $\epsilon_i = E_i/M_i$ and $\delta_i = D_i/M_i$, for the entrainment and detrainment by

$$\frac{1}{\eta_i} \frac{\partial}{\partial z} \eta_i = \epsilon_i - \delta_i. \tag{1.3}$$

Vertical integration leads to

$$\eta_i = \exp \left[\int_{z_B}^{z} (\epsilon_i - \delta_i) dz' \right] \tag{1.4}$$

such that the normalization of the mass flux profile is obtained with $\eta_i(z = z_B) = 1$.

The present paper presents a general formulation for mass-flux convection parameterization closure (*i.e.*, for the calculation of $M_{B,i}$) as an extension of the common current closures which are based on CAPE (convective available potential energy) and moisture convergence. A general statement of the problem is made in the next section and the formulation is developed over Sections 3–6, gradually increasing its generalizabilty and providing examples of how existing closure methods fit into the overall structure. The most general case is presented in Section 6. The two earlier sections may be considered preparations for presenting this general result. Nevertheless, these sections also contain their own useful results for some simple but important examples.

The results are summarized and further examined in Section 7. The generalized convective quasi-equilibrium principle presented herein naturally does not cover all of the existing closure hypotheses. These more general aspects are discussed in the last section in conclusions.

2. General statement of the problem

A common approach for defining the mass-flux convection parameterization closure is to assume that a certain vertical integral, I, is quasi-stationary under interactions between convection and the large-scale dynamics. Thus,

$$\frac{\partial}{\partial t}I = 0, \tag{2.1}$$

where

$$I = \int_{z_B}^{z_T} f dz. \tag{2.2}$$

Various possibilities for the integrand f will be specified in the following sections. The integral may be performed from the convection base, z_B, to the convection top, z_T, but this can be modified as in Section 3.2 below. Eq. (2.1) along with Eq. (2.2) may be considered as a generalization of the AS convective quasi-equilibrium principle, as will be demonstrated in the following.

Here, note that the definition for the bottom and the top of convection is itself an open question. The convection base, z_B, would be most conveniently defined at the top of the planetary boundary layer (assuming a well-mixed convective boundary layer), as assumed by AS. In many CRM/LES (cloud-resolving model/large-eddy simulation) diagnostic analyses, the convection base is simply defined as a cloud base, or a lifting condensation level. The ECMWF model, for example, also takes this latter definition. Alternatively, we may take the convection base simply at the surface. The convection top, z_T, is relatively straightforward to define if we follow the basic ideas of convective plumes (*cf.*, de Rooy et al., 2013; Yano, 2014b): it would be equated with the level of neutral buoyancy. However, this is not a unique option, and one may for example wish to consider the possibility of convective overshooting.

The basic idea behind this closure formulation may be understood by explicitly writing down a budget equation in the form:

$$\frac{\partial I}{\partial t} = \left(\frac{\partial I}{\partial t}\right)_L + \left(\frac{\partial I}{\partial t}\right)_c \tag{2.3}$$

Here, $(\partial I/\partial t)_L$ is the rate at which the quantity I is generated by large-scale processes (*i.e.*, the large-scale forcing), and $-(\partial I/\partial t)_c$ is the rate at which I is consumed by convection (*i.e.*, convective damping). The precise meanings of these two terms will be specified step by step as the formulation is developed in the following. The closure of Eq. (2.1) implies that whenever I is generated by a large-scale process, it is consumed by convection almost immediately so that a balance

$$\left(\frac{\partial I}{\partial t}\right)_L + \left(\frac{\partial I}{\partial t}\right)_c = 0 \tag{2.4}$$

is maintained. The idea was originally proposed by AS, and their specific formulation (convective quasi-equilibrium closure) is presented in Section 6.2. In this respect, Eq. (2.4) may be considered a generalization of the AS convective quasi-equilibrium principle (*cf.*, Yano and Plant, 2012a).

Note that generally, the two terms, $(\partial I/\partial t)_L$ and $(\partial I/\partial t)_c$, on the right-hand side of Eq. (2.3) are not necessarily positive and negative definite, respectively. Thus, some modifications of the physical interpretation of the balance condition (2.4) may be required. Nevertheless, it is reasonable to expect that the condition (2.1) or (2.4) remains a useful guiding principle for convection-parameterization closure.

3. Closures depending only on the large-scale variables: $f = f(\overline{\varphi})$

The simplest choice for the vertical integral, I, is

$$I = \int_{z_B}^{z_T} f(\overline{\varphi}) dz, \tag{3.1}$$

where f is an unspecified function of an unspecified physical variable, $\overline{\varphi}$, that is defined as a grid-box average (*i.e.*, a “large-scale” variable).

The function f may, in general, depend on multiple such variables, in which case, $\overline{\varphi}$ must be replaced by a vector representing those variables. The possibility for this generalization is kept implicit in much of the following analysis, because it introduces only a minor modification of the notation in so far as the derivations are concerned. The important point for now is that the function, f, is taken to depend solely on large-scale variables. As it turns out, the two most commonly adopted types of closure fall into this category.

3.1. Equation for the large-scale variable

In order to find a more explicit expression for Eq. (2.1), we substitute Eq. (2.2), and perform the temporal derivative:

$$\frac{\partial I}{\partial t} = \int_{z_B}^{z_T} \left(\frac{\partial f}{\partial \overline{\varphi}}\right) \frac{\partial \overline{\varphi}}{\partial t} dz + \dot{z}_T f_T - \dot{z}_B f_B \tag{3.2}$$

by invoking Leibniz's theorem. Here and hereafter, the subscripts T and B denote values at the top and the bottom of the integration limits.

As already remarked, it is convenient to separate the temporal tendency, $\partial\overline{\varphi}/\partial t$, into terms due to convective activity and terms due to large-scale processes:

$$\frac{\partial \overline{\varphi}}{\partial t} = \left(\frac{\partial \overline{\varphi}}{\partial t}\right)_c + \left(\frac{\partial \overline{\varphi}}{\partial t}\right)_L \tag{3.3}$$

Substitution of Eq. (3.3) into Eq. (3.2) leads to:

$$\left(\frac{\partial I}{\partial t}\right)_c = \int_{z_B}^{z_T} \left(\frac{\partial f}{\partial \overline{\varphi}}\right) \left(\frac{\partial \overline{\varphi}}{\partial t}\right)_c dz, \tag{3.4a}$$

$$\left(\frac{\partial I}{\partial t}\right)_L = \int_{z_B}^{z_T} \left(\frac{\partial f}{\partial \overline{\varphi}}\right) \left(\frac{\partial \overline{\varphi}}{\partial t}\right)_L dz + \dot{z}_T f_T - \dot{z}_B f_B. \tag{3.4b}$$

Note that the top and the bottom contributions are mostly conveniently assigned to be a part of the large-scale forcing for now. However, in later developments, it turns out that a part of these terms may be better considered as corresponding to a convective-scale contribution, as discussed in Section 7.1.

In the mass flux parameterization framework, the prognostic equation for a large-scale variable $\overline{\varphi}$ can be written as in Eq. (7.9) of Yano (2014a), with the convective and large-scale tendencies being given respectively by:

$$\left(\frac{\partial \overline{\varphi}}{\partial t}\right)_c = \frac{1}{\rho}\sum_i \left[D_i(\varphi_i^D - \overline{\varphi}) + M_i \frac{\partial \overline{\varphi}}{\partial z}\right] \tag{3.5a}$$

$$\left(\frac{\partial \overline{\varphi}}{\partial t}\right)_L = -\overline{\nabla}\cdot\overline{\varphi\mathbf{u}} - \frac{1}{\rho}\frac{\partial}{\partial z}\rho\overline{w\varphi} + F_e. \tag{3.5b}$$

Here, the superscript D is added to φ_i to indicate the value on detrainment, and F_e is the forcing on φ averaged over the environment.

By substituting Eq. (3.5a) into Eq. (3.4a) we obtain:

$$\left(\frac{\partial I}{\partial t}\right)_c = \sum_i K_i M_{B,i} \tag{3.6}$$

where

$$K_i = \int_{z_B}^{z_T} \frac{\eta_i}{\rho}\frac{\partial f}{\partial \overline{\varphi}} \left[\delta_i(\varphi_i - \overline{\varphi}) + \frac{\partial \overline{\varphi}}{\partial z}\right] dz \tag{3.7}$$

It is obvious that this type of closure depending only on the large-scale variables cannot define a spectrum of convective types, since only a single constraint is available. In order to emphasize this point, where a bulk formulation is necessary, we replace the index i by the subscript c as required. Thus,

$$\frac{\partial I}{\partial t} = KM_B + \left(\frac{\partial I}{\partial t}\right)_L \tag{3.8a}$$

where

$$K = \int_{z_B}^{z_T} \frac{\eta_c}{\rho}\frac{\partial f}{\partial \overline{\varphi}} \left[\delta_c(\varphi_c - \overline{\varphi}) + \frac{\partial \overline{\varphi}}{\partial z}\right] dz. \tag{3.8b}$$

By substituting Eq. (3.8a) into the closure condition (2.1), we obtain

$$M_B = -\frac{1}{K}\left(\frac{\partial I}{\partial t}\right)_L. \tag{3.9}$$

3.2. Kuo's (1974) moisture-based closure

The moisture-based closure proposed by Kuo (1974) is probably the best known example of a closure solely based on large-scale variables. It sets

$$f = \rho\overline{q} \tag{3.10}$$

with q the moisture mixing ratio. In applying the general formulation derived above, we note that the vertical eddy flux in the boundary layer, especially the surface evaporation rate, is important for the moisture budget. In order to see this contribution explicitly, we set

$$F_e = -\frac{1}{\rho}\frac{\partial}{\partial z}\rho\overline{w'q'}. \tag{3.11}$$

The closure condition is then given by Eq. (3.9) with

$$K = \int_{z_B}^{z_T} \eta_c \left[\delta_c(q_c - \overline{q}) + \frac{\partial\overline{q}}{\partial z}\right] dz \tag{3.12a}$$

$$\left(\frac{\partial I_q}{\partial t}\right)_L = -\int_{z_B}^{z_T} \left[\rho\nabla\cdot\overline{\mathbf{u}q} + \frac{\partial}{\partial z}\rho\overline{wq}\right] dz + H_E \tag{3.12b}$$

Here we have added a subscript q to I to indicate results specific to the moisture-based closure. The vertical integral of the forcing term has been written as H_E, which is simply the surface evaporation rate if z_B is taken at the Earth's surface. The top of the integral, z_T, is taken at the top of the atmosphere for now, so that there is no contribution from that limit to the eddy moisture flux.

The major interest of Kuo (1974) is to obtain the convective moisture tendency, $(\partial\overline{q}/\partial t)_c$, which may be expressed as:

$$\left(\frac{\partial\overline{q}}{\partial t}\right)_c = -\frac{\tilde{f}(z)}{K}\left(\frac{\partial I_q}{\partial t}\right)_L, \tag{3.13a}$$

where

$$\tilde{f}(z) = \frac{\eta_c}{\rho}\left[\frac{\partial\overline{q}}{\partial z} + \delta_c(q_c^D - \overline{q})\right]. \tag{3.13b}$$

In Kuo's (1974) original formulation, the vertical profile, $\tilde{f}(z)$, is determined in a rather arbitrary manner. However, once Kuo's (1974) closure is re-cast into the mass-flux framework as presented here, this issue simply reduces to that of determining a vertical profile for the mass flux, η_c, as seen from Eq. (3.13b). Kuo's (1974) formulation has been further pursued by *e.g.*, Krishnamurti et al. (1976), Anthes (1977), Molinari (1985).

3.3. CAPE-based closure

Many current operational models adopt CAPE (*cf.*, Roff and Yano, 2002) as the basis for their closure. This is perhaps a less obvious example, but it does belong to the same closure category. It amounts to setting

$$f = b \tag{3.14}$$

in Eq. (2.2), where b is the lifting-parcel buoyancy defined in terms of the virtual temperature, T_v, by:

$$b = g\frac{T_{vp} - \overline{T}_v}{\overline{T}_v}. \tag{3.15}$$

Here, T_{vp} is the lifting-parcel virtual potential temperature, in which no mixing with the environment is assumed. The virtual temperature may be defined by:

$$T_v = (1 + \hat{\delta}q - l)T,$$

with $\hat{\delta} = R_v/R_d - 1$ defined in terms of the gas constants for dry air, R_d, and water vapour, R_v. The cloud liquid water is denoted by l. Alternatively, it is convenient to express the lifting-parcel buoyancy as:

$$b = \rho\alpha(s_{vp} - \overline{s}_v), \tag{3.16}$$

in terms of the virtual static energy:

$$s_v = C_pT_v + gz, \tag{3.17}$$

where C_p is the specific heat at constant pressure, and

$$\alpha = \frac{g}{\rho C_p \overline{T}_v}. \tag{3.18}$$

The CAPE closure can be considered as a case where the function *f* only depending on large-scale variables, given that the lifting-parcel virtual potential temperature, T_{vp}, may be interpreted as a large-scale variable. Specifically, T_{vp} does not follow the standard rule for convective-scale variables as being influenced by entrainment, as reviewed below in Section 4.2. By following the same procedure as for the moisture-closure case (Section 3.2), the CAPE closure is again given by Eq. (3.9), setting I = CAPE and $\overline{\varphi} = s_{vp} - \overline{s}_v$ to produce:

$$K = \int_{z_B}^{z_T} \alpha \frac{\partial}{\partial z}(s_{vp} - \overline{s}_v) dz \tag{3.19a}$$

$$\left(\frac{\partial \text{CAPE}}{\partial t}\right)_L = -\int_{z_B}^{z_T} \rho\alpha \left[\frac{\partial}{\partial t}(s_{vp} - \overline{s}_v)\right]_L dz. \tag{3.19b}$$

Note that $\eta_c = 1$ for non-entraining parcel ascent. In writing Eq. (3.19b), it is also assumed that the integral limits are set where the lifting-parcel buoyancy vanishes. Eq. (3.19a) can be further simplified by neglecting any changes to the lifting-parcel moist virtual static energy s_{vp} during a non-entraining ascent, so that

$$K \approx -C_p \int_{z_B}^{z_T} \alpha \left(\frac{\partial \overline{T}_v}{\partial z} + \frac{g}{C_p}\right) dz \tag{3.20}$$

which is a typically adopted formulation in operations (*cf.*, Bechtold et al., 2014).

In many current operational implementations, however, the large-scale tendency, $(\partial \text{CAPE}/\partial t)_L$, is usually not directly calculated but instead replaced by the term

$$\frac{\text{CAPE}}{\tau} \tag{3.21}$$

where τ is known as the closure timescale. This replacement is necessary in practice, because otherwise a parameterization "underestimates convective activity in situations where the large-scale forcing is weak, and where convective heating precedes the dynamic adjustment" (Bechtold et al., 2014). Various other examples and further discussions of such an implementation are found in Bechtold et al. (2001), Emanuel (1993), Fritsch and Chappell (1980), Gregory (1997), Gregory et al. (2000), Kain (2004), and Zhang and McFarlane (1995).

By putting all those approximations and assumptions together, a final expression for the closure is given by

$$M_B = \frac{\text{CAPE}}{\tau}\left[C_p \int_{z_B}^{z_T} \alpha \left(\frac{\partial \overline{T}_v}{\partial z} + \frac{g}{C_p}\right) dz\right]^{-1}. \tag{3.22}$$

3.3.1. Boundary-layer and parcel-environment based closure

The CAPE tendency may, furthermore, be divided into two contributions: one coming from the parcel virtual temperature, T_{vp}, and the other directly from the environmental state, $\overline{T}_v$. These are given by

$$\left(\frac{\partial \text{CAPE}}{\partial t}\right)_{\text{BL}} = \int_{z_B}^{z_T} \frac{g}{\overline{T}_v}\frac{\partial T_{vp}}{\partial t} dz \tag{3.23a}$$

and

$$\left(\frac{\partial \text{CAPE}}{\partial t}\right)_{\text{env}} = -\int_{z_B}^{z_T} \frac{g}{\overline{T}_v}\frac{\partial \overline{T}_v}{\partial t} dz \tag{3.23b}$$

respectively. We have neglected any contributions arising from changes to $\overline{T}_v$ in the denominator of the integrand: this factor may be absorbed into a part of the integration variable if the integral is performed in terms of pressure.

The contribution from the parcel virtual temperature, T_{vp} is considered as arising due to boundary-layer (BL) processes, because the parcel originates from the boundary layer and, by definition, does not interact with the environmental air aloft. On the other hand, the contribution from the large-scale virtual temperature, $\overline{T}_v$, is labelled as environmental (env).

Our physical intuition would suggest that most of the CAPE variability originates from the boundary layer, so that

$$\frac{\partial \text{CAPE}}{\partial t} \simeq \left(\frac{\partial \text{CAPE}}{\partial t}\right)_{\text{BL}} \tag{3.24}$$

Emanuel (1995) and Raymond (1995), thus, argue that the CAPE closure can be well approximated by considering only its boundary-layer contribution:

$$\left(\frac{\partial \mathrm{CAPE}}{\partial t}\right)_{\mathrm{BL}} \simeq 0. \tag{3.25}$$

This idea is called boundary-layer quasi-equilibrium.

However, the observational data analyses by Zhang (2002, 2003) and Donner and Phillips (2003) lead to rather unexpected conclusions. The data both from the tropics and the mid-latitudes does not support boundary-layer quasi-equilibrium observationally (see Yano et al., 2013 for detailed discussions). These authors instead propose that the CAPE closure should be replaced by that for the parcel environment, *i.e.*,

$$\left(\frac{\partial \mathrm{CAPE}}{\partial t}\right)_{\mathrm{env}} \simeq 0. \tag{3.26}$$

This leads to the idea of the parcel-environment based closure. For an operational implementation of the parcel-environment based closure, see Bechtold et al. (2014).

An important variant on CAPE is to replace the integrated buoyancy b by the density-weighted integrated buoyancy, choosing $f = \rho b$. The resulting integral is named PCAPE by Bechtold et al. (2014), who also show that this modification is a key ingredient for simulating the diurnal cycle of convection along with the adoption of the parcel-environment closure.

4. Closures depending only on the convective-scale variables: $f_i = f(\varphi_i)$

4.1. General formulation

An alternative possibility, considered now, is to assume that the integral function, f, depends only on convective-scale variables, φ_i. This assumption has some physical appeal, because the properties of the convection are expected to reach quasi-equilibrium (or quasi-stationarity: Yano and Plant, 2012a) against the large-scale state, and not the other way round. Here, note that the convective-scale variables are already slaved to the large-scale variables under the steady-plume hypothesis (*cf.*, Section 7.3 in Yano, 2014a), as suggested by Eq. (1.1) above, and as will be fully elucidated in Section 4.2 below. Thus, this attempt should not be confused with that of closing a convection parameterization solely in terms of convective-scale variables. The latter attempt would be ill-posed.

In this case, a vertical integral, I_i, is defined for each convection type, i, and thus it can more readily be applied to a spectral formulation. Specifically, in this case the vertical integral may be defined by

$$I_i = \int_{z_B}^{z_{Ti}} f(\varphi_i) dz, \tag{4.1}$$

and the quasi-equilibrium condition becomes

$$\frac{\partial}{\partial t} I_i = 0. \tag{4.2}$$

The convection top, z_{Ti}, is considered to depend on the convection type, i, whereas the convection base, z_B, is assumed to be common to the all types.

Taking Eq. (4.1) to define I_i, we can re-write Eq. (4.2) as:

$$\frac{\partial I_i}{\partial t} = \int_{z_B}^{z_{Ti}} \frac{\partial f_i}{\partial \varphi_i} \frac{\partial \varphi_i}{\partial t} dz + \dot{z}_{Ti} f_{Ti} - \dot{z}_B f_B \tag{4.3}$$

in analogy with Eq. (3.2).

In order to derive a full expression for Eq. (4.3), however, we need an explicit expression for the tendency, $\partial\varphi_i/\partial t$, which is the subject of the next subsection.

4.2. Convective-scale variables

4.2.1. Diagnostic solution

The convective-scale variables are dealt with diagnostically under the standard mass-flux formulation as carefully discussed in Section 7 of Yano (2014a). This diagnostic equation is given by Eq. (1.1) for a convective-scale variable, φ_i. That may be re-written as

$$\left(\frac{\partial}{\partial z} + \tilde{\epsilon}_{\varphi_i}\right) \varphi_i = \epsilon_i \tilde{\bar{\varphi}}_i, \tag{4.4}$$

where

$$\tilde{\bar{\varphi}}_i = \bar{\varphi} + \frac{\rho\sigma_i}{\epsilon_i M_i}\tilde{F}_i, \tag{4.5a}$$

and

$$\tilde{\epsilon}_{\varphi_i} = \epsilon_i - \frac{\rho\sigma_i}{M_i\varphi_i}\hat{F}_i. \tag{4.5b}$$

Here and hereafter, the subscript φ_i is added whenever it is necessary to indicate a definition depending on φ_i. The final terms in both of the expressions (4.5a) and (4.5b) are obtained by dividing the forcing term, F_i, into two arbitrary contributions:

$$F_i = \tilde{F}_i + \hat{F}_i. \tag{4.6}$$

The division can be made in any manner desired so as to obtain a more convenient analytic expression for the particular variable φ_i. The ideal division would be to make the parameter $\tilde{\epsilon}_{\varphi_i}$ a function of height only (with a possible extension to the case with additional dependence on φ_i), and for $\tilde{\bar{\varphi}}_i$ to have a simple closed expression (ideally independent of the convective-scale variables: see immediately below).

In general, $\tilde{\bar{\varphi}}_i$ *may* depend on other physical variables such as χ_i and $\bar{\chi}$ so that Eq. (4.5a) takes the form

$$\tilde{\bar{\varphi}}_i = \bar{\varphi} + \tilde{\mathcal{F}}_{L,i}(\chi_i, \bar{\chi}), \tag{4.7a}$$

where

$$\tilde{\mathcal{F}}_{L,i} = \frac{\rho\sigma_i}{\epsilon_i M_i}\tilde{F}_i. \tag{4.7b}$$

This possibility will be considered only later in Section 5.2. For now, however, $\tilde{\bar{\varphi}}_i$ is assumed to be a function of height only with no additional functional dependence.

Eq. (4.4) is readily solved, and the solution is

$$\varphi_i = \frac{1}{\tilde{\eta}_{\varphi_i}}\left[\varphi_{iB} + \int_{z_B}^{z} \epsilon_i \tilde{\eta}_{\varphi_i}\tilde{\bar{\varphi}}_i dz'\right], \tag{4.8}$$

where

$$\tilde{\eta}_{\varphi_i} = \exp\left[\int_{z_B}^{z} \tilde{\epsilon}_{\varphi_i} dz'\right]. \tag{4.9}$$

Note that $\tilde{\eta}_{\varphi_i} \neq \eta_i$ (compare Eqs. (1.4) and (4.9)) even when $\tilde{\epsilon}_{\varphi_i} = \epsilon_i$, unless a purely entraining plume is assumed. Keep in mind that this paper pursues a general formulation without this assumption. It is also convenient to introduce

$$\hat{\eta}_i = \exp\left[\int_{z_B}^{z} \epsilon_i dz'\right] \tag{4.10}$$

for later use.

4.2.2. Prognostic equations

A prognostic equation for a convective variable, φ_i, is obtained by taking a time derivative of Eq. (4.8). This procedure is consistent with the spirit of the bounded-derivative method (Kreiss, 1980; Browning et al., 1980): *i.e.*, when a balance condition (diagnostic relation) is assumed for a given variable, its prognostic equation is obtained by taking a time derivative of the given balance condition.

In order to proceed towards this direction, we first need to note that

$$\frac{\partial \tilde{\eta}_{\varphi_i}}{\partial t} = -\dot{\tilde{z}}_B \tilde{\epsilon}_{\varphi i,B}\tilde{\eta}_{\varphi_i} \tag{4.11a}$$

where

$$\dot{\tilde{z}}_B = \dot{z}_B - \frac{1}{\tilde{\epsilon}_{\varphi i,B}}\int_{z_B}^{z} \frac{\partial \tilde{\epsilon}_{\varphi_i}}{\partial t} dz'. \tag{4.11b}$$

Before taking the time derivative of Eq. (4.8), we first re-write it as

$$\tilde{\eta}_{\varphi_i}\varphi_i = \varphi_{iB} + \int_{z_B}^{z} \epsilon_i \tilde{\eta}_{\varphi_i}\tilde{\bar{\varphi}}_i dz'.$$

The time derivative of the left hand side is:

$$\frac{\partial}{\partial t}\tilde{\eta}_{\varphi_i}\varphi_i = \tilde{\eta}_{\varphi_i}\frac{\partial \varphi_i}{\partial t} - \dot{\tilde{z}}_B \tilde{\epsilon}_{\varphi i,B}\tilde{\eta}_{\varphi_i}\varphi_i.$$

The time derivative of the integral on the right hand side gives:

$$\frac{\partial}{\partial t}\int_{z_B}^{z}\epsilon_i\tilde{\eta}_{\varphi_i}\tilde{\tilde{\varphi}}_i dz' = \int_{z_B}^{z}\epsilon_i\tilde{\eta}_{\varphi_i}\left(\frac{\partial}{\partial t}-\dot{\tilde{z}}_B\tilde{\epsilon}_{\varphi i,B}\right)\tilde{\tilde{\varphi}}_i dz' - \dot{z}_B\epsilon_{iB}\tilde{\tilde{\varphi}}_{iB}$$

Putting these two expressions together, and simplifying the result with the help of Eqs. (4.8) and (4.11b) we obtain

$$\tilde{\eta}_{\varphi_i}\frac{\partial\varphi_i}{\partial t} = -\dot{z}_B\epsilon_{iB}\Delta\tilde{\varphi}_{iB} + \left[\frac{\partial}{\partial t}-\dot{z}_B\left(\frac{\rho\sigma_i}{M_i\varphi_i}\hat{F}_i\right)_B - \int_{z_B}^{z}\frac{\partial\tilde{\epsilon}_{\varphi_i}}{\partial t}dz'\right]\varphi_{iB} + \int_{z_B}^{z}\epsilon_i\tilde{\eta}_{\varphi_i}\frac{\partial\tilde{\tilde{\varphi}}_i}{\partial t}dz' \tag{4.12a}$$

where

$$\Delta\tilde{\varphi}_{i,B} = \tilde{\tilde{\varphi}}_{iB} - \varphi_{iB}. \tag{4.12b}$$

For further developments, we also need the following prognostic equation:

$$\frac{\partial\tilde{\tilde{\varphi}}_i}{\partial t} = \frac{1}{\rho}\left[\sum_j D_j(\varphi_j^D-\overline{\varphi}) + M_c\frac{\partial\overline{\varphi}}{\partial z}\right] + \left(\frac{\partial\tilde{\tilde{\varphi}}_i}{\partial t}\right)_L, \tag{4.13a}$$

where

$$\left(\frac{\partial\tilde{\tilde{\varphi}}_i}{\partial t}\right)_L = \left(\frac{\partial\overline{\varphi}}{\partial t}\right)_L + \frac{\partial}{\partial t}\left(\frac{\rho\sigma_i}{\epsilon_i M_i}\tilde{F}_i\right), \tag{4.13b}$$

which follows immediately from Eqs. (3.3) and (4.5a). In general, the second term on the right hand side of Eq. (4.13b) may depend on convective-scale variables, but it is assumed for now to depend only on the large-scale variables, with modifications to be considered later in Section 5.2.

The final result is obtained by substituting from Eqs. (4.12a) and (4.13a) into Eq. (4.3). In this process, an additional key step is to exchange the order of integration, *e.g.*,

$$\int_{z_B}^{z_{Ti}}\frac{1}{\tilde{\eta}_{\varphi_i}}\frac{\partial f_i}{\partial\varphi_i}\int_{z_B}^{z}\epsilon_i\tilde{\eta}_{\varphi_i}\frac{\partial\tilde{\tilde{\varphi}}_i}{\partial t}dz'dz = \int_{z_B}^{z_{Ti}}\epsilon_i\tilde{\eta}_{\varphi_i}\frac{\partial\tilde{\tilde{\varphi}}_i}{\partial t}\int_{z}^{z_{Ti}}\frac{1}{\tilde{\eta}_{\varphi_i}}\frac{\partial f_i}{\partial\varphi_i}dz'dz. \tag{4.14}$$

We finally obtain

$$\frac{\partial I_i}{\partial t} = \sum_j K_{ij}M_{jB} + \left(\frac{\partial I_i}{\partial t}\right)_L \tag{4.15}$$

where

$$K_{ij} = \int_{z_B}^{z_{Ti}}\frac{1}{\rho}\tilde{a}_{\varphi_i}\epsilon_i\tilde{\eta}_{\varphi_i}\eta_j\left[\delta_j(\varphi_j^D-\overline{\varphi})+\frac{\partial\overline{\varphi}}{\partial z}\right]dz \tag{4.16a}$$

$$\left(\frac{\partial I_i}{\partial t}\right)_L = \int_{z_B}^{z_{Ti}}\epsilon_i\tilde{a}_{\varphi_i}\tilde{\eta}_{\varphi_i}\left(\frac{\partial\overline{\varphi}}{\partial t}\right)_L dz + \int_{z_B}^{z_{Ti}}\epsilon_i\tilde{a}_{\varphi_i}\tilde{\eta}_{\varphi_i}\frac{\partial}{\partial t}\left(\frac{\rho\sigma}{\epsilon_i M_i}\tilde{F}_i\right)dz + \left[\tilde{a}_{\varphi i,B}\frac{\partial}{\partial t}-\tilde{c}_{\varphi_i}\right]\varphi_{iB} + \dot{z}_{Ti}G_{Ti} + \dot{z}_B G_{Bi} \tag{4.16b}$$

and

$$G_{Ti} = f_{Ti} \tag{4.17a}$$

$$G_{Bi} = -\tilde{a}_{\varphi i,B}\left[\epsilon_{iB}\Delta\tilde{\varphi}_{iB} + \left(\frac{\rho\sigma_i}{M_i\varphi_i}\hat{F}_i\right)_B\varphi_{iB}\right] - f_{Bi}. \tag{4.17b}$$

The coefficients in Eq. (4.16) are defined by:

$$\tilde{a}_{\varphi_i} = \int_{z}^{z_{Ti}}\frac{1}{\tilde{\eta}_{\varphi_i}}\frac{\partial f_i}{\partial\varphi_i}dz' \tag{4.18a}$$

$$\tilde{c}_{\varphi_i} = \int_{z_B}^{z_{Ti}}\tilde{a}_{\varphi_i}\frac{\partial\tilde{\epsilon}_{\varphi_i}}{\partial t}dz. \tag{4.18b}$$

4.3. Two-part vertical integral

In vertically integrating convective-scale variables, it often becomes convenient to divide the integration range into two parts, in order to adopt a different form for the integrand when crossing the condensation level, z_{ci}. Thus,

$$I_i = \int_{z_B}^{z_{ci}}f_1(\varphi_i)dz + \int_{z_{ci}}^{z_{Ti}}f_2(\varphi_i)dz. \tag{4.19}$$

We also denote $f_1(\varphi_i) = f_{1i}$ and $f_2(\varphi_i) = f_{2i}$. We assume that the two functions are continuous over the interface, $z = z_{ci}$, so that

$$f_{1i}(z = z_{ci}) = f_{2i}(z = z_{ci}). \tag{4.20}$$

Even with this separation of the integral, the derivation of the budget for I_i proceeds in a similar manner as before. The starting point is

$$\frac{\partial I_i}{\partial t} = \int_{z_B}^{z_{ci}} \frac{\partial f_{1i}}{\partial \varphi_i} \frac{\partial \varphi_i}{\partial t} dz + \int_{z_{ci}}^{z_{Ti}} \frac{\partial f_{2i}}{\partial \varphi_i} \frac{\partial \varphi_i}{\partial t} dz + \dot{z}_{Ti} f_{2i}(z = z_{Ti}) - \dot{z}_B f_{1i}(z = z_B). \tag{4.21}$$

Here the assumption of continuity (Eq. (4.20)) of the two functions over $z = z_{ci}$ ensures that contributions at the integral boundary $z = z_{ci}$ cancel out. However, some care is required in changing the order of the integrals. In place of Eq. (4.14) above, we need to use:

$$\int_{z_B}^{z_{ci}} \frac{1}{\tilde{\eta}_{\varphi_i}} \frac{\partial f_{1i}}{\partial \varphi_i} \int_{z_B}^{z} \epsilon_i \tilde{\eta}_{\varphi_i} \frac{\partial \tilde{\overline{\varphi}}_i}{\partial t} dz' dz = \int_{z_B}^{z_{ci}} \epsilon_i \tilde{\eta}_{\varphi_i} \frac{\partial \tilde{\overline{\varphi}}_i}{\partial t} \int_{z}^{z_{ci}} \frac{1}{\tilde{\eta}_{\varphi_i}} \frac{\partial f_{1i}}{\partial \varphi_i} dz' dz, \tag{4.22a}$$

$$\int_{z_{ci}}^{z_{Ti}} \frac{1}{\tilde{\eta}_{\varphi_i}} \frac{\partial f_{2i}}{\partial \varphi_i} \int_{z_B}^{z} \epsilon_i \tilde{\eta}_{\varphi_i} \frac{\partial \tilde{\overline{\varphi}}_i}{\partial t} dz' dz = \int_{z_B}^{z_{Ti}} \epsilon_i \tilde{\eta}_{\varphi_i} \frac{\partial \tilde{\overline{\varphi}}_i}{\partial t} \int_{\max(z, z_{ci})}^{z_{Ti}} \frac{1}{\tilde{\eta}_{\varphi_i}} \frac{\partial f_{2i}}{\partial \varphi_i} dz' dz. \tag{4.22b}$$

Following a similar reduction, we arrive at the same general form as Eq. (4.15), although with different definitions for the terms:

$$K_{ij,c} = \int_{z_B}^{z_{ci}} \frac{\epsilon_i \tilde{a}_{1,\varphi_i} \tilde{\eta}_{\varphi_i}}{\rho} \eta_j \left[\delta_j (\varphi_j^D - \overline{\varphi}) + \frac{\partial \overline{\varphi}}{\partial z} \right] dz + \int_{z_B}^{z_{Ti}} \frac{\epsilon_i \tilde{a}_{2,\varphi_i} \tilde{\eta}_{\varphi_i}}{\rho} \eta_j \left[\delta_j (\varphi_j^D - \overline{\varphi}) + \frac{\partial \overline{\varphi}}{\partial z} \right] dz \tag{4.23a}$$

$$\begin{aligned} \left(\frac{\partial I_i}{\partial t} \right)_{L,c} = & \left[(\tilde{a}_{1,\varphi_i B} + \tilde{a}_{2,\varphi_i B}) \frac{\partial}{\partial t} - (\tilde{c}_{1,\varphi_i} + \tilde{c}_{2,\varphi_i}) \right] \varphi_{iB} + \int_{z_B}^{z_{ci}} \epsilon_i \tilde{a}_{1,\varphi_i} \tilde{\eta}_{\varphi_i} \left(\frac{\partial \tilde{\overline{\varphi}}_i}{\partial t} \right)_L dz \\ & + \int_{z_B}^{z_{Ti}} \epsilon_i \tilde{a}_{2,\varphi_i} \tilde{\eta}_{\varphi_i} \left(\frac{\partial \tilde{\overline{\varphi}}_i}{\partial t} \right)_L dz + \dot{z}_{Ti} G_{Ti} + \dot{z}_B G_{Bi} \end{aligned} \tag{4.23b}$$

with

$$G_{Ti} = f_{2Ti} \tag{4.24a}$$

$$G_{Bi} = -(\tilde{a}_{1,\varphi_i B} + \tilde{a}_{2,\varphi_i B}) \left[\epsilon_{iB} \Delta \tilde{\varphi}_{iB} + \left(\frac{\rho \sigma_i}{M_i \varphi_i} \hat{F}_i \right)_B \varphi_{iB} \right] - f_{1Bi}. \tag{4.24b}$$

For later ease of reference, a further subscript c has been added to the definitions in Eq. (4.23) in order to indicate that these terms arise from an integrand with dependence on convective-scale variables.

The new definitions of the coefficients are

$$\tilde{a}_{1,\varphi_i} = \int_{z}^{z_{ci}} \frac{1}{\tilde{\eta}_{\varphi_i}} \frac{\partial f_{1i}}{\partial \varphi_i} dz' \tag{4.25a}$$

$$\tilde{a}_{2,\varphi_i} = \int_{\max(z, z_{ci})}^{z_{Ti}} \frac{1}{\tilde{\eta}_{\varphi_i}} \frac{\partial f_{2i}}{\partial \varphi_i} dz' \tag{4.25b}$$

$$\tilde{c}_{1,\varphi_i} = \int_{z_B}^{z_{ci}} \tilde{a}_{1,\varphi_i} \frac{\partial \tilde{\epsilon}_{\varphi_i}}{\partial t} dz \tag{4.25c}$$

$$\tilde{c}_{2,\varphi_i} = \int_{z_B}^{z_{Ti}} \tilde{a}_{2,\varphi_i} \frac{\partial \tilde{\epsilon}_{\varphi_i}}{\partial t} dz. \tag{4.25d}$$

It is easy to check that the results of this subsection reduce to those of Section 4.2 by setting $z_{ci} = z_{Ti}$ or $z_{ci} = z_B$.

4.4. Convective-scale moisture closure

As a variant of the standard moisture closure presented in Section 3.2, we can consider a closure based on a quasi-equilibrium of the column-integrated *convective-scale* water vapour. Such a possibility is identified by a cloud-resolving model analysis by Yano et al. (2012). In particular, their Fig. 10(a) demonstrates that this quantity satisfies the quasi-equilibrium. Thus, we may set $f_i = \rho q_i$.

Since the moisture is not a conserved quantity above the condensation level, it is convenient to look for an alternative expression, which may be found as Eq. (56) in AS:

$$q_i \simeq \overline{q}^* + \frac{\gamma}{1+\gamma}\left(\frac{h_i - \overline{h}^*}{L_v}\right), \tag{4.26}$$

where γ is defined by

$$\gamma = \frac{L_v}{C_p}\left(\frac{\partial \overline{q}^*}{\partial \overline{T}}\right)_{\overline{p}}, \tag{4.27a}$$

and

$$h = C_p T + L_v q + gz \tag{4.27b}$$

is the moist static energy with the latent heat of condensation, L_v. The above expression (4.26) is obtained from a Taylor expansion of q_i about $\overline{q}^* = q^*(\overline{T}, \overline{p})$. By applying the same procedure for a infinitesimal change in $\overline{q}^*$ in time, it is straightforward to obtain

$$\frac{\partial \overline{q}^*}{\partial t} = \frac{\gamma}{1+\gamma}\frac{1}{L_v}\frac{\partial \overline{h}^*}{\partial t}. \tag{4.28a}$$

By invoking this relation, we further find that

$$\frac{\partial q_i}{\partial t} = \frac{\gamma}{1+\gamma}\frac{1}{L_v}\frac{\partial h_i}{\partial t}. \tag{4.28b}$$

Thus, we set

$$f_{1i} = \rho q_i \tag{4.29a}$$

$$f_{2i} = \rho\left[\overline{q}^* + \frac{\gamma}{1+\gamma}\frac{h_i - \overline{h}^*}{L_v}\right]. \tag{4.29b}$$

In applying the general results (4.23)–(4.25) to this closure, we need to keep in mind that the dependent variable, φ_i, changes when crossing the condensation level. Apart from this caveat, the application is relatively straightforward as q_i and h_i are conserved quantities below and above the condensation level respectively. Although Eq. (4.29b) contains the two large-scale variables $\overline{q}^*$ and $\overline{h}^*$ in its definition, they do not contribute in the following due to the constraint of Eq. (4.28a).

The final results are:

$$K_{ij,c} = \int_{z_B}^{z_{ci}} \frac{\epsilon_i}{\rho}\hat{\eta}_i\eta_j\left[\delta_j(q_j - \overline{q}) + \frac{\partial \overline{q}}{\partial z}\right]\int_z^{z_{ci}} \frac{\rho}{\hat{\eta}_i}dz'dz + \int_{z_B}^{z_{Ti}} \frac{\epsilon_i}{\rho}\hat{\eta}_i\eta_j\left[\delta_j(h_j - \overline{h}) + \frac{\partial \overline{h}}{\partial z}\right]\int_{\max(z,z_{ci})}^{z_{Ti}} \frac{\rho}{\hat{\eta}_i L_v}\frac{\gamma}{1+\gamma}dz'dz \tag{4.30a}$$

$$\begin{aligned}\left(\frac{\partial I_i}{\partial t}\right)_{L,c} = {} & \frac{\partial q_{iB}}{\partial t}\int_{z_B}^{z_{ci}} \frac{\rho}{\hat{\eta}_i}dz + \frac{\partial h_{iB}}{\partial t}\int_{z_{ci}}^{z_{Ti}} \frac{\rho}{\hat{\eta}_i L_v}\frac{\gamma}{1+\gamma}dz + \int_{z_B}^{z_{ci}} \epsilon_i\hat{\eta}_i\left(\frac{\partial \overline{q}}{\partial t}\right)_L\int_z^{z_{ci}} \frac{\rho}{\hat{\eta}_i}dz'dz \\ & + \int_{z_B}^{z_{Ti}} \epsilon_i\hat{\eta}_i\left(\frac{\partial \overline{h}}{\partial t}\right)_L\int_{\max(z,z_{ci})}^{z_{Ti}} \frac{\rho}{\hat{\eta}_i L_v}\frac{\gamma}{1+\gamma}dz'dz\end{aligned} \tag{4.30b}$$

This is an attractive alternative closure because unlike classical closures based on large-scale variables as considered in Section 3, it does not lose the predictability of the large-scale variable (*e.g.*, column-integrated moisture, CAPE) that is chosen to be in quasi-equilibrium.

5. The mixed closure: $f = f(\varphi_i, \overline{\varphi})$

5.1. General formulation

In general, the vertical integral may depend on both large-scale variables, $\overline{\varphi}$ and convective-scale variables, φ_i. Under this generalization, the vertical integral may be defined by

$$I_i = \int_{z_B}^{z_{Ti}} f(\varphi_i, \overline{\varphi})dz. \tag{5.1}$$

Here, we take the same notation for the convective-scale variable as for the large-scale variable solely for the sake of the simplicity. In general, the two could be different variables. As already discussed in the last section, more generally, the vertical integral may also be separated into two parts:

$$I_i = \int_{z_B}^{z_{ci}} f_1(\varphi_i, \overline{\varphi})dz + \int_{z_{ci}}^{z_{Ti}} f_2(\varphi_i, \overline{\varphi})dz. \tag{5.2}$$

In this section, we consider closures based on Eq. (5.2). As before, we denote $f_1(\varphi_i, \overline{\varphi}) = f_{1i}$ and $f_2(\varphi_i, \overline{\varphi}) = f_{2i}$.

Here, the time derivatives can be expanded as, for example for f_{1i},

$$\frac{\partial f_{1i}}{\partial t} = \frac{\partial f_{1i}}{\partial \varphi_i}\frac{\partial \varphi_i}{\partial t} + \frac{\partial f_{1i}}{\partial \overline{\varphi}}\frac{\partial \overline{\varphi}}{\partial t} \tag{5.3}$$

with a similar expression for f_{2i}. From this expression, it is seen that the present case can be considered a linear combination of the cases considered in the previous two sections. Thus, we write

$$K_{ij} = K_{ij,L} + K_{ij,c} \tag{5.4a}$$

$$\left(\frac{\partial I_i}{\partial t}\right)_L = \left(\frac{\partial I_i}{\partial t}\right)_{L,L} + \left(\frac{\partial I_i}{\partial t}\right)_{L,c} \tag{5.4b}$$

The terms $K_{ij,c}$ and $(\partial I_i/\partial t)_{L,c}$ are defined by Eqs. (4.23a) and (4.23b) respectively. The terms $K_{ij,L}$ and $(\partial I_i/\partial t)_{L,L}$ are easily obtained as modest generalizations of Eqs. (3.8b) and (3.4b) respectively, and are:

$$K_{ij,L} = \int_{z_B}^{z_{ci}} \frac{\eta_j}{\rho}\frac{\partial f_{1i}}{\partial \overline{\varphi}}\left[\delta_j(\varphi_j - \overline{\varphi}) + \frac{\partial \overline{\varphi}}{\partial z}\right]dz + \int_{z_{ci}}^{z_{Ti}} \frac{\eta_j}{\rho}\frac{\partial f_{2i}}{\partial \overline{\varphi}}\left[\delta_j(\varphi_j - \overline{\varphi}) + \frac{\partial \overline{\varphi}}{\partial z}\right]dz \tag{5.5a}$$

$$\left(\frac{\partial I_i}{\partial t}\right)_{L,L} = \int_{z_B}^{z_{ci}} \frac{\partial f_{1i}}{\partial \overline{\varphi}}\left(\frac{\partial \overline{\varphi}}{\partial t}\right)_L dz + \int_{z_{ci}}^{z_{Ti}} \frac{\partial f_{2i}}{\partial \overline{\varphi}}\left(\frac{\partial \overline{\varphi}}{\partial t}\right)_L dz. \tag{5.5b}$$

5.2. Dependence of forcing on convective-scale variables

As discussed in Section 4.2.1, for the derivations so far we have assumed that the pseudo large-scale tendency, $(\partial\tilde{\overline{\varphi}}_i/\partial t)_L$, can be treated as a part of the large-scale forcing. In general, this is not the case, and a dependence on convective-scale variables may be present, as indicated by Eq. (4.7). This subsection considers further modifications of the closure formulation in order to accommodate this generalization.

Moreover, the tendency of the pseudo-fractional entrainment rate, $\tilde{\epsilon}_i$, could also depend on convective-scale variables, as indicated by Eq. (4.5b). Such a further generalization is in fact straightforward. However, with the convective buoyancy, b_i, as a specific example in mind, it turns out that only the generalized treatment of $(\partial\tilde{\overline{\varphi}}_i/\partial t)_L$ is necessary. Thus we do not explicitly consider a generalized treatment of $\partial\tilde{\epsilon}_{\varphi_i}/\partial t$ in this paper since it would serve only to complicate the final results presented.

The generalization means that $(\partial\tilde{\overline{\varphi}}_i/\partial t)_L$ does not solely represent a large-scale tendency, but also contains some convective contributions that stem from the second term on the right hand side of Eq. (4.13b). That term should therefore be separated into the contributions associated with the large scale and convection:

$$\frac{\partial}{\partial t}\left(\frac{\rho\sigma_i}{\epsilon_i M_i}\tilde{F}_i\right) = \left(\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial t}\right)_L + \left(\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial t}\right)_c \tag{5.6}$$

by referring to the definition (4.7b). As a result, Eqs. (4.13a) and (4.13b) read:

$$\frac{\partial\tilde{\overline{\varphi}}_i}{\partial t} = \frac{1}{\rho}\left[\sum_j D_j(\varphi_j^D - \overline{\varphi}) + M_c\frac{\partial \overline{\varphi}}{\partial z}\right] + \left(\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial t}\right)_c + \left(\frac{\partial\tilde{\overline{\varphi}}_i}{\partial t}\right)_L, \tag{5.7a}$$

and

$$\left(\frac{\partial\tilde{\overline{\varphi}}_i}{\partial t}\right)_L = \left(\frac{\partial \overline{\varphi}}{\partial t}\right)_L + \left(\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial t}\right)_L, \tag{5.7b}$$

respectively.

A new type of term (the second term on the right hand side) appears in Eq. (5.7a) and leads to a corresponding new type of contribution, $\Delta(\partial I_i/\partial t)_L$, in the budget for I_i, which may be written as

$$\Delta\left(\frac{\partial I_i}{\partial t}\right)_L = \int_{z_B}^{z_{ci}} \epsilon_i\tilde{\alpha}_{1,\varphi_i}\tilde{\eta}_{\varphi_i}\left(\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial t}\right)_c dz + \int_{z_B}^{z_{Ti}} \epsilon_i\tilde{\alpha}_{2,\varphi_i}\tilde{\eta}_{\varphi_i}\left(\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial t}\right)_c dz. \tag{5.8}$$

by analogy with the second and third terms on the right hand side of Eq. (4.23b). In order to express this contribution in the form of the other convective terms within Eq. (4.21) we begin by writing:

$$\left(\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial t}\right)_c = \frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial \chi_i}\frac{\partial \chi_i}{\partial t} + \frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial \overline{\chi}}\left(\frac{\partial \overline{\chi}}{\partial t}\right)_c. \tag{5.9}$$

Here, the tendencies $\partial\chi_i/\partial t$ and $(\partial\overline{\chi}/\partial t)_c$ can be expressed using the equivalent equations to Eqs. (4.12a) and (3.3) for the convective-scale and large-scale variables χ_i and $\overline{\chi}$, respectively. Note that the total tendency is considered for χ_i, whereas only the convective tendency is considered for $\overline{\chi}$ so that the necessary new contributions are properly accounted for. In the following, we will assume that χ is a conserved variable so that no further forcing terms applying to χ must be added. This assumption serves only to simplify the final expression, but a further generalization or the inclusion of a further dependence of $\tilde{\mathcal{F}}_{\varphi_i}$ on the mass-flux profile, η_i, is also possible if desired.

With these assumptions, the tendency for χ_i reads

$$\frac{\partial \chi_i}{\partial t} = \frac{1}{\hat{\eta}_i}\left[-\dot{z}_B\epsilon_{iB}\Delta\chi_{iB} + \frac{\partial \chi_{iB}}{\partial t} + \int_{z_B}^{z}\epsilon_i\hat{\eta}_i\frac{\partial\overline{\chi}}{\partial t}dz'\right]. \tag{5.10}$$

Recall that $\hat{\eta}_i$ is defined by Eq. (4.10). This tendency may furthermore be divided (somewhat arbitrarily) into the term containing the convective tendency for $\overline{\chi}$ and the remaining terms, adding the subscripts c and L, respectively:

$$\left(\frac{\partial \chi_i}{\partial t}\right)_c = \frac{1}{\hat{\eta}_i}\int_{z_B}^{z}\epsilon_i\hat{\eta}_i\left(\frac{\partial\overline{\chi}}{\partial t}\right)_c dz' \tag{5.11a}$$

$$\left(\frac{\partial \chi_i}{\partial t}\right)_L = \frac{1}{\hat{\eta}_i}\left[-\dot{z}_B\epsilon_{iB}\Delta\chi_{iB} + \frac{\partial \chi_{iB}}{\partial t} + \int_{z_B}^{z}\epsilon_i\hat{\eta}_i\left(\frac{\partial\overline{\chi}}{\partial t}\right)_L dz'\right]. \tag{5.11b}$$

Accordingly, the tendency, $(\partial\tilde{\mathcal{F}}_{\varphi_i}/\partial t)_c$, may also be divided into the two major contributions:

$$\left(\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial t}\right)_c = \left(\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial t}\right)_{c,c} + \left(\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial t}\right)_{c,L} \tag{5.12}$$

where

$$\left(\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial t}\right)_{c,c} = \frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial \chi_i}\left(\frac{\partial \chi_i}{\partial t}\right)_c + \frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial \overline{\chi}}\left(\frac{\partial \overline{\chi}}{\partial t}\right)_c, \tag{5.13a}$$

$$\left(\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial t}\right)_{c,L} = \frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial \chi_i}\left(\frac{\partial \chi_i}{\partial t}\right)_L. \tag{5.13b}$$

Substituting into Eq. (5.13a) using Eq. (5.11a) for $(\partial\chi_i/\partial t)_c$ and Eq. (3.5a) for $(\partial\overline{\chi}/\partial t)_c$, and reversing the order of integration for the double integrals, the required correction due to $(\partial\tilde{\mathcal{F}}_{\varphi_i}/\partial t)_{c,c}$ can be reduced to have the same form as the other convective terms in Eq. (4.15):

$$\Delta\left(\frac{\partial I_i}{\partial t}\right)_{L,c} = \sum_j \Delta K_{ij}M_{jB} \tag{5.14a}$$

with the correction to the interaction matrix given by

$$\begin{aligned}\Delta K_{ij} = &\int_{z_B}^{z_{ci}}\frac{\epsilon_i}{\rho}\left[\tilde{\eta}_{\varphi_i}\tilde{a}_{1,\varphi_i}\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial \overline{\chi}} + \hat{\eta}_i\int_{z}^{z_{ci}}\epsilon_i\tilde{a}_{1,\varphi_i}\frac{\tilde{\eta}_i}{\hat{\eta}_i}\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial \chi_i}dz'\right]\eta_j\left[\delta_j(\chi_j^D-\overline{\chi}) + \frac{\partial\overline{\chi}}{\partial z}\right]dz\\ &+\int_{z_B}^{z_{Ti}}\frac{\epsilon_i}{\rho}\left[\tilde{\eta}_i\tilde{a}_{2,\varphi_i}\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial \overline{\chi}} + \hat{\eta}_i\int_{\max(z,z_{ci})}^{z_{Ti}}\epsilon_i\tilde{a}_{2,\varphi_i}\frac{\tilde{\eta}_{\varphi_i}}{\hat{\eta}_i}\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial \chi_i}dz'\right]\eta_j\left[\delta_j(\chi_j^D-\overline{\chi}) + \frac{\partial\overline{\chi}}{\partial z}\right]dz.\end{aligned} \tag{5.14b}$$

The term, $(\partial\tilde{\mathcal{F}}_{\varphi_i}/\partial t)_{c,L}$, on the other hand, contributes as an additional term for the large-scale forcing:

$$\begin{aligned}\Delta\left(\frac{\partial I_i}{\partial t}\right)_L = &\dot{z}_B\Delta G_{Bi,\chi} + \int_{z_B}^{z_{ci}}\epsilon_i\tilde{a}_{1,\varphi_i}\frac{\tilde{\eta}_{\varphi_i}}{\hat{\eta}_i}\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial \chi_i}\left[\frac{\partial \chi_{iB}}{\partial t} + \int_{z_B}^{z}\epsilon_i\hat{\eta}_i\left(\frac{\partial\overline{\chi}}{\partial t}\right)_L dz'\right]dz\\ &+\int_{z_B}^{z_{Ti}}\epsilon_i\tilde{a}_{2,\varphi_i}\frac{\tilde{\eta}_{\varphi_i}}{\hat{\eta}_i}\frac{\partial \tilde{\mathcal{F}}_{\varphi_i}}{\partial \chi_i}\left[\frac{\partial \chi_{iB}}{\partial t} + \int_{z_B}^{z}\epsilon_i\hat{\eta}_i\left(\frac{\partial\overline{\chi}}{\partial t}\right)_L dz'\right]dz\end{aligned} \tag{5.14c}$$

Here, this term includes contributions from changes of the convection base with

$$\Delta G_{Bi,\chi} = -\epsilon_{iB}\Delta\chi_{iB}\left[\int_{z_B}^{z_{ci}} \epsilon_i\tilde{a}_{1,\varphi_i}\frac{\tilde{\eta}_{\varphi_i}}{\hat{\eta}_i}\frac{\partial\tilde{\mathcal{F}}_{\varphi_i}}{\partial\chi_i}dz + \int_{z_B}^{z_{Ti}} \epsilon_i\tilde{a}_{2,\varphi_i}\frac{\tilde{\eta}_{\varphi_i}}{\hat{\eta}_i}\frac{\partial\tilde{\mathcal{F}}_{\varphi_i}}{\partial\chi_i}dz\right] \tag{5.14d}$$

Consequently, the interaction matrix and the large-scale forcing are given by

$$K_{ij} = K_{ij,L} + K_{ij,c} + \Delta K_{ij} \tag{5.15a}$$

$$\left(\frac{\partial I_i}{\partial t}\right)_L = \left(\frac{\partial I_i}{\partial t}\right)_{L,L} + \left(\frac{\partial I_i}{\partial t}\right)_{L,c} + \Delta\left(\frac{\partial I_i}{\partial t}\right)_L \tag{5.15b}$$

as generalizations from Eqs. (5.4a) and (5.4b) respectively.

5.3. Dilute CAPE-based closure (based on convective-scale buoyancy)

As an alternative to the standard CAPE, we may wish to consider b_i, the actual buoyancy felt by the ith convection type, so that

$$b_i = \rho\alpha(s_{v,i} - \bar{s}_v). \tag{5.16a}$$

The generalization of CAPE is then defined by

$$\mathrm{CAPE}_i = \int_{z_B}^{z_{Ti}} b_i dz \tag{5.16b}$$

with the subscript i designating the convection type. This definition is sometimes referred to as "dilute CAPE".

In practice, the virtual static energy, s_v, is not a convenient variable to work with above the condensation level, z_{ci}, because it is no longer conserved. For this reason, above the condensation level, we re-write the convective buoyancy, b_i, by invoking a relation

$$b_i = \rho\beta(h_i - \bar{h}^*) + \rho\alpha\varepsilon L_v[\hat{\delta}(\bar{q}^* - \bar{q}) - l_i], \tag{5.17}$$

as given by Eq. (B3) of AS. Here,

$$\beta = \alpha\left(\frac{1+\gamma\varepsilon\hat{\delta}}{1+\gamma}\right), \tag{5.18a}$$

with

$$\varepsilon = \frac{C_p\bar{T}}{L_v}. \tag{5.18b}$$

while α was already defined by Eq. (3.18).

Thus, the dilute CAPE reduces to a vertical integral of the form of Eq. (5.2) with the two functions defined by

$$f_{1i} = \rho\alpha(s_{v,i} - \bar{s}_v), \tag{5.19a}$$

$$f_{2i} = \rho\beta(h_i - \bar{h}^*) + \rho\alpha\varepsilon L_v[\hat{\delta}(\bar{q}^* - \bar{q}) - l_i]. \tag{5.19b}$$

Here, note $\rho\alpha\varepsilon L_v = g/(1+\hat{\delta}\bar{q})$. In the following, the variables, α and β, associated with the large-scale virtual temperature profile, $\bar{T}_v$, are treated as constant with time, as in Section 3.3.

The temporal tendency of the dilute CAPE is given by

$$\frac{\partial\mathrm{CAPE}_i}{\partial t} = \int_{z_B}^{z_{ci}}\left[\frac{\partial f_{1i}}{\partial s_{vi}}\frac{\partial s_{vi}}{\partial t} + \frac{\partial f_{1i}}{\partial\bar{s}_v}\frac{\partial\bar{s}_v}{\partial t}\right]dz + \int_{z_{ci}}^{z_{Ti}}\left[\frac{\partial f_{2i}}{\partial h_i}\frac{\partial h_i}{\partial t} + \frac{\partial f_{2i}}{\partial l_i}\frac{\partial l_i}{\partial t} + \frac{\partial f_{2i}}{\partial\bar{h}^*}\frac{\partial\bar{h}^*}{\partial t} + \frac{\partial f_{2i}}{\partial\bar{q}^*}\frac{\partial\bar{q}^*}{\partial t} + \frac{\partial f_{2i}}{\partial\bar{q}}\frac{\partial\bar{q}}{\partial t}\right]dz$$
$$-\dot{z}_B f_{1Bi} + \dot{z}_{Ti} f_{2Ti}. \tag{5.20}$$

The forms of the functional derivatives are straightforward to derive from the definitions of f_{1i} and f_{2i}. In order to simplify the result it is useful to invoke Eq. (B9) of AS, which reads

$$-\rho\alpha\frac{\partial\bar{s}_v}{\partial t} = \rho\beta\frac{\partial\bar{h}^*}{\partial t} + \rho\alpha\varepsilon L_v\hat{\delta}\frac{\partial}{\partial t}(\bar{q}^* - \bar{q})$$

and from which it follows that:

$$\frac{\partial f_{2i}}{\partial\bar{h}^*}\frac{\partial\bar{h}^*}{\partial t} + \frac{\partial f_{2i}}{\partial\bar{q}^*}\frac{\partial\bar{q}^*}{\partial t} + \frac{\partial f_{2i}}{\partial\bar{q}}\frac{\partial\bar{q}}{\partial t} = \frac{\partial f_{1i}}{\partial\bar{s}_v}\frac{\partial\bar{s}_v}{\partial t}.$$

This relation allows us to simplify Eq. (5.20) to

$$\frac{\partial \mathrm{CAPE}_i}{\partial t} = \int_{z_B}^{z_{ci}} \frac{\partial f_{1i}}{\partial s_{vi}} \frac{\partial s_{vi}}{\partial t} dz + \int_{z_{ci}}^{z_{Ti}} \left[\frac{\partial f_{2i}}{\partial h_i} \frac{\partial h_i}{\partial t} + \frac{\partial f_{2i}}{\partial l_i} \frac{\partial l_i}{\partial t} \right] dz + \int_{z_B}^{z_{Ti}} \frac{\partial f_{1i}}{\partial \bar{s}_v} \frac{\partial \bar{s}_v}{\partial t} dz - \dot{z}_B f_{1Bi} + \dot{z}_{Ti} f_{2Ti}. \tag{5.21}$$

In order to follow the full recipe of Section 5.2, we now need to consider further the treatment of forcing terms for the convective-scale variables. The moist static energy, h_i, is the simplest since this is conserved, and so the associated entrainment, $\tilde{\epsilon}_{h_i} = \epsilon_i$, the hat-forcing, $\hat{F}_{h_i} = 0$ and $\tilde{c}_{2,h_i} = 0$. For the virtual static energy, s_{vi}, there may be a forcing due to evaporation and we partition this as being an effective modification of s_{vi} rather than as an effective entrainment. Thus, the associated entrainment, $\tilde{\epsilon}_{s_{vi}} = \epsilon_i$, the hat-forcing $\hat{F}_{s_{vi}} = 0$ and $\tilde{c}_{1,s_{vi}} = 0$ while the profile of s_{vi} itself is determined from Eq. (4.4) as

$$\left(\frac{\partial}{\partial z} + \epsilon_i \right) s_{vi} = \epsilon_i \tilde{\bar{s}}_{vi}, \tag{5.22a}$$

where

$$\tilde{\bar{s}}_{vi} = \bar{s}_v - L_v(1 - \varepsilon\hat{\delta}) \frac{\rho\sigma_i}{\epsilon_i M_i} e_i. \tag{5.22b}$$

Here, e_i is the evaporation rate from the ith convective type, and note that AS assume $e_i = 0$.

A major additional hidden contribution from the convective-scale arises from a term involving $(\partial \tilde{\bar{l}}_i / \partial t)_L$. A closer look at the convective-scale cloud-water budget is required in order to obtain an explicit form for this term. This is facilitated by examining the convective total-water budget. The only sink term for convective total-water, q_{ti}, is the precipitation, r_i. Thus,

$$\left(\frac{\partial}{\partial z} + \epsilon_i \right) q_{ti} = \epsilon_i \bar{q}_t - \frac{\rho\sigma_i}{\epsilon_i M_i} r_i, \tag{5.23a}$$

where $q_{ti} = q_i + l_i$ and $\bar{q}_t = \bar{q} + \bar{l}$.

We consider a division for the precipitation rate, r_i, of the ith convective type into two contributions by setting

$$\frac{\rho\sigma_i}{\epsilon_i M_i} r_i = c_0 l_i + \frac{\rho\sigma_i}{M_i} \tilde{r}_i. \tag{5.23b}$$

Thus, the precipitation is treated as being potentially an effective entrainment, potentially as an external forcing modifying q_{ti}, and also potentially as a combination of the two. AS took the first of these options and set c_0 to be a constant as well as $\tilde{r}_i = 0$. The additional term $\tilde{r}_i \neq 0$ is introduced here to provide a possible further freedom for the convective precipitation rate formulation. Rewriting Eq. (5.23a) as an equation for the convective cloud-water l_i, it takes the form

$$\left(\frac{\partial}{\partial z} + \tilde{\epsilon}_{l_i} \right) l_i = \epsilon_i \tilde{\bar{l}}_i, \tag{5.24}$$

where

$$\tilde{\epsilon}_{l_i} = \epsilon_i + c_0 \tag{5.25a}$$

and

$$\tilde{\bar{l}}_i = \bar{q} + \bar{l} - \left(\frac{1}{\epsilon_i} \frac{\partial}{\partial z} + 1 \right) q_i - \frac{\rho\sigma_i}{\epsilon_i M_i} \tilde{r}_i. \tag{5.25b}$$

Partitioning the forcing in this way means that, recalling Eq. (4.9),

$$\tilde{\eta}_{\varphi_i} = \hat{\eta}_i \tag{5.26a}$$

for $\varphi_i = s_{vi}, h_i$, but for l_i we have

$$\tilde{\eta}_{l_i} = \exp\left[\int_{z_B}^{z} (\epsilon_i + c_0) dz' \right]. \tag{5.26b}$$

According to Eq. (5.25b) above, $\tilde{\bar{l}}_i$ depends on $\bar{q}$ and q_i. Thus the tendencies in these variables must be taken into account in computing the tendency for $\tilde{\bar{l}}_i$. Specifically, the recipe of Section 5.2 requires that $(\partial \tilde{\mathcal{F}}_{l_i} / \partial t)_c$ must be evaluated, which is given by

$$\left(\frac{\partial \tilde{\mathcal{F}}_{l_i}}{\partial t} \right)_c = \frac{\partial}{\partial t} \left[\bar{q} - \left(\frac{1}{\epsilon_i} \frac{\partial}{\partial z} + 1 \right) q_i \right]_c$$

Working directly on those tendencies is not quite convenient, because the moisture is *not* a conserved quantity. Rather, we re-write these tendencies in terms of those for dry and moist static energies, obtaining:

$$\frac{\partial}{\partial t}\left[\bar{q}-\left(\frac{1}{\epsilon_i}\frac{\partial}{\partial z}+1\right)q_i\right]=-\frac{1}{\epsilon_i L_v}\frac{\partial}{\partial z}\left(\frac{\gamma}{1+\gamma}\right)\frac{\partial h_i}{\partial t}+\frac{1}{L_v(1+\gamma)}\frac{\partial\bar{h}}{\partial t}-\frac{1}{L_v}\frac{\partial\bar{s}}{\partial t}. \tag{5.27}$$

This result is straightforward to verify by starting from Eqs. (4.26) and (4.27b) and taking appropriate derivatives. It immediately follows that

$$\left(\frac{\partial\tilde{\mathcal{F}}_{l_i}}{\partial t}\right)_{c,c}=-\frac{1}{\epsilon_i L_v}\frac{\partial}{\partial z}\left(\frac{\gamma}{1+\gamma}\right)\left(\frac{\partial h_i}{\partial t}\right)_c+\frac{1}{L_v(1+\gamma)}\left(\frac{\partial\bar{h}}{\partial t}\right)_c-\frac{1}{L_v}\left(\frac{\partial\bar{s}}{\partial t}\right)_c,$$

and corresponding to Eq. (5.8) in Section 5.2, we obtain

$$\Delta\left(\frac{\partial I_i}{\partial t}\right)_L=\int_{z_B}^{z_{Ti}}\rho\epsilon_i\left[\hat{\eta}_i\mathfrak{c}_i-\frac{\mathfrak{d}_i}{1+\gamma}\tilde{\eta}_{l_i}\right]\left(\frac{\partial\bar{h}}{\partial t}\right)_c dz+\int_{z_B}^{z_{Ti}}\epsilon_i\rho\mathfrak{d}_i\tilde{\eta}_{l_i}\left(\frac{\partial\bar{s}}{\partial t}\right)_c dz \tag{5.28}$$

for a correction to the forcing term. Note that in order to obtain this final result, $(\partial h_i/\partial t)_c$ is expressed in an analogous manner to Eq. (5.11a). Here, some coefficients are introduced by

$$\mathfrak{d}_i=\frac{g}{\rho L_v}\int_{\max(z,z_{ci})}^{z_{Ti}}\frac{1}{1+\hat{\delta}\bar{q}}\exp\left[-\int_{z_B}^{z'}(\epsilon_i+c_0)dz''\right]dz', \tag{5.29a}$$

$$\mathfrak{c}_i=\frac{1}{\rho}\int_z^{z_{Ti}}\rho\mathfrak{d}_i\frac{\tilde{\eta}_{l_i}}{\hat{\eta}_i}\frac{\partial}{\partial z}\left(\frac{\gamma}{1+\gamma}\right)dz'. \tag{5.29b}$$

These two definitions may be considered as generalizations of Eqs. (B20) and (B19) of AS, respectively, specifically formulated for the entraining plumes. Here, we note that

$$\tilde{a}_{2,l_i}=-L_v\rho\mathfrak{d}_i,$$

where $\tilde{a}_{2,l_i}$ is obtained from Eq. (4.25b) by setting $\varphi_i=l_i$. Furthermore, we recall that

$$\left(\frac{\partial\bar{h}}{\partial t}\right)_c=\frac{1}{\rho}\sum_j M_{jB}\eta_j\left[\delta_j(h_j^D-\bar{h})+\frac{\partial\bar{h}}{\partial z}\right],$$

$$\left(\frac{\partial\bar{s}}{\partial t}\right)_c=\frac{1}{\rho}\sum_j M_{jB}\eta_j\left[\delta_j(s_j^D-\bar{s})+\frac{\partial\bar{s}}{\partial z}\right]$$

as particular cases of Eq. (3.5a). Substituting these relations into Eq. (5.28) enables us to then determine the corrections to the interaction matrix associated with the precipitation.

The corresponding correction to the large-scale forcing term, $\Delta(\partial I_i/\partial t)_{L,L}$, can also be evaluated by following the method shown by Eqs. (5.11b) and (5.14c) in Section 5.2.

After putting all of these calculations together, the final result is:

$$\begin{aligned}K_{ij}=&\int_{z_B}^{z_{Ti}}\eta_j\left(-\alpha+\epsilon_i\mathfrak{a}_i\hat{\eta}_i\right)\left[\delta_j(s_{vj}^D-\bar{s}_v)+\frac{\partial\bar{s}_v}{\partial z}\right]dz+\int_{z_B}^{z_{Ti}}\epsilon_i\hat{\eta}_i\eta_j\left[\mathfrak{b}_i+\mathfrak{c}_i-\frac{\mathfrak{d}_i}{1+\gamma}\left(\frac{\tilde{\eta}_{l_i}}{\hat{\eta}_i}\right)\right]\left[\delta_j(h_j^D-\bar{h})+\frac{\partial\bar{h}}{\partial z}\right]dz\\&+\int_{z_B}^{z_{Ti}}\epsilon_i\mathfrak{d}_i\tilde{\eta}_{l_i}\eta_j\left[\delta_j(s_j^D-\bar{s})+\frac{\partial\bar{s}}{\partial z}\right]dz-\int_{z_B}^{z_{Ti}}\epsilon_i L_v\mathfrak{d}_i\tilde{\eta}_{l_i}\eta_j\left[\delta_j(l_j^D-\bar{l})+\frac{\partial\bar{l}}{\partial z}\right]dz\end{aligned} \tag{5.30a}$$

$$\begin{aligned}\left(\frac{\partial\mathrm{CAPE}_i}{\partial t}\right)_L=&\ \rho_B\left[\mathfrak{a}_{i,B}\frac{\partial}{\partial t}s_{viB}+(\mathfrak{b}_{i,B}+\mathfrak{c}_{i,B})\frac{\partial}{\partial t}h_{iB}-L_v\mathfrak{d}_{i,B}\frac{\partial}{\partial t}l_{iB}\right]+\left[\int_{z_B}^{z_{Ti}}\rho L_v\mathfrak{d}_i\dot{c}_0 dz\right]l_{iB}\\&+\int_{z_B}^{z_{Ti}}\left[\epsilon_i\mathfrak{a}_i\hat{\eta}_i\left(\frac{\partial\tilde{\bar{s}}_{vi}}{\partial t}\right)_L-\alpha\left(\frac{\partial\bar{s}_v}{\partial t}\right)_L\right]\rho dz+\int_{z_B}^{z_{Ti}}\epsilon_i\hat{\eta}_i\left[\mathfrak{b}_i+\mathfrak{c}_i-\frac{\mathfrak{d}_i}{1+\gamma}\left(\frac{\tilde{\eta}_{l_i}}{\hat{\eta}_i}\right)\right]\left(\frac{\partial\bar{h}}{\partial t}\right)_L\rho dz\\&+\int_{z_B}^{z_{Ti}}\epsilon_i\mathfrak{d}_i\tilde{\eta}_{l_i}\left(\frac{\partial\bar{s}}{\partial t}\right)_L\rho dz-\int_{z_B}^{z_{Ti}}\epsilon_i L_v\mathfrak{d}_i\tilde{\eta}_{l_i}\left(\frac{\partial\bar{l}}{\partial t}\right)_L\rho dz+\int_{z_B}^{z_{Ti}}\epsilon_i L_v\mathfrak{d}_i\tilde{\eta}_{l_i}\frac{\partial}{\partial t}\left(\frac{\rho\sigma_i}{\epsilon_i M_i}\tilde{r}_i\right)\rho dz\\&+\int_{z_B}^{z_{Ti}}\epsilon_i L_v\mathfrak{d}_i\tilde{\eta}_{l_i}\int_z^{z_{Ti}}\dot{c}_0 dz'\rho dz+\dot{z}_{Ti}G_{Ti}+\dot{z}_B G_{Bi}\end{aligned} \tag{5.30b}$$

where

$$G_{Ti} = -\rho\alpha(\bar{s}_v - s_{vi})|_{z=z_{Ti}} \tag{5.31a}$$

$$G_{Bi} = -\rho_B[\epsilon_{iB}\{\mathfrak{a}_{i,B}\Delta\tilde{s}_{viB} + (\mathfrak{b}_{i,B} + \mathfrak{c}_{i,B})\Delta h_{iB} - L_v\mathfrak{d}_{i,B}\Delta\tilde{l}_{iB}\} - \alpha_B\Delta s_{viB} + c_0 L_v \mathfrak{d}_{i,B} l_{iB}] \tag{5.31b}$$

in which further coefficients have been introduced by

$$\mathfrak{a}_i = \frac{1}{\rho}\int_{\min(z,z_{ci})}^{z_{ci}} \rho\frac{\alpha}{\hat{\eta}_i}dz' \tag{5.32a}$$

$$\mathfrak{b}_i = \frac{1}{\rho}\int_{\max(z,z_{ci})}^{z_{Ti}} \rho\frac{\beta}{\hat{\eta}_i}dz' \tag{5.32b}$$

6. Closures also depending on the mass-flux profile: $f = f(\eta_i, \overline{\varphi}, \varphi_i)$

6.1. General formulation

The vertical profile of the mass flux, η_i, may play an important role in order to constrain the intensity of convection for a given component, i. For this reason, η_i may also be added as an additional dependence in the function f. Thus, the most general case to be considered is to set:

$$I_i = \int_{z_B}^{z_{ci}} f_1(\eta_i, \varphi_i, \overline{\varphi})dz + \int_{z_{ci}}^{z_{Ti}} f_2(\eta_i, \varphi_i, \overline{\varphi})dz. \tag{6.1}$$

The original convective quasi-equilibrium hypothesis of AS is a special case of a vertical integral with this form. In the following, we derive the closure condition under a constraint (4.2) for such a vertical integral I_i.

The procedure remains the same as in the last section, but we have to add a new term, $(\partial f_1/\partial\eta_i)(\partial\eta_i/\partial t)$, to the right hand side of Eq. (5.3). There is a similar term arising from the f_2 derivative and hence two additional integral terms

$$\int_{z_B}^{z_{ci}} \frac{\partial f_{1i}}{\partial\eta_i}\frac{\partial\eta_i}{\partial t}dz + \int_{z_{ci}}^{z_{Ti}} \frac{\partial f_{2i}}{\partial\eta_i}\frac{\partial\eta_i}{\partial t}dz \tag{6.2}$$

must be added to the expression of $\partial I_i/\partial t$.

The tendency, $\partial\eta_i/\partial t$, may be obtained directly from the definition of Eq. (1.4) and is

$$\frac{\partial\eta_i}{\partial t} = -\dot{z}_B(\epsilon_{iB} - \delta_{iB})\eta_i. \tag{6.3}$$

Thus the first integral from Eq. (6.2) can be re-written as

$$\int_{z_B}^{z_{ci}} \frac{\partial f_{1i}}{\partial\eta_i}\frac{\partial\eta_i}{\partial t}dz = -\dot{z}_B(\epsilon_{iB} - \delta_{iB})\int_{z_B}^{z_{Ci}} \eta_i\frac{\partial f_i}{\partial\eta_i}dz \tag{6.4}$$

with a similar expression for the second integral. Thus, it leads to a change of the bottom boundary contribution

$$\Delta G_{Bi,\eta} = -(\epsilon_{iB} - \delta_{iB})\left[\int_{z_B}^{z_{ci}} \eta_i\frac{\partial f_{1i}}{\partial\eta_i}dz + \int_{z_{ci}}^{z_{Ti}} \eta_i\frac{\partial f_{2i}}{\partial\eta_i}dz\right], \tag{6.5}$$

which is to be added to the right hand side of Eq. (4.24b). This is the only modification required in order to add an η_i-dependency to the closure integral. Here, the subscript η is added in the definition (6.5) in order to distinguish it from the modifications defined by Eq. (5.14d) associated with forcings that depend on ceonvective-scale variables.

A case of particular interest for this form of integral is one for which the integrands take the form $f_{\nu i} = \eta_i\tilde{f}_\nu(\varphi_i, \overline{\varphi})$ with ν = 1, 2, because this produces a convective-profile weighting of $\tilde{f}_{\nu i}$. The formulation development proceeds just as in the previous section, except for the introduction of factors η_i in the integrands and adding the new terms given by Eq. (6.5). The AS convective quasi-equilibrium closure considered explicitly in the next subsection falls into this category.

The final, and most general result of our calculations is given by bringing together Eqs. (4.15), (4.23a), (4.23b), (4.24a), (4.24b), (5.5a), (5.5b), (5.14b), (5.14c), (5.14d), (6.5) to produce:

$$K_{ij} = \int_{z_B}^{z_{ci}} \frac{1}{\rho}\left(\epsilon_i \tilde{a}_{1,\varphi_i}\tilde{\eta}_{\varphi_i} + \frac{\partial f_{1i}}{\partial\overline{\varphi}}\right)\eta_j\left[\delta_j(\varphi_j^D - \overline{\varphi}) + \frac{\partial\overline{\varphi}}{\partial z}\right]dz + \int_{z_B}^{z_{Ti}} \frac{\epsilon_i\tilde{a}_{2,\varphi_i}\tilde{\eta}_{\varphi_i}}{\rho}\eta_j\left[\delta_j(\varphi_j^D - \overline{\varphi}) + \frac{\partial\overline{\varphi}}{\partial z}\right]dz$$
$$+ \int_{z_{ci}}^{z_{Ti}} \frac{1}{\rho}\frac{\partial f_{2i}}{\partial\overline{\varphi}}\eta_j\left[\delta_j(\varphi_j^D - \overline{\varphi}) + \frac{\partial\overline{\varphi}}{\partial z}\right]dz + \int_{z_B}^{z_{ci}} \frac{\epsilon_i}{\rho}\left[\tilde{\eta}_{\varphi_i}\tilde{a}_{1,\varphi_i}\frac{\partial\tilde{\mathcal{F}}_{\varphi_i}}{\partial\overline{\chi}} + \hat{\eta}_i\int_z^{z_{ci}}\epsilon_i\tilde{a}_{1,\varphi_i}\frac{\tilde{\eta}_{\varphi_i}}{\hat{\eta}_i}\frac{\partial\tilde{\mathcal{F}}_{\varphi_i}}{\partial\chi_i}dz'\right]\eta_j$$
$$\left[\delta_j(\chi_j^D - \overline{\chi}) + \frac{\partial\overline{\chi}}{\partial z}\right]dz + \int_{z_B}^{z_{Ti}}\frac{\epsilon_i}{\rho}\left[\tilde{\eta}_{\varphi_i}\tilde{a}_{2,\varphi_i}\frac{\partial\tilde{\mathcal{F}}_{\varphi_i}}{\partial\overline{\chi}} + \hat{\eta}_i\int_{\max(z,z_{ci})}^{z_{Ti}}\epsilon_i\tilde{a}_{2,\varphi_i}\frac{\tilde{\eta}_{\varphi_i}}{\hat{\eta}_i}\frac{\partial\tilde{\mathcal{F}}_{\varphi_i}}{\partial\chi_i}dz'\right]\eta_j\left[\delta_j(\chi_j^D - \overline{\chi}) + \frac{\partial\overline{\chi}}{\partial z}\right]dz \tag{6.6a}$$

$$\left(\frac{\partial I_i}{\partial t}\right)_L = \left[(\tilde{a}_{1,\varphi_i B} + \tilde{a}_{2,\varphi_i B})\frac{\partial}{\partial t} - (\tilde{c}_{1,\varphi_i} + \tilde{c}_{2,\varphi_i})\right]\varphi_{iB} + \int_{z_B}^{z_{ci}}\left[\epsilon_i\tilde{a}_{1,\varphi_i}\tilde{\eta}_{\varphi_i}\left(\frac{\partial\tilde{\overline{\varphi}}_i}{\partial t}\right)_L + \frac{\partial f_{1i}}{\partial\overline{\varphi}}\left(\frac{\partial\overline{\varphi}}{\partial t}\right)_L\right]dz$$
$$+ \int_{z_B}^{z_{Ti}}\epsilon_i\tilde{a}_{2,\varphi_i}\tilde{\eta}_{\varphi_i}\left(\frac{\partial\tilde{\overline{\varphi}}_i}{\partial t}\right)_L dz + \int_{z_{ci}}^{z_{Ti}}\frac{\partial f_{2i}}{\partial\overline{\varphi}}\left(\frac{\partial\overline{\varphi}}{\partial t}\right)_L dz + \int_{z_B}^{z_{ci}}\epsilon_i\tilde{a}_{1,\varphi_i}\frac{\tilde{\eta}_{\varphi_i}}{\hat{\eta}_i}\frac{\partial\tilde{\mathcal{F}}_{\varphi_i}}{\partial\chi_i}\left[\frac{\partial\chi_{iB}}{\partial t} + \int_{z_B}^{z}\epsilon_i\hat{\eta}_i\left(\frac{\partial\overline{\chi}}{\partial t}\right)_L dz'\right]dz$$
$$+ \int_{z_B}^{z_{Ti}}\epsilon_i\tilde{a}_{2,\varphi_i}\frac{\tilde{\eta}_{\varphi_i}}{\hat{\eta}_i}\frac{\partial\tilde{\mathcal{F}}_{\varphi_i}}{\partial\chi_i}\left[\frac{\partial\chi_{iB}}{\partial t} + \int_{z_B}^{z}\epsilon_i\hat{\eta}_i\left(\frac{\partial\overline{\chi}}{\partial t}\right)_L dz'\right]dz + \dot{z}_{Ti}G_{Ti} + \dot{z}_B G_{Bi} \tag{6.6b}$$

where

$$G_{Ti} = f_{2Ti} \tag{6.7a}$$

$$G_{Bi} = -(\epsilon_{iB} - \delta_{iB})\left[\int_{z_B}^{z_{ci}}\eta_i\frac{\partial f_{1i}}{\partial\eta_i}dz + \int_{z_{ci}}^{z_{Ti}}\eta_i\frac{\partial f_{2i}}{\partial\eta_i}dz\right] - (\tilde{a}_{1,\varphi_i B} + \tilde{a}_{2,\varphi_i B})\left[\epsilon_{iB}\Delta\tilde{\varphi}_{iB} + \left(\frac{\rho\sigma_i}{M_i\varphi_i}\hat{F}_i\right)_B\varphi_{iB}\right]$$
$$-\epsilon_{iB}\Delta\chi_{iB}\left[\int_{z_B}^{z_{ci}}\epsilon_i\tilde{a}_{1,\varphi_i}\frac{\tilde{\eta}_{\varphi_i}}{\hat{\eta}_i}\frac{\partial\tilde{\mathcal{F}}_{\varphi_i}}{\partial\chi_i}dz + \int_{z_B}^{z_{Ti}}\epsilon_i\tilde{a}_{2,\varphi_i}\frac{\tilde{\eta}_{\varphi_i}}{\hat{\eta}_i}\frac{\partial\tilde{\mathcal{F}}_{\varphi_i}}{\partial\chi_i}dz\right] - f_{1Bi}. \tag{6.7b}$$

6.2. *The AS convective quasi-equilibrium formulation*

The core of the AS convective quasi-equilibrium closure is to take the cloud work function as the vertical-integral quantity under the general formulation presented above. The cloud work function is defined by

$$A_i = \int_{z_B}^{z_{Ti}}\eta_i b_i dz \tag{6.8}$$

in terms of the convective buoyancy, b_i, for the ith convection type defined by Eq. (5.16a) above. The cloud work function constitutes a measure of the capacity of a convective ensemble for generating convective kinetic energy, which may be called the potential energy convertibility (PEC; *cf.*, Yano et al., 2005).

By referring to the general formulae already obtained, the interaction matrix, K_{ij}, and the large-scale forcing, $(\partial A_i/\partial t)_L$, are given by:

$$K_{ij} = \int_{z_B}^{z_{Ti}}\eta_i\eta_j\left(-\alpha + \epsilon_i\tilde{\mathfrak{a}}_i\frac{\hat{\eta}_i}{\eta_i}\right)\left[\delta_j(s_{vj}^D - \bar{s}_v) + \frac{\partial\bar{s}_v}{\partial z}\right]dz + \int_{z_B}^{z_{Ti}}\epsilon_i\hat{\eta}_i\eta_j\left[\tilde{\mathfrak{b}}_i + \tilde{\mathfrak{c}}_i - \frac{\tilde{\mathfrak{d}}_i}{1+\gamma}\left(\frac{\tilde{\eta}_{l_i}}{\hat{\eta}_i}\right)\right]\left[\delta_j(h_j^D - \overline{h}) + \frac{\partial\overline{h}}{\partial z}\right]dz$$
$$+ \int_{z_B}^{z_{Ti}}\epsilon_i\tilde{\mathfrak{d}}_i\tilde{\eta}_{l_i}\eta_j\left[\delta_j(s_j^D - \bar{s}) + \frac{\partial\bar{s}}{\partial z}\right]dz - \int_{z_B}^{z_{Ti}}\epsilon_i L_v\tilde{\mathfrak{d}}_i\tilde{\eta}_{l_i}\eta_j\left[\delta_j(l_j^D - \bar{l}) + \frac{\partial\bar{l}}{\partial z}\right]dz \tag{6.9a}$$

$$\left(\frac{\partial A_i}{\partial t}\right)_L = \rho_B\left[\tilde{\mathfrak{a}}_{iB}\frac{\partial}{\partial t}s_{viB} + (\tilde{\mathfrak{b}}_{iB} + \tilde{\mathfrak{c}}_{iB})\frac{\partial}{\partial t}h_{iB} - L_v\tilde{\mathfrak{d}}_{iB}\frac{\partial}{\partial t}l_{iB}\right] + \left[\int_{z_B}^{z_{Ti}}\rho L_v\tilde{\mathfrak{d}}_i\left(\frac{\eta_i}{\tilde{\eta}_{l_i}}\right)\dot{c}_0 dz\right]l_{iB}$$
$$+ \int_{z_B}^{z_{Ti}}\left[\epsilon_i\tilde{\mathfrak{a}}_i\hat{\eta}_i\left(\frac{\partial\tilde{\bar{s}}_{vi}}{\partial t}\right)_L - \alpha\eta_i\left(\frac{\partial\bar{s}_v}{\partial t}\right)_L\right]\rho dz + \int_{z_B}^{z_{Ti}}\epsilon_i\hat{\eta}_i\left[\tilde{\mathfrak{b}}_i + \tilde{\mathfrak{c}}_i - \frac{\tilde{\mathfrak{d}}_i}{1+\gamma}\left(\frac{\eta_i}{\hat{\eta}_i}\right)\right]\left(\frac{\partial\overline{h}}{\partial t}\right)_L\rho dz$$

$$+\int_{z_B}^{z_{Ti}} \epsilon_i \tilde{\mathfrak{d}}_i \eta_i \left(\frac{\partial \bar{s}}{\partial t}\right)_L \rho dz - \int_{z_B}^{z_{Ti}} \epsilon_i L_v \tilde{\mathfrak{d}}_i \eta_i \left(\frac{\partial \bar{l}}{\partial t}\right)_L \rho dz + \int_{z_B}^{z_{Ti}} \epsilon_i L_v \tilde{\mathfrak{d}}_i \eta_i \frac{\partial}{\partial t}\left(\frac{\rho\sigma_i}{\epsilon_i M_i}\tilde{r}_i\right)\rho dz$$

$$+\int_{z_B}^{z_{Ti}} \epsilon_i L_v \tilde{\mathfrak{d}}_i \eta_i \int_z^{z_{Ti}} \dot{c}_0 dz' \rho dz + \dot{z}_{Ti} G_{Ti} + \dot{z}_B G_{Bi} \tag{6.9b}$$

where

$$G_{Ti} = -\rho\alpha\eta_i(\bar{s}_v - s_{vi})|_{z=z_{Ti}} \tag{6.10a}$$

$$G_{Bi} = -(\epsilon_{iB} - \delta_{iB})A_i - \rho_B[\epsilon_{iB}\{\tilde{\mathfrak{a}}_{iB}\Delta\tilde{s}_{viB} + (\tilde{\mathfrak{b}}_{iB} + \tilde{\mathfrak{c}}_{iB})\Delta h_{iB} - L_v \mathfrak{d}_{iB}\Delta\tilde{l}_{iB}\} - \alpha_B\eta_{iB}\Delta s_{viB} + c_0 L_v \tilde{\mathfrak{d}}_{iB} l_{iB}] \tag{6.10b}$$

in which further coefficients have been introduced by

$$\tilde{\mathfrak{a}}_i = \frac{1}{\rho}\int_{\min(z,z_{ci})}^{z_{ci}} \rho\alpha\frac{\eta_i}{\hat{\eta}_i}dz' \tag{6.11a}$$

$$\tilde{\mathfrak{b}}_i = \frac{1}{\rho}\int_{\max(z,z_{ci})}^{z_{ci}} \rho\beta\frac{\eta_i}{\hat{\eta}_i}dz' \tag{6.11b}$$

$$\tilde{\mathfrak{c}}_i = \frac{1}{\rho}\int_{z}^{z_{Ti}} \rho\tilde{\mathfrak{d}}_i\frac{\eta_i}{\hat{\eta}_i}\frac{\partial}{\partial z}\left(\frac{\gamma}{1+\gamma}\right)dz' \tag{6.12a}$$

$$\tilde{\mathfrak{d}}_i = \frac{g}{\rho L_v}\int_{\max(z,z_{ci})}^{z_{Ti}} \frac{1}{1+\hat{\delta}\bar{q}}\exp\left[\int_z^{z'}(\delta_i - c_0)dz''\right]dz' \tag{6.12b}$$

The expressions obtained above generalize Eqs. (B17) and (B18) of AS in several ways. For example, AS restricted their attention to an entraining plume, so that $\tilde{\eta}_{\varphi_i} = \hat{\eta}_i = \eta_i$. Also note that $G_{Ti} = 0$ when the convection top is defined by the level of neutral buoyancy as in AS. Moreover, according to AS, c_0 is a fixed constant, and thus $\dot{c}_0 = 0$, although again such contributions are retained for generality above.

7. Summary and discussions

7.1. Summary

The major finding of the present analysis is that, regardless of many details of the vertical integral, a closure condition defined by a stationarity (or equilibrium) of a vertically-integrated quantity, as given by Eq. (2.1) or Eq. (4.2) always reduces to a form given by Eq. (4.15), which constitutes a generalization of the AS convective quasi-equilibrium.

However, the formulation is not quite closed yet, because we still require expressions for $\dot{z}_{Ti}$ and $\dot{z}_B$ in $(\partial I_i/\partial t)_L$ (Eq. (6.6b)). In the case of AS, they assume that

$$\dot{z}_{Ti} = 0$$

and

$$\dot{z}_B = -\frac{1}{\rho_B}M_{cB} + (\dot{z}_B)_L$$

where $(\dot{z}_B)_L$ is a large-scale tendency for z_B. Note that the latter formula is derived based on their own boundary-layer formulation. The result changes when a different boundary-layer formulation is adopted.

As their example suggests, also in general, these two terms proportional to $\dot{z}_{Ti}$ and $\dot{z}_B$ can be partitioned into convective and large-scale terms so long as a certain linearity is satisfied. As a result the general closure condition can be reduced to Eq. (4.15), with now no dependence of $(\partial I_i/\partial t)_L$ on the mass flux M_j, by re-defining these two terms accordingly. Comparing Eq. (4.15) to Eq. (2.4) we see that the convective consumption term is defined by

$$\left(\frac{\partial I_i}{\partial t}\right)_c = \sum_j K_{ij}M_{jB}. \tag{7.1}$$

The closure condition may equally be presented in a vector-matrix form as:

$$\mathbf{K}\mathbf{M}_B + \left(\frac{\partial \mathbf{I}}{\partial t}\right)_L = 0.$$

In principle, the closure condition can be solved by inverting the matrix, **K**, to obtain:

$$\mathbf{M}_B = -\mathbf{K}^{-1}\left(\frac{\partial \mathbf{I}}{\partial t}\right)_L. \tag{7.2}$$

The idea of quasi-equilibrium closure, generalized here, is schematically summarized in Fig. 2 of Yano and Plant (2012a).

7.2. Operational implementation issues

Although the solution (7.2) may appear straightforward, we have to take into account technical aspects such as the positiveness of the mass flux,

$$M_{B,i} \geq 0.$$

In order to overcome this difficulty, a rather involved procedure for solving Eq. (4.15) was proposed by Lord (1982), and Lord et al. (1982).

As an alternative approach, Moorthi and Suarez (1992) propose to consider only the diagonal terms of **K** in order to simplify the procedure, and thus the solution (7.2) is replaced by

$$M_{i,B} = -\frac{1}{K_{i,i}}\left(\frac{\partial I_i}{\partial t}\right)_L. \tag{7.3}$$

They call this procedure the relaxed Arakawa and Schubert (RAS).

7.3. Choice of I: physical considerations

The generality of the convective quasi-equilibrium principle presented allows us to examine many existing closure hypotheses, and several popular examples have been presented. The advantages and disadvantages of existing closure hypotheses can then be discussed and their validity may be analyzed in detail through evaluation of the various terms in the budgets, perhaps from cloud-resolving model data. However, the presented general principle does not specify which variable is the most physical to be taken for the vertical integral(s), **I**, in Eq. (7.2).

In order to address this last question, we propose the following dictum:

Dictum 1. *Any physically-based diagnostic convective closure must have a prognostic counterpart.*

This dictum essentially says that a closure should be derived from a budget equation. If not, then a given closure hypothesis must be deemed to be unphysical. This statement is rather trivial if a closure condition is derived based on the generalized convective quasi-equilibrium principle as presented herein. However, in the literature, there are various closure hypotheses proposed which are not necessarily consistent with the above dictum.

For example, Bougeault (1985) assumes the stationarity of the convective tendency (rather than the total tendency) for the moist static energy as a closure condition for defining a height-independent detrainment rate (*cf.*, his Eq. (8)). However, this diagnostic closure does not have a prognostic counterpart (*i.e.*, knowing the convective tendency is not enough for a prognostic evaluation).

A stronger version of this dictum may be stated as:

Dictum 2. *Any physically-based diagnostic closure condition must have a prognostic counterpart that can be integrated in time in a self-contained manner.*

This dictum may much narrow down the possibilities.

Under the mass-flux formulation, the goal of the closure is to define M_{Bi}. A first point to be emphasized is that so long as any variable controlled by convection is chosen for the vertical integral (and assumed to be steady) a given closure condition can define the mass-flux magnitude (provided certain mathematical conditions, such as invertibility of a matrix are satisfied). However, it is natural to expect that in a prognostic treatment, such a self-contained description should produce a self-contained prognostic equation for the mass flux or an equivalent quantity.

AS's choice of the cloud work function as the vertically-integrated quantity, I_i, based on the convective kinetic-energy budget, is consistent with the stronger version of the dictum. Although they do not explicitly remark on the possibility of integrating this energy-cycle system in time self-consistently, arguably that idea was implicit. This possibility was first taken up by Randall and Pan (1993), and Pan and Randall (1998). More recently, Yano and Plant (2012b,c), and Plant and Yano (2013) proposed a different version, which can explain a basic life-cycle of convective systems consisting of discharge and recharge (or trigger and suppression) as well as interactions between shallow and deep convection.

A self-consistent closure framework can also be developed simply by writing down a prognostic equation for the convection-base mass flux, which can essentially be derived by vertically integrating a prognostic mass-flux equation (*i.e.*, physically a convective vertical-velocity equation). It is straightforward to show that in this case, the evolution of the vertically-integrated mass flux is controlled by the vertically-integrated convective buoyancy. By then constructing a prognostic equation for the vertical integral of convective buoyancy, we obtain a self-contained prognostic system for describing the evolution of mass-flux amplitude. From this perspective, the stationarity of the vertically-integrated convective buoyancy may be seen as a logical choice for an equilibrium convective closure under the mass-flux formulation.

On the other hand, the consistency of Kuo's (1974) moisture closure with the second, stronger dictum is not obvious. It is widely believed that atmospheric moist convection is controlled by moisture, but there is no known self-contained prognostic description under a coupling with the moisture closure. A strong objection to moisture closure from this point of view was expressed by Emanuel et al. (1994).

Some potential issues with quasi-equilibrium closures are listed in Section 3.4 of Yano and Plant (2012a). Note that these issues equally apply to the generalized formulation developed herein.

7.4. Prognostic formulations for the moisture-based closure

As just stated above, it is less obvious how to proceed to a prognostic version of the moisture-based closure. Nevertheless, it appears to be a good idea to maintain a certain predictability of the column-integrated moisture. Two possibilities are considered here.

7.4.1. Kuo's (1974) solution

In his original formulation, Kuo (1974) introduces a major provision in making the column-integrated water-vapour tendency slightly non-stationary by setting

$$\frac{\partial I_q}{\partial t} = b_q \left(\frac{\partial I_q}{\partial t} \right)_L. \tag{7.4}$$

Here, b_q is a small positive parameter that controls this weak unsteadiness ($0 < b_q \ll 1$). After this modification, the closure changes to

$$M_B = -\frac{1-b_q}{K} \left(\frac{\partial I_q}{\partial t} \right)_L. \tag{7.5}$$

Note that under this generalization, Eq. (3.13a) is re-written as

$$\left(\frac{\partial \bar{q}}{\partial t} \right)_c = -\tilde{f}(z) \frac{1-b_q}{K} \left(\frac{\partial I_q}{\partial t} \right)_L. \tag{7.6}$$

It is often criticized that the small parameter, b_q, remains arbitrary. However, the introduction of the parameter, b_q, may be viewed more positively as a simple attempt to overcome the limit of a strictly stationary closure condition.

7.4.2. Bougeault (1985)

Bougeault (1985) proposed an alternative approach for making the moisture-based closure prognostic. His main proposal is to modify Kuo's (1974) closure so that the moistening, $D_c(q_c^D - \bar{q})$, by detrained air does not contribute as part of the closure balance as in Eq. (3.13b), but simply acts to increase the large-scale moisture. Thus,

$$\frac{\partial \bar{q}}{\partial t} = D_c(q_c^D - \bar{q}).$$

By substituting this expression into the moisture budget equation in the form (2.3), we obtain

$$M_B \int_{z_B}^{z_T} \eta_c \frac{\partial \bar{q}}{\partial z} dz = \left(\frac{\partial I_q}{\partial t} \right)_L.$$

7.5. Precipitation forcing

In order to treat the non-conservative nature of convective-scale physical variables in a general analytic manner, we have proposed to divide the non-conservative term (forcing) into the two components (Section 4.3). As a result, a part of convective-scale forcing may be externalized into a part of large-scale forcing. Specific examples are found in Section 5.3, where the dilute CAPE closure is considered, and in Section 6.2, where the AS convective quasi-equilibrium closure is examined under generalizations. Our general consideration of the precipitation formulation has led to the possibility of an externalized forcing term, which may be called *precipitation forcing*.

By assuming a generality of the precipitation formula (5.23b), we find an additional term due to $\tilde{r}_i \neq 0$ in the forcing term: the temporal tendency, $\partial \tilde{r}_i / \partial t$, of the precipitation rate becomes a part of large-scale forcing. The possibility merits further investigation because the order of magnitude of precipitation forcing is comparable to that of other aspects of the standard large-scale forcing, as is shown now.

According to Eq. (6.9b) in Section 6.2, the precipitation forcing is defined by

$$L_v \int_{z_B}^{z_{Ti}} \epsilon_i \tilde{\delta}_i \eta_i \frac{\partial}{\partial t}\left(\frac{\tilde{r}_i}{\epsilon_i w_i}\right) \rho dz. \tag{7.7}$$

An equivalent term can also be found in Eq. (5.30b) from Section 5.3. Recall that $\tilde{r}_i$ measures a convective-scale precipitation formation rate as defined by Eq. (5.23b), and $w_i = M_i/\rho\sigma_i$ is the convective vertical velocity. However, $\tilde{r}_i/w_i$ is a rather non-trivial variable to interpret, with a unit of m^{-1} or $\mathrm{g\,kg^{-1}\,m^{-1}}$ depending on the unit taken for the water mixing ratio. This is essentially a vertical gradient of the precipitating water generation rate. Only after multiplying by w_i does the quantity reduce to a rate at which precipitating water is being generated at a given vertical level per unit time (with the unit of s^{-1} or $\mathrm{g\,kg^{-1}\,s^{-1}}$).

The corresponding total convective precipitation is given by

$$P = \frac{1}{\rho_w} \int_{z_B}^{z_T} \rho_a \sum_i \tilde{r}_i dz \tag{7.8}$$

where ρ_a and ρ_w are the air and liquid water densities. A typical tropical precipitation rate is $P \sim 10\,\mathrm{mm\,h^{-1}} \sim 3 \times 10^{-6}\,\mathrm{ms^{-1}}$. The precipitation rate due to the ith convective type may be, to an order of magnitude, estimated from Eq. (7.8) as

$$P_i \sim \frac{\rho_a}{\rho_w} \tilde{r}_i H_i$$

with H_i providing a vertical scale for the convection. A typical value for $\tilde{r}_i/w_i$ is then estimated as

$$\frac{\tilde{r}_i}{w_i} \sim \frac{P_i}{(\rho_a/\rho_w) w_i H} \sim \frac{3 \times 10^{-6}\,\mathrm{ms^{-1}}}{10^{-3} \times 1\,\mathrm{ms^{-1}} \times 10^4\,\mathrm{m}} \sim 3 \times 10^{-7}\,\mathrm{m^{-1}}.$$

Here, we have assumed that $w_i \sim 1\,\mathrm{ms^{-1}}$ and $H \sim 10^4\,\mathrm{m}$. A crucial assumption behind this estimate is that the order of magnitude of the ith convective precipitation is of the same order as the total.

Next, note that the precipitation forcing given by Eq. (7.7) is controlled by a temporal change, $\partial(\tilde{r}_i/w_i)/\partial t$, of the precipitation formation measure. In order to estimate this, we introduce a characteristic timescale, τ, for convective precipitation formation. We consider the two possible values, $\tau \sim 1\,\mathrm{h} \sim 3 \times 10^3\,\mathrm{s}$ and $\tau \sim 1\,\mathrm{day} \sim 10^5\,\mathrm{s}$. With the respective values, we obtain the estimates:

$$\frac{\partial}{\partial t}\frac{\tilde{r}_i}{w_i} \sim \frac{\tilde{r}_i/w_i}{\tau} \sim \frac{3 \times 10^{-7}\,\mathrm{m^{-1}}}{3 \times 10^3\,\mathrm{s}} \sim 10^{-10}\,\mathrm{m^{-1}\,s^{-1}}$$

and

$$\frac{\partial}{\partial t}\frac{\tilde{r}_i}{w_i} \sim \frac{\tilde{r}_i/w_i}{\tau} \sim \frac{3 \times 10^{-7}\,\mathrm{m^{-1}}}{10^5\,\mathrm{s}} \sim 3 \times 10^{-12}\,\mathrm{m^{-1}\,s^{-1}}.$$

Additionally, we need an order of magnitude estimate for $L_v\tilde{\delta}_i$ defined by Eq. (6.12b), which is given by

$$L_v \tilde{\delta}_i \sim \frac{g}{\rho} H_i \sim \frac{10\,\mathrm{ms^{-2}} \times 10^4\,\mathrm{m}}{1\,\mathrm{kg\,m^{-3}}} \sim 10^5\,\mathrm{m^5\,s^{-2}\,kg^{-1}}$$

assuming $c_0 = 0$ and $\delta_i \sim 0$ (assuming an entraining plume, this term contribute only a factor of unity to the integrand).

Finally, we obtain the order of magnitude estimate for precipitation forcing as

$$L_v \int \rho \eta_i \tilde{\delta}_i \frac{\partial}{\partial t}\left(\frac{\tilde{r}_i}{w_i}\right) dz \sim \rho L_v \tilde{\delta}_i \frac{\partial}{\partial t}\left(\frac{\tilde{r}_i}{w_i}\right) H_i$$

$$\sim 1\,\mathrm{kg\,m^{-3}} \times 10^5\,\mathrm{m^5\,s^{-2}\,kg^{-1}} \times 10^{-10}\,\mathrm{m^{-1}\,s^{-1}} \times 10^4\,\mathrm{m}$$

$$\sim 10^{-1}\,\mathrm{J\,kg^{-1}\,s^{-1}} \sim 10^4\,\mathrm{J\,kg^{-1}\,day^{-1}}$$

with $\tau \sim 1\,\mathrm{h}$, and

$$L_v \int \rho \eta_i \tilde{\delta}_i \frac{\partial}{\partial t}\left(\frac{\tilde{r}_i}{w_i}\right) dz \sim \rho L_v \tilde{\delta}_i \frac{\partial}{\partial t}\left(\frac{\tilde{r}_i}{w_i}\right) H_i$$

$$\sim 1\,\mathrm{kg\,m^{-3}} \times 10^5\,\mathrm{m^5\,s^{-2}\,kg^{-1}} \times 3 \times 10^{-12}\,\mathrm{m^{-1}\,s^{-1}} \times 10^4\,\mathrm{m}$$

$$\sim 3 \times 10^{-3}\,\mathrm{J\,kg^{-1}\,s^{-1}} \sim 300\,\mathrm{J\,kg^{-1}\,day^{-1}}$$

with $\tau \sim 1$ day. These estimates are comparable to an order of magnitude estimate for large-scale forcing $F \sim 10^3$ J kg^{-1} day^{-1} (*cf.*, Yano and Plant, 2012b). The very last estimate can also be obtained by recalling a typical value for CAPE $\sim 10^3$ J kg^{-1} for the tropics as well as assuming a characteristic timescale of 1 day.

In general, when a sophisticated convective precipitation formulation is adopted, it becomes increasingly difficult to incorporate this process as a part of the convective response within the interaction matrix, $K_{i,j}$. It may be more straightforward to treat it as a part of the large-scale forcing from the point of view of studying the closure relation. Generally speaking, such a precipitation forcing cannot be fully determined until the full convective response is known and thus the procedure for solving the closure problem and evaluating precipitation forcing becomes an iterative procedure.

8. Concluding remarks

The present paper has introduced a general principle of convective quasi-equilibrium, which also constitutes a generalized diagnostic convection-parameterization closure. The generalization is based on a dictum that any diagnostic closure must have a prognostic counterpart. Thus, the general closure is constructed under a stationarity condition of the budget equation of a vertical integral of a general function of physical variables. The formulation may also be considered as a generalization of the AS convective quasi-equilibrium hypothesis, by taking a mathematically analogous formulation.

A very general structure is required in order to incorporate the AS quasi-equilibrium within the same structure as other common closures such as moisture convergence, CAPE and dilute CAPE. The final expressions derived may appear rather involved but they have been obtained here in a stepwise manner in order to make plain the origin of the various contributions to the budget. A very general structure is required in order to treat forcing terms in anything other than a grossly simplified manner. Forcing terms in the equations for a convective-scale variable equation have a rather subtle role, as shown in Section 5.2. As stated therein, we did not even attempt a full generalization. Further generalizations would introduce further complications to the budget equations, and we did not see any immediate benefit in doing so.

In the literature, moisture and CAPE-based closures are often taken as major counterparts (*cf.*, Emanuel et al., 1994). However, under the general closure formulation presented herein, both fall into the same category in which the closure only depends on the large-scale variables. Thus, the moisture-based closure may be considered as a type of convective quasi-equilibrium condition, a perspective which is also supported by observations (*cf.*, John L. McBride, unpublished manuscript, ca., 1990). We also note that Kuo's (1974) formulation may be presented under the mass-flux formulation in a self-consistent manner.

Section 4 presents the major alternative possibility of taking a convective-scale variable, *e.g.*, convective moisture, as a closure variable. An important advantage of such a closure is that as a result, the predictability of the large-scale variables is not lost even in a vertically integrated sense. This possibility is worthwhile to pursue further. Here, recall that a convective-scale variable is determined in terms of the large-scale variables, as detailed in Section 4.2, as a consequence of the steady-plume hypothesis. Thus, we merely take a convective-scale variable as a medium for controlling convection by the large-scale variables.

Under this general perspective, the original AS quasi-equilibrium closure is the most complex case of the categories considered: it includes both large-scale and convective-scale variables in the closure. The advantage of this closure is the relative ease of developing a self-contained prognostic version (Randall and Pan, 1993; Pan and Randall, 1998; Yano and Plant, 2012b,c; Plant and Yano, 2013). We propose the existence of a self-contained prognostic version as a stronger dictum for justifying a physical basis for convection closure. Unfortunately, none of the other closures based on quasi-equilibrium principle in the literature has been shown to satisfy this stronger dictum. Here, a possibility of developing another self-contained prognostic formulation by considering the budget of vertically-integrated convective mass flux is suggested in Section 7.3 in discussing the dicta.

The generalized convective quasi-equilibrium principle has also suggested that some of the convective-scale non-conservative processes may be more conveniently considered a part of large-scale forcing. The precipitation process is specifically identified as such an example. We expect that many other microphysical processes, which are fairly involved in their formulations, might also be more conveniently represented within closure budgets as being a part of large-scale forcing. Our preliminary estimate suggests that the strength of such precipitation forcing could be comparable to other important terms in the "proper" large-scale forcing.

The possibilities for many other alternative closure formulations would be needless to emphasize (*cf.*, Yano et al., 2013). The present general formulation for quasi-equilibrium hardly covers all of those, but the generality presented under this sub-class should not be underestimated. Equivalent studies for other closure types are awaited. Many new closures of interest could nonetheless be constructed within the framework presented. For example, we raised the possibility of using a convective-scale moisture variable to provide a closure and it would be useful to investigate numerically the budgets derived for that possibility, alongside the directly equivalent budget for the large-scale moisture. Our calculations also enable more directly comparable and much more detailed numerical analyses of the budgets for CAPE, dilute CAPE and cloud work function than have been conducted thus far.

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