An optimization problem concerning multiplicative functions


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An optimization problem concerning multiplicative functions

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Abstract

In this paper we study the problem of maximizing a quadratic form $\langle Ax, x \rangle$ subject to $\|x\|_q = 1$, where $A$ has matrix entries $f(\frac{p_i}{p_j})$ with $i,j \mid k$ and $q \geq 1$. We investigate when the optimal is achieved at a ‘multiplicative’ point; i.e. where $x_1x_{mn} = x_mx_n$. This turns out to depend on both $f$ and $q$, with a marked difference appearing as $q$ varies between 1 and 2. We prove some partial results and conjecture that for $f$ is multiplicative such that $0 < f(p) < 1$, the solution is at a multiplicative point for all $q \geq 1$.

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§1. Introduction

In optimization problems involving multiplicative structure, there is a tendency for multiplicative functions to play a crucial role. This can appear in various ways; the optimal may itself be multiplicative, or the point where the optimal occurs may be multiplicative.

For instance in [3], Codecá and Nair considered (amongst others) the problem of minimizing a quadratic form $\langle Bx, x \rangle$ subject to $\|x\|_2 = 1$ where $B$ is the $d(k) \times d(k)$ matrix with entries $h(\frac{i,j}{k})$ where $i,j \mid k$, $(i,j)$ is the gcd of $i$ and $j$, and $k$ is squarefree. They proved that any real multiplicative function $f$ with $0 < f(p) < 1$ (for primes $p$) can be realised as such as minimum. Further, they explicitly determined this minimum when $h$ is multiplicative and of the form $h = 1 + q$, with $q \geq 0$.

Another example comes from [7], where Perelli and Zannier considered the problem of minimizing $\langle Ax, x \rangle$ subject to $\|x\|_2 = 1$ where $A$ is the $d(k) \times d(k)$ matrix (again with $k$ squarefree) with entries $f(\frac{i,j}{k})$ (here $[i,j]$ is the lcm of $i$ and $j$) in the special case that $f(n) = \frac{1}{2} + \frac{1}{12n}$. They show that the minimum is $\frac{\sqrt{\mu(d)}}{\mu(k)}$ and that this is achieved at the point $x_d = \frac{\mu(d)}{\sqrt{\mu(k)}}$.

In [6], it was noted that the operation $c \circ d = \frac{[c,d]}{\gcd(c,d)}$ is a group operation on $D(k) = \{d : d \mid k\}$ if $k$ is squarefree and, as an application of this algebraic structure, the problem of maximizing $\langle A_{f}x, x \rangle$ was considered, where $A_f = (f(c \circ d))_{c,d \mid k}$ but now subject to $\|x\|_q = 1$ with $q \geq 2$. It was found that for any $f : D(k) \rightarrow (0, \infty)$, the optimal is

$$d(k)^{1-\frac{q}{2}} \sum_{d \mid k} f(d),$$

and that it occurs at $x_d$ constant. Notice that in both of the above examples, $\frac{\mu(d)}{\sqrt{\mu(k)}}$ is multiplicative at the optimal, even if $f$ is not. In the latter, the optimal itself is also multiplicative precisely when $f$ is.

In this paper we consider the above optimization problem for the range $1 < q < 2$, which turns out to be highly non-trivial. This has its origin in a problem concerning gcd sums. Briefly, one wishes to maximize the sum

$$F_\alpha(S) = \sum_{m,n \in S} \frac{1}{(m \circ n)^\alpha}$$

over all sets $S$ of size $N$ (see [5] for the case $\alpha = 1$ and [4] and [1] for other values of $\alpha > 0$). For $\alpha \geq \frac{1}{2}$ good bounds for this maximum have been established (sharp for $\alpha = 1$ [5] and close to best
possible for $\frac{1}{2} \leq \alpha < 1$ see [1], [2]), but for $0 < \alpha < \frac{1}{2}$ little is as yet known, except for rather crude upper and lower bounds. Thus it is known that in this range

$$N^{2-2\alpha} \ll \max_{|S|=N} F_\alpha(S) \ll N^{2-2\alpha} \exp\left\{ c\sqrt{\frac{\log N \log \log N}{\log \log N}} \right\}$$

for some absolute constant $c$ (see [2]), but the true order is far from settled. In work in progress, a new lower bound $N^{2-2\alpha} (\log \log N)^{2\alpha}$ can be established which may also turn out to be the correct order of magnitude. This hinges (in part) on maximizing $\langle A_f x, x \rangle$ with $f(n) = n^{-\alpha}$ over $\|x\|_q = 1$, where $q = \frac{1}{1-\alpha} \in (1, 2)$. This motivates studying the following

**Optimization problem:** Let $f: D(k) \to (0, \infty)$ where $k$ is squarefree. Find the supremum of

$$\langle A_f x, x \rangle = \sum_{c,d|k} f(c \circ d)x_c x_d \text{ subject to } \|x\|_q = 1.$$ 

Throughout the article, $k$ is squarefree, $q \geq 1$ and $\|x\|_q$ is the usual $q$-norm: with $x = (x_d)_{d|k}$, $\|x\|_q = (\sum_{d|k} |x_d|^q)^{1/q}$. Also let $F(k) = \sum_{d|k} f(d)$.

**Remarks 1**

(a) Note the following symmetry: let $x' = (x'_d)$ where $x'_d = x_{cd}$ for some $c|k$ (for all $d|k$); then $\langle A_f x', x' \rangle = \langle A_f x, x \rangle$, and $\|x'\|_q = \|x\|_q$. Thus if $x$ is optimal, then so is $x'$. Also, as $f > 0$, the maximum occurs for $x \geq 0$. Hence, without loss of generality, by permuting the $x_d$, we may always assume that at the optimal, $x_1 \geq x_d \geq 0$ for every $d|k$.

(b) For $A_f$ positive definite, $A_f = B^*B$ for some $B$, so that $\langle A_f x, x \rangle = \|Bx\|^2$ and the problem becomes one of evaluating the norm $\|B\|_{q,2}$. We discuss the details in §5.

For $q = 2$ the problem is standard: optimizing a (Hermitian) quadratic form. The optimal is just the largest eigenvalue of $A_f$, which is $F(k) = \sum_{d|k} f(d)$. As mentioned earlier, for $q > 2$ the answer is also relatively straightforward as shown in [6], and we briefly outline the proof. Our main interest shall be the range $1 < q < 2$.

Let $\Lambda$ (or $\Lambda_q$ if we wish to emphasize the dependence on $q$) denote the optimum, indeed maximum. Also let

$$M_q = \max \left\{ \langle A_f x, x \rangle : \|x\|_q = 1 \text{ and } \frac{x_d}{x_1} \text{ is multiplicative} \right\}$$

denote the maximum over ‘multiplicative’ $x$; i.e. when $x_1 x_{mn} = x_m x_n$ for $(m, n) = 1$.

Our main results are the following:

**Theorem 1**

Let $f: D(k) \to (0, \infty)$. Then there exists $c > 0$, depending on $f$ and $k$, such that for $q \geq 2 - c$, the optimal solution occurs at $x_d$ constant and $\Lambda_q = d(k)^{1-2/q} F(k)$.

**Theorem 2**

Let $f$ be multiplicative on $D(k)$ such that $0 < f(p) < 1$ for all $p|k$. Then there exists $c > 0$, depending on $f$ and $k$, such that for $q \in [1, 1 + c)$, the optimal solution occurs at a multiplicative point; i.e. where $x_1 x_{mn} = x_m x_n$ whenever $(m, n) = 1$.

Combining these, we see that for $f$ multiplicative, $M_q = \Lambda_q$ for $q \in [1, 1+c_1) \cup (2-c_2, \infty)$ for some $c_1, c_2 > 0$, depending on $f$ and $k$. However, we believe that the result is true throughout $[1, \infty)$. In other words, we make the following

**Conjecture:** Let $f$ be multiplicative on $D(k)$ such that $0 < f(p) < 1$ for all $p|k$. Then the optimal solution occurs at a multiplicative point and so $M_q = \Lambda_q$ for all $q \geq 1$. 

2
Briefly we outline the rest of the paper. In §2, we indicate how the method of Lagrange multipliers deals with the \( q \geq 2 \) case and what it tells us about the range \( 1 < q < 2 \). We take a particular look at the first non-trivial case \( k = 6 \).

In §3, we evaluate \( M_q \) explicitly, while in §4 we give the proofs of our main results. In §5, we show how we can view the problem as a problem of determining a norm, giving an equivalent form of the above conjecture.

§2. The method of Lagrange multipliers

To find the optimal, we use the method of Lagrange multipliers. We observe that, for \( q > 0 \) of the above conjecture.

Proof. Let \( \epsilon = (\epsilon^0, \epsilon^1, \ldots, \epsilon^m) \) denote the optimal and \( \epsilon^q \) chosen so that \( \|x + h\|_q = 1 \). As such

\[
\epsilon^q = x_b - (x_b^q - \epsilon^q)^{1/2} \sim \frac{\epsilon^q}{q x_b^q} = o(\epsilon),
\]

as \( \epsilon \to 0 \). Now

\[
G(x + h) - G(x) = \sum_{c,d|k} f(c \circ d) \left\{(x_c + h_c)(x_d + h_d) - x_c x_d \right\}
\]

and consider \( G(x + h) - G(x) \) with \( h = (h_d) = (\ldots, \epsilon, \ldots, -\epsilon', \ldots) \) where there is an \( \epsilon > 0 \) in the \( a^\text{th} \) place and \( -\epsilon' \) in the \( b^\text{th} \) place and zeros elsewhere, with \( \epsilon' \) chosen so that \( \|x + h\|_q = 1 \). As such

\[
\epsilon' = x_b - (x_b^q - \epsilon^q)^{1/2} \sim \frac{\epsilon^q}{q x_b^q} = o(\epsilon),
\]

for \( \epsilon \) sufficiently small and positive. Thus \( G(x) \) cannot be maximal.

For \( x = (x_d)_{d|k} \in \mathbb{R}_{\geq 0}^{d|k} \), let \( H(x) = G(x) - 2A(\sum_{d|k} x_d^q - 1) \), where \( A \) is to be determined. Then at the optimal solution, we must have \( \frac{\partial H}{\partial x_d} = 0 \) for every \( d|k \); i.e.

\[
Ax_d^{q-1} = \sum_{c|k} f(c \circ d)x_c \quad (\forall d|k).
\]

Multiplying through by \( x_d \) and summing over \( d \) shows that we must take \( A = \Lambda \). Thus, at the optimal,

\[
Ax_d^{q-1} = \sum_{c|k} f(c \circ d)x_c \quad \text{for every } d|k. \tag{2.1}
\]

2.1 The case \( q \geq 2 \)

Using equations (2.1), the case \( q \geq 2 \) can be easily dealt with.

Theorem A (see [6])

Let \( k \) be squarefree, \( f : D(k) \to (0, \infty) \) and \( q \geq 2 \). Then \( \Lambda = d(k)^{1 - \frac{q}{2}} F(k) \), where the optimal occurs for \( x_d \) constant; i.e. \( x_d = \frac{1}{\sqrt{d(k)}} \).

Proof. Let \( x = (x_d) \) denote the optimal and \( \overline{x} \) and \( \underline{x} \) the minimum and maximum of \( x_d \) respectively. By (2.1), for some \( d|k \),

\[
\Lambda x_d^{q-1} = \sum_{c|k} f(c \circ d)x_c \geq \overline{x} \sum_{c|k} f(c \circ d) = \overline{x} F(k)
\]
since \((D(k), \circ)\) is a group. On the other hand, for some \(d'|k\),

\[
\Lambda \pi^{q-1} = \sum_{c|k} f(c \circ d')x_c \leq \pi \sum_{c|k} f(c \circ d') = \pi F(k).
\]

Combining these gives \(\Lambda \pi^{q-2} \geq F(k) \geq \Lambda \pi^{q-2}\). For \(q = 2\) this forces \(\Lambda = \sum_{d|k} f(d)\). For \(q > 2\), we must have \(\pi \leq x^2\); i.e. \(x_d\) must be constant. As \(\sum_{d|k} x_d^q = 1\), this forces \(x_d = 1/\sqrt{d(k)}\). This must give the maximum value of \(G\) as it exists and it lies in the interior of the region. Hence \(\Lambda = d(k)^{1 - \frac{q}{2}} F(k)\) follows.

\[
\square
\]

### 2.2 The case \(1 < q < 2\)

If \(q \in (1, 2)\), the above analysis using Lagrange Multipliers leading to (2.1) is still valid, but the conclusion that \(x_d\) is constant at the optimum no longer holds in general. However, as we shall prove in Theorem 1, this constant solution continues to hold in an interval \(q \in (2 - c, 2)\) for some \(c > 0\), depending on both \(f\) and \(k\).

For smaller \(q\) though, the optimal changes. Indeed, looking at the behaviour of the optimal solution when \(q\) is close to 1, shows precisely what is required for multiplicativity. Indeed, for \(q = 1\), one can construct examples with \(f > 1\) where the optimal is not multiplicative, even if \(f\) is (see Remarks 2). By continuity, this shows it also fails for some \(q > 1\). However, if \(f(n) \leq f(1) = 1\) for all \(n\), then the optimal when \(q = 1\) occurs at \(x = (1, 0, \ldots, 0)\). For \(q\) close to 1, we shall see that in this case (taking \(x_1 = x_d\))

\[
x_d^{q-1} \sim f(d) \quad \text{as } q \to 1^+, \text{for every } d|k.
\]

Thus for \(x_d/x_1\) to be multiplicative, we need \(f\) to be multiplicative.

However, there are indications that it is also sufficient. Note that for \(f\) multiplicative, the eigenvalues of \(A_f\) are \(\prod_{p|k}(1 \pm f(p))\) (where any combination of \(\pm\) is possible – see [6]) and \(A_f\) is positive definite precisely when \(-1 < f(p) < 1\) for all prime divisors \(p\) of \(k\). The condition that \(f\) is at most 1 in Theorem 2 is therefore quite natural.

### 2.3 The simplest non-trivial case; \(k = 6\)

The reason why we expect multiplicativity at the optimum may not be clear at this stage. That it is true in a fairly trivial way for \(q \geq 2\) is not sufficient reason. Also it is vacuously true when \(k\) is prime. A look at the first non-trivial case gives some indication why multiplicativity is expected.

Writing \(f(2) = a\) and \(f(3) = b\) (so that \(f(6) = ab\)), the problem for the \(k = 6\) case now becomes: maximize

\[
x_2^q + x_3^q + x_6^q + 2a(x_1x_2 + x_3x_6) + 2b(x_1x_3 + x_2x_6) + 2ab(x_1x_6 + x_2x_3)
\]

subject to \(x_1, x_2, x_3, x_6 \geq 0\) and \(x_1^q + x_2^q + x_3^q + x_6^q = 1\).

The Conjecture says that, if \(0 < a, b < 1\) then, at the maximum, \(x_1x_6 = x_2x_3\). Let us see why this is plausible. Equations (2.1) give

\[
\begin{align*}
\Lambda x_1^{q-1} &= x_1 + ax_2 + bx_3 + abx_6 \\
\Lambda x_2^{q-1} &= ax_1 + x_2 + abx_3 + bx_6 \\
\Lambda x_3^{q-1} &= bx_1 + abx_2 + x_3 + ax_6 \\
\Lambda x_6^{q-1} &= abx_1 + bx_2 + ax_3 + x_6.
\end{align*}
\]

Multiplying the cases \(d = 1\) and \(d = 6\) together and subtracting the product of \(d = 2\) and \(d = 3\) gives (after some cancellation)

\[
\Lambda^2 \left( (x_1x_6)^{q-1} - (x_2x_3)^{q-1} \right) = (1 - a^2)(1 - b^2)(x_1x_6 - x_2x_3).
\]
This indicates the special role played by the quantity \( x_1x_6 - x_2x_3 \).

If \( x_1x_6 \neq x_2x_3 \), then we may divide through:

\[
\Lambda^2 = (1 - a^2)(1 - b^2) \frac{x_1x_6 - x_2x_3}{(x_1x_6)^{q-1} - (x_2x_3)^{q-1}} < \frac{x_1x_6 - x_2x_3}{(x_1x_6)^{q-1} - (x_2x_3)^{q-1}},
\]

It is not difficult to show that the RHS has its supremum (over all \( x \) such that \( \|x\|_q = 1 \) and \( x_1x_6 \neq x_2x_3 \)) when \( x_d \) is constant, interpreted in the limit as \( x_1x_6 \to x_2x_3 \). (We omit the details.) As a result,

\[
\Lambda^2 < \frac{(1/4)^{2(q-2)}}{q-1}.
\]

But \( \Lambda \geq 1 \) (by taking \( x_1 = 1 \) and \( x_d = 0 \) for \( d > 1 \)). Thus \( x_1x_6 \neq x_2x_3 \) implies

\[
(q-1)^{2(q-2)} < 1.
\]

But this is (fairly easily) shown to be false for \( q \in (1.1076, 2] \). Thus the conjecture holds when \( k = 6 \) for \( q \in (1.1076, 2] \) at least. By Theorem 2, it also holds for \( q \) in an interval \([1, 1 + c]\) but, unfortunately, \( c \) is not an absolute constant, depending as it does on \( a \) and \( b \). So the case \( k = 6 \) is still open.

§3. The maximum over multiplicative \( x \) for \( f \) multiplicative

Now we calculate the maximum over ‘multiplicative’ \( x \) (i.e. evaluate \( M_q \)) when \( f \) is multiplicative. We shall require some preliminaries. For \( 1 \leq q < 2, a \in (0, 1) \) and \( x \geq 0 \), define the functions

\[
h_q(a, x) = ax^q + x^{q-1} - a - x
\]

\[
L_q(a, x) = \frac{1 + 2ax + x^2}{(1 + x^q)^{2/q}}.
\]

Note that \( h_q(a, 1) = 0 \), and for \( x > 0, h_q(a, \frac{1}{q}) = -x^{-q}h_q(a, x) \) and \( L_q(a, \frac{1}{q}) = L_q(a, x) \).

**Lemma 3.1**

Fix \( q \in (1, 2) \) and \( a \in (0, 1) \) and let \( \gamma = \frac{2}{q} - 1 \), so that \( \gamma \in (0, 1) \). Then

(a) if \( a \geq \gamma \), then \( h_q(a, x) < 0 \) in \([0, 1]\);

(b) if \( a \gamma = 0 \) and \( h_q(a, x) < 0 \) in \([0, 1]\);

(b) if \( a < \gamma \), then \( h_q(a, x) \) has precisely one root in \([0, 1]\).

**Proof.** We have \( h_q(a, 0) = -a < 0 \), \( h_q(a, 1) = 0 \) and \( h_q'(a, 1) = q(a - \gamma) \). Thus we have a zero at 1 in any case, while if \( a < \gamma \) we must have (at least) one more in \((0, 1)\). But also

\[
h_q''(a, x) = q(q - 1)x^{q-3}(ax - \gamma).
\]

If \( a < \gamma \), then \( h \) is concave in \([0, 1]\) and so there is precisely one zero in \((0, 1)\). If \( a \geq \gamma \), then \( h' \) is decreasing on \([0, \frac{1}{q}]\) and increasing on \([\frac{1}{q}, 1]\). Thus

\[
\min_{0 \leq x \leq 1} h_q'(a, x) = h_q'(a, \frac{1}{q}) = \left( \frac{a}{\gamma} \right)^{2-q} - 1 \geq 0
\]

and so \( h_q(a, x) \) is (strictly) increasing in \([0, 1]\).

Now let \( r_q(a) \) denote the unique root of \( h_q(a, x) \) in \((0, 1)\) for \( a < \gamma \). Thus

\[
r_q(a)^{q-1} = \frac{a + r_q(a)}{1 + ar_q(a)}.
\]
Also extend to $(0, 1)$ by defining $r_q(a) = 1$ for $\gamma \leq a < 1$. Let

$$Q_q(a) = \sup_{x > 0} L_q(a, x) = \max_{0 \leq x \leq 1} L_q(a, x).$$

Since $L_q'(a, x) = -\frac{2h_q(a, x)}{(1 + x^2)^{1/q}}$, it is quickly seen that for $q > 1$, $Q_q(a) = L_q(a, r_q(a))$ while

$$Q_1(a) = \begin{cases} 1 - a^{1/2} & \text{if } a \leq 1 \\ 1 & \text{if } a > 1 \end{cases}.$$

**Lemma 3.2**

Fix $a \in (0, 1)$. Then, as $q \to 1^+$, $r_q(a) \to 0$. More precisely, for $a < \gamma = \frac{q}{q} - 1$,

$$r_q(a) \leq \frac{q - 1}{1 - aq}.$$ Hence (3.1) implies $r_q(a)^{q-1} \to a$ as $q \to 1^+$.

**Proof.** For $a < \gamma$, $h_q(a, x)$ has one turning point in $(0, 1)$, say at $s(a)$. This is necessarily a maximum and $r(a) < s(a)$. We have $aq(s(a))^{q-1} + (q - 1)s(a)^{q-2} = 1$. In particular, $1 \leq (q - 1)s(a)^{q-2} + aq$. Thus

$$r(a) \leq r(a)^{q-1} \leq s(a)^{q-2} \leq \frac{q - 1}{1 - aq}.$$

\[\square\]

**Proposition 3.3**

Let $f$ be multiplicative and positive on $D(k)$. Then, with $\gamma = \frac{q}{q} - 1$

$$M_k = \prod_{p \mid q} Q_q(f(p)) = \prod_{f(p) < q} Q_q(f(p)) \prod_{f(p) \geq q} \frac{1 + f(p)}{2^{\gamma}}.$$ In particular for $q = 1$,

$$M_1 = \prod_{f(p) > 1} \frac{1 + f(p)}{2}.$$

**Proof.** For $x = (x_d)$ such that $\|x\|_q = 1$, we may write

$$x_d = \frac{g(d)}{G(k)}, \quad \text{where } g \geq 0 \text{ is multiplicative and } G(k) = (\sum_{d \mid k} g(d)^q)^{1/q}.$$

We recall from [6] that with $F \otimes G$ defined on $D(k)$ by $(F \otimes G)(n) = \sum_{d \mid k} F(d)G(n \circ d)$, then $(F \otimes G)(n) := \frac{(F \otimes G)(n)}{(F \otimes G)(1)}$ is multiplicative whenever $F$ and $G$ are, provided that $(F \otimes G)(1) \neq 0$. Further, $(F \otimes G)(p) = \frac{F(p) + G(p)}{1 + F(p)G(p)}$ for a prime $p$. As such,

$$\langle Af, x \rangle = \frac{1}{G(k)^2} \sum_{c \mid d, k} f(c \circ d)g(c)g(d) = \frac{1}{G(k)^2} \sum_{d \mid k} g(d)(f \otimes g)(d)$$

$$= \frac{1}{G(k)^2} \sum_{d \mid k} f(c)g(c) \sum_{d \mid k} g(d)(f \otimes g)(d)$$

$$= \prod_{p \mid k} \left\{ \frac{1 + f(p)g(p)}{(1 + g(p)^q)^{2/q}} : (1 + g(p)(f \otimes g)(p)) \right\}$$

(by multiplicativity)

$$= \prod_{p \mid k} \left\{ \frac{1 + 2f(p)g(p) + g(p)^2}{(1 + g(p)^q)^{2/q}} \right\} = \prod_{p \mid k} L_q(f(p), g(p)).$$
In order to maximize this, we maximize each factor independently of the others. Since there is no restriction on \( g(p) \), we need to maximize \( L_q(f(p), t) \) over \( t \) in \((0, 1)\). Thus we take \( g(p) = r_q(f(p)) \) giving the maximum \( Q_q(f(p)) \), and so

\[
M_q = \prod_{p \mid k} Q_q(f(p)).
\]

The second formula follows on using \( Q_q(f(p)) = \frac{1 + f(p)}{2} \) whenever \( f(p) \geq \gamma \).

\[\Box\]

Remarks 2

(a) Note that if \( f(p) < 1 \) for each \( p \) then, for \( q \) close to \( 1 \), \( g(p)^{q-1} = f(p) + O(q - 1) \) by Lemma 3.2 and, by multiplicativity, \( g(d)^{q-1} = f(d) + O(q - 1) \).

(b) From the formula for \( M_1 \) we can show that the maximum need not necessarily occur at a ‘multiplicative’ point, even if \( f \) is multiplicative. As an example, take \( k = 6 \) and let \( f \) be multiplicative with \( f(2), f(3) > 1 \). Then

\[
M_1 = \frac{(1 + f(2))(1 + f(3))}{4}.
\]

But at \( x = (\frac{1}{2}, 0, 0, \frac{1}{2}) \), \( \langle A_f x, x \rangle = \frac{1 + f(2)f(3)}{2} \), which is larger. (Indeed this can be shown to be the maximum.) By continuity, for this \( f \), \( M_q < \Lambda_q \) if \( q \) is a little larger than 1.

§4. Proof of Theorems 1 and 2

Proof of Theorem 1. We need only consider \( q < 2 \). Since \( \Lambda_q \) varies continuously with \( q \) and \( \Lambda_2 = F(k) \), we must have

\[
\Lambda_q = F(k) + o(1) \quad \text{as } q \to 2^-.
\]

Let \( x \) be such that \( \|x\|_q = 1 \) and \( x_1 \geq x_d \) without loss of generality. Then we have

\[
1 = \sum_{d \mid k} x_d^q \leq x_1^q d(k),
\]

so that \( x_1 \geq d(k)^{-1/q} = \frac{1}{\sqrt{d(k)}} + o(1) \). Now put \( d = 1 \) in (2.1). Thus

\[
\sum_{c \mid k} f(c) x_c = \Lambda_q x_1^{q-1} \sim F(k) x_1.
\]

It follows that, for every \( d \mid k \),

\[
0 \leq f(d) (x_1 - x_d) \leq \sum_{c \mid k} f(c) (x_1 - x_c) = F(k) x_1 - \Lambda_q x_1^{q-1} \to 0
\]

as \( q \to 2^- \). Thus \( x_d = x_1 + o(1) \) for every \( d \mid k \). We may therefore write

\[
x_d = x_1 e^{-\eta_d}, \quad \text{where } 0 \leq \eta_d \to 0 \text{ as } q \to 2^-.
\]

Let \( \eta = \max_{d \mid k} \eta_d \) and \( H = \frac{1}{d(k)} \sum_{d \mid k} \eta_d \). Note that \( H \leq \eta \), and \( \eta \to 0 \) as \( q \to 2^- \). Then

\[
1 = \sum_{d \mid k} x_d^q = x_1^q \sum_{d \mid k} e^{-\eta_d} = x_1^q \sum_{d \mid k} (1 - q \eta_d + O(\eta^2)) = x_1^q d(k) (1 - qH + O(\eta^2)).
\]

Thus

\[
x_1 = 1 + H + O(\eta^2) d(k)^{1/q}, \quad (4.1)
\]


Next,
\[ \Lambda_q = x_1^2 \sum_{c,d \mid k} f(c \circ d)e^{-\eta_c - \eta_d} = x_1^2 \sum_{c,d \mid k} f(c \circ d)(1 - \eta_c - \eta_d + O(\eta^2)) \]
\[ = x_1^2 \left( \sum_{c \mid k} \sum_{d \mid k} f(c \circ d) - 2 \sum_{c \mid k} \eta_c \sum_{d \mid k} f(c \circ d) + O(\eta^2) \right) \]
\[ = x_1^2 F(k)d(k)(1 - 2H + O(\eta^2)). \]

Inserting (4.1) gives
\[ \Lambda_q = F(k)d(k)^{1 - 2/q}(1 + O(\eta^2)). \]

Now, with \( d = 1 \) in (2.1), and dividing through by \( x_1 \),
\[ \Lambda_q x_1^{q - 2} = \sum_{c \mid k} f(c)e^{-\eta_c} = \sum_{c \mid k} f(c)(1 - \eta_c + O(\eta^2)) = F(k) - \sum_{c \mid k} f(c)\eta_c + O(\eta^2). \]

Rearranging and inserting (4.1) and (4.2),
\[ \sum_{c \mid k} f(c)\eta_c = F(k) - \Lambda_q x_1^{q - 2} + O(\eta^2) = F(k) - F(k)(1 + (q - 2)\eta) + O(\eta^2) \]
\[ = (2 - q)HF(k) + O(\eta^2) \leq (2 - q)\eta F(k) + O(\eta^2). \]

But the left-hand side is at least \( f(d)\eta \) for some \( d \). If \( \eta > 0 \), we may divide through to get
\[ f(d) \leq (2 - q)F(k) + O(\eta). \]

This is a contradiction for all \( \eta \) sufficiently close to \( 2 \). Thus \( \eta = 0 \) and \( x_d \) is constant.

For the proof of Theorem 2, we first determine the asymptotic behaviour of the solution and \( \Lambda_q \) as \( q \to 1 \). For the following result we do not require \( f \) to be multiplicative, only to be bounded by \( 1 \).

**Proposition 4.1**

Let \( f : D(k) \to (0, 1] \) such that \( f(d) = 1 \) at \( d = 1 \) only. Then, at the optimal, as \( q \to 1^+ \)
\[ \Lambda_q = 1 + O(q - 1) \quad \text{and} \quad x_d^{q - 1} = f(d) + O(q - 1). \]

**Proof.** Since \( f \leq 1 \), we have for \( \|x\|_q = 1 \),
\[ 1 \leq \Lambda_q \leq \left( \sum_{d \mid k} x_d \right)^2 \leq \left( \sum_{d \mid k} x_d^q \right)^{\frac{2}{q}} \left( \sum_{d \mid k} 1 \right)^{2(1 - \frac{1}{q})} = d(k)^{\frac{2(1 - \frac{1}{q})}{q}} = 1 + O(q - 1). \]

Also \( 1 = \sum_{d \mid k} x_d^q \leq d(k)x_1^q \leq d(k)x_1 \), so that \( \frac{1}{x_1} \leq x_1 \leq 1 \) and hence \( x_1^{q - 1} = 1 + O(q - 1) \). Now (2.1) with \( d = 1 \) implies
\[ \sum_{c \mid k} f(c)x_c = \Lambda_q x_1^{q - 1} = 1 + O(q - 1). \]

But \( \sum_{c \mid k} x_c = 1 + O(q - 1) \) also, and subtracting gives
\[ \sum_{c \mid k} (1 - f(c))x_c = O(q - 1). \]

As \( f(c) < 1 \) whenever \( c > 1 \), we see that \( x_d = O(q - 1) \) for each \( d > 1 \), and hence \( x_1 = 1 + O(q - 1) \). This implies
\[ \Lambda_q x_d^{q - 1} = \sum_{c \mid k} f(c \circ d)x_c = f(d) + O(q - 1), \]

8
with \( c = 1 \) giving the main term. Thus \( x_d^{q-1} = f(d) + O(q - 1) \) as required.

\[ \square \]

**Proof of Theorem 2.** Again we may assume that at the optimal solution \( x_1 \geq x_d > 0 \) for all \( d/k \). We shall also assume that \( q > 1 \), the \( q = 1 \) case being trivial, so that the method of Lagrange multipliers is valid and equations (2.1) hold.

These may be rewritten by letting \( h(d) = \frac{f}{x_1} \) as follows. Then dividing (2.1) through by the \( d = 1 \) case gives

\[
h(d)^{q-1} \sum_{c|k} f(c)h(c) = \sum_{c|k} f(c \circ d)h(c) \quad \text{or} \quad h(d)^{q-1} = (f \otimes h)(d). \tag{4.3}
\]

The aim is now to show that \( h(d) = g(d) \), where \( g(d) \) is the optimal chosen in the multiplicative case in Proposition 3.3. There we found that

\[
g(p)^{q-1} = \frac{f(p) + g(p)}{1 + f(p)g(p)} = (f \circ g)(p).
\]

Since \( f \) and \( g \) are multiplicative, it follows that

\[
g(d)^{q-1} = (f \circ g)(d).
\]

Thus \( g(d) \) also satisfies (4.3).

Furthermore, both \( g(d)^{q-1} = f(d) + O(q - 1) \) and \( h(d)^{q-1} = f(d) + O(q - 1) \) as \( q \to 1^+ \) (from Remarks 2(a) and Proposition 4.1 respectively). Thus \( h(d) \approx g(d) \approx f(d)^{\frac{q}{q-1}} \) and we may write

\[
h(d) = g(d)e^{\eta_d},
\]

where \( \eta_d = O(1) \). As such, (4.3) becomes

\[
\sum_{c|k} \left( f(c \circ d) - f(c)g(d)^{q-1}e^{\eta_d(q-1)} \right) h(c) = 0.
\]

Splitting \( e^{\eta_d(q-1)} \) into \( 1 + (e^{\eta_d(q-1)} - 1) \) and using (4.3) for \( g \) leads to

\[
\sum_{c|k} \left( f(c \circ d) - f(c)g(d)^{q-1} \right) g(c)(e^{\eta_c} - 1) = g(d)^{q-1}(e^{\eta_d(q-1)} - 1) \sum_{c|k} f(c)h(c). \tag{4.4}
\]

Choose \( d \) such that \( |\eta_d| \geq |\eta_c| \) for all \( c|k \) and suppose for a contradiction that \( |\eta_d| > 0 \). Then the RHS in (4.4) is, in modulus, at least

\[
g(d)^{q-1}e^{\eta_d(q-1)} - 1 \sim f(d)|\eta_d|(q - 1).
\]

But on the left of (4.4), the \( c = 1 \) term is zero, while for \( c > 1 \), \( g(c) \) is exponentially small, as \( g(c)^{q-1} \to f(c) < 1 \). Thus the LHS of (4.4) is, in modulus,

\[
\ll |\eta_d| \sum_{c > 1} g(c) \ll |\eta_d| (\max_{c > 1} f(c))^{\frac{1}{q-1}} = o(|\eta_d|(q - 1)).
\]

We have our desired contradiction, and so \( h = g \), making \( h \) multiplicative.

\[ \square \]

§5. **Problem transposed into one of norms**

If \( A_f \) is positive definite, which is our main interest, then \( A_f = B^*B \) for some \( B \), so that \( \langle A_f x, x \rangle = \|Bx\|^2 \) and the problem becomes one of evaluating the norm

\[
\|B\|_{q,2} = \sup_{x \neq 0} \frac{\|Bx\|^2}{\|x\|^q}.
\]
Such norms are generally difficult to find, there being no general formulae. Indeed, for bounded linear operators \( \varphi : l^p \rightarrow l^q \), a general formula (in terms of the associated matrix entries) is only known for the cases \( p = 1 \) or \( q = \infty \) (see for example [8], Chapter 4).

Now if \( f \) is multiplicative, then \( A_f \) is positive definite precisely when \( f(p) \in (-1, 1) \) for all \( p \mid k \).

We can give a precise form for \( B \) in this case. We require some concepts from [6].

Every \( f : D(k) \rightarrow \mathbb{C} \) has a Fourier series

\[
f(n) = \frac{1}{d(k)} \sum_{\chi \in D(k)} \hat{f}(\chi)\chi(n),
\]

where \( \chi \) ranges over the characters of \( D(k) \) and \( \hat{f}(\chi) \) are the Fourier coefficients of \( f \), given by

\[
\hat{f}(\chi) = \sum_{d \mid k} \chi(d)f(d) \quad \left( = \prod_{p \mid k} (1 + \chi(p)f(p)) \text{ if } f \text{ is multiplicative} \right).
\]

If \( \hat{f}(\chi) \geq 0 \) for all \( \chi \), we may define for \( \alpha > 0 \),

\[
f^{\otimes \alpha}(n) = \frac{1}{d(k)} \sum_{\chi \in D(k)} \hat{f}(\chi)\alpha\chi(n). 
\] (5.1)

Equivalently, we may write \( A_f = U^* DU \) where \( U \) is the unitary matrix with entries \( (\chi(d))_{d \mid k, \chi \in \hat{D(k)}} \) and \( D = \text{diag}(\hat{f}(\chi))_{\chi \in \hat{D(k)}} \), in which case \( A_f^2 = A_{f^{\otimes \alpha}} \).

Also let \( f^{\otimes \alpha}(n) = \frac{f^{\otimes \alpha}(n)}{f^{\otimes \alpha}(1)} \) whenever the denominator is non-zero.

**Proposition 5.1**

Let \( f \) be multiplicative on \( D(k) \) such that \( 0 < f(p) < 1 \) for all primes \( p \mid k \). Then \( f^{\otimes \alpha} \) is multiplicative for every \( \alpha > 0 \), and furthermore for each \( n \mid k \),

\[
f^{\otimes \alpha}(n) = \prod_{p \mid n} \frac{(1 + f(p))^\alpha - (1 - f(p))^\alpha}{(1 + f(p))^\alpha + (1 - f(p))^\alpha}.
\]

**Proof.** Denote the \( d(k) \) characters of \( \hat{D(k)} \) by \( \chi_d(\cdot) = \mu(\cdot, d) \) where \( d \mid k \) and \( \mu(\cdot) \) is the Möbius function. We prove by induction on \( w(k) \) (the number of prime factors of \( k \)) that

\[
f^{\otimes \alpha}(n) = \frac{1}{d(k)} \prod_{p \mid k} \left( 1 + f(p) \right)^\alpha + \chi_p(n)(1 - f(p))^\alpha \right). 
\] (5.2)

For if (5.2) holds, then dividing through by the \( n = 1 \) case and using \( \chi_p(n) = -1 \) if \( p \mid n \) and 1 otherwise, gives the result.

Now if \( w(k) = 2 \), then \( k \) is prime and \( \hat{D(k)} \) consists of two characters 1 and \( \mu \). Thus by (5.1)

\[
f^{\otimes \alpha}(n) = \frac{1}{2} \left( (1 + f(1))^\alpha + (1 + f(\mu)^\alpha \mu(n)(1 - f(\mu))^\alpha \right)
\]

which is the RHS of (5.2).

For the inductive step, suppose (5.2) holds for some \( k \) squarefree and all \( n \mid k \). Let \( q \) be prime and such that \( q \not\mid k \), and consider (5.2) for \( qk \).

Observe that (i) \( D(qk) = D(k) \cup qD(k) \) (since every divisor \( d \mid qk \) satisfies either \( d \mid k \) or \( d = qd' \), \( d' \mid k \)), and (ii) \( \chi \in \hat{D(qk)} \Leftrightarrow \chi = \chi_d \) or \( \chi = \chi_{qd} = \chi_q \chi_d \) for \( d \mid k \) since \( (q, d) = 1 \).

Thus for \( \chi \in \hat{D(qk)} \), we have

\[
\hat{f}(\chi) = \prod_{p \mid k} (1 + \chi(p)f(p)) = (1 + \chi(q)f(q)) \prod_{p \mid k} (1 + \chi(p)f(p))
\]

\[
= \begin{cases} 
(1 + f(q)) \prod_{p \mid k} (1 + \chi(p)f(p)) & \text{if } \chi = \chi_d \\
(1 - f(q)) \prod_{p \mid k} (1 + \chi(p)f(p)) & \text{if } \chi = \chi_{qd} \\
\end{cases} (d \mid k),
\]
using the fact that $\chi_q(p) = 1$ if $p|k$ and $-1$ otherwise. Thus

$$
\sum_{\chi \in \hat{D}(q^k)} \chi(n)\hat{f}(\chi)^a = \sum_{\chi \in \hat{D}(k)} \chi(n)(1 + f(q))^a \prod_{p|k} (1 + \chi(p)f(p))^a
$$

$$
+ \sum_{\chi \in \hat{D}(k)} \chi_q(n)\chi(n)(1 - f(q))^a \prod_{p|k} (1 + \chi(p)f(p))^a
$$

$$
= \left((1 + f(q))^a + \chi_q(n)(1 - f(q))^a\right) \sum_{\chi \in \hat{D}(k)} \chi(n) \prod_{p|k} (1 + \chi(p)f(p))^a
$$

$$
= \left((1 + f(q))^a + \chi_q(n)(1 - f(q))^a\right) \prod_{p|k} \left((1 + f(p))^a + \chi_p(n)(1 - f(p))^a\right)
$$

(by assumption)

$$
= \prod_{p|q^k} \left((1 + f(p))^a + \chi_p(n)(1 - f(p))^a\right).
$$

Note also that $0 < f^\otimes a(p) < 1$ for all $p|k$.

It follows from Proposition 5.1 that for $f$ multiplicative on $D(k)$ satisfying $0 < f(p) < 1$ for $p|k$, we have

$$A_f = A_f^2 = g(1^2 A_h^2),$$

where $g = f^\otimes \frac{1}{2}$ and $h$ is the multiplicative function $f^\otimes \frac{1}{2}$. Thus

$$\Lambda_q = f^\otimes \frac{1}{2}(1^2 A_h)^2,$$

and an equivalent problem is therefore to evaluate $\|A_h\|_{q,2}$ for a general multiplicative function $h$.

As such, let $h_p : D(k) \rightarrow (0, \infty)$ denote the function restricted to $D(p)$; i.e.

$$h_p(n) = \begin{cases} h(n) & \text{if } n = 1, p \\ 0 & \text{otherwise} \end{cases}.$$  

Using the above relation to $\Lambda_q$, it is readily seen\(^1\) that $\|A_{h_p}\|_{q,2} = \sqrt{1 + h(p)^2} \sqrt{Q_q(f(p))}$ with $Q_q$ as in section 3. But also Proposition 3.3 gives

$$\max\left\{ \frac{\|A_x\|_{2,q}}{\|x\|_q} : x \text{ is multiplicative} \right\} = \frac{\sqrt{M_q}}{f^\otimes \frac{1}{2}(1)} = \prod_{p|k} \|A_{h_p}\|_{q,2},$$

by using (5.2). On replacing $h$ by $f$, the conjecture (made after the statement of Theorem 2) is therefore equivalent to

**Conjecture:** Let $f$ be multiplicative on $D(k)$ such that $0 < f(p) < 1$ for all $p|k$. Then

$$\|A_f\|_{q,2} = \prod_{p|k} \|A_{f_p}\|_{q,2},$$

(5.3)

and the norm is achieved at a multiplicative point.

Note that since $A_f = \prod_{p|k} A_{f_p}$ (see Theorem 3.3, [6]), (5.3) may equally be written as

$$\left\|\prod_{p|k} A_{f_p}\right\|_{q,2} = \prod_{p|k} \|A_{f_p}\|_{q,2}.$$  

\(^1\)Use the formula $\sqrt{1 + f(p)^2} + \sqrt{1 - f(p)^2} = \sqrt{1 + h(p)^2}$. 

11
References


