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A note on structured pseudospectra of block matrices

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ABSTRACT

In this note we consider the question of equivalence of pseudospectra and structured pseudospectra of block matrices. The structures we study are all so called double structures; that is, the blocks of the given matrix are of the same structure as the block matrix. The approach is based on that of non-block matrices, which are also briefly studied, and the use of distance to singularity. We also list some open problems and conjectures.

1. Introduction

In perturbation analysis it is natural to ask what happens to the spectrum of a matrix when it is perturbed by matrices of the same structure. While pseudospectra is well-studied and vast literature exists on the subject (see, e.g., [1] and the references therein), much less is known about structured pseudospectra of matrices, and especially of matrices that possess block structures.

Motivation for structured pseudospectra comes from applications, such as floating-point error analysis; situations where the entries are affected by experimental uncertainty; backward error analysis, numerical algorithms and other spectral problems in linear algebra; problems in control theory; and stability theory for dynamical systems; see, e.g., [2–6,1]. Further motivation to study block matrices can be found in the works of Doyle.

Let T be a bounded linear operator on a Banach space X. For $\epsilon > 0$, the ϵ-*pseudospectrum* of T is defined by

$$\sigma_\epsilon(T) = \{\sigma \in \mathbb{C} : \|(\sigma I - T)^{-1}\| \geq \epsilon^{-1}\}.$$

Note here we have used the weak inequality to define the pseudospectrum because our main interest is in finite matrices; compare this with the definitions equipped with strict inequalities in [1].

An equivalent definition is given by

$$\sigma_\epsilon(T) = \{\sigma \in \mathbb{C} : \sigma \in \sigma(T+E) \text{ for } \|E\| \leq \epsilon\}, \tag{1}$$

where $\sigma(T)$ is the usual spectrum of T. For two more equivalent formulations of pseudospectra, see [1]. The definition in (1) can be modified to take into account the structure of perturbation matrices E. Suppose that T is of some structure

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(e.g., Toeplitz, Hankel or symmetric) which we denote by struct. For a list of structures that we consider in this note, see Table 1. The *structured ϵ-pseudospectrum* of T is defined by

$$\sigma_\epsilon^{\text{struct}}(T) = \{\sigma \in \mathbb{C} : \sigma \in \sigma(T+E) \text{ for } E \in M_n^{\text{struct}} \text{ and } \|E\| \le \epsilon\}, \tag{2}$$

where M_n^{struct} stands for the space of all $n \times n$ matrices of the given structure. This definition of the structured pseudospectra was first given in [7] in the context of Toeplitz matrices. It seems that the other definitions of pseudospectra have no analogy for structured matrices equivalent to the one in (2).

We are concerned with matrices that have certain block structures, and extend the definition of the structured pseudospectrum as follows.

Definition 1. For two matrix structures s_1 and s_2 and $\epsilon > 0$, we define the structured ϵ-pseudospectrum of a block matrix A in $M_n^{s_1}(M_m^{s_2}(\mathbb{F}))$ by setting

$$\sigma_\epsilon^{s_1,s_2}(A) = \{\sigma \in \sigma(A+E) : E \in M_n^{s_1}(M_m^{s_2}(\mathbb{F})) \text{ and } \|E\| \le \epsilon\}, \tag{3}$$

where n is the size of the block matrix and m stands for the size of the blocks. Also we write $M_n^*(\mathbb{F}) = M_n(\mathbb{F})$, so for example $\sigma_\epsilon^{\text{Toep},*}(A)$ is used for block Toeplitz matrices whose blocks have no specific structure.

Most results on structured pseudospectra only involve matrices with scalar entries and there are very few results on block matrices. In both cases, we clearly have

$$\sigma_\epsilon^{\text{struct}}(T) \subset \sigma_\epsilon(T). \tag{4}$$

It turns out that for several classes of matrices with scalar entries, the pseudospectrum of a matrix T can be obtained by considering only perturbations of the same structure as that of T. The following result can be found in [8]; see also [9].

Theorem 2. *Let $\epsilon > 0$. If*

$$\text{struct} \in \{\textit{Toep}, \textit{Hank}, \textit{circ}, \textit{sym}, \textit{symToep}, \textit{persym}, \textit{persymHank}\}$$

and $A \in M_n^{\text{struct}}(\mathbb{C})$, then

$$\sigma_\epsilon^{\text{struct}}(A) = \sigma_\epsilon(A).$$

For $A \in M_n^{\text{Herm}}(\mathbb{C})$,

$$\sigma_\epsilon^{\text{Herm}}(A) = \sigma_\epsilon(A) \cap \mathbb{R}.$$

For $A \in M_n^{\text{skewHerm}}(\mathbb{C})$,

$$\sigma_\epsilon^{\text{skewHerm}}(A) = \sigma_\epsilon(A) \cap i\mathbb{R}.$$

To our knowledge this question of whether $\sigma_\epsilon^{\text{struct}}(T) = \sigma_\epsilon(T)$ has not been answered for any block structures or norms other than the spectral norm. Our aim is to initiate a similar study for matrices with block structures when $s_1 = s_2$, which we call the case of double structures. In what follows, we assume that $\|\cdot\|$ is the spectral norm.

When studying the equivalence of the structured and unstructured pseudospectra of a matrix, it is often useful to consider the (structured) distance to singularity. For a nonsingular matrix $A \in M_n(\mathbb{C})$, the distance to singularity $\delta(A)$ is defined by

$$\delta(A) = \min\{\|E\| : E \in M_n(\mathbb{C}), A+E \text{ is singular}\}, \tag{5}$$

and the structured distance to singularity $\delta^{\text{struct}}(A)$ is defined by

$$\delta^{\text{struct}}(A) = \min\{\|E\| : E \in M_n^{\text{struct}}(\mathbb{C}), A+E \text{ is singular}\}. \tag{6}$$

For a block matrix A in $M_n^{s_1}(M_m^{s_2}(\mathbb{F}))$, we set

$$\delta^{s_1,s_2}(A) = \min\{\|E\| : E \in M_n^{s_1}(M_m^{s_2}(\mathbb{F})), A+E \text{ is singular}\} \tag{7}$$

and we use $\delta^{s_1,*}(A)$ to denote the case in which blocks have no specific structure.

As far as we know, distance to singularity of block matrices has not been studied before and all known results on the equivalence in the non-block case are affirmative; that is, there are no "mainstream" matrix structures for which the structured and unstructured distance to singularity differ; see Table 1. However, we give an example of linear structures of the opposite effect below in the next section.

When dealing with the distance to singularity, it is useful to use the following well known identities:

$$\delta(A) = \|A^{-1}\|^{-1} = \sigma_{\min}(A),$$

where $\sigma_{\min}(A)$ stands for the smallest singular value of A; see [1]. The following table lists the structures with their definitions and known results with their references.

Table 1
Comparison of the structured and unstructured distance to singularity of $A \in M_n^{\text{struct}}$.

Struct	$A = (a_{i,j})$	$\delta^{\text{struct}}(A) = \delta(A)$
Toep($\mathbb{F}$)	a_{i-j}	[5]
Hankel($\mathbb{F}$)	a_{i+j}	[5]
circ($\mathbb{F}$)	see (8)	[5]
sym($\mathbb{R}$)	$A^T = A$	[5]
sym($\mathbb{F}$)		Theorem 10
Herm($\mathbb{C}$)	$A^* = A$	[5]
skewHerm($\mathbb{C}$)	$A^* = -A$	[5]
skewsym($\mathbb{R}$)	$A^T = -A$	[5]
skewsym($\mathbb{C}$)		Not known
symToep($\mathbb{R}$)		[5]
symToep($\mathbb{C}$)		Proposition 6
persym($\mathbb{R}$)	$a_{n+1-j,n+1-i}$	[5]
persym($\mathbb{C}$)		Proposition 6
persymHank($\mathbb{R}$)		[5]
persymHank($\mathbb{C}$)		Proposition 6
centrosym($\mathbb{F}$)	$a_{n+1-i,n+1-j}$	Not known

1.1. Notation

We write $\sigma_\epsilon(A)$, $\sigma_\epsilon^{\text{struct}}(A)$, and $\sigma_\epsilon^{s_1,s_2}(A)$ for the (structured) pseudospectra of A; see (1), (2), and (3), respectively. Three types of (structured) distances to singularity are denoted by $\delta(A)$, $\delta^{\text{struct}}(A)$, and $\delta^{s_1,s_2}(A)$; see (5), (6), and (7). The smallest singular value of A is denoted by $\sigma_{\min}(A)$ and the spectrum of A by $\sigma(A)$. We denote by $M_n(\mathbb{F})$ the set of all $n \times n$ matrices with their entries in $\mathbb{F}$, where $\mathbb{F}$ is the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. We denote by $M_n^{\text{struct}}(\mathbb{F})$ the set of all matrices of the given structure. If A is an $n \times n$ block matrix of structure s_1 with $m \times m$ entries of structure s_2, we write $A \in M_n^{s_1}(M_m^{s_2}(\mathbb{F}))$. We denote the transpose of A by A^T and the conjugate transpose by A^*. Particular structures that we consider are defined in Table 1. Finally, we denote the spectral norm by $\|\cdot\|$ and write I for the identity matrix and J for the reversal matrix; see (9).

2. Preliminary results

When dealing with the structured distance to singularity, the following result (see [5, Lemma 10.1]) is one of the key ingredients for Hankel and symmetric structures.

Lemma 3. *Let $\mathbb{F}$ be the field of real or complex numbers. For $x \in \mathbb{F}^n$, there is a Hankel matrix in $M_n(\mathbb{F})$ such that $Hx = \bar{x}$ and $\|H\| \le 1$.*

We need the following direct consequence when we study structured block matrices.

Lemma 4. *Let $x \in \mathbb{F}^{nm}$. If $nm = jk$, then there exists a block Hankel matrix in $M_j^{\text{Hank}}(M_k^{\text{Hank}}(\mathbb{F}))$ such that*

$$Hx = \bar{x} \quad \text{and} \quad \|H\| \le 1;$$

that is, H is a $j \times j$ block Hankel matrix whose blocks are $k \times k$ Hankel matrices.

There is a version of Lemma 3 for Toeplitz matrices but it works only for some vectors x; see [9, Lemma 2.4].

Lemma 5. *If $x = \alpha Jx \in \mathbb{F}^n$ with $\alpha \in \{-1, 1\}$, then there is a $T \in M_n^{\text{symToep}}(\mathbb{F})$ such that $Tx = \bar{x}$ and $\|T\| = 1$. If x is real, T can be chosen to be real.*

The following result is in [5, Theorem 12.1] in the real case, but with the preceding lemma it is easy to see it also holds in the complex case.

Proposition 6. *Let $A \in M_n^{\text{struct}}(\mathbb{C})$, where*

$$\text{struct} \in \{symToep, persym, persymHank\}.$$

Then $\delta(A) = \delta^{\text{struct}}(A)$.

Proof. It is trivial that $\delta(A) \le \delta^{\text{struct}}(A)$. First let A be symmetric Toeplitz. Then A is symmetric and persymmetric, and hence $Ax = \sigma_{\min}(A)\bar{x}$ for some $x \in \mathbb{C}^n$ satisfying $x = \alpha Jx$, where $\alpha \in \{-1, 1\}$ (see [9, Lemma 4.2]). By Lemma 5, there is a symmetric Toeplitz matrix T such that $Tx = \bar{x}$ and $\|T\| = 1$. Thus,

$$(A - \sigma_{\min}(A)T)x = 0$$

and $\|\sigma_{\min}(A)T\| = \sigma_{\min}(A)$, and so $\delta^{\text{symToep}}(A) \le \sigma_{\min}(A) = \delta(A)$.

Next we suppose that $A \in M_n^{\text{persym}}(\mathbb{C})$. Then there is an $x \in \mathbb{C}^n \setminus \{0\}$ such that $Ax = \sigma_{\min}(A)J\bar{x}$ (see [9, Lemma 4.2]). By Lemma 3, there is an $H \in M_n^{\text{Hank}}(\mathbb{C})$ such that $Hx = \bar{x}$ and $\|H\| = 1$. Now JH is Toeplitz and hence persymmetric. It remains to note that $(A - \sigma_{\min}JH)x = 0$ and $\|\sigma_{\min}(A)JH\| = \sigma_{\min}(A)$.

We reduce the case of persymmetric Hankel matrices to that of symmetric Toeplitz matrices. Let $A \in M_n^{\text{persymHank}}(\mathbb{C})$. Observe that $AJ \in M_n^{\text{symToep}}(\mathbb{C})$, $\|AJ\| = \|A\|$ and B is singular if and only if BJ is singular. Thus,

$$\begin{aligned}\delta(A) &= \delta(AJ) = \delta^{\text{symToep}}(AJ)\\ &= \min\{\|E\| : \det(AJ+E) = 0,\ E \in M_n^{\text{symToep}}(\mathbb{C})\}\\ &= \min\{\|EJ\| : \det(AJ+EJ) = 0,\ E \in M_n^{\text{persymHank}}(\mathbb{C})\}\\ &= \min\{\|E\| : \det(A+E) = 0,\ E \in M_n^{\text{persymHank}}(\mathbb{C})\}\\ &= \delta^{\text{persymHank}}(A). \quad \square\end{aligned}$$

It is not difficult to show that there are linear structures for which the preceding theorem fails as seen in the following example.

Example 7. Consider matrices of the form $\begin{pmatrix} a & a \\ b & a \end{pmatrix}$, where $a, b \in \mathbb{F}$. The set of such matrices forms a linear structure. Let A be of this structure with $a = 1$ and $b = 0$. Then $A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and

$$\|A^{-1}\| = \sup_{x \neq 0} \frac{\|A^{-1}x\|}{\|x\|} = \sup_{x \neq 0} \left(\frac{(x_1 - x_2)^2 + x_2^2}{x_1^2 + x_2^2} \right)^{1/2} \geq \sqrt{\frac{5}{2}}.$$

Thus, $\delta(A) = \sigma_{\min}(A) = \|A^{-1}\|^{-1} \leq \sqrt{\frac{2}{5}}$. Given a perturbation $\Delta A = \begin{pmatrix} u & u \\ v & u \end{pmatrix}$, the matrix $A + \Delta A$ is singular if and only if $(1+u)^2 - v(1+u) = 0$; that is,

$$\Delta A = \begin{pmatrix} u & u \\ 1+u & u \end{pmatrix} \quad \text{or} \quad \Delta A = \begin{pmatrix} -1 & -1 \\ u & -1 \end{pmatrix}.$$

Note

$$\left\| \begin{pmatrix} u & u \\ 1+u & u \end{pmatrix} \right\| = |u| \left\| \begin{pmatrix} 1 & 1 \\ \frac{1}{u} + 1 & 1 \end{pmatrix} \right\|$$

and

$$\begin{aligned}\left\| \begin{pmatrix} 1 & 1 \\ \frac{1}{u} + 1 & 1 \end{pmatrix} \right\| &= \sup_{(a,b) \neq 0} \left(\frac{(a+b)^2 + \left(\frac{a}{u} + a + b\right)^2}{a^2 + b^2} \right)^{1/2}\\ &\geq \sqrt{\frac{a^2}{2a^2u^2}} = \frac{1}{\sqrt{2}|u|},\end{aligned}$$

so $\| \begin{pmatrix} u & u \\ 1+u & u \end{pmatrix} \| \geq 1/\sqrt{2}$ for all $u \in \mathbb{R}$. Also,

$$\left\| \begin{pmatrix} -1 & -1 \\ u & -1 \end{pmatrix} \right\| = \sup_{(a,b) \neq 0} \left(\frac{(-a-b)^2 (au-b)^2}{a^2 + b^2} \right)^{1/2} \geq \sqrt{2}.$$

Thus, $\delta^{\text{struct}}(A) \geq 1/\sqrt{2}$ and hence $\delta^{\text{struct}}(A) > \delta(A)$.

The following simple observations about the correspondence between matrices of certain structure and double structured matrices is useful in what follows.

Proposition 8. *For $n, m \in \mathbb{N}$, the following inclusions are proper*

$$M_{nm}^{\text{Toep}} \subset M_n^{\text{Toep}}(M_m^{\text{Toep}}(\mathbb{F})),$$
$$M_{nm}^{\text{Hank}} \subset M_n^{\text{Hank}}(M_m^{\text{Hank}}(\mathbb{F})),$$

and

$$M_n^{\text{sym}}(M_m^{\text{sym}}(\mathbb{F})) \subset M_{nm}^{\text{sym}}(\mathbb{F}).$$

An example of a structure where we have no inclusion either way is given by circulant matrices:

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_n & a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & & & \vdots \\ a_1 & a_2 & \cdots & a_n & a_0 \end{pmatrix}; \tag{8}$$

that is, each row of A is the previous row cycled forward one step.

The following result is well known but as it is quite central to the treatment of most (double) structures in our work we recall its simple proof.

Lemma 9. *If A is symmetric, then*

$$A\bar{x} = \sigma_{\min}(A)x,$$

where x is the last column vector of the unitary matrix U in the factorization $A = U\Sigma U^T$. If A is real, then x can be chosen to be real.

Proof. Since A is symmetric, there is a unitary matrix U and a diagonal matrix Σ whose entries consist of the singular values of A in a nonincreasing order (see [10, Corollary 4.4.4]). Using the fact that the columns of U are orthonormal (see [10, Theorem 2.1.4]), it is easy to see that $A\bar{x} = \sigma_{\min}(A)x$. □

3. Main results

We are concerned with block Toeplitz, Hankel and symmetric matrices whose blocks have the same structure as the given block matrix. We call these matrices *double structured* block matrices. We first show that the structured and unstructured distances to singularity of a matrix A are equal when A is a symmetric, Hankel, persymmetric or Toeplitz double structured block matrix. Our approach is based on similar considerations of structured matrices in [8,5] which strongly rely on symmetric structures.

We consider first double structures that we can deal with both real and complex cases.

Theorem 10. *If $A \in M_n^s(M_m^s(\mathbb{F}))$, where*

$$s \in \{\text{Toep}, \text{Hank}, \text{sym}, \text{persym}\},$$

then

$$\delta^{s,s}(A) = \delta(A) = \sigma_{\min}(A) = \|A^{-1}\|^{-1}.$$

Proof. Since we are using the spectral norm, it is clear that we have $\delta^{s,s}(A) \geq \delta(A) = \sigma_{\min}(A)$ for any structure s.

For the reverse inequality, it suffices to show that there exists a matrix $\Delta A \in M_n^s(M_m^s(\mathbb{F}))$ such that $A + \Delta A$ is singular and $\|\Delta A\| = \sigma_{\min}(A)$. Suppose first that $s = \text{sym}$. By Lemma 9, there is an x such that $A\bar{x} = \sigma_{\min}(A)x$. By Lemma 4, there is an $H \in M_n^{\text{Hank}}(M_m^{\text{Hank}}(\mathbb{F}))$ such that $H\bar{x} = x$ and $\|H\| \leq 1$. Clearly, H is in $M_n^{\text{sym}}(M_m^{\text{sym}}(\mathbb{F}))$. Let $\Delta A = -\sigma_{\min}(A)H$. Then ΔA is of the same structure as A and $\|\Delta A\| \leq \sigma_{\min}(A)$. Since $(A + \Delta A)\bar{x} = 0$, it follows that $\delta^{\text{sym},\text{sym}}(A) \leq \sigma_{\min}(A) = \delta(A)$. The same proof works for $s = \text{Hank}$.

We define the reversal matrix J by setting

$$(J)_{i,nm-i+1} = 1 \quad \text{for } i = 1, \ldots, nm \tag{9}$$

and other entries equal to zero. Let $A \in M_n^{\text{Toep}}(M_m^{\text{Toep}}(\mathbb{F}))$. Then $AJ \in M_n^{\text{Hank}}(M_m^{\text{Hank}}(\mathbb{F}))$ and $\|AJ\| = \|A\|$. Therefore,

$$\begin{aligned} \delta(A) = \delta(AJ) &= \delta^{\text{Hank},\text{Hank}}(AJ) \\ &= \min\{\|E\| : \det(AJ + E) = 0, E \in M_n^{\text{Hank}}(M_m^{\text{Hank}}(\mathbb{F}))\} \\ &= \min\{\|EJ\| : \det(AJ + EJ) = 0, E \in M_n^{\text{Toep}}(M_m^{\text{Toep}}(\mathbb{F}))\} \\ &= \min\{\|E\| : \det(A + E) = 0, E \in M_n^{\text{Toep}}(M_m^{\text{Toep}}(\mathbb{F}))\} \\ &= \delta^{\text{Toep},\text{Toep}}(A). \end{aligned}$$

Similarly, if $A \in M_n^{\text{persym}}(M_m^{\text{persym}}(\mathbb{F}))$, then $AJ \in M_n^{\text{sym}}(M_m^{\text{sym}}(\mathbb{F}))$ and we get

$$\delta(A) = \delta(AJ) = \delta^{\text{sym},\text{sym}}(AJ) = \delta^{\text{persym},\text{persym}}(A). \quad \square$$

Remark 11. It follows from the proof of the preceding theorem that for $S \in M_n^{\text{sym}}(M_m^{\text{sym}}(\mathbb{F}))$,

$$\delta(S) = \delta^{\text{Hank,Hank}}(S);$$

that is, to compute the distance to singularity of a symmetric matrix, it suffices to consider only perturbations that are Hankel.

Remark 12. Obviously we cannot apply the procedure of Proposition 6 to deal with $A \in M_n^{\text{symToep}}(M_m^{\text{symToep}}(\mathbb{C}))$ because the matrix A may not be Toeplitz.

Theorem 13. *Let $A \in M_n^s(M_m^s(\mathbb{R}))$, where*

$$s \in \{\text{circ}, \text{symToep}, \text{persymHank}\},$$

then

$$\delta^{s,s}(A) = \delta(A) = \sigma_{\min}(A) = \|A^{-1}\|^{-1}.$$

Proof. Let $s = $ symToep and $A \in M_n^s(M_m^s(\mathbb{R}))$. By Lemma 9, there is a real x such that

$$Ax = \sigma_{\min}(A)x.$$

Choose $\Delta A = -\sigma_{\min}(A)I$. Then $\Delta A \in M_n^s(M_m^s(\mathbb{R}))$, $A + \Delta A$ is singular and $\|\Delta A\| = \sigma_{\min}(A)$, so $\delta^{s,s}(A) \le \sigma_{\min}(A) = \delta(A)$.

Let $s = $ persymHank and $A \in M_n^s(M_m^s(\mathbb{R}))$. Then

$$AJ \in M_n^s(M_m^s(\mathbb{R}))$$

and we can proceed as in the proof of Theorem 10 for the Toeplitz structure.

If

$$A \in M_n^{\text{circ}}(M_m^{\text{circ}}(\mathbb{F})),$$

then A is normal and

$$A = F^* \Lambda F,$$

where Λ is a diagonal matrix and F is the two-dimensional unitary Fourier transform matrix. Similarly to Lemma 9, we see that there is a real x such that $Ax = \sigma_{\min}(A)x$ and we can proceed as before. □

One reason to study (structured) distance to singularity is its connection to (structured) pseudospectra. In the unstructured case, we have the following result; see [11].

Lemma 14. *Let s_1 and s_2 be linear structures and $A \in M_n^{s_1}(M_m^{s_2}(\mathbb{F}))$. If $\epsilon > 0$ and if the identity matrix is of the same structure as A, then*

$$\sigma_\epsilon^{s_1,s_2}(A) = \{\lambda \in \mathbb{C} : \delta^{s_1,s_2}(A - \lambda I) \le \epsilon\}.$$

Proof. This follows from the observation that $\lambda \in \sigma(A + E)$ for some $E \in M_n^{s_1}(M_m^{s_2}(\mathbb{F}))$ with $\|E\| \le \epsilon$ if and only if $\det(A + E - \lambda I) = 0$ for some $E \in M_n^{s_1}(M_m^{s_2}(\mathbb{F}))$ with $\|E\| \le \epsilon$. □

Theorem 15. *Let $\epsilon > 0$. If*

$$s \in \{\text{Toep}, \text{Hank}, \text{sym}, \text{persym}\}.$$

and if $A \in M_n^s(M_m^s(\mathbb{F}))$, then

$$\sigma_\epsilon^{s,s}(A) = \sigma_\epsilon(A). \tag{10}$$

In the real case, if

$$s \in \{\text{symToep}, \text{persymHank}, \text{circ}\},$$

and if $A \in M_n^s(M_m^s(\mathbb{R}))$, then $\sigma_\epsilon^{s,s}(A) = \sigma_\epsilon(A)$.

Proof. If $s_1 = s_2 \in \{\text{Toep}, \text{sym}, \text{persym}\}$, then $s\lambda I \in M_n^{s_1}(M_m^{s_2}(\mathbb{F}))\}$, and we can apply Theorem 10 together with the previous lemma to conclude that (10) holds for these three structures. Symmetric Toeplitz matrices and circulant matrices can be dealt with similarly.

Suppose that $s_1 = s_2 = $ Hank and let $\lambda \in \sigma_\epsilon(A)$. Then $\lambda \in \sigma(A + E)$ for some $E \in M_{nm}(\mathbb{F})$ with $\|E\| \le \epsilon$. Since $A - \lambda I$ is symmetric, there is an x such that $(A - \lambda I)\bar{x} = \sigma_{\min}(A - \lambda I)x$ (see Lemma 9). By Lemma 4, there is an $H \in M_n^{\text{Hank}}(M_m^{\text{Hank}}(\mathbb{F}))\}$ such that $Hx = \bar{x}$ and $\|H\| \le 1$. Put $\Delta A = -\sigma_{\min}(A - \lambda I)H$. Then $(A + \Delta A - \lambda I)\bar{x} = 0$, and so $\lambda \in \sigma_\epsilon^{\text{Hank}}(A)$. □

4. Open problems and conjectures

As stated above, in the case of non-block matrices, the structured distance to singularity and structured pseudospectra are known for many structures but there are still some important ones left to study. For example, matrices that possess certain centrosymmetric structures; that is, matrices that are symmetric about their geometric center. More precisely, A is centrosymmetric if $JA = AJ$, where J is the reversal matrix. Obviously the identity matrix is centrosymmetric and hence the study of its structured pseudospectrum can be reduced to determining its distance to singularity via Lemma 14. Another useful observation is that when $n = 2m$, the matrix A can be written as a block matrix:

$$A = \begin{pmatrix} B & JCJ \\ C & JBJ \end{pmatrix},$$

where $B, C \in M_m(\mathbb{F})$. When $n = 2m + 1$, we have

$$A = \begin{pmatrix} B & Jy & JCJ \\ x^T & \alpha & x^T J \\ C & y & JBJ \end{pmatrix}$$

for some $B, C \in M_m(\mathbb{F})$, $x, y \in \mathbb{F}^m$, and $\alpha \in \mathbb{F}$. Other similar structures are skew centrosymmetric matrices ($JA = -AJ$), centrohermitian ($JA = \bar{A}J$), and skew centrohermitian ($JA = -\bar{A}J$).

Regarding block matrices, for $s \in \{\text{Herm}, \text{skewHerm}\}$, the description of $\sigma_\epsilon^{s,s}(A)$ remains open. Also, we have not been able to answer the question for symmetric Toeplitz, persymmetric Hankel or circulant matrices in the complex case. It may well be that something like Lemma 5 is needed.

We have only considered block matrices that are double structured in this note. It is of considerable interest to answer these questions also for block matrices whose blocks have no special structure. Since the identity matrix in $M_{nm}(\mathbb{F})$ can be viewed as a block Toeplitz matrix, we can invoke Lemma 14 again and we conjecture that for a block Toeplitz matrix A, $\delta^{\text{Toep},*}(A) = \delta(A)$ and hence $\sigma_\epsilon^{Toep,*}(A) = \sigma_\epsilon(A)$ as in the double structured case. This would also imply that the structured and unstructured pseudospectra of a block Hankel matrix are equal.

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References

[1] L.N. Trefethen, M. Embree, Spectra and Pseudospectra, Princeton University Press, Princeton, NJ, 2005, The behavior of nonnormal matrices and operators.
[2] B. Adhikari, R. Alam, Structured mapping problems for linearly structured matrices, Linear Algebra Appl. 444 (2014) 132–145.
[3] D. Hinrichsen, A.J. Pritchard, On spectral variations under bounded real matrix perturbations, Numer. Math. 60 (4) (1992) 509–524.
[4] M. Karow, Geometry of Spectral Value Sets (Ph.D. thesis), Universität Bremen, 2003.
[5] S.M. Rump, Structured perturbations. I. Normwise distances, SIAM J. Matrix Anal. Appl. 25 (1) (2003) 1–30 (electronic).
[6] F. Tisseur, S. Graillat, Structured condition numbers and backward errors in scalar product spaces, Electron. J. Linear Algebra 15 (2006) 159–177.
[7] A. Böttcher, S. Grudsky, A. Kozak, On the distance of a large Toeplitz band matrix to the nearest singular matrix, in: Toeplitz Matrices and Singular Integral Equations (Pobershau, 2001), in: Oper. Theory Adv. Appl., vol. 135, Birkhäuser, Basel, 2002, pp. 101–106.
[8] S. Graillat, A note on structured pseudospectra, J. Comput. Appl. Math. 191 (1) (2006) 68–76.
[9] S.M. Rump, Eigenvalues, pseudospectrum and structured perturbations, Linear Algebra Appl. 413 (2–3) (2006) 567–593.
[10] R.A. Horn, C.R. Johnson, Matrix Analysis, second ed., Cambridge University Press, Cambridge, 2013.
[11] L.N. Trefethen, Computation of pseudospectra, in: Acta Numerica, 1999, in: Acta Numer., vol. 8, Cambridge Univ. Press, Cambridge, 1999, pp. 247–295.