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M-PERT. A manual project duration estimation technique for teaching scheduling basics

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ABSTRACT

The Program Evaluation and Review Technique (PERT) has become a classic Project Management tool for estimating project duration when the activities have uncertain durations. However, despite its simplicity and widespread adoption, the original PERT, in neglecting the merge event bias, significantly underestimated the duration average and overestimated the duration variance of real-life projects. To avoid these and other shortcomings, many authors have worked over the last 60 years at producing interesting alternative PERT extensions. This paper proposes joining the most relevant of those to create a new reformulated PERT, named M-PERT.

M-PERT is quite accurate when estimating real project duration, while also allowing for a number of interesting network modelling features the original PERT lacked: probabilistic alternative paths, activity self-loops, minima of activity sets and correlation between activities. However, unlike similar scheduling methods, M-PERT allows manual calculation through a recursive merging procedure that downsizes the network until the last standing activity represents the whole (or remaining) project duration. Hence, M-PERT constitutes an attractive tool for teaching scheduling basics to engineering students in a more intuitive way, with or without the assistance of computer-based simulations or software. One full case study will also be proposed and future research paths suggested.

KEYWORDS

Scheduling, PERT; GERT; Stochastic Network Analysis; Project duration.

Introduction

In 1959, an Operations Research team formed by D.G. Malcom, J.H. Roseboom and C.E. Clark from the company *Booz, Allen and Hamilton* in Chicago, along with W. Fazar, from the *US Navy Special Projects Office* in Washington, published a technique for measuring and controlling the progress of the Polaris Fleet Ballistic Missile (FBM) program (Malcolm et al. 1959). This technique was initially named the *Program Evaluation Research Task* and later renamed the *Program Evaluation and Review Technique* (PERT). It has become one of the most popular Project Management (PM) scheduling tools since then.

The initial idea was to provide the project manager with an integrated and quantitative management methodology that allowed him/her to evaluate '(a) [the] progress to date and [the] outlook for accomplishing the objectives of the FBM program, (b) [the] validity of established plans and schedules for accomplishing the program objectives, and (c) [the] effects of changes proposed in established plans' (Malcolm et al., 1959, p.646). These three aims were quite ambitious indeed, and despite one imagining PERT doing all of these with a leap of faith, what PERT actually and basically does is to estimate the probabilistic duration of any project or the remaining part of a project when the activities have uncertain (not deterministic) durations. Also, PERT can be implemented for cost estimation purposes with some technical variations (e.g. (Asmar et al. 2011)); however, for the sake of clarity we will just deal with the time dimension in this paper.

In its favor, this technique is very easy to implement and was the first of its kind for dealing with projects whose activity durations are modeled by probability distributions, or Stochastic Network Analysis (SNA) as it known as nowadays. Against it, the procedure proposed by PERT underestimates the project duration when there are multiple parallel paths, which unfortunately is the norm in construction projects. Indeed, the authors were aware of this shortcoming since they very briefly mentioned in the original paper that 'this simplification gives biased estimates such that the estimated expected time of events are always too small' (Malcolm et al. 1959). Two publications by C.E. Clark indeed followed the publication of PERT (Clark 1961, 1962), but the solution proposed was not easy to implement, so it opened interesting avenues of research for the scientific community that have continued to this day, mostly in the Operations Research discipline.

Since the inception of PERT almost 60 years ago, the number of fields in which this technique has been applied has been quite varied. It is not only applied to military, research & development and civil engineering projects as was originally intended (Stauber et al. 1959), but also to fields as diverse as medicine (e.g. Woolf et al. (1968)), exports (e.g. Tatterson (1974)), contracting (e.g. Mummolo (1997)) and rural development (e.g. Tavares (2002)), to cite just a few. In fact, its reach and application are virtually unlimited since it is useful for any project duration estimation purposes where activities have non-deterministic durations.

However, another collateral problem has been that despite PERT being a celebrated PM technique, it also probably is one of the most widely misunderstood. In particular, many engineering and scheduling students, as well as professionals, still think that PERT is either a type of network representation, as one of the first versions of the Project Management Body of Knowledge (PMBoK) pointed out explicitly (PMI 1996), or that it is just a three-point time and/or cost estimation technique (PMI 2008). The latter misrepresentation is the consequence of the second half of the PERT method, which deals with calculating the expected project duration and its variance by means of critical path project activities, being left out of this publication.

The purpose of this paper is to update PERT and propose a newly redefined technique named M-PERT, which deals with the most relevant shortcoming of the original technique (the merge event bias), allows manual calculation, and adds a series of new interesting features that the original technique lacked. M-PERT will allow the modelling of real-life projects far more representatively and help scheduling students to understand more intuitively basic concepts of scheduling when activities have uncertain durations.

The number of PERT-related publications might be counted in the thousands nowadays, so we cannot reference them all. Instead, as in any other mature fields, we will undertake a systematic review of the most relevant sources that have shaped this new technique. This systematic review will be presented in the *Background* section under three thematic subsections. Then, an outline of the model will be presented followed by an application in the *M-PERT outline* and *Case study* sections. A *Results* section will gather the application outcomes while comparing the accuracy improvements achieved, whereas a final and brief *Discussions and conclusions* section will summarize the most important aspects and possible research directions for the model proposed.

Background

<u>An overview of PERT and its limitations</u>

In a nutshell, PERT is a scheduling tool that is applied in two stages. First, after identifying all the activities involved in a project and stating their precedence relationships, the first stage comprises modelling their probable durations. For that purpose, the original PERT authors proposed a three-point estimation procedure. The user (or another expert on hand) is asked to state the Optimistic (O), most Likely (L) and Pessimistic (P) durations that each activity is foreseen to have before those activities take place. Then, the mean duration (μ) and the standard deviation (σ) from each activity are calculated by using these three duration estimates (O, L and P) by means of these straightforward expressions:

$$= \left(O + 4L + P\right)/6 \tag{1}$$

$$= \left(P \quad O\right)/6 \tag{2}$$

With the values of μ and σ for each activity, a Beta distribution which varies between [*O*, *P*] is fitted for modelling the activity duration. However, despite the highly controversial and still ongoing debate about the accuracy of both the three-point estimate procedure and the choice of the Beta distribution to model the activity durations, the PERT method is only interested in the μ and σ values of each activity for the second stage of the application. This second stage of PERT (not mentioned by any Project Management standard) comprises: (a) identifying the activities that are in the critical path, and (b) assuming that the Project duration mean (μ_p) and the standard deviation (σ_p) are equal:

$$\mu_p = \sum_{i \in critical\ path} \mu_i \tag{3}$$

$$\sigma_p = \sqrt{\sum_{i \in critical\ path} \sigma_i^2} \tag{4}$$

Now, if the project scheduler wants to know what the probabilities of ending a project in *X* days are, he/she just needs to know that the project total duration is supposed to be following a Normal distribution with mean μ_p and standard deviation σ_p. Then, the specific probability of *X* happening can be calculated by means of either the mathematical expressions of the cumulative Normal distribution, by resorting to a spreadsheet or, when the calculations are made by hand, by standardizing the project duration as in Equation 5 and then looking up the cumulative probability value of *z* in a standard Normal table.

$$z = (X - \mu_p)/\sigma_p \tag{5}$$

This is the essence of the PERT method. The logical lack of trust displayed by the research community since PERT was published has manifested itself as a determination to see what kind of errors are involved by implementing it. In this regard, MacCrimmon and Ryavec (1964) broke down PERT into every possible piece and developed the most thorough analytical review that anyone has made of PERT since its publication. This piece of research was actually outsourced by the US Air Force to the Rand Corporation due to the consistent and systematic errors that PERT was evidencing in their project forecasts, and they divided their memorandum into two main sections. In the first section, they studied the activity duration-level errors, that is, '(1) the assumption of the Beta distribution, (2) imprecise time estimates, and (3) the assumption of the standard deviation (one-sixth of the range) and the approximation formula for calculating the mean time' (MacCrimmon and Ryavec 1964). The second section checked the accuracy of the project duration mean, variance and the (Normal) probability statements.

The results summary from the first section was that using the three-point estimates could cause a range estimation error of between 10% and 20% of the range [O, P], but, despite the error of estimation of the mean (μ) and standard deviation (σ) being likely to be bigger (from 15% to 30%), a degree of cancellation was expected to occur depending on the network. Indeed, there is generally a high degree of cancellation as will be proven later. But the other noteworthy statement that the authors made at that time was that, to a certain extent, it was pointless having a discussion about what the right kind of distribution is – Beta or another – for modelling an activity duration. Indeed, they reinforced this idea by saying that if PERT had used triangular distributions instead of Beta distributions, their analysis would have given almost identical results (MacCrimmon and Ryavec 1964). A recent study only focusing on determining the importance of the specific distribution selection in PERT also supported this claim and stated that the maximum deviation by using alternative distributions in the project duration an estimation is generally well below 10% (Hajdu and Bokor 2014).

Nevertheless, the results from the second section of the PERT model analysis evidenced the real problem. They confirmed that unless there is one clear dominant critical path (a chain of activities whose total duration is significantly longer than the second longest path), the PERT-calculated mean (equation 3) will be biased optimistically (underestimating the real project duration), while the PERT-calculated variance (equation 4) will be biased in the other direction (overestimating the actual duration variance). At the time this study was published, Extreme Value theory had not been properly developed, but the authors clearly identified that the higher the number of parallel paths, the more positively skewed the distribution modelling the project duration would become, making the Normally-distributed assumption of the ‘total’ project duration of questionable validity with multiple parallel paths.

<u>What matters and what does not matter in PERT</u>

The first part of the systematic review by MacCrimmon and Ryavec (1964) focused on the activity level duration modelling. They stated that the particular choice of the distribution was hardly relevant. Instead, it was the distribution (duration) mean and variance that really matters. This was an important outcome that, apparently, a big part of the research community had tried to ignore due to the extraordinary amount of research focused on improving the particular statistical distribution for better modelling of the activity duration.

In a similar vein, a recent study by Herrerías-Velasco et al. (2011), seeking to close the unending debate about whether the approximations originally proposed by the PERT authors to estimate whether the activity duration mean and variance were accurate enough, concluded that, in general, the expression of the mean (Equation 1) is very accurate, whereas the expression for the variance (Equation 2) requires to be multiplied by a correction factor K whose expression is:

$$K = \sqrt{\frac{5}{7} + \frac{16}{7}\,\frac{(L - O)(P - L)}{(P - O)^2}} \tag{6}$$

Concerning the (un)importance of the specific distribution chosen for modelling the activity duration, Figure 1 is divided into two vertical blocks. The top half shows how the statistical distribution resulting from one, two and three activities in series evolves when the activity duration is modeled by a Uniform distribution (first row) or a Beta (PERT-like) distribution (second row). The case with just one activity (first column) illustrates indeed the original distribution, but when more independent and identically distributed (*iid*) activities are added in series, the resulting distribution quickly changes its shape to resemble a Normal distribution. In fact, if we observed the result for five activities in series, it would be quite difficult to tell them apart from a Normal distribution. Therefore, the particular statistical distribution chosen hardly matters since it is difficult to find more extreme examples than a Uniform and a Beta distribution, both equally and quickly converging to a Normal distribution.

<Figure 1>

However, what is relevant is the original activity duration mean and standard deviation. Both the Uniform and the Beta distribution had the same duration mean of μ=5 (time units) and it is easy to see that as we add more activities in series, the resulting mean is $n{\cdot}\mu$, where n is the number of activities in the series ($n{\cdot}\mu$ =10 for two activities, 15 for three activities). On the other hand, the Uniform distribution above has a higher variance (higher standard deviation) compared to the Beta distribution below which, on top of that, is positively skewed and has a range of variation from 0 to 15. It is again evident that as we add more activities in series, the resulting distribution of the Uniform is sparser (since it originally had a bigger variance), whereas the result from the Beta-distributed activities has almost removed the skewness (degree of asymmetry) while it is narrower (duration values are more concentrated near the mean).

Therefore, in any project that has three or more activities in series (basically any real project), the selection of the specific statistical distribution is not relevant. What matters is to specify as accurately as possible the activity mean and standard deviation. The original PERT technique proposed doing this by a three-point estimate, but this caused an error of between 15 and 35%, particularly concentrated in the variance estimation (when departing from the *O*, *L* and *P* estimates). The obvious solution is to get rid of the intermediate step and ask the scheduler (or the field expert) to directly provide a mean and a standard deviation for each activity duration, and if he/she feels unsure about quantifying particularly the variance, then resort to the original PERT expressions (equations 1 and 2) with the correction factor stated in Equation 6.

With the activity-level duration issues clarified, it is time to address another PERT shortcoming: the result of having multiple parallel paths in a network. To exemplify this, we will make use of the bottom half of Figure 1. But first, let us propose an example.

Imagine that the activity duration can be modeled with a fair six-sided die. Then, the activity duration might be {1, 2, 3, 4, 5, 6}, with equal probability each of 1/6 (a discrete Uniform distribution). The average of these six possible outcomes would be 3.5 (time units), So, if we cast two dice, the average duration would be two times 3.5, that is, 7. If we cast three dice the average duration would be three times 3.5, that is, 10.5. This is the effect of adding activities in series; we are just 'adding' distributions (convolution of distributions), as in the upper half of Figure 1. But when there are several parallel paths, we are not doing sums anymore: we are taking the 'maxima' (computing the Cumulative Distribution Function of a maximum) of the different path durations. In our dice example, imagine that we have two parallel paths the duration of which are modeled by a six-sided die. The next activity after these two paths merge in one path (gray nodes in Figure 1) will start only after the path with the maximum duration is finished (the one with the maximum die cast value). If two dice are cast, the maximum of both dice will not be 3.5 anymore, it will be higher. Analogously, if we had more and more parallel paths (imagine 10 paths with one activity each, for instance), the probabilities of getting at least one six in one of the casts would be quite high. Therefore, what is happening is that when several parallel paths converge, the mean project duration is shifted to the right (the project ends later than expected) and the variance contracts. Additionally, the skewness and kurtosis also momentarily deviate from the Normal distribution (resembling more at that point an Extreme Value distribution). But unlike the mean and the variance, they will dissipate again when more activities in the series are added after the activities merger point or before the paths are separated.

This same effect can be visualized in the bottom half of Figure 1. The same two distributions were chosen to illustrate how as more and more paths converge, the resulting distribution (which actually is the highest order statistic of several distributions) shifts its mean towards the right (it is increasingly higher than 5, which was the original mean duration value), whereas the dispersion (variance) also decreases (the distribution becomes more compressed). Overall, this phenomenon has been named *merge bias* or *merge-event bias* (Khamooshi and Cioffi 2013; Vanhoucke 2012), and it was the biggest problem that the original PERT had and the one that the scientific community has struggled most to solve. Indeed, this interesting phenomenon is hardly known by most professional construction schedulers nowadays.

The reason why this has been an enduring problem (indeed, no exact analytical solution has been found to date) is that despite apparently any distribution could be chosen to model the activity durations, that distribution would need to be sum-stable and max-stable. Sum-stable means that the distribution of the convolution (sum) of several activity durations (activities in series) should again belong to the same kind of distribution. An example is the Normal distribution, in which, after being summed, the result is again Normal and with a mean and standard deviation as in equations 3 and 4. Max-stable means that after computing the maximum from some distributions of the same type (activities in parallel), the result is again another distribution of the same type. The only two existing max-stable distributions are the Gumbel distribution and the Fréchet distribution, and neither one is sum-stable. Hence, the ideal situation in which both operations can be performed with just one type of distribution is not feasible.

To sum up, it is not possible to resort to an exact analytical approach for solving the PERT major shortcoming, and the only alternative is to work with mathematical approximations that keep the mean and variance relatively intact towards the end of the project. The logical alternative will be to resort to the Normal distribution, since at least it is sum-stable (which is the most frequent operation in any network due to the generally dominant number of activities in series), it is very well documented in the PERT-related literature, and it is still easy to handle.

Review of the most relevant PERT-related extensions

The study of project completion times when activities have uncertain durations mostly started attracting interest when PERT was published by Malcolm et al. (1959), but it has continued up to the present. Seminal works that followed PERT publication have been Anklesaria and Drezner's (1986) multivariate approach to estimating the project completion time for stochastic networks, Elmaghraby's (1989) study, review and critique of the estimation of activity network parameters, Kamburowski's (1989) network analysis in situations when probabilistic information is incomplete, and, more recently, the computational studies by Ludwig et al. (2001) on bounding the makespan distribution of stochastic project networks. These works are considered, without exception, classics in the stochastic network analysis (SNA) field and have established the foundations of most of the later PERT-related research.

However, computer-based (Monte Carlo) simulations have been to date probably the best alternative for obtaining a highly accurate representation of the actual project duration distribution no matter what the SNA topology and size are (e.g. (Douglas 1978; Hajdu 2013; Khamooshi and Cioffi 2013; Nelson et al. 2016)). Simulations will also be used here for comparing M-PERT outputs with the actual solution. But, obviously, the main reason why approximations like M-PERT are developed is because implementing computer simulations in schedule networks with special attributes (e.g., self-loops, alternative paths, correlated activities) is not usually straightforward for most construction students and practitioners (Ballesteros-Pérez et al. 2015). Also, manual procedures like M-PERT allow step-by-step incremental calculations, offering a more intuitive vision than simulation results.

Likewise, there have been other attempts to create pieces of software to develop PERT-like approaches for Stochastic Network Analysis (SNA) (e.g., Pontrandolfo (2000); Trietsch and Baker (2012)) that basically have the same aim as the Monte Carlo simulation, but allow a higher user interaction mostly oriented to enhancing the Decision Support process and progress monitoring, but with some accuracy loss.

In parallel, many recent algorithms (e.g., Mouhoub et al. (2011)) and other more advanced statistical techniques (Cho 2009; Węglarz et al. 2011), mostly involving Markov chains (e.g., Creemers et al. (2010); Magott and Skudlarski (1993); Xiangxing et al. (2010)), have also been developed for handling (optimizing) some project outcomes like activity time-cost trade-offs (e.g., Azaron and Tavakkoli-Moghaddam (2007)), the total project duration (e.g., Baradaran et al. (2010)) or the project Net Present Value (NPV) (Creemers et al. 2010) while implementing PERT, but particularly in Resource-Constrained Project Schedules (e.g., Azaron et al. (2006); Baradaran et al. (2012); Yaghoubi et al. (2015)). These extensions, while worth noting, deal with computer implementations, unlike M-PERT, which adopts a manual approach.

Furthermore, other techniques like fuzzy logic (e.g., Chen (2007)) and Artificial Neural Network (ANN) analysis (e.g., Lu (2002)) have also been applied to improve activity duration estimation and critical path(s) determination for later ranking purposes. We think that these last two techniques and others in a similar fashion (e.g., Kuklan et al. (1993)), despite having proved to be successful in other fields, do not offer a significant advantage versus Monte Carlo simulations – commonly materialized in Schedule Risk Analysis (SRA) (Vanhoucke 2012) – which are equally (or less) computer-intensive and give almost exact results.

Also, concerning the monitoring and control dimension of PERT, some research has been carried out to improve PERT for use as a project progress tool (e.g., Castro et al. (2007); Wenying and Xiaojun (2011)), for example, by means of intersecting buffers after key activities in networks considering both the time and cost dimensions (Khamooshi and Cioffi 2013), sometimes referred as Dynamic Planning and Scheduling (Azaron and Tavakkoli-Moghaddam 2007; Yaghoubi et al. 2015). Also in this vein, some research has been developed connected to crashing the PERT activities in order to fast-track project execution (e.g., Abbasi and Mukattash (2001); Foldes and Soumis (1993)). The scope of these PERT extensions, however, is related to control and schedule compression, which deviates from the aim of the current paper (proposing a technique for estimating the total or remaining project duration).

Interesting, but certainly minority research has also been devoted to studying the possible effects of assuming independence versus the existence of (partial) correlation between activities (e.g., Banerjee and Paul (2008); Cho (2009); Mehrotra et al. (1996); Sculli and Shum (1991)). Some of their principles will be used here in M-PERT too, and two brief examples are provided later in Figure 5.

Concerning the application of PERT to situations where the activities are of a highly repetitive nature, other PERT extensions have been published (e.g., RPERT (Aziz 2014)). Repetition in activities will also be included among the M-PERT modelling capabilities later by means of activities (or groups of activities) self-loops.

Concerning the number of point estimates to obtain the activity duration average and standard deviation, as in expressions 1 and 2, extensive research has also been conducted. In particular, the 'three-point estimates' from expressions 1 and 2, as this approach is commonly known, make use of the minimum (optimistic), most likely (mode) and maximum (pessimistic) durations that an activity duration following a Beta distribution can have. Other attempts, which are actually as complementary to M-PERT as the original three-point estimate proposal, have been proposed using two parameters (e.g., the mode with either the minimum or the maximum duration (Mohan et al. 2006)), three parameters but with some degree of reparametrization (e.g., Herrerías-Velasco et al. 2011), or even up to four parameters (e.g., (Hahn 2008)). However, normally, when the parameters have been changed, the distribution type modelling the activity duration changes too.

Another minor but relevant subfield of research to this study is the work that deals with the PERT network reducibility problem. Essentially, M-PERT is a technique that recursively downsizes the schedule network by merging activities in series and in parallel. With a similar aim, but with a strong emphasis on the computational point of view, the original works by Ringer (1969), Elmaghraby (1989) and Bein et al. (1992) studied this challenging problem in extensive detail. As M-PERT lends itself to a manual reduction approach, the reader interested in computer implementations is referred to these remarkable works.

Finally, as was anticipated earlier, an overwhelming amount of research has been devoted to trying to improve the original Beta distribution fitness (Hahn 2008; Premachandra 2001), or just try to find another probability distribution that better fits the activity durations (e.g., Lau and Somarajan (1995); López Martín et al. (2012)). The variety of distributions that can be found as an alternative to the Beta distribution is overwhelming. They include the doubly truncated normal distribution (Kotiah and Wallace 1973), the triangular distribution (Johnson 1997), the log-normal distribution (Mohan et al. 2006), the mixed beta and uniform distribution (Hahn 2008), and the Parkinson distribution (Trietsch et al. 2012), to cite a few.

As expected, however, this line of research has caused quite a lot of controversy, with some researchers in favor of pursuing the fit-for-all distribution (e.g., Clark (1962); Golenko-Ginzburg (1989); Grubbs (1962); Healy (1961); Sasieni (1986)) and others against (e.g., Herrerías-Velasco et al. (2011); Kamburowski (1997); Pleguezuelo et al. (2003)). As MacCrimmon and Ryavec (1964) anticipated and we again proved in Figure 1, the particular distribution is hardly relevant, so no more references will be made to this discussion.

With the most relevant PERT-related lines of research identified, it is time to identify the subset of works upon which M-PERT has been built, which are those that deal with the merge event bias in some way or another. These works are summarized in Table 1 by columns, and all of them are approximate techniques for solving some flaws that the original PERT technique had. Unfortunately, these works are not widely known by the research community. Even more surprisingly, most of them were unaware of the others, even though all but the first one were contemporaneous. For the sake of brevity, some details have been given by the rows in Table 1 and some more will be specified later in the *M-PERT outline* section.

<Table 1>

Of particular interest perhaps, is the work by Pritsker (1966), who developed the Graphical Evaluation and Review Technique (GERT), which would have fulfilled the promise of devising a new PERT without most of its problems if it had not been for the mathematical complexity that did not allow the authors to finally implement it without resorting to Monte Carlo simulations. However, this technique introduced very interesting features, some of which M-PERT has inherited, like self-loops and probabilistic (alternative) paths.

Last of all, another small subset of approximate PERT variants sought to tackle the merge bias problem by obtaining the upper and/or lower bounds of the project total duration. Without seeking to be exhaustive, maybe some of the most relevant findings were the Clark's (one of the PERT creators) bias correction procedure (Clark 1961, 1962), the 'f' estimate (Fulkerson 1962) and the Modified Network Evaluation Technique (PNET) (Ang et al. 1975). A recent exhaustive study by González et al. (2014) comparing the accuracy achieved by these methods when approximating the Project duration mean and variance in networks with a varied array of topological indicators showed that the best methods were the 'f' estimate for the mean (with an average error of 2.6%) and the PNET for the variance (with an average error of 29.8%). Anticipating some results presented later, the methods summarized in Table 1, on which M-PERT was built, exhibited smaller errors in the benchmark networks tested, which has allowed M-PERT to currently have average errors below 2% for the mean and below 10% for the variance. Therefore, these methods, despite deserving acknowledgment, will not be referred to any more.

M-PERT outline

M-PERT has been built upon the five models presented in Table 1, most of which shared some common traits (evidenced in lines with several '✓' marks). All five methods made use of Activity-on-Arc (AoA) networks, whereas M-PERT resorts to Activity-on-Node (AoN) representation since nowadays it is more user-friendly for practitioners and more commonly found in software (e.g., Microsoft Project, Oracle Primavera).

Consequently, the precedence relationships are necessarily represented by means of the arrows connecting the nodes. Initially, it is well known that, given two activities *i* and *j*, there can be four precedence relationships: Finishes *i* – Starts *j* (the most common by far), Starts *i* – Starts *j*, Finishes *i* – Finishes *j*, and Starts *i* – Finishes *j*, all of them with or without time lags between them.

Representation of the last three, despite being possible, significantly complicates the network visualization and the critical path analysis when the number of activities is high since some arrows depart the nodes from their back and/or reach the nodes at their front. Fortunately, as recently found by Lu and Lam (2009), these three precedence types can be reformulated as a Finish-Start relationship, as Figure 2 illustrates, and with which scheduling students can be trained. Hence, from now on, M-PERT will make exclusive use of AoN networks with Finish-Start (FS) precedence relationships, and should the user wish to establish different types of precedence, he/she should refer to Figure 2, or, alternatively, to Lu and Lam's (2009) comprehensive treatment of non-finish-to-start relationships in project networks, and transform them to FS relationships.

<Figure 2>

Essentially, M-PERT is a reduction technique in which project activities are merged by groups of two or more, resulting in a new single merged activity, and this process is repeated until there is just one activity left, which represents the total project duration. An idea of this sequential merging procedure can be observed later in Figure 4 and the supplemental data.

The merging procedure was proposed by Cox (1995), and it is an intuitive and relatively quick approach for reducing any (complex) network into a simple one. Of course, the challenge is how to exactly *merge* different activities (which in the end correspond to duration distributions). For that purpose and as justified earlier, it is assumed that the activity durations follow a Normal distribution. Being aware that the specific probability distribution was hardly important, the Normal assumption was also made by three out of the five methods stated in Table 1, and this obeys the need of simplifying as much as possible the merging operation of activities in series, which is the most frequent in any network. The merger operations in the serial activities are represented in the first row of Figure 3.

<Figure 3>

Figure 3 is, in general, quite self-explanatory and basically represents the most common operations (mergers) that can be found in any network – mostly serial activities (first row) and maxima of multiple paths (bottom row) – along with other interesting operations that GERT (Pritsker 1966) proposed (probabilistic paths, self-loops and activity minima) but was not able to implement without ad hoc computational programmes. For illustrative purposes, the case study shown later in Figure 4 will provide examples of these logic operations.

Particularly in construction projects, operations like probabilistic (alternative) paths might seem quite straightforward for modelling uncertain courses of action that will mostly depend on information that is still unknown, or at least inaccurate, at the conceptual stage. Examples are the choice between different types of foundations depending on the ground conditions or the soil-bearing capacity, or just the need to take special measures if the project uncovers archeological remains.

Self-loops, on the other hand, are helpful for representing activities being repeated after an unsuccessful outcome. These activities can correspond to isolated activities (e.g., a load test, a calibration) or even groups of activities (e.g., a whole project drawing up process, a series of defective structural elements, a non-compliant subnetwork within a water supply system). It is up to the scheduler, however, to decide how many loops, or even how many multiple nested loops, could be feasibly implemented before the construction manager or the client stops the iterations.

Similarly, activity minima, despite certainly being much less frequent than activity maxima, can also be useful at times to allow the project continuation after one (or some) of the predecessors are finished. A distinctive quality of activity minima versus probabilistic paths is that all the activities involved in the minima merger must be executed at some point before the project finishes, whereas in probabilistic paths only one (or a subset of) path(s) will be finally carried out. Examples of activity minima are the supply of any electromechanical equipment among a series of any other equipment that needs to be received in a worksite before the mechanical engineers can start working, or the clearance of any stocking areas of a linear worksite before a new pipeline section can be supplied.

Overall, for every operation by rows in Figure 3, their representation (second column) is shown; the input parameters (third column) necessary for defining the activities and performing the merger into a single activity (which is generically named *k*) are provided; the output parameters (fourth column) that define the new duration mean and variance of the resulting merged activity *k* are provided; a representation of the result (fifth column) is shown; and, finally, some observations (last column on the right) that clarify how to apply the merger operations in particular (more complex) contexts.

It is worth highlighting that M-PERT requires that the activity duration means (μ_i) and variances (σ_i) have been specified for each activity beforehand, a requirement that forces the student or scheduler to elicit them or, if he/she wanted to start from the three-point estimates (O, L and P), it would require a first stage of application of equations 1 and 2 (with the correction proposed in equation 6) for each activity, like the original PERT.

Concerning the origin of the mathematical expressions stated for all the mergers (stated in the *Output parameters* column), the serial activities merger just corresponds to the sum of means and variances of several Normal distributions (as in equations 3 and 4), whereas the expressions for the probabilistic alternative paths and the self-loops are easily derived from the mathematical expressions of the mixture (Union) of *n* non-overlapping (Normal) population samples (X_i=X_1 … X_n), which are:

$$\mu_k = \frac{\sum_i W_{X_i}\mu_{X_i}}{\sum_i W_{X_i}} \tag{7}$$

$$\sigma_k^2 = \frac{\sum_i W_{X_i}\sigma_{X_i}^2}{\sum_i W_{X_i}} + \frac{\sum_{i<j} W_{X_i}W_{X_j}\left(\mu_{X_i}-\mu_{X_j}\right)^2}{\left(\sum_i W_{X_i}\right)^2} \tag{8}$$

Where the W_{Xi} are the weights of each Normal population merged, that is, the proportion by which that Normal distribution (representing a set of previously merged activities' duration) will be present in the resulting distribution after the next merger. For example, for the probabilistic paths, the weights are the probabilities of each path occurring, that is, $W_{X1}=p$ and $W_{X2}=1-p$ for two paths; or $W_{Xi}=p_x$ for $i=x=1,2,\ldots n$ paths, with $\sum_{x=1}^{n} W_x = 1$ always. For the self-loops, an activity *i* is merged with itself – activity *i*'s mean $\mu_{X1}=\mu_i$ and standard deviation $\sigma_{X1}=\sigma_i$ when the activity happens just once with probability $W_{X1}=1-p_i$, and $\mu_{X2}=2\mu_i$ and $\sigma_{X2}=\sqrt{2}\,\sigma_i$ when it is repeated with probability $W_{X2}=p_i$.

Finally, the case of the maximum from two Normal distributions (the durations of two activities) was coincidentally proposed by Clark (1961) but at that time it required intensive use of tables and did not allow for correlation (ρ) between both activities. In particular, the expression for the maximum was taken from Sculli and Shum (1991) and was also used by Cox (1995), whereas the expression for the minimum was taken from Nadarajah and Kotz (2008). Both expressions, despite looking long, are very easy to calculate.

The obvious problem with the maxima and minima from several activities is that the formulae only allow for the merger of two of them at the same time. Then these expressions can be applied recursively until there is just one activity left, a moment in which the activity will be in series again and can keep being merged with its neighbor activities in series. This recursive merging procedure is not new either, as it was successfully applied by Cox (1995), Gong and Hugsted (1993) and Sculli and Shum (1991). However, it is also true that when more than two activities are merged, there is a small error in the final mean and variance calculation, which is also dependent (but to a minimum extent) on the exact merger order too. Simulations performed that try to estimate the maximum error magnitude with M-PERT identified how this error is maximized when the different paths' duration mean is the same, and variances are also identical between the parallel paths. However, even in that case, the error obtained when merging "eight" activities under the same node (which is a higher number than most of the real projects have under one converging node) causes an error of 6.7‰ in the mean and 7.3% in the standard deviation, which are considered, in general, good approximations.

Therefore, M-PERT consists of merging activities until there is one activity left that models the total project duration by means of μ_p and σ_p , which correspond to the project duration mean and standard deviation, respectively. The order in which merger operations are performed is relevant too. For example, if a self-loop comprised several activities, those inner activities should first be merged, and then the self-loop resolved; or, if there are several activities in series within a parallel path, those activities need to be merged before the path can be merged with other paths. Overall, there must be an overriding use of common sense when performing the mergers, and unless there is correlation among activities (a situation that will be treated separately later), the merging procedure is quite straightforward.

There is just one last step left in the application of M-PERT. Obviously, the resulting distribution modelling the total project duration is forced to be Normal in M-PERT, but, in reality, this distribution might be more similar to an Extreme Value distribution when there is a "high number of parallel paths converging all of them to the same node (activity)". This is not a secret, but it was not properly incorporated in a PERT model until Dodin and Sirvanci

(1990). He proposed using the Gumbel distribution (despite it being generically named the Extreme Value distribution) for modelling the total project duration.

In M-PERT, however, we think the Normal distribution constitutes a reasonable approximation. This is because most construction projects, despite usually including multiple paths, they do not "all converge in the same single activity", rather they have multiple paths which converge at different activities. In other words, there is not a dominance of maxima computation, but some maxima calculations mixed with a high number of activities in series or activity maxima in series, both of which quickly degenerate (as shown in Figure 1) in Normal distributions. Furthermore, students who are exposed to learning M-PERT are much more familiar with the Normal distribution than with Extreme Value distributions.

Case study

<u>A fictitious and simplified bridge project</u>

M-PERT can be equally applied to simple and complex networks, the only difference being the number of mergers that are required to be performed before the network is reduced to a single activity representing the project duration. In this regard, Figure 4 presents the application in a simple but representative bridge project construction with three piles (three parallel paths), in which, purposefully, all possible kinds of merging operations have been included. The activities 'Start' and 'End' are just dummy activities (null duration).

<Figure 4>

On the lower part of Figure 4, the sequential merging procedure has been exemplified. Note, however, that the activities represent the standard deviation, not the variance, and this is to be considered when applying all the expressions stated in Figure 3. Furthermore, the type of merger operation has been stated in the five-step (solution) reduction procedure, replacing the original activity descriptions and allowing the reader to understand where each newly merged activity came from. All these indications however, are not necessary when we try to expedite the manual calculation process.

Few explanations are needed at this stage, since the merger procedure just requires the application of simple mathematical expressions already introduced mostly in Figure 3. It is worth highlighting that, unlike the original PERT with fixed paths (in which what was described in the network had to be necessarily carried out and excluded the possibility of going backwards), the introduction of probabilistic paths, minima and self-loops allows for a closer representation of construction projects. In this sense, M-PERT allows for some paths happening while others do not, and that a group (or even the whole project if need be) is repeated thanks to the inclusion of a self-loop. Overall, the modelling possibilities of M-PERT are significantly more powerful than the original PERT.

Networks with correlation between activities

Sometimes there are networks (project schedules) in which the merger procedure ends in a knot that cannot be untangled. This is quite usual indeed when there are several parallel paths that do not depart from the same node and/or do not converge in the same node (Mehrotra et al. 1996). In those situations, the partially reduced network can end up in (sub)networks similar to the ones shown in Figure 5, although maybe with more nodes.

<Figure 5>

For solving these subnetworks that usually represent a small portion of the whole network, authors like Mehrotra et al. (1996) and Sculli and Shum (1991) proposed a sequential recursive procedure that involves calculating all the variance and covariance values between the ‘untangled’ paths. Implementing this procedure basically requires identifying all the possible paths and then merging them, first, two of them, and then one by one with the already merged group. However, the user needs to be aware that there is a correlation (*ρ≠0*) between activities when applying the maxima (or minima) expressions from Figure 3, that is, between paths that have some activities in common (as the covariance between those two paths will not generally be zero).

As can be seen in Figure 5, identifying all the possible paths is not difficult unless the network knot size involves a lot of activities and is very dense in precedence relationships (number of arrows). The only tricky part is to calculate the correct value of ρ between the paths. However, Sculli and Shum (1991) derived the generic expression for obtaining it, but since that expression is somehow difficult to understand for the beginner, the ρ expressions for up to four parallel paths (normally ordered from higher to lower mean duration μ_k and sequentially named A, B, C and D) are given here:

$$ {}_{AB} = \frac{COV_{AB}}{{}_{A}\ {}_{B}} \tag{9}$$

$$ {}_{ABC} = \frac{COV_{AC}\Phi(\ {}_{AB}) + COV_{BC}(1 \quad \Phi(\ {}_{AB}))}{{}_{AB}\ {}_{C}} \tag{10}$$

$$ {}_{ABCD} = \frac{[COV_{AD}\Phi(\ {}_{AB}) + COV_{BD}(1 \quad \Phi(\ {}_{AB}))]\Phi(\ {}_{ABC}) + COV_{CD}(1 \quad \Phi(\ {}_{ABC}))}{{}_{ABC}\ {}_{D}} \tag{11}$$

Where ρ_{AB}, ρ_{ABC}, ρ_{ABCD} representing, respectively, the correlation coefficients as paths A with B, AB with C, and ABC with D are merged one by one with the previously merged paths. COV_{XY} simply corresponds to the sum of variances of the activities that paths X and Y have in common, and σ_X and σ_Y, the standard deviations of the paths X and Y respectively, with $X, Y = A, B, C, D$. Examples of calculation of the variance and covariance values have been provided in Figure 5 in the covariance matrices.

Finally, for the sake of clarity, the complete step-by-step procedure for implementing expressions 9 to 11 in a project network with four paths with correlation has been described in the supplemental data under Alternative 2 in the example project 0, along with other multiple examples of M-PERT exercises.

Results

The five methods from which M-PERT has been built were tested in several benchmark networks. There is no reason to believe that M-PERT, which makes use of their most meritorious contributions, even including new features for better modelling real-life projects, will have a lower accuracy. Anyhow, M-PERT has been applied to one network in this paper and its accuracy comparison is presented in Figure 6.

<Figure 6>

It must be borne in mind, though, that Figure 6 shows a comparison of four statistical distributions of which the distribution parameters are perfectly known, not a comparison of datasets against another dataset or a known distribution, as is more usual in research analyses. Hence, we will not be testing how 'significant' the difference between any two distributions is, as it is already known that they are different distributions. Instead, a graphical approach is adopted here for illustrating the location and dispersion deviations between the distributions.

Figure 6, in particular, shows four curves. The two continuous gray curves show the exact solution of the project duration distribution (obtained through computer simulations) by modelling the activity durations with Normal distributions and Uniform distributions respectively. As stated before, the choice of one or other distribution hardly has any relevance, an outcome proven again by observing how close both gray curves are. The average (project duration) of these two simulated distributions is close to 147.5 days despite they being lightly positively skewed and leptokurtic.

The dashed curve, on the other hand, represents the approximation offered by M-PERT by using the Normal distribution modelling the project duration distribution, whose parameters were obtained at the bottom of Figure 4. The bridge project, with three dominant parallel paths, does not deviate much from the Normal approximation. It is also easy to appreciate how the M-PERT project duration mean is close to 147 days (estimation error below 1% when compared to the simulated curves, whereas the variance or dispersion error is also below 10%).

Last of all, the continuous black curve representing the classical PERT approximation significantly falls behind, with an average project duration of 133 days (error above 10%). Again, this evidences how inaccurate the original PERT is when there are parallel paths and the merge event bias is neglected.

Finally, for reassurance, results in other networks exemplifying step-by-step calculations, along with the (simulated) actual project duration distribution, can be found as supplemental data.

Discussion and conclusions

A new scheduling technique named M-PERT has been proposed. This technique takes advantage of several methods that were proposed since PERT was devised for correcting several shortcomings that the original technique had. In particular, in situations with multiple parallel paths (the norm in construction projects), the original PERT resulted in a project duration underestimation and a variance overestimation.

A systematic literature review has been developed in order to justify why M-PERT concentrates on the really significant weaknesses of PERT and why, by neglecting other accessorial aspects, the resulting tool is still accurate, but not complex. Indeed, the method proposed is easy to implement and easy to learn due to its intuitive nature and simplifying assumptions. This makes M-PERT an attractive tool for teaching scheduling basics to construction and project management students, especially since its calculations can be developed manually or by means of very simple spreadsheets. Finally, an application has been included in which the new features offered by M-PERT, such as the minima of activities, probabilistic (alternative) paths or activity or groups of activities self-loops evidence a higher resemblance between scheduled networks and real construction projects.

However, there is still a long way to go. Indeed, despite this tool really departing from the original PERT conception, it is still open to improvement. The cost dimension has not been included in M-PERT, but it should follow a very similar approach. However, the inclusion of resources in the analysis really represents a challenging route that should probably be incorporated in the next versions of this scheduling technique.

Data availability

All data generated or analyzed during the study are included in the submitted article or supplemental materials files.

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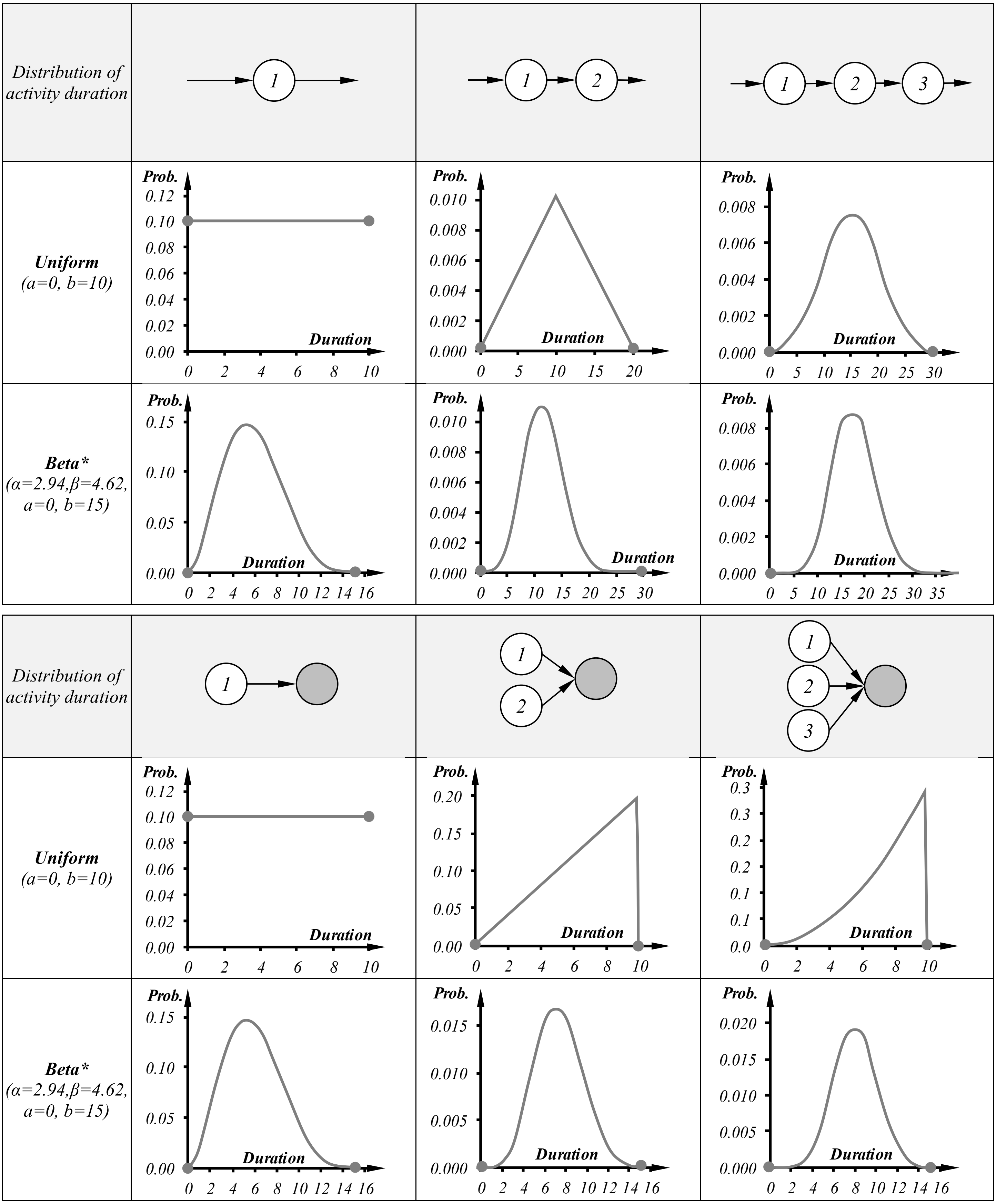
▼*Highlights* *Work* ►	Pritsker (1966)	Sculli and Shum (1991)	Gong & Hugsted (1993)	Cox (1995)	Mehrotra et al. (1996)
Avoids having to calculate all the possible paths (critical and non-critical) ***(allows big networks)***	✓		✓	✓	
Allows for recursive application of activity maxima and/or minima ***(time-saving simplification)***		✓	✓	✓	
Normal distributions for modelling the activity durations ***(only mean and variance matter)***		✓	✓		✓
Allows for correlation between different activities ***(allows complex projects)***	✓	✓	✓		✓
Use of simple expressions for solving the activity mergers ***(calculation simplicity)***				✓	✓
Allows for extra features (activity minima, self-loops or probabilistic paths, etc) ***(resemblance to reality)***	✓				
Number of benchmark networks in which the model was tested ***(Number of validation tests)***	4	2	3	2	10

Table 1. Key works on which PERT 2.0 has been based



*According to the original PERT, with Optimistic duration: 0, Most likely: 5, and Pessimistic: 15 time units.

Fig. 1. Comparison of two probability distributions alteration as more activities are added

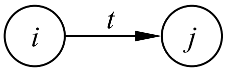
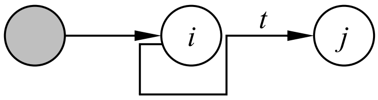
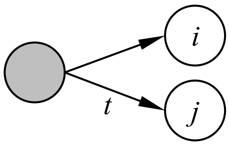
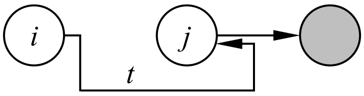
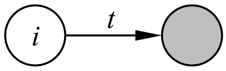
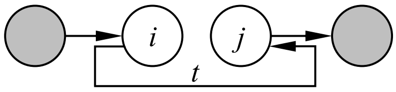
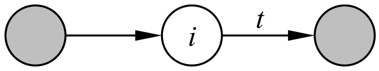
Precedence type	***Activity-on-Node (AoN) representation***	***Finish-Start equivalent***
Finishes i, Starts j with a time lag t (omit if t=0)		
Starts i, Starts j with a time lag t		
Finishes i, Finishes j with a time lag t		 (j is not relevant)
Starts i, Finishes j with a time lag t		 (j is not relevant)

Fig. 2. Array of activity precedences in M-PERT

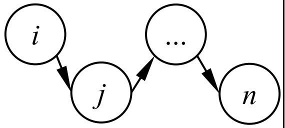
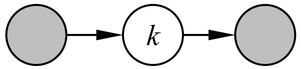
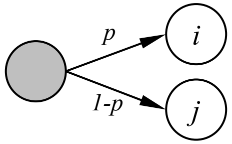
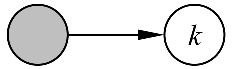
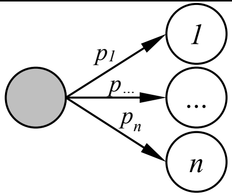
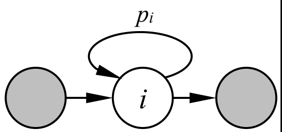
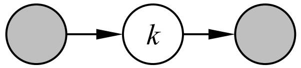
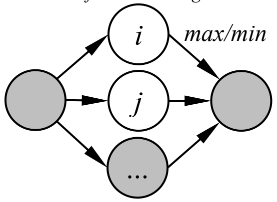
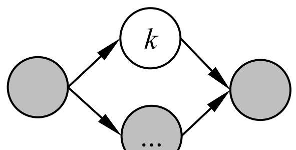
Operation	*AoN Representation*	*Input par.*	*Output parameters*	*Result*	*Observations*
Serial activities		$\mu_i,\ \sigma_i^2$ $\mu_j,\ \sigma_j^2$ ... $\mu_n,\ \sigma_n^2$	$\mu_k = \sum_{x=i,j,..n} \mu_x$ $\sigma_k^2 = \sum_{x=i,j,..n} \sigma_x^2$		When there is a time lag t between any two activities, then $\mu_k = \mu_i + \mu_j + t$ and $\sigma_k^2 = \sigma_i^2 + \sigma_j^2$
Probabilistic (alternative) paths		$\mu_i,\ \sigma_i^2,\ p$ $\mu_j,\ \sigma_j^2,\ q=1-p$	$\mu_k = p\,\mu_i + (1-p)\,\mu_j$ $\sigma_k^2 = p\,\sigma_i^2 + (1-p)\,\sigma_j^2 + p(1-p)(\mu_i-\mu_j)^2$		If there are more activities after any of the first activities from the paths, resolve those mergers first, then merge into a single activity k
		$\mu_x,\ \sigma_x^2,\ p_x$ for $x=1,2,3...n$, with $\sum_{x=1}^{n} p_x = 1$	$\mu_k = \sum_{x=1}^{n} p_x \mu_x$ $\sigma_k^2 = \sum_{x=1}^{n} p_x\left(\sigma_x^2 + \mu_x^2\right) - \mu_k^2$		
Self-loops		$\mu_i,\ \sigma_i^2,\ p_i$	$\mu_k = (1+p_i)\,\mu_i$ $\sigma_k^2 = (1+p_i)\,\sigma_i^2 + p(1-p)\mu_i^2$		When the self-loop comprises more than one activity, resolve (merge) the activities within the loop first. If there are nested (or multiple) self-loops, resolve inner self-loops first.
(maximum or minimum) Parallel paths	*Maximum* *(all activities are to finish before continuing)* *or* *Minimum* *(just one activity needs to finish before continuing* 	$\mu_i,\ \sigma_i^2$ $\mu_j,\ \sigma_j^2$ ρ (if i and j are correlated, otherwise $\rho=0$)	<u>*Maximum:*</u> $\mu_k = \mu_i\Phi(\delta) + \mu_j\left(1-\Phi(\delta)\right) + \theta\phi(\delta)$ $\sigma_k^2 = \left(\sigma_i^2+\mu_i^2\right)\Phi(\delta) + \left(\sigma_j^2+\mu_j^2\right)\left(1-\Phi(\delta)\right) + \left(\mu_i+\mu_j\right)\theta\phi(\delta) - \mu_k^2$ <u>*Minimum:*</u> $\mu_k = \mu_i\left(1-\Phi(\delta)\right) + \mu_j\Phi(\delta) - \theta\phi(\delta)$ $\sigma_k^2 = \left(\sigma_i^2+\mu_i^2\right)\left(1-\Phi(\delta)\right) + \left(\sigma_j^2+\mu_j^2\right)\Phi(\delta) - \left(\mu_i+\mu_j\right)\theta\phi(\delta) - \mu_k^2$ Where $\delta = \left(\mu_i-\mu_j\right)/\theta$; $\theta = \sqrt{\sigma_i^2+\sigma_j^2-2\rho\sigma_i\sigma_j}$ and $\varphi(\cdot)$ and $\Phi(\cdot)$ are the Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of the Standard Normal distribution (use Supplemental Figure 1 from the Supplemental Online material for quick calculation)	*max/min* 	When there are several activities on, at least, one path, resolve (merge) the activities within that/those path/s first, then apply the maximum/minimum merger. When there are more than two alternative paths, keep recursively resolving (merging) the paths by groups of two activities, until finishing with just one path (activity k will be then in series)

Fig. 3. Array of activity mergers in M-PERT

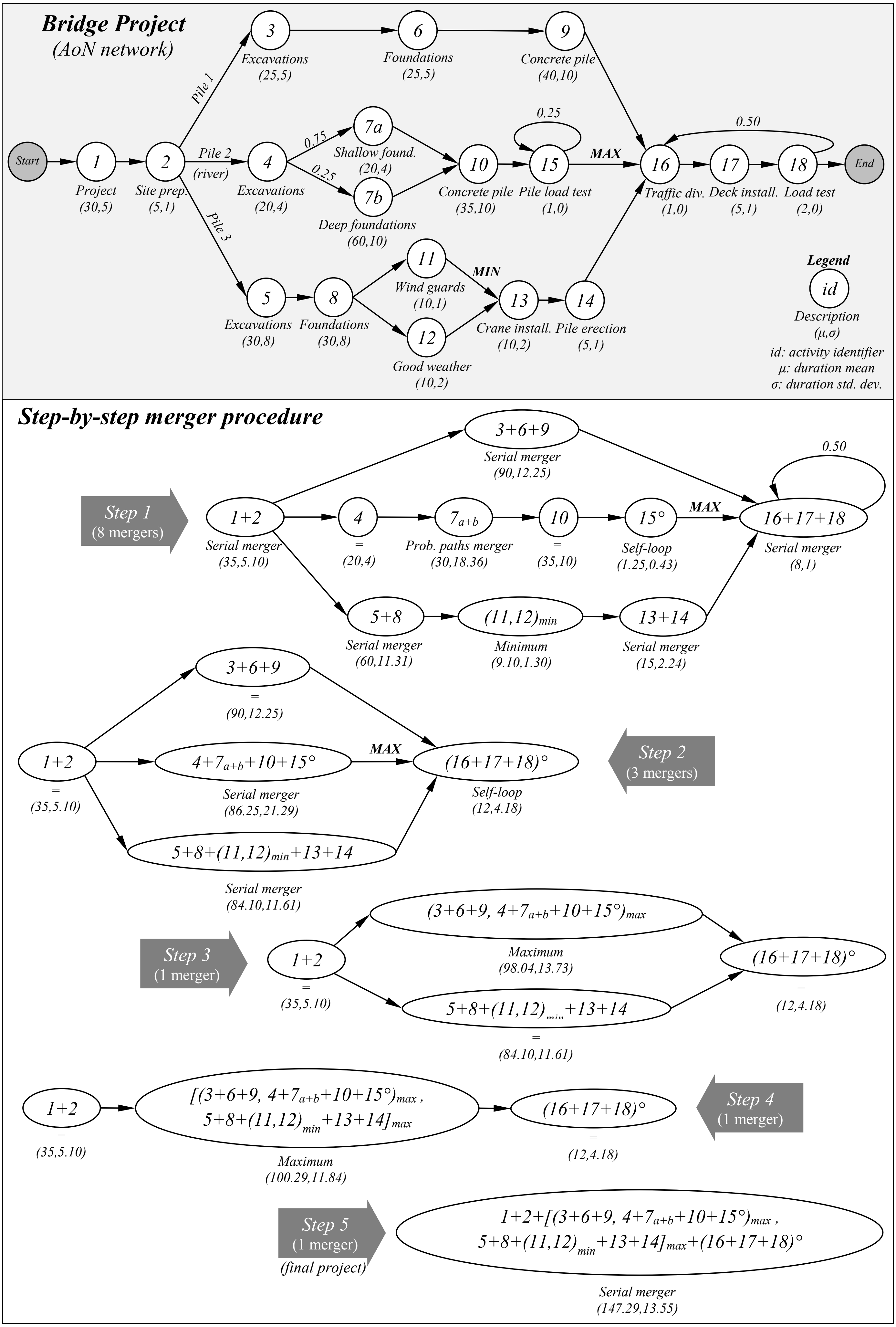
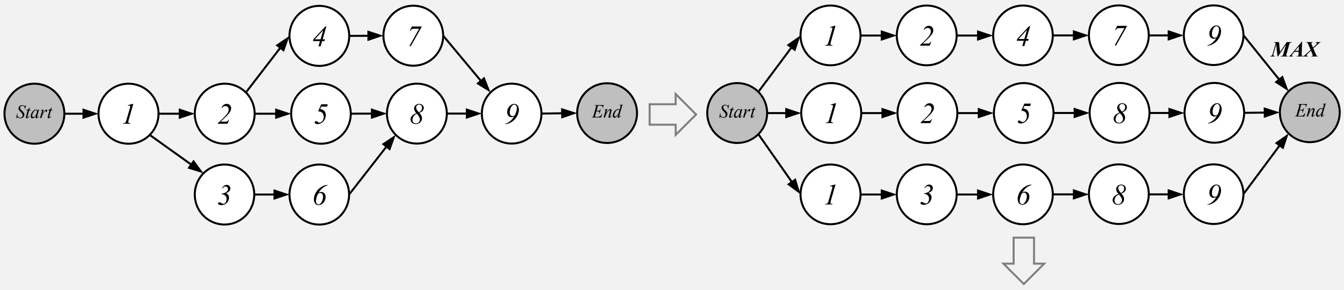
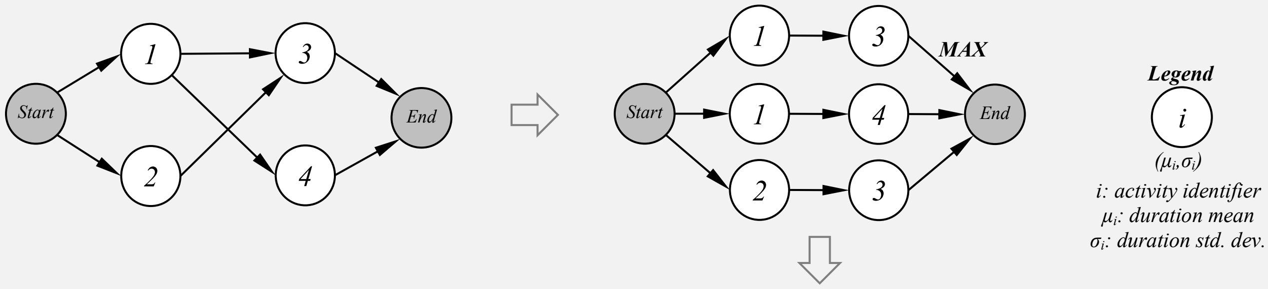


Fig. 4. M-PERT application in a fictitious simplified bridge project



Path	Activities	μ_k	A	B	C
A	1-2-4-7-9	$\mu_A=$ $\mu_1+\mu_2+\mu_4+\mu_7+\mu_9$	$\sigma^2_A=$ $\sigma^2_1+\sigma^2_2+\sigma^2_4+\sigma^2_7+\sigma^2_9$	$COV_{AB}=$ $\sigma^2_1+\sigma^2_2+\sigma^2_9$	$COV_{AC}=$ $\sigma^2_1+\sigma^2_9$
B	1-2-5-8-9	$\mu_B=$ $\mu_1+\mu_2+\mu_5+\mu_8+\mu_9$	$COV_{BA}=COV_{AB}$	$\sigma^2_B=$ $\sigma^2_1+\sigma^2_2+\sigma^2_5+\sigma^2_8+\sigma^2_9$	$COV_{BC}=$ $\sigma^2_1+\sigma^2_8+\sigma^2_9$
C	1-3-6-8-9	$\mu_C=$ $\mu_1+\mu_3+\mu_6+\mu_8+\mu_9$	$COV_{CA}=COV_{AC}$	$COV_{CB}=COV_{BC}$	$\sigma^2_C=$ $\sigma^2_1+\sigma^2_3+\sigma^2_6+\sigma^2_8+\sigma^2_9$



Path	Activities	μ_k	A	B	C
A	1-3	$\mu_A=\mu_1+\mu_3$	$\sigma^2_A=\sigma^2_1+\sigma^2_3$	$COV_{AB}=\sigma^2_1$	$COV_{AC}=\sigma^2_3$
B	1-4	$\mu_B=\mu_1+\mu_4$	$COV_{BA}=COV_{AB}$	$\sigma^2_B=\sigma^2_1+\sigma^2_4$	$COV_{BC}=0$
C	2-3	$\mu_C=\mu_2+\mu_3$	$COV_{CA}=COV_{AC}$	$COV_{CB}=COV_{BC}$	$\sigma^2_C=\sigma^2_2+\sigma^2_3$

Fig. 5. Two examples of AoN networks with correlation and the calculation of their covariance matrix

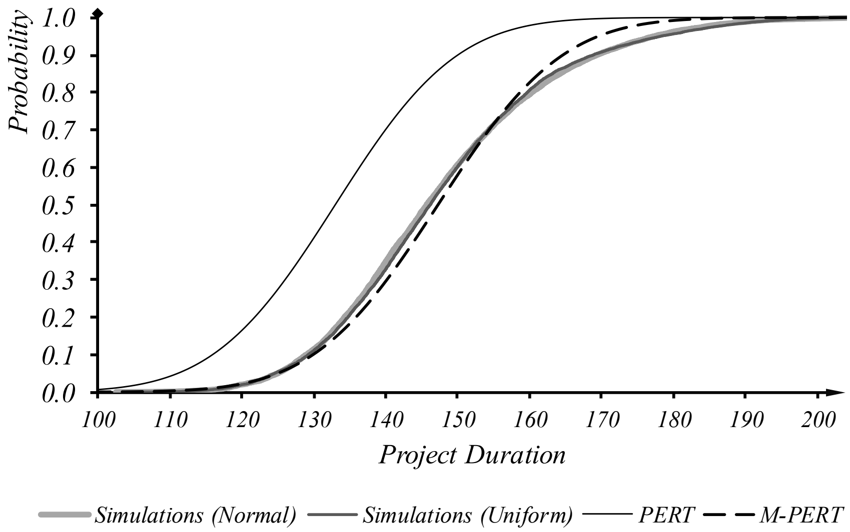


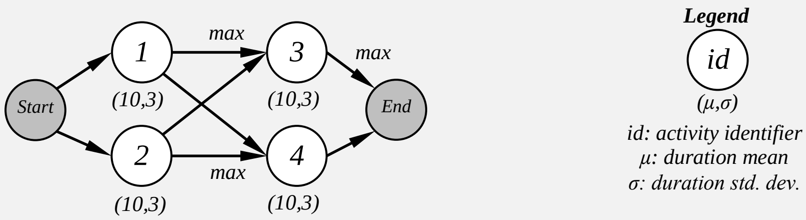
Fig. 6. Comparison of project duration estimation accuracy between PERT and M-PERT for the simplified bridge project

Supplemental data

Contents

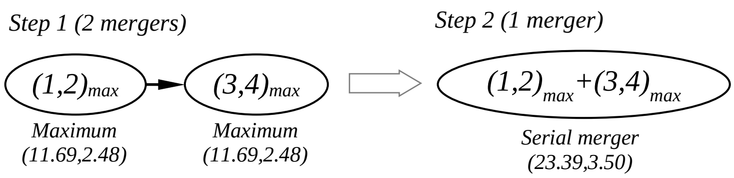
Detailed step-by-step calculations for a (Sub)network with correlation.

Example Project 0

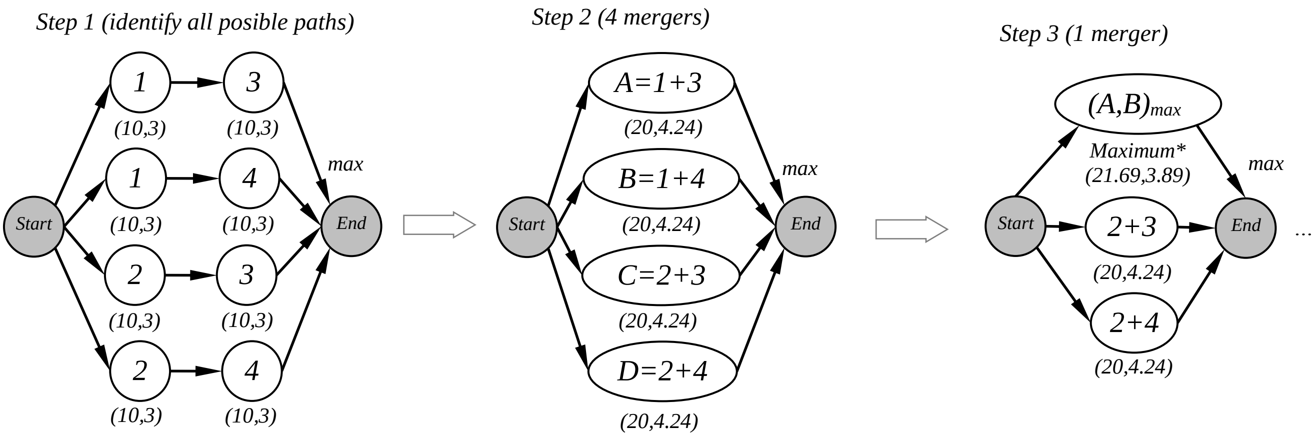


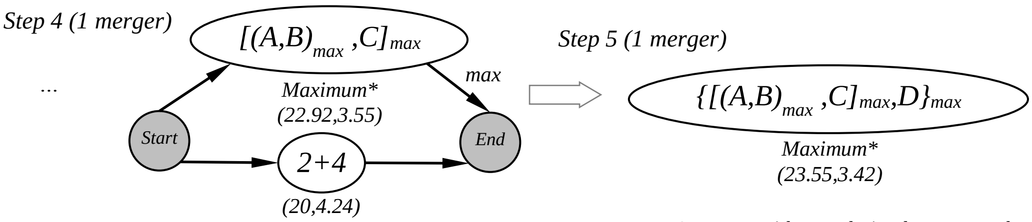
Step-by-step merger procedure

Alternative 1 (sometimes possible):



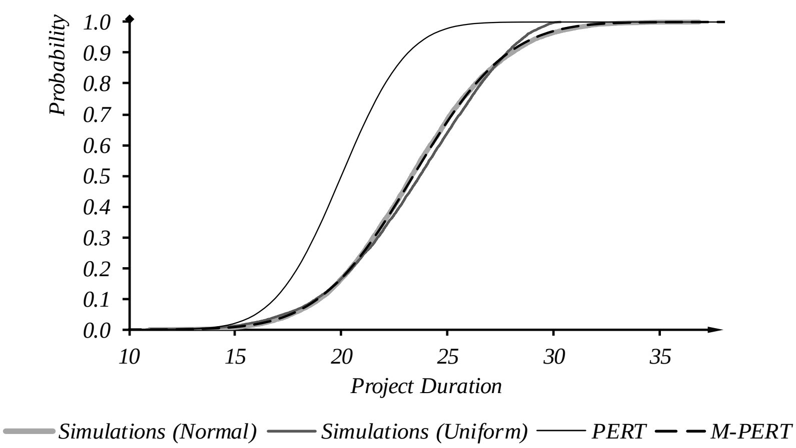
Alternative 2 (general procedure):





* merger with correlation between paths

Results



Alternative 1

This alternative is possible due to the problem symmetry. If, for instance, the Finish-Start precedence relationship between nodes 2 and 4 had not existed, this alternative would have not been possible, only Alternative 2. However, when Alternative 1 is possible is always preferable since it is quicker and more accurate (since normally it entails fewer recursive applications of the maximum merger formulae)

<u>*Step 1 (2 mergers)*</u>

■ Maximum merger between activities i=1 (with μ_i=10 and σ_i=3) and j=2 (with μ_j=10 and σ_j=3), then $k=(1,2)_{max}$. Values of $\Phi(\cdot)$ and $\varphi(\cdot)$ can be calculated with the Supplemental Figure 1.

$\theta = \sqrt{\sigma_i^2 + \sigma_j^2 - 2\rho\sigma_i\sigma_j} = \sqrt{3^2 + 3^2 - 2\cdot 0\cdot 3\cdot 3} = \sqrt{18} = 4.243$ (there is no correlation between 1 and 2, i.e. $\rho=0$)

$$\delta = (\mu_i - \mu_j)/\theta = (10-10)/4.243 = 0$$

$$\mu_k = \mu_{(1,2)\max} = \mu_i \Phi(\delta) + \mu_j(1-\Phi(\delta)) + \theta\phi(\delta) = 10\Phi(0) + 10(1-\Phi(0)) + 4.243\phi(0) = 10\cdot 0.5 + 10\cdot(1-0.5) + 4.243\cdot 0.399 = 11.693$$

$$\sigma_k^2 = \sigma_{(1,2)\max}^2 = (\sigma_i^2 + \mu_i^2)\Phi(\delta) + (\sigma_j^2 + \mu_j^2)(1-\Phi(\delta)) + (\mu_i + \mu_j)\theta\phi(\delta) - \mu_k^2 = (3^2 + 10^2)\Phi(0) + (3^2 + 10^2)(1-\Phi(0)) + (10+10)4.243\phi(0) - 11.693^2 = 109\cdot 0.5 + 109\cdot(1-0.5) + 84.86\cdot 0.399 - 136.726 = 6.135 \rightarrow \sigma_k = \sqrt{6.135} = 2.477$$

■ Maximum merger between activities i=3 (with μ_i=10 and σ_i=3) and j=4 (with μ_j=10 and σ_j=3), then $k=(3,4)_{max}$.

In this case, it is obvious that since the means and variances of all the activities are the same, the result coincides with the maximum merger for i=1 and j=2. Therefore, μ_k= $\mu_{(3,4)max}$ =11.70 and σ_k= $\sigma_{(3,4)max}$=2.46

<u>*Step 2 (1 mergers)*</u>

■ Serial merger between activities $i=(1,2)_{max}$. (with μ_i=11.693 and σ_i=2.477) and $j=(3,4)_{max}$ (with μ_j=11.693 and σ_j=2.477), then $k=(1,2)_{max}+(3,4)_{max}$.

$$\mu_k = \mu_{(1,2)\max+(3,4)\max} = \sum_{x=i,j,\dots n} \mu_x = \mu_i + \mu_j = 11.693 + 11.693 = 23.385 \rightarrow \textit{Project Duration mean}$$

$$\sigma_k^2 = \sigma_{(1,2)\max+(3,4)\max}^2 = \sum_{x=i,j,\dots n} \sigma_x^2 = \sigma_i^2 + \sigma_j^2 = 2.477^2 + 2.477^2 = 12.270 \rightarrow \sigma_k = \sqrt{12.270} = 3.503 \rightarrow \textit{Project}$$

Duration standard deviation

Now that μ_p=23.385 and σ_p=3.503 are known, the probabilities of any possible project duration (p) can be calculated by using the $\Phi(z)$ curve in Supplemental Figure 1 with $z = (D - \mu_p)/\sigma_p$. Or the other way around, that is, given a known probability $\Phi(z)=P$ (Y axis value), obtain the respective z value and then $D = \mu_p + z\sigma_p$.

Alternative 2 (general approach)

When alternative 1 option is not possible, a recursive (maximum) merger procedure for untangling the network is always possible between paths with shared activities. The only caution needed is to account for the correlation among the different paths when they are merged one by one until there is no one left to merge.

<u>Step 1 (identify all possible paths)</u>

■ There are four possible paths in this (sub)network 1-3, 1-4, 2-3 and 2-4, which we will name for the sake of brevity for later calculations as *A*, *B*, *C* and *D*, respectively.

It is also necessary to calculate all the activities means and variances, as well as their covariances. The covariance between any two paths is just the sum of the variances of those activities which are common between those two paths (unless another additional source of correlation is stated).

Path (i)	Activities	μ_i	*A*	*B*	*C*	*D*
A	1,3	10+10=20	$3^2+3^2=18$	$3^2=9$	$3^2=9$	0
B	1,4	10+10=20	$3^2=9$	$3^2+3^2=18$	0	$3^2=9$
C	2,3	10+10=20	$3^2=9$	0	$3^2+3^2=18$	$3^2=9$
D	2,4	10+10=20	0	$3^2=9$	$3^2=9$	$3^2+3^2=18$

Values in the diagonal correspond to the variances, and those outside to the covariances between couples of paths. Finally, when the time comes later, the coefficient of correlation between two paths *i* and *j* will be calculated as $\rho = COV_{ij} / \sigma_i \sigma_j$ where all the values are taken from the table above.

<u>Step 2 (4 mergers)</u>

■ There are four serial mergers, as indicated in the table above. Detailed calculations are given for path *A* applying the serial activities merger from Figure 3:

$$\mu_A = \mu_1 + \mu_3 = 20 \qquad \sigma_A = \sqrt{\sigma_1^2 + \sigma_3^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = 4.243$$

Since the rest of the paths (in this example) have the same duration means and standard deviations, then: $\mu_A = \mu_B = \mu_C = \mu_D = 20$ and $\sigma_A = \sigma_B = \sigma_C = \sigma_D = 4.243$.

<u>Step 3 (1 merger)</u>

As a general rule all paths will be ordered from longest to shortest mean project duration and then merged. First, those two longest paths will be merged, and then one by one, in a recursive fashion, the rest will also be merged/added. This one-by-one order rule (unlike the quickest approach by couples of paths first, then by couples of couples of paths, and so on, when there is no correlation between paths) is necessary due to how the mathematical expressions of the correlation coefficients ρ between maxima of paths are stated, since it is not possible (not easily at least) to calculate covariance between two different groups with multiple paths each.

■ Maximum merger between paths i=A (with μ_i=20 and σ_i=4.243) and j=B (with μ_j=20 and σ_j=4.243), then $k=(A,B)_{max}$. Values of $\Phi(\cdot)$ and $\varphi(\cdot)$ can be calculated with the Supplemental Figure 1.

$$\rho_{AB} = \frac{COV_{ij}}{\sigma_i \sigma_j} = \frac{COV_{AB}}{\sigma_A \sigma_B} = \frac{9}{4.243 \cdot 4.243} = 0.5$$

$$\theta_{AB}=\sqrt{\sigma_i^2+\sigma_j^2-2\rho\sigma_i\sigma_j}=\sqrt{\sigma_A^2+\sigma_B^2-2\rho_{AB}\sigma_A\sigma_B}=\sqrt{4.243^2+4.243^2-2\cdot 0.5\cdot 4.243\cdot 4.243}=\sqrt{18}=4.243$$

$$\delta_{AB}=(\mu_i-\mu_j)/\theta=(\mu_A-\mu_B)/\theta_{AB}=(20-20)/4.243=0$$

$$\mu_k=\mu_{(A,B)\max}=\mu_i\Phi(\delta)+\mu_j(1-\Phi(\delta))+\theta\phi(\delta)=\mu_A\Phi(\delta_{AB})+\mu_j(1-\Phi(\delta_{AB}))+\theta_{AB}\phi(\delta_{AB})=$$
$$=20\Phi(0)+20(1-\Phi(0))+4.243\phi(0)=20\cdot 0.5+20\cdot(1-0.5)+4.243\cdot 0.399=21.693\rightarrow\mu_{AB}$$

$$\sigma_k^2=\sigma_{(A,B)\max}^2=(\sigma_i^2+\mu_i^2)\Phi(\delta)+(\sigma_j^2+\mu_j^2)(1-\Phi(\delta))+(\mu_i+\mu_j)\theta\phi(\delta)-\mu_k^2=$$
$$=(\sigma_A^2+\mu_A^2)\Phi(\delta_{AB})+(\sigma_B^2+\mu_B^2)(1-\Phi(\delta_{AB}))+(\mu_A+\mu_B)\theta_{AB}\phi(\delta_{AB})-\mu_{AB}^2=$$
$$=(4.243^2+20^2)\Phi(0)+(4.243^2+20^2)(1-\Phi(0))+(20+20)4.243\phi(0)-21.693^2=$$
$$=418\cdot 0.5+418\cdot(1-0.5)+169.72\cdot 0.399-470.586=15.132\rightarrow\sigma_k=\sqrt{15.132}=3.890\rightarrow\sigma_{AB}$$

Note that, for ease of notation, both results have been renamed as μ_{AB} and σ_{AB}.

Step 4 (1 merger)

■ Maximum merger between paths $i=(A,B)_{max}$=AB (with μ_i=21.693 and σ_i=3.890) and j=C (with μ_j=20 and σ_j=4.243), then $k=[(A,B)_{max},C]=ABC$. The expression for the coefficient of correlation calculation is derived from Sculli & Shum (1991):

$$\rho_{ABC}=\frac{COV_{AC}\Phi(\delta_{AB})+COV_{BC}(1-\Phi(\delta_{AB}))}{\sigma_{AB}\sigma_C}=\frac{9\Phi(0)+0(1-\Phi(0))}{3.890\cdot 4.243}=\frac{9\cdot 0.5+0}{16.505}=0.273$$

$$\theta_{ABC}=\sqrt{\sigma_{AB}^2+\sigma_C^2-2\rho_{ABC}\sigma_{AB}\sigma_C}=\sqrt{3.890^2+4.243^2-2\cdot 0.273\cdot 3.890\cdot 4.243}=\sqrt{24.120}=4.913$$

$$\delta_{ABC}=(\mu_{AB}-\mu_C)/\theta_{ABC}=(21.693-20)/4.913=0.345$$

$$\mu_{ABC}=\mu_{AB}\Phi(\delta_{ABC})+\mu_C(1-\Phi(\delta_{ABC}))+\theta_{ABC}\phi(\delta_{ABC})=21.693\Phi(0.345)+20(1-\Phi(0.345))$$
$$+4.912\phi(0.345)=21.693\cdot 0.635+20\cdot(1-0.635)+4.913\cdot 0.376=22.921\rightarrow\mu_{ABC}$$

$$\sigma_{ABC}^2=(\sigma_{AB}^2+\mu_{AB}^2)\Phi(\delta_{ABC})+(\sigma_C^2+\mu_C^2)(1-\Phi(\delta_{ABC}))+(\mu_{AB}+\mu_C)\theta_{ABC}\phi(\delta_{ABC})-\mu_{ABC}^2=$$
$$=(3.890^2+21.693^2)\Phi(0.345)+(4.243^2+20^2)(1-\Phi(0.345))+(21.693+20)4.913\phi(0.345)$$
$$-22.921^2=485.718\cdot 0.635+418\cdot(1-0.635)+204.754\cdot 0.376-525.372=12.617\rightarrow$$
$$\sigma_k=\sqrt{12.617}=3.548\rightarrow\sigma_{ABC}$$

Step 5 (1 merger)

■ Maximum merger between paths $i=[(A,B)_{max},C]_{max}$=ABC (with μ_i=22.921 and σ_i=3.548) and j=D (with μ_j=20 and σ_j=4.243), then $k=\{[(A,B)_{max},C]_{max},D\}_{max}=ABCD$. The expression for the coefficient of correlation calculation is derived from Sculli & Shum (1991):

$$\rho_{ABCD}=\frac{[COV_{AD}\Phi(\delta_{AB})+COV_{BD}(1-\Phi(\delta_{AB}))]\Phi(\delta_{ABC})+COV_{CD}(1-\Phi(\delta_{ABC}))}{\sigma_{ABC}\sigma_D}=$$
$$=\frac{[0\Phi(0)+9(1-\Phi(0))]\Phi(0.345)+9(1-\Phi(0.345))}{3.548\cdot 4.243}=\frac{9\cdot(1-0.5)\cdot 0.635+9\cdot(1-0.635)}{15.054}=0.408$$

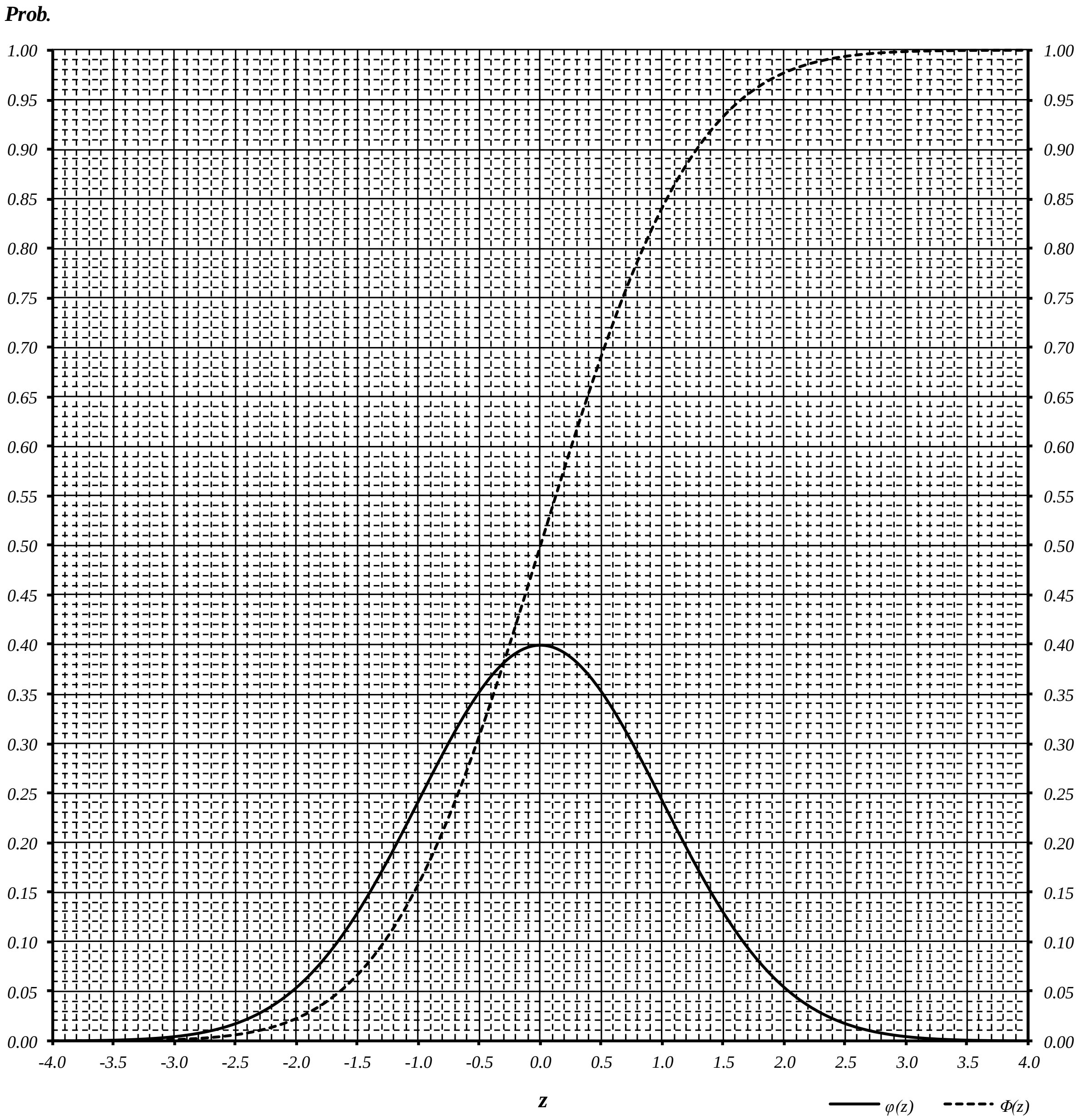
$$\theta_{ABCD}=\sqrt{\sigma_{ABC}^2+\sigma_D^2-2\rho_{ABCD}\sigma_{ABC}\sigma_D}=\sqrt{3.548^2+4.243^2-2\cdot 0.408\cdot 3.548\cdot 4.243}=\sqrt{18.305}=4.278$$

$$\delta_{ABCD}=(\mu_{ABC}-\mu_D)/\theta_{ABCD}=(22.921-20)/4.278=0.683$$

$$\mu_{ABCD} = \mu_{ABC}\Phi(\delta_{ABCD}) + \mu_D(1-\Phi(\delta_{ABCD})) + \theta_{ABCD}\phi(\delta_{ABCD}) = 22.921\Phi(0.683) + 20(1-\Phi(0.683))$$
$$+ 4.278\phi(0.683) = 22.921 \cdot 0.753 + 20 \cdot (1-0.753) + 4.278 \cdot 0.316 = 23.551 \rightarrow \textit{Project Duration mean}$$

$$\sigma^2_{ABCD} = (\sigma^2_{ABC} + \mu^2_{ABC})\Phi(\delta_{ABCD}) + (\sigma^2_D + \mu^2_D)(1-\Phi(\delta_{ABCD})) + (\mu_{ABC} + \mu_D)\theta_{ABCD}\phi(\delta_{ABCD}) - \mu^2_{ABCD} =$$
$$= (3.548^2 + 22.921^2)\Phi(0.683) + (4.243^2 + 20^2)(1-\Phi(0.683)) + (22.921 + 20)4.278\phi(0.683)$$
$$- 23.551^2 = 537.960 \cdot 0.753 + 418 \cdot (1-0.753) + 183.616 \cdot 0.316 - 554.650 = 11.695$$
$$\rightarrow \sigma_k = \sqrt{11.695} = 3.420 \rightarrow \textit{Project Duration standard deviation}$$

It is easy to see that, due to the recursive application of the maximum merger for the four paths, these results (μ_p=23.551 and σ_p=3.420) do not exactly coincide with the previous from alternative 1 (μ_p=23.385 and σ_p=3.503), but they are very close. Again, the probabilities of any possible project duration (p) can be calculated exactly as was explained for the Alternative 1.

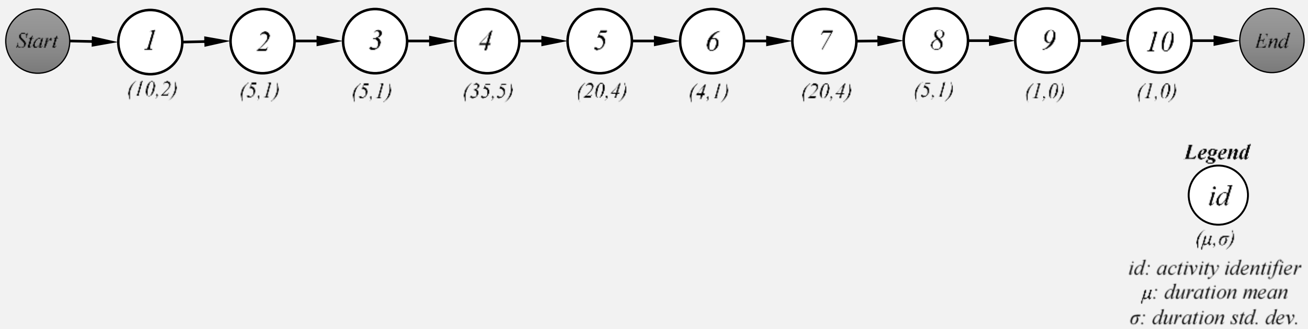


Supplemental Figure 1. Abacus for calculating the probability density function *φ(z)* and the cumulative distribution function *Φ(z)* of the Standard Normal distribution

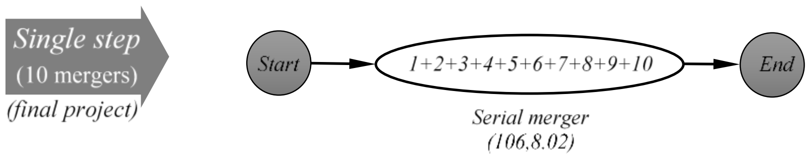
Sample of 7 exercises (project schedules) of increasing difficulty for learning M-PERT

Note: *for the calculations of the original PERT technique, self-loops are ignored (as they cannot be calculated with PERT) and it has been assumed that the Critical Path goes through the path with higher probabilities of happening in presence of probabilistic paths.*

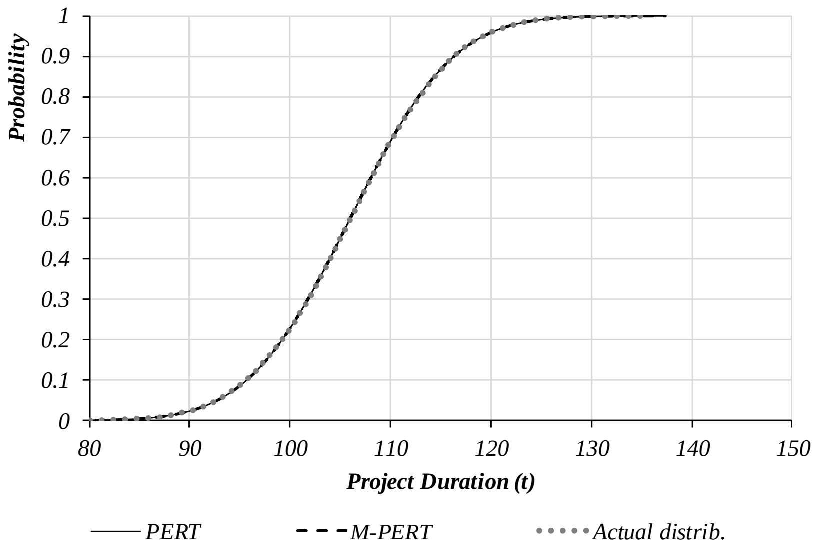
Example Project 1

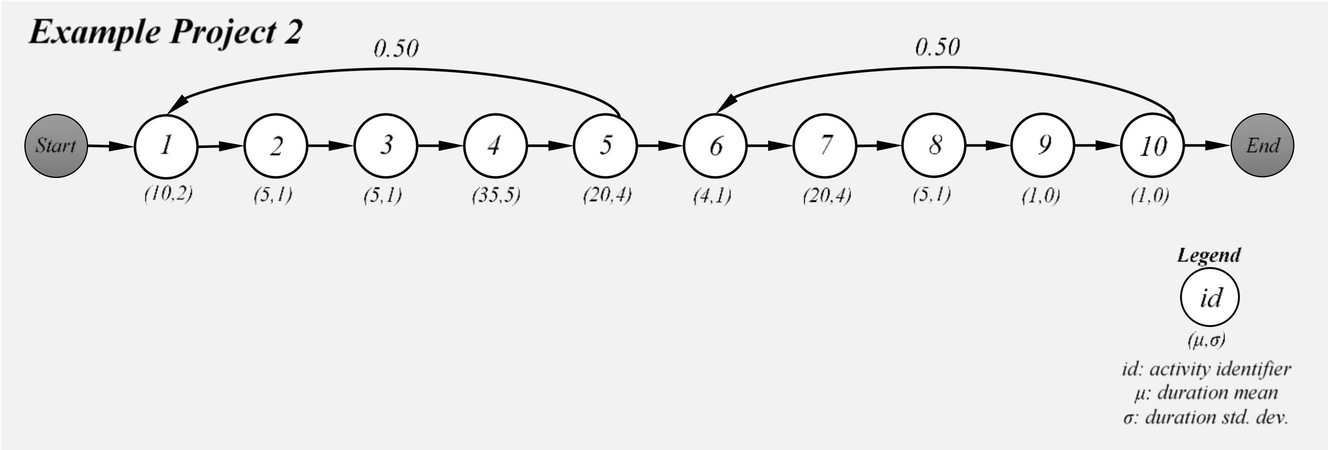


Step-by-step merger procedure

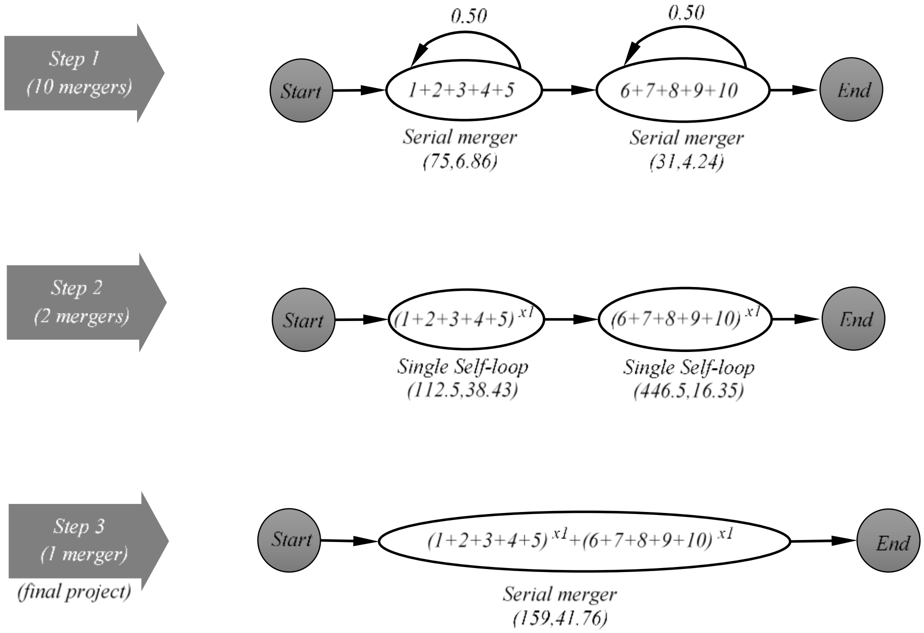


Results





Step-by-step merger procedure



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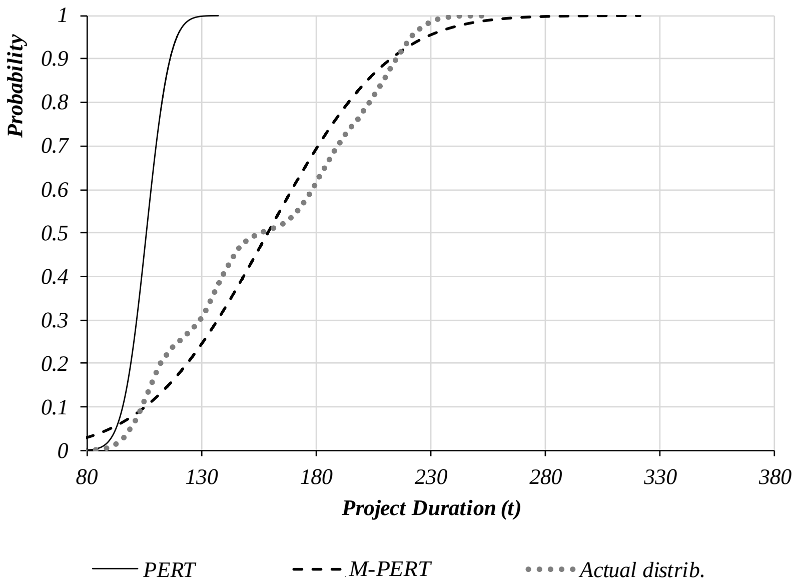
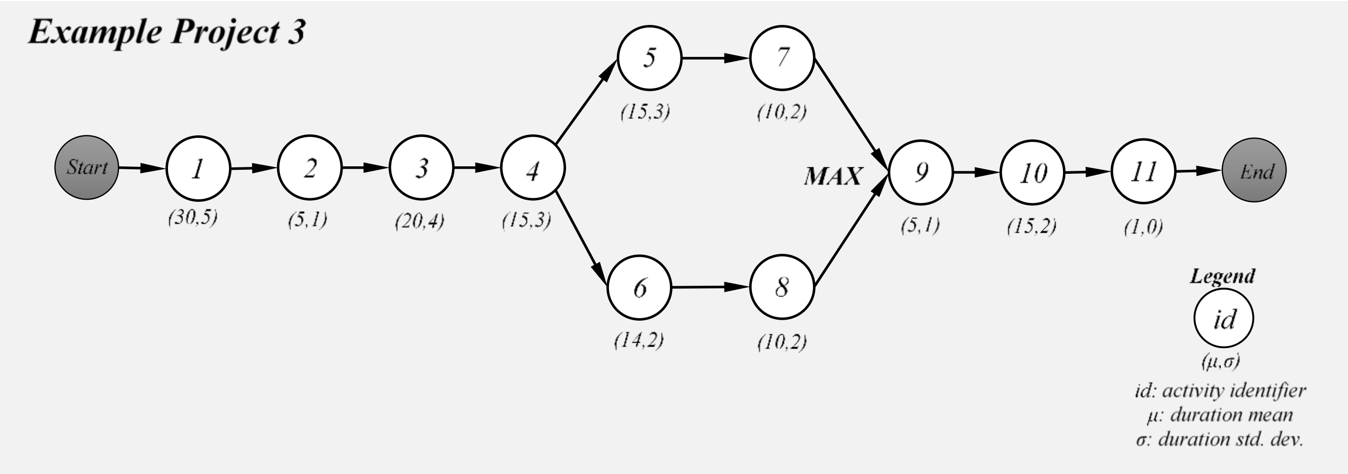
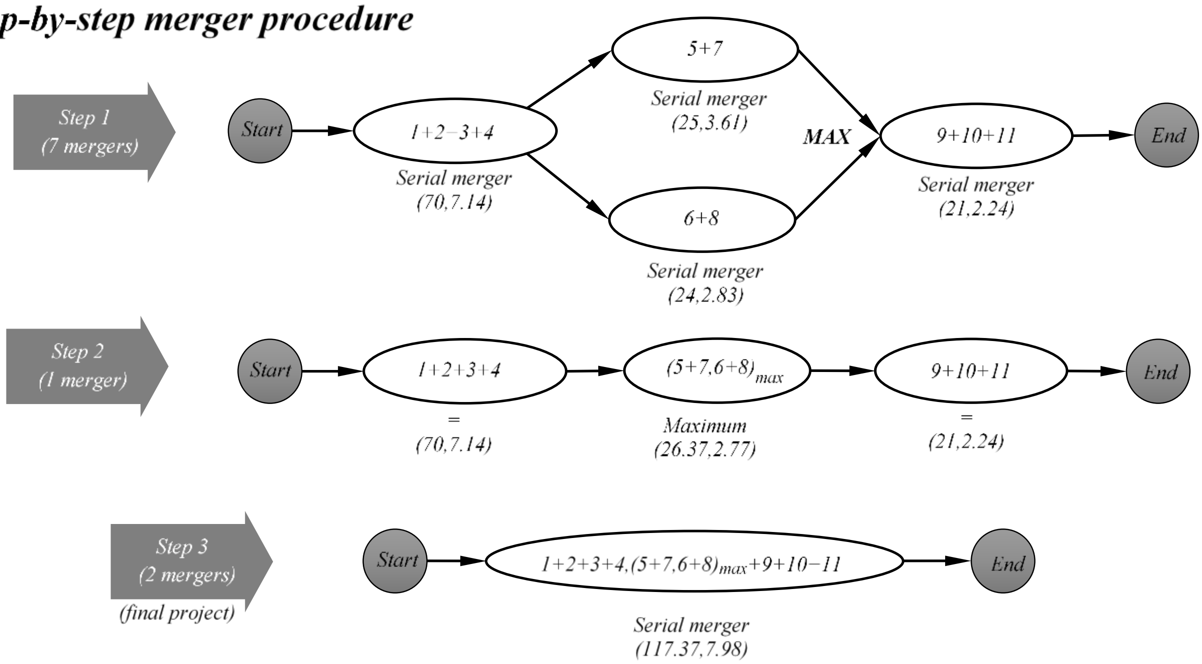
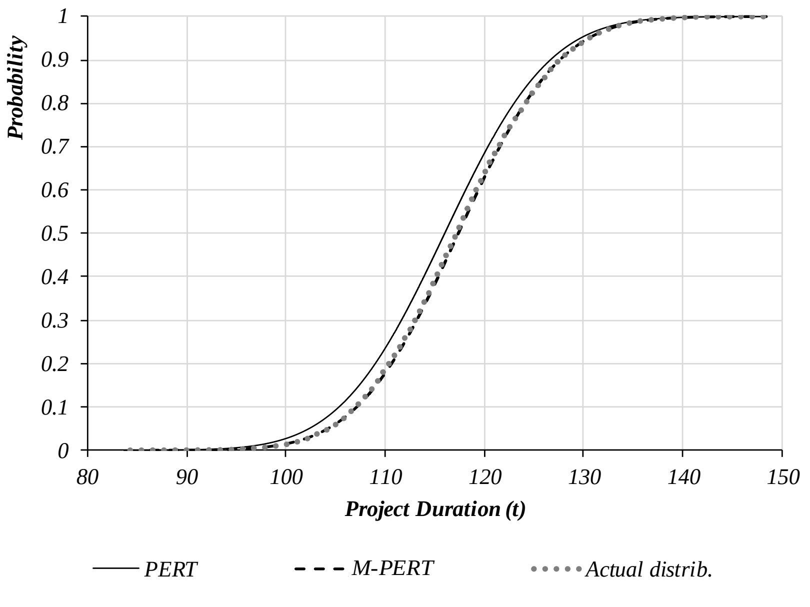


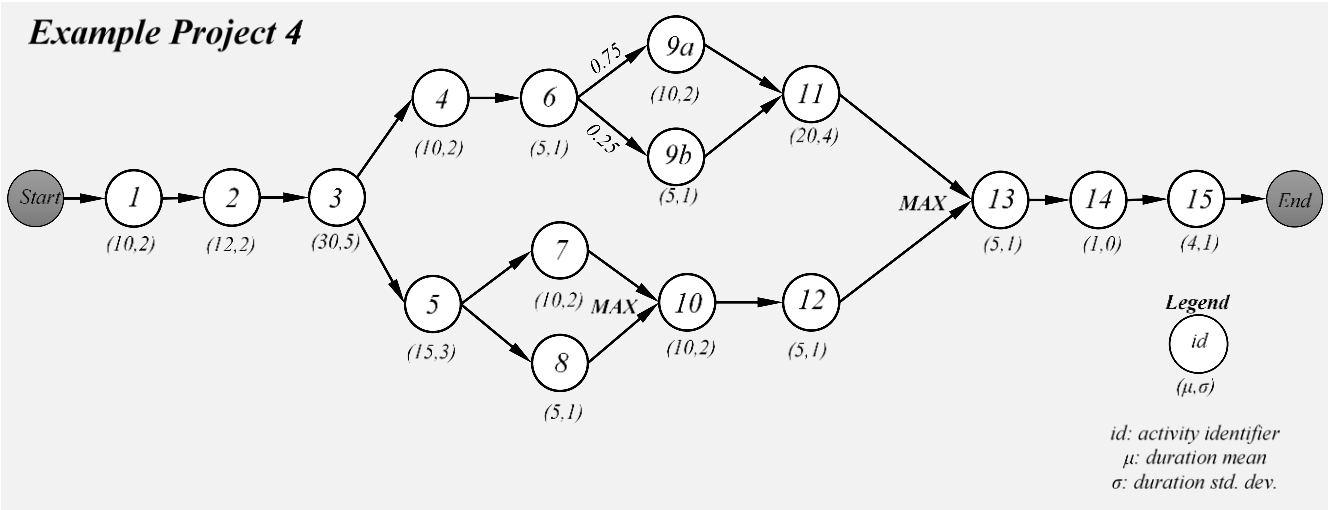


Step-by-step merger procedure

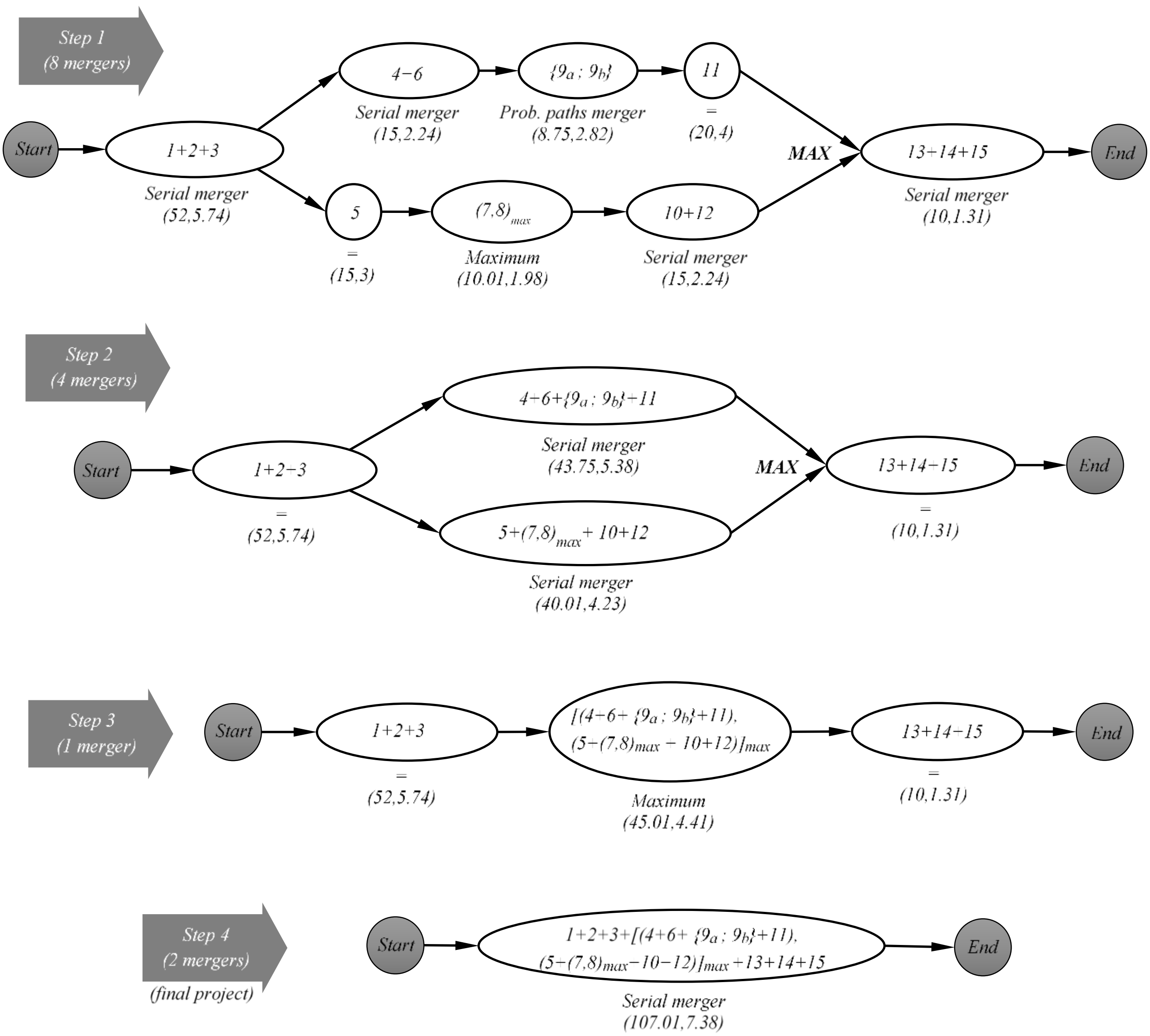


Results

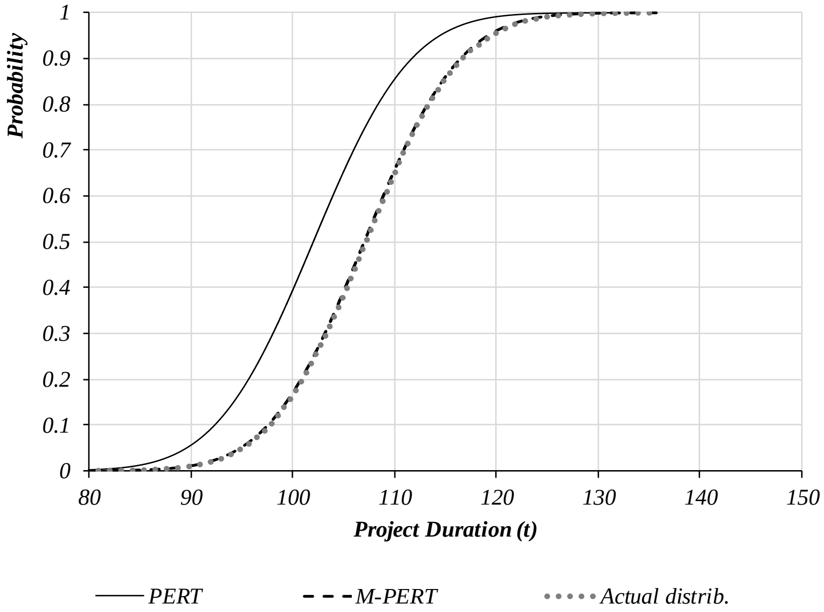




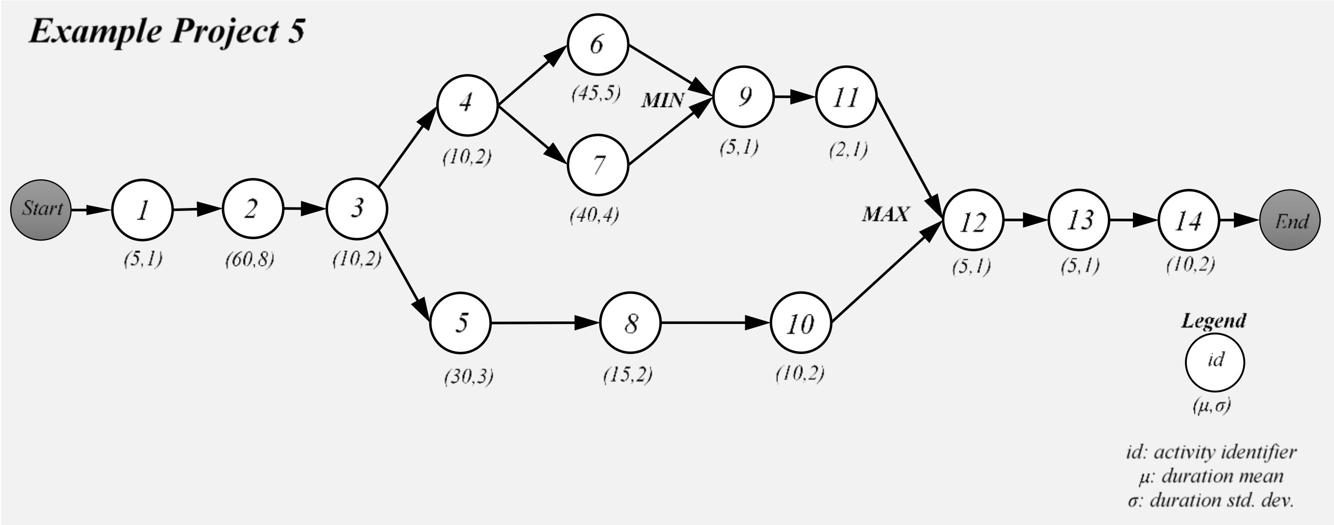
Step-by-step merger procedure



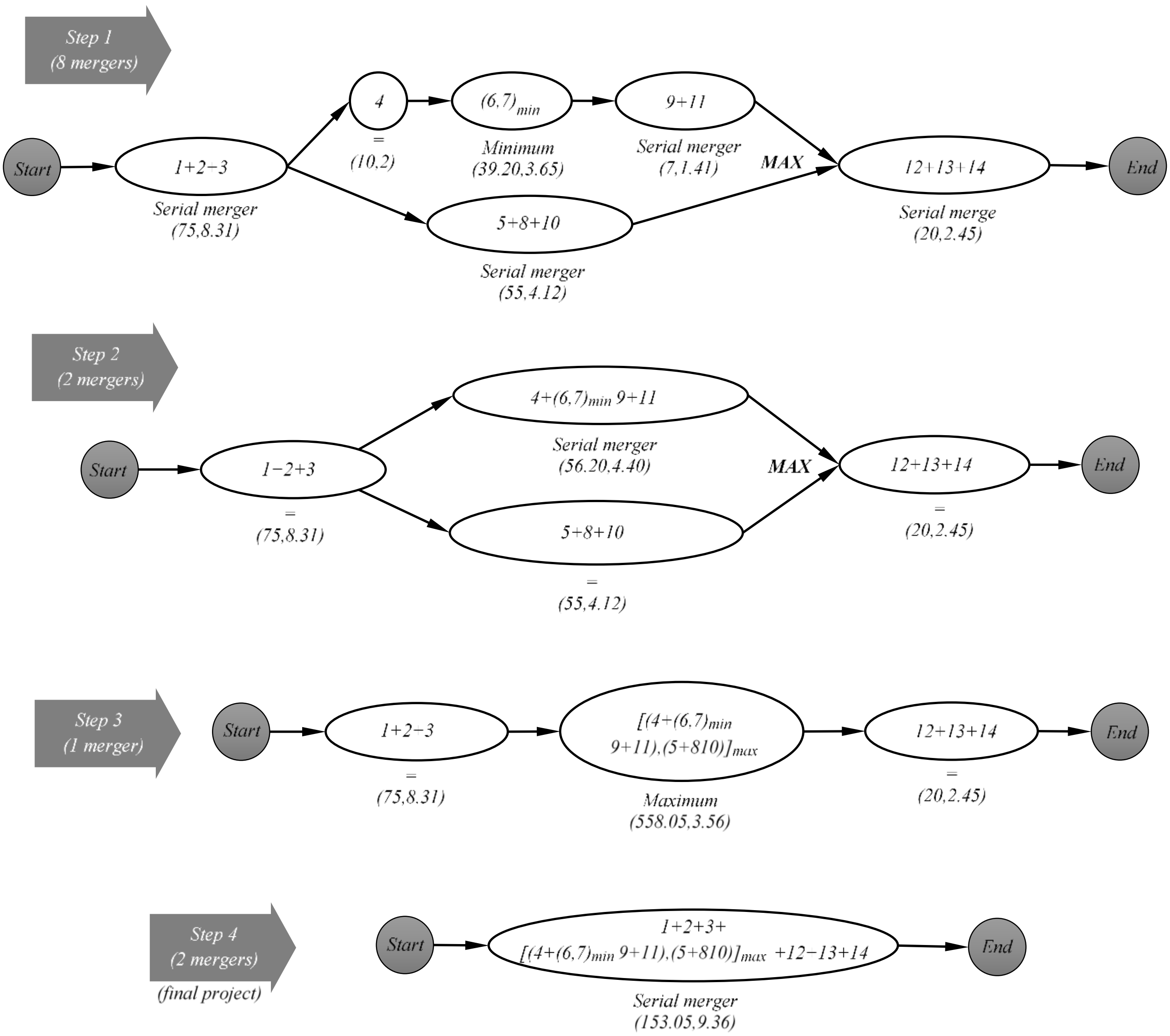
Results



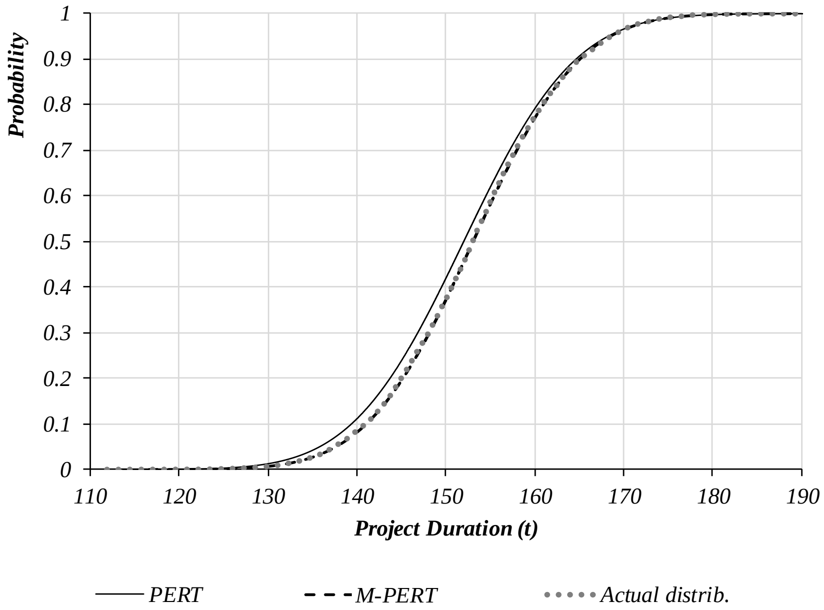
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Probability
80
90
100
110
120
130
140
150
Project Duration (t)
PERT
M-PERT
Actual distrib.



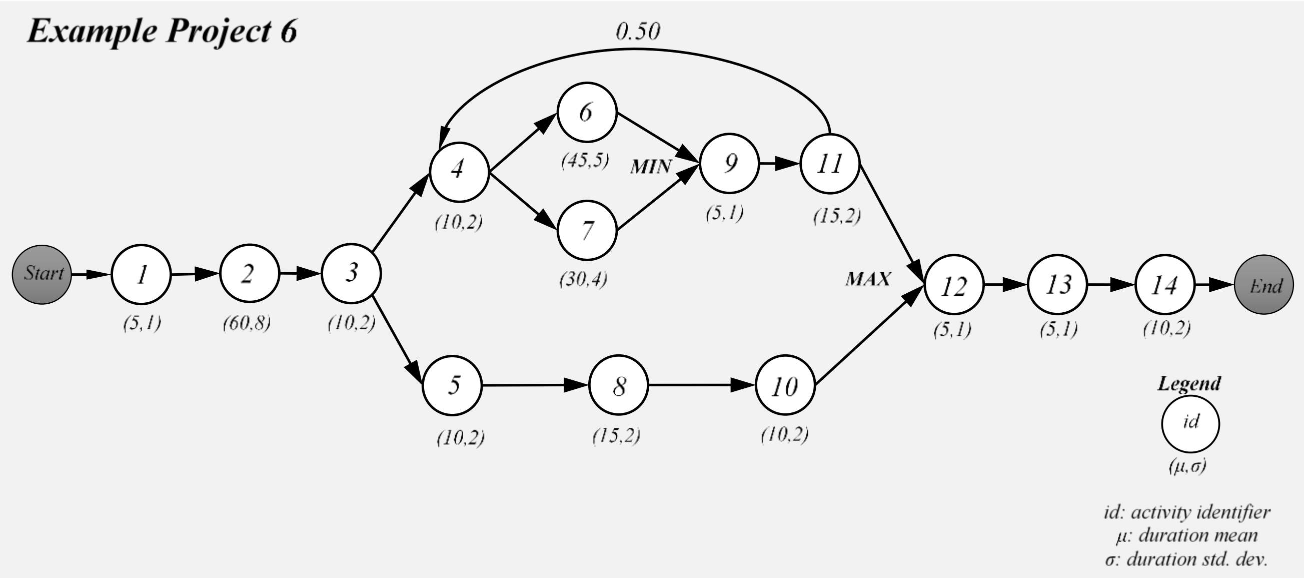
Step-by-step merger procedure



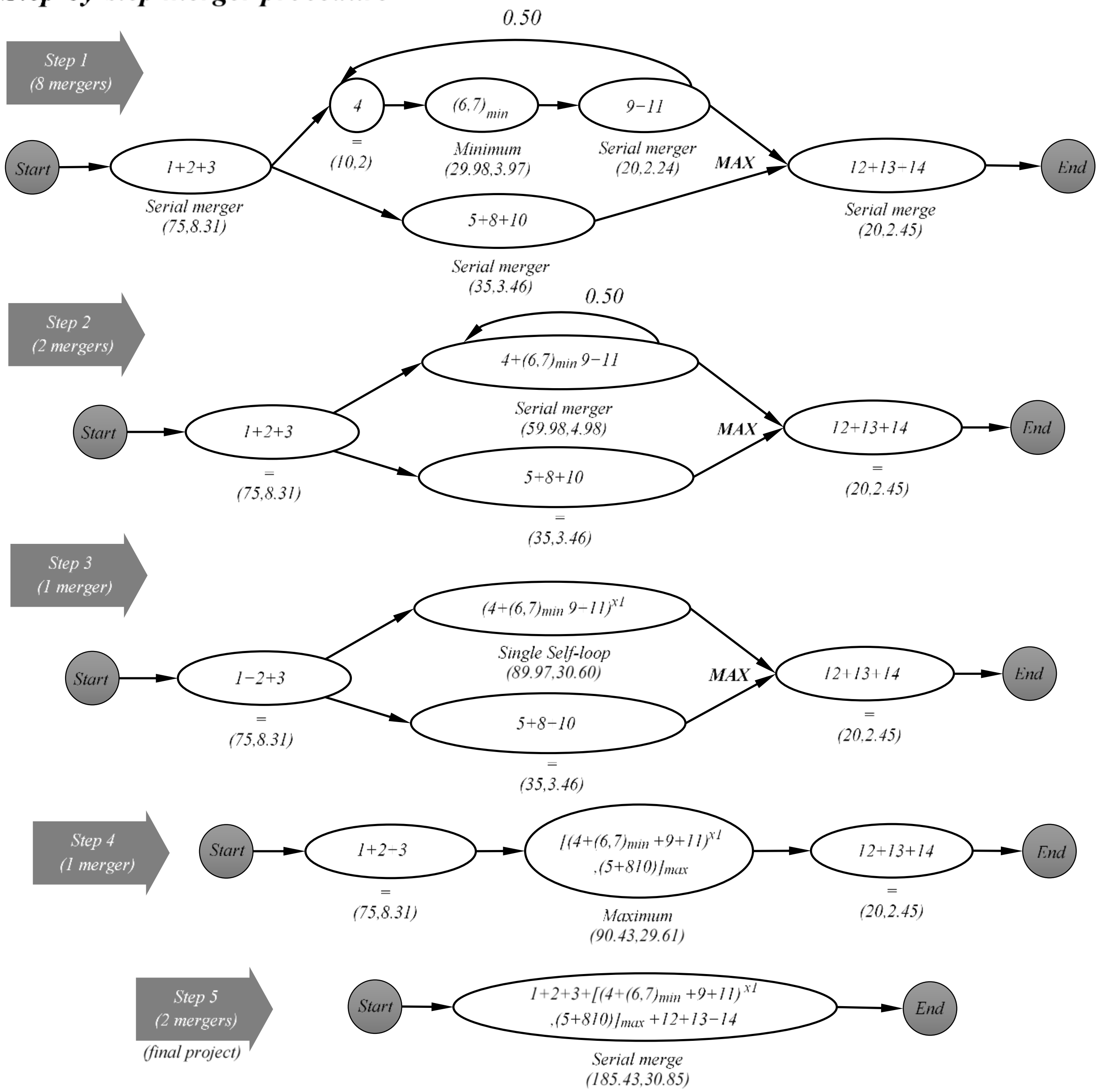
Results



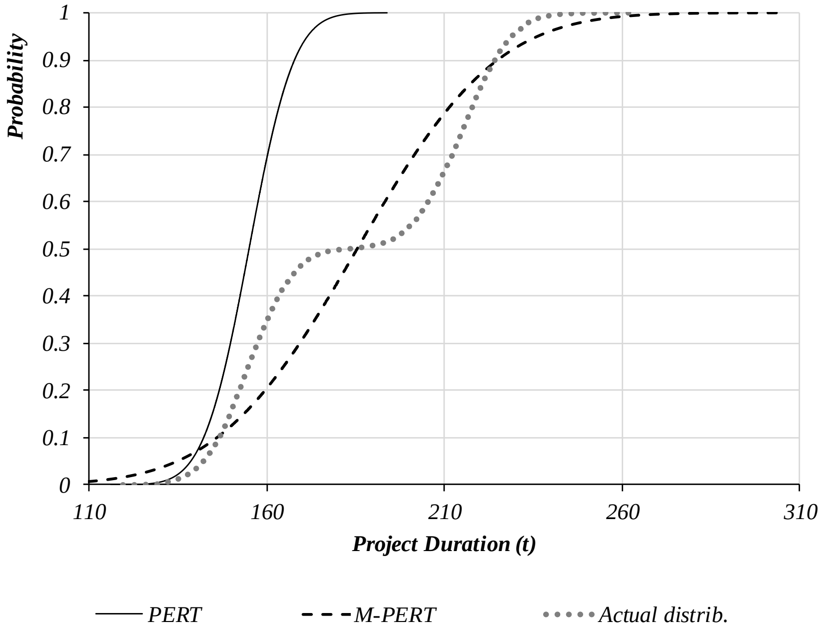
Probability
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
110
120
130
140
150
160
170
180
190
Project Duration (t)
PERT
M-PERT
Actual distrib.



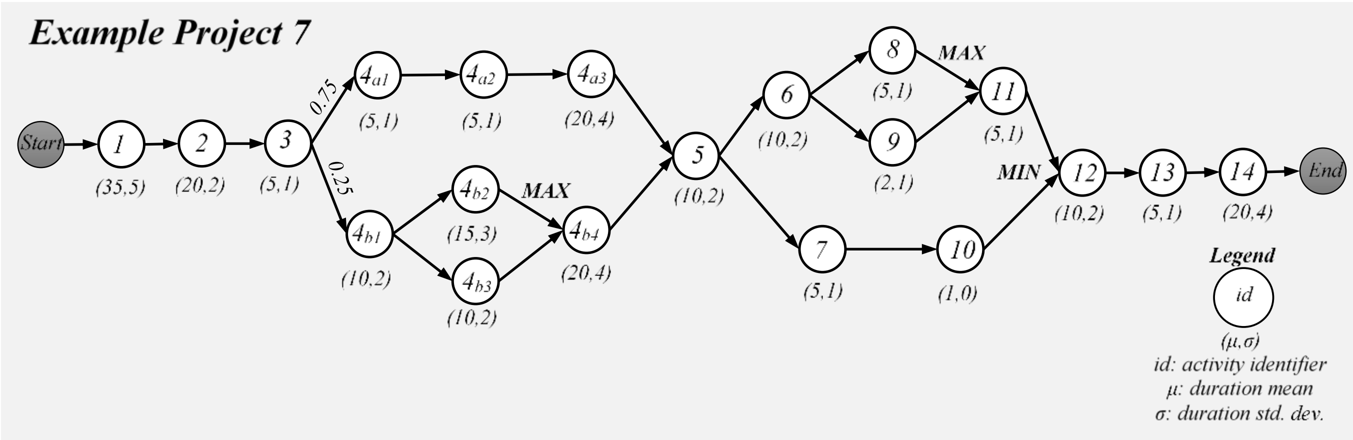
Step-by-step merger procedure



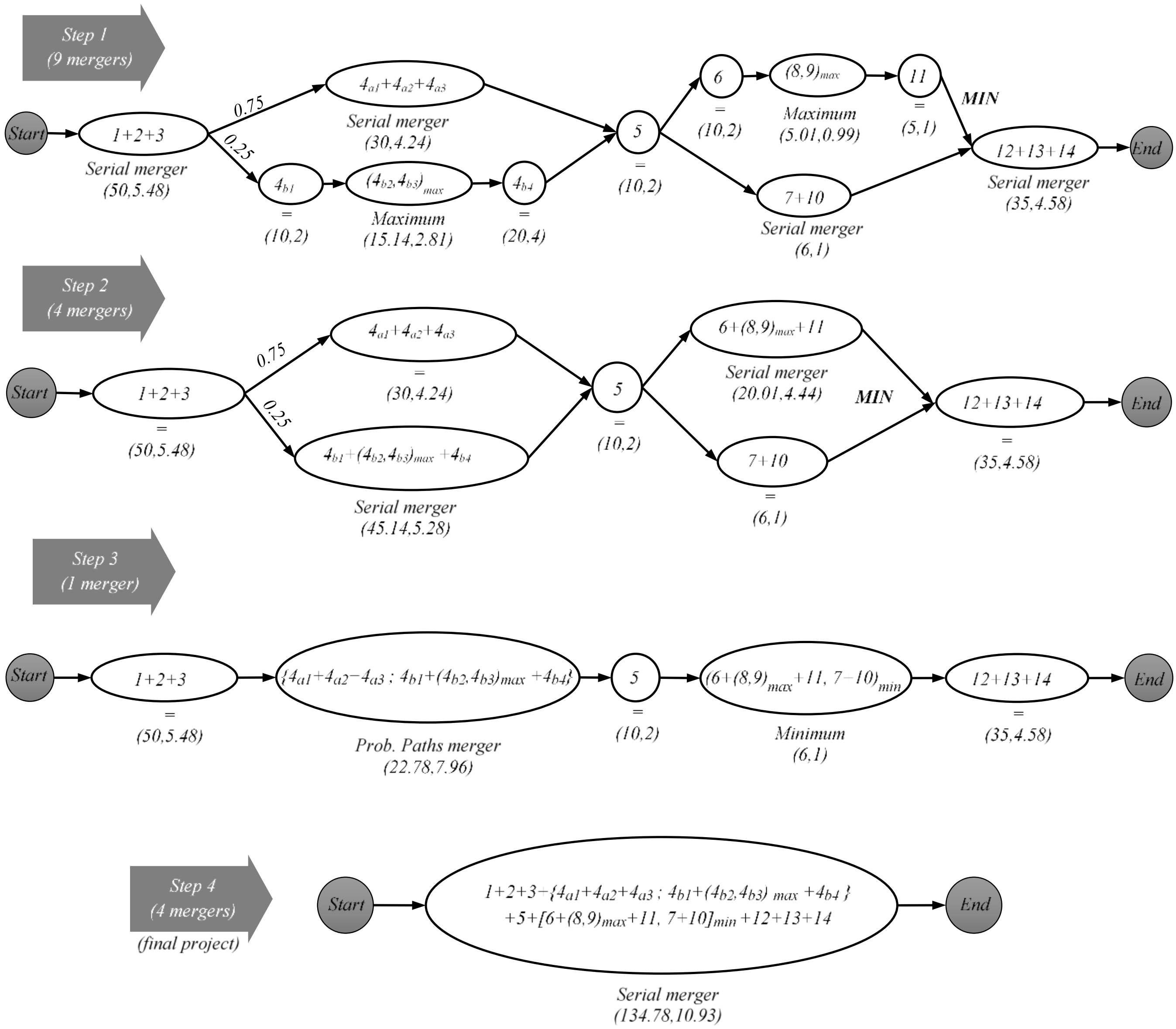
Results



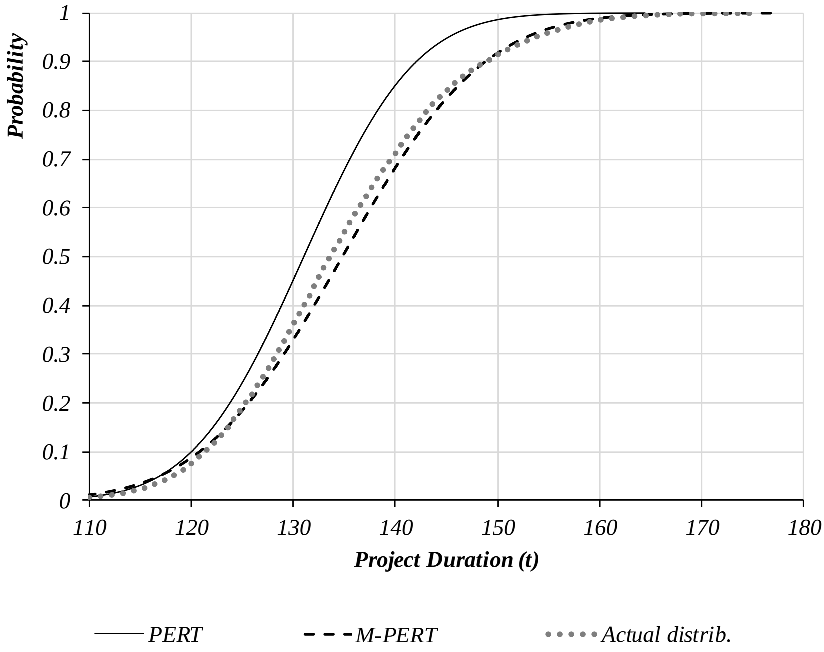
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Probability
110
160
210
260
310
Project Duration (t)
PERT
M-PERT
Actual distrib.



Step-by-step merger procedure



Results



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Probability
110
120
130
140
150
160
170
180
Project Duration (t)
PERT
M-PERT
Actual distrib.