Duality and distance formulas in spaces defined by means of oscillation

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Duality and distance formulas in spaces defined by means of oscillation

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Abstract. For the classical space of functions with bounded mean oscillation, it is well known that $\mathrm{VMO}^{**}=\mathrm{BMO}$ and there are many characterizations of the distance from a function f in BMO to VMO. When considering the Bloch space, results in the same vein are available with respect to the little Bloch space. In this paper such duality results and distance formulas are obtained by pure functional analysis. Applications include general Möbius invariant spaces such as Q_K-spaces, weighted spaces, Lipschitz–Hölder spaces and rectangular BMO of several variables.

1. Introduction

It is well known that the bidual of VMO is BMO, that is, the second dual of the space of functions on the unit circle $\mathbb{T}$ (or the line $\mathbb{R}$) with vanishing mean oscillation can be naturally represented as the space of functions with bounded mean oscillation. The same holds true for the respective subspaces VMOA and BMOA of those functions in VMO or BMO whose harmonic extensions are analytic, and there has been considerable interest in estimating the distance from a function $f\in\mathrm{BMOA}$ to VMOA, starting with Axler and Shapiro [5], continuing with Carmona and Cufí [8] and Stegenga and Stephenson [26]. For the Bloch space B and little Bloch space B_0 the situation is similar. $B_0^{**}=B$ and the distance from $f\in B$ to B_0 has been characterized by Attele [4] and Tjani [27].

Concerning weighted spaces of analytic functions, numerous people have explored the validity of the biduality $Hv_0(\Omega)^{**}=Hv(\Omega)$, see for example Rubel and Shields [24] and Anderson and Duncan [2]. Bierstedt and Summers [6] expanded upon these results and characterized the weights v for which the biduality holds for a general domain Ω.

The purpose of this paper is to obtain such duality results and distance formulas in a very general setting. Working with a Banach space M defined by a big-O condition and a corresponding "little space" M_0, we will under mild assumptions show that $M_0^{**}=M$ and prove an isometric formula for the distance from $f\in M$ to M_0 in terms of the defining condition for M_0. Using a theorem of Godefroy [18], we will additionally obtain as a corollary that M_0^* is the unique isometric predual of M. When $M=B$ this specializes to a result of Nara [23]. The methods involved are purely operator-theoretic, appealing to embeddings into spaces of continuous vector-valued functions rather than analyticity, invariance properties or geometry.

The power of these general results is illustrated in the final section. Many examples will be given there, where the main theorems are applied to general Möbius invariant spaces of analytic functions including a large class of so-called Q_K-spaces, weighted spaces, rectangular BMO of several variables and Lipschitz–Hölder spaces.

The paper is organized as follows: Section 2 formulates the main results, while Section 3 contains their proofs. Applications of the theory are given in Section 4.

2. Statements of main results

In this section the statements of the main theorems are given. Fixing the notation for this, and for the rest of this paper, X and Y will be two Banach spaces, with X separable and reflexive. $\mathcal{L}$ will be a given collection of bounded operators $L\colon X\to Y$ that is accompanied by a σ-compact locally compact Hausdorff topology τ such that for every $x\in X$, the map $T_x\colon \mathcal{L}\to Y$ given by $T_xL=Lx$ is continuous. Here Y is considered with its norm topology. Note that we impose no particular algebraic structure on $\mathcal{L}$. Z^* will denote the dual of a Banach space Z and we shall without mention identify Z as a subset of Z^{**} in the usual way.

Our main objects of study are the two spaces

$$M(X,\mathcal{L})=\Big\{x\in X : \sup_{L\in\mathcal{L}} \|Lx\|_Y<\infty\Big\}$$

and

$$M_0(X,\mathcal{L})=\Big\{x\in M(X,\mathcal{L}) : \varlimsup_{\mathcal{L}\ni L\to\infty} \|Lx\|_Y=0\Big\},$$

where the limit $L\to\infty$ is taken in the sense of the one-point compactification of $(\mathcal{L},\tau)$. By replacing X with the closure of $M(X,\mathcal{L})$ in X and making appropriate modifications to the setting just described, we may as well assume that $M(X,\mathcal{L})$ is dense in X. Furthermore, we assume that $\mathcal{L}$ is such that

$$\|x\|_{M(X,\mathcal{L})}=\sup_{L\in\mathcal{L}} \|Lx\|_Y$$

defines a norm which makes $M(X,\mathcal{L})$ into a Banach space continuously contained in X. Note that $M_0(X,\mathcal{L})$ is then automatically a closed subspace of $M(X,\mathcal{L})$.

These assumptions are mostly for convenience and will hold trivially in all examples to come.

Example 2.1. Let $X=L^2(\mathbb{T})/\mathbb{C}$, $Y=L^1(\mathbb{T})$ and

$$\mathcal{L}=\left\{L_I : L_I f=\chi_I \frac{1}{|I|}(f-f_I) \text{ and } \varnothing\neq I\subset\mathbb{T} \text{ is an arc}\right\},$$

where χ_I is the characteristic function of I, $|I|$ is its length and $f_I=\int_I f\,ds/|I|$ is the average of f on I. Each arc I is given by its midpoint $a\in\mathbb{T}$ and length b, $0<b\leq 2\pi$. We give $\mathcal{L}$ the quotient topology τ of $\mathbb{T}\times(0,2\pi]$ obtained when identifying all pairs (a_1,b_1) and (a_2,b_2) with $b_1=b_2=2\pi$. Then $M(X,\mathcal{L})=\mathrm{BMO}(\mathbb{T})$ is the space of functions of bounded mean oscillation on the circle. $L_I\to\infty$ in τ means exactly that $|I|\to 0$, so it follows that $M_0(X,\mathcal{L})=\mathrm{VMO}(\mathbb{T})$ are the functions of vanishing mean oscillation (see Garnett [17], Chapter VI).

$M_0(X,\mathcal{L})$ may be trivial even when $M(X,\mathcal{L})$ is not. This happens for example when $M(X,\mathcal{L})$ is the space of Lipschitz-continuous functions f on $[0,1]$ with $f(0)=0$ (see Example 4.6). In the general context considered here we shall not say anything about this, but instead make one of the following two assumptions. They say that $M_0(X,\mathcal{L})$ is dense in X (under the X-norm) with additional norm-control when approximating elements of $M(X,\mathcal{L})$. This is a natural hypothesis that is easy to verify in the examples we have in mind. In fact, the assumptions are necessary for the respective conclusions of Theorem 2.2.

Assumption A. For every $x\in M(X,\mathcal{L})$ there is a sequence $\{x_n\}_{n=1}^\infty$ in $M_0(X,\mathcal{L})$ such that $x_n\to x$ in X and $\sup_n \|x_n\|_{M(X,\mathcal{L})}<\infty$.

Assumption B. For every $x\in M(X,\mathcal{L})$ there is a sequence $\{x_n\}_{n=1}^\infty$ in $M_0(X,\mathcal{L})$ such that $x_n\to x$ in X and $\sup_n \|x_n\|_{M(X,\mathcal{L})}\leq\|x\|_{M(X,\mathcal{L})}$.

Note that the assumptions could have equivalently been stated with the sequence $\{x_n\}_{n=1}^\infty$ tending to x only weakly in X. The main theorems are as follows.

Theorem 2.2. *Suppose that Assumption A holds. Then X^* is continuously contained and dense in $M_0(X,\mathcal{L})^*$. Denoting by*

$$I\colon M_0(X,\mathcal{L})^{**}\longrightarrow X$$

the adjoint of the inclusion map $J\colon X^*\to M_0(X,\mathcal{L})^*$, the operator I is a continuous isomorphism of $M_0(X,\mathcal{L})^{**}$ onto $M(X,\mathcal{L})$ which acts as the identity on $M_0(X,\mathcal{L})$. Furthermore, I is an isometry if Assumption B holds.

Theorem 2.3. *Assume that Assumption A holds. Then, for any $x\in M(X,\mathcal{L})$, it holds that*

$$\operatorname{dist}(x, M_0(X,\mathcal{L}))_{M(X,\mathcal{L})} = \overline{\lim_{\mathcal{L}\ni L\to\infty}} \|Lx\|_Y\,. \tag{1}$$

Example 2.4. Let X, Y, $\mathcal{L}$ and τ be as in Example 2.1. Then Assumption B holds by letting $f_n=f*P_{1-1/n}$ for $f\in M(X,\mathcal{L})$, where P_r is the Poisson kernel for the unit disc, $P_r(\theta)=(1-r^2)/|e^{i\theta}-r|^2$. The theorems say that $\mathrm{VMO}(\mathbb{T})^{**}\simeq \mathrm{BMO}(\mathbb{T})$ isometrically via the $L^2(\mathbb{T})$-pairing, and that

$$\operatorname{dist}(f,\mathrm{VMO})_{\mathrm{BMO}} = \overline{\lim_{|I|\to 0}} \frac{1}{|I|}\int_I |f-f_I|\,ds. \tag{2}$$

This improves upon a result in [26]. Note that if we repeat the construction with $Y=L^p(\mathbb{T})$ for some $1<p<\infty$, we still obtain that $M(X,\mathcal{L})=\mathrm{BMO}(\mathbb{T})$ with an equivalent norm, due to the John–Nirenberg theorem (see [17]). This gives us a distance formula, corresponding to (2), involving the p-norm on the right-hand side.

We say that Z is a *unique* (*isometric*) *predual* if for any Banach space W, W^* isometric to Z^* implies that W is isometric to Z. Note that the canonical decomposition

$$Z^{***} = Z^*\oplus Z^{\perp} \tag{3}$$

induces a projection $\pi\colon Z^{***}\to Z^*$ with kernel $Z^{\perp}$,

$$(\pi z^{***})(z) = z^{***}(z), \quad z\in Z.$$

We say that Z is a *strongly unique predual* if this is the only projection π from Z^{***} to Z^* of norm one with $\operatorname{Ker}\pi$ weak-star closed. An excellent survey of these matters can be found in Godefroy [18].

Corollary 2.5. *Suppose that Assumption B holds. Then $M_0(X,\mathcal{L})^*$ is the strongly unique predual of $M(X,\mathcal{L})$.*

3. Proofs of main results

3.1. Preliminaries

One of our main tools will be the isometric embedding $V\colon M(X,\mathcal{L})\to C_b(\mathcal{L},Y)$ of $M(X,\mathcal{L})$ into the space $C_b(\mathcal{L},Y)$ of bounded continuous Y-valued functions on $\mathcal{L}$, given by

$$Vx = T_x,$$

where $T_x L = Lx$ as before. $C_b(\mathcal{L}, Y)$ is normed by the usual supremum norm, so that V indeed is an isometry. Note that V embeds $M_0(X, \mathcal{L})$ into the space $C_0(\mathcal{L}, Y)$, consisting of those functions $T \in C_b(\mathcal{L}, Y)$ vanishing at infinity.

In order to study duality via this embedding, we will make use of vector-valued integration theory. Of central importance will be the Riesz–Zinger theorem [28] for $C_0(\mathcal{L}, Y)$, representing the dual of $C_0(\mathcal{L}, Y)$ as a space of measures. Let $\mathcal{B}_0$ be the σ-algebra of all Baire sets of $\mathcal{L}$, generated by the compact G_δ-sets. By $\operatorname{cabv}(\mathcal{L}, Y^*)$ we shall denote the Banach space of countably additive vector Baire measures $\mu\colon \mathcal{B}_0 \to Y^*$ with bounded variation

$$\|\mu\| = \sup \sum_{i=1}^{\infty} \|\mu(\mathcal{E}_i)\|_{Y^*} < \infty,$$

where the supremum is taken over all finite partitions of $\mathcal{L} = \bigcup_{i=1}^{\infty} \mathcal{E}_i$ into disjoint Baire sets $\mathcal{E}_i$. Excellent references for these matters are found for example in Dobrakov [11], [12] and [13].

Theorem 3.1. ([13]) *For every bounded linear functional $\ell \in C_0(\mathcal{L}, Y)^*$ there is a unique measure $\mu \in \operatorname{cabv}(\mathcal{L}, Y^*)$ such that*

$$\ell(T) = \int_{\mathcal{L}} T(L)\, d\mu(L), \quad T \in C_0(\mathcal{L}, Y). \tag{4}$$

Furthermore, every $\mu \in \operatorname{cabv}(\mathcal{L}, Y^*)$ defines a continuous functional on $C_0(\mathcal{L}, Y)$ via (4), and $\|\ell\| = \|\mu\|$.

Remark 3.2. In our situation, $(\mathcal{L}, \tau)$ being σ-compact, every continuous function $T\colon \mathcal{L} \to Y$ is Baire measurable ([19], pp. 220–221). In particular, every $T \in C_b(\mathcal{L}, Y)$ induces a bounded functional $m \in \operatorname{cabv}(\mathcal{L}, Y^*)^*$ via

$$m(\mu) = \int_{\mathcal{L}} T(L)\, d\mu(L).$$

Clearly, $\|m\| = \|T\|_{C_b(\mathcal{L}, Y)}$. Hence, $C_b(\mathcal{L}, Y)$ isometrically embeds into $\operatorname{cabv}(\mathcal{L}, Y^*)^*$ in a way that extends the canonical embedding of $C_0(\mathcal{L}, Y)$ into $C_0(\mathcal{L}, Y)^{**} = \operatorname{cabv}(\mathcal{L}, Y^*)^*$.

In the proof of Theorem 2.3 we shall require the following simple lemma.

Lemma 3.3. *Suppose that $m \in C_b(\mathcal{L}, Y)^*$ annihilates $C_0(\mathcal{L}, Y)$. Then*

$$|m(T)| \le \|m\| \varlimsup_{L \to \infty} \|T(L)\|_Y, \quad T \in C_b(\mathcal{L}, Y). \tag{5}$$

Proof. Let $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \dots$ be an increasing sequence of compact subsets of $(\mathcal{L}, \tau)$ such that $\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{K}_n$. Denote by $\alpha\mathcal{L} = \mathcal{L} \cup \{\infty\}$ the one-point compactification of $\mathcal{L}$. For each n, let $s_n : \alpha L \to [0,1]$ be a continuous function such that $s_n^{-1}(0) \supset \mathcal{K}_n$ and $s_n(\infty) = 1$. Then

$$|m(T)| = |m(s_n T)| \leq \|m\| \sup_{L \in \mathcal{L} \setminus \mathcal{K}_n} \|T(L)\|_Y .$$

In the limit we obtain (5). □

Corollary 2.5 will follow as an application of a result in [18].

Theorem 3.4. ([18], Theorem V.1) *Let Z be a Banach space. Suppose that for every $z^{**} \in Z^{**}$ it is true that $z^{**} \in Z$ if and only if*

$$z^{**}(z^*) = \lim_{n \to \infty} z^{**}(z_n^*)$$

for every weak Cauchy sequence $\{z_n^*\}_{n=1}^{\infty}$ in Z^* with weak-star limit z^*. Then Z is the strongly unique predual of Z^*.

3.2. Main proofs

We shall now proceed to prove the main theorems and Corollary 2.5.

Proof of Theorem 2.2. As $M_0(X, \mathcal{L})$ is continuously contained in X, every $x^* \in X^*$ is clearly continuous also on $M_0(X, \mathcal{L})$. Assumption A implies that $M_0(X, \mathcal{L})$ is dense in X, so that each element of X^* induces a unique functional on $M_0(X, \mathcal{L})$. This proves that X^* is continuously contained in $M_0(X, \mathcal{L})^*$.

We shall now demonstrate that X^* is dense in $M_0(X, \mathcal{L})^*$. Thus, let $\ell \in M_0(X, \mathcal{L})^*$. By Theorem 3.1 and the Hahn–Banach theorem there is a measure $\mu \in \mathrm{cabv}(\mathcal{L}, Y^*)$ such that

$$\ell(x) = (\ell \circ V^{-1})(T_x) = \int_{\mathcal{L}} Lx \, d\mu(L), \quad x \in M_0(X, \mathcal{L}).$$

Let $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \dots$ be an increasing sequence of compact subsets of $(\mathcal{L}, \tau)$ such that $\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{K}_n$. By [19], Section 50, Theorem D, we can choose the $\mathcal{K}_n$ to be G_δ and hence Baire sets. Put $\mu_n = \mu|_{\mathcal{K}_n}$ and let ℓ_n be the corresponding functionals

$$\ell_n(x) = \int_{\mathcal{L}} Lx \, d\mu_n(L).$$

The operators $L\in\mathcal{K}_n$ are uniformly bounded by the Banach–Steinhaus theorem, from which it is clear that $\ell_n\in X^*$. Finally, note that $\lim_{n\to\infty}\|\mu_n-\mu\|=0$, which implies that the functionals ℓ_n converge to ℓ in $M_0(X,\mathcal{L})^*$.

X^* being dense ensures the injectivity of $I=J^*$. Moreover, it is clear that I acts as the identity on $M_0(X,\mathcal{L})$. Let $m\in M_0(X,\mathcal{L})^{**}$ and $x=Im\in X$. Note that the unit ball of $M_0(X,\mathcal{L})$ is weak-star dense in the unit ball of $M_0(X,\mathcal{L})^{**}$ ([9], Proposition 4.1). Furthermore, the weak-star topology of $M_0(X,\mathcal{L})^{**}$ is metrizable on the unit ball, since $M_0(X,\mathcal{L})^*$ was just proven to be separable. Accordingly, choose a sequence $\{x_n\}_{n=1}^{\infty}\subset M_0(X,\mathcal{L})$ with $\sup_n\|x_n\|\le\|m\|$ such that $x_n\to m$ weak-star. Then, for $y^*\in Y^*$ and $L\in\mathcal{L}$,

$$\begin{aligned} y^*(Lx) = (L^*y^*)(x) = m(JL^*y^*) &= \lim_{n\to\infty}(JL^*y^*)(x_n) \\ &= \lim_{n\to\infty}(L^*y^*)(x_n) = \lim_{n\to\infty} y^*(Lx_n). \end{aligned}$$

It follows that $x\in M(X,\mathcal{L})$ and

$$\|x\|_{M(X,\mathcal{L})} = \|Im\|_{M(X,\mathcal{L})} \le \|m\|_{M_0(X,\mathcal{L})^{**}}, \tag{6}$$

since

$$\|Lx\|_Y = \sup_{\|y^*\|=1}|y^*(Lx)| \le \sup_n\|x_n\|_{M(X,\mathcal{L})} \le \|m\|, \quad L\in\mathcal{L}.$$

We have thus proved that I maps $M_0(X,\mathcal{L})^{**}$ into $M(X,\mathcal{L})$ contractively.

Given $x\in M(X,\mathcal{L})$ choose a sequence $\{x_n\}_{n=1}^{\infty}\subset M_0(X,\mathcal{L})$ such that $x_n\to x$ in X and $\sup_n\|x_n\|_{M(X,\mathcal{L})}<\infty$ ($\le\|x\|_{M(X,\mathcal{L})}$ if Assumption B holds). Define $\hat{x}\in M_0(X,\mathcal{L})^{**}$ by $\hat{x}(Jx^*)=x^*(x)=\lim_{n\to\infty}(Jx^*)(x_n)$ for $x^*\in X^*$. It is clear from the last equality that this defines $\hat{x}$ as a bounded linear functional on $M_0(X,\mathcal{L})^*$ and if Assumption B holds, then

$$\|\hat{x}\|_{M_0(X,\mathcal{L})^{**}} \le \|x\|_{M(X,\mathcal{L})}. \tag{7}$$

Obviously, $I\hat{x}=x$. This proves that I is onto. If Assumption B holds, then we obtain from (6) and (7) that I is an isometry. $\square$

Proof of Theorem 2.3. Let $m\in M(X,\mathcal{L})^*$. Then $m\circ V^{-1}$ acts on $VM(X,\mathcal{L})$. As in Remark 3.2 we naturally view $C_b(\mathcal{L},Y)$ as a subspace of $\operatorname{cabv}(\mathcal{L},Y^*)^*$. With this identification, $m\circ V^{-1}$ extends by Hahn–Banach's theorem to a functional $\overline{m}\in\operatorname{cabv}(\mathcal{L},Y^*)^{**}$ with $\|\overline{m}\|=\|m\|$. Applying the decomposition (3) with $Z=C_0(\mathcal{L},Y)$ we obtain

$$\operatorname{cabv}(\mathcal{L},Y^*)^{**} = \operatorname{cabv}(\mathcal{L},Y^*)\oplus C_0(\mathcal{L},Y)^{\perp},$$

and we decompose $\overline{m}=\overline{m}_{\omega^*}+\overline{m}_s$ accordingly. Let $\mu\in\mathrm{cabv}(\mathcal{L},Y^*)$ be the measure corresponding to $\overline{m}_{\omega^*}$, so that, in particular,

$$\overline{m}_{\omega^*}(T)=\int_{\mathcal{L}} T(L)\,d\mu(L),\quad T\in C_b(\mathcal{L},Y).$$

Let $I\colon M_0(X,\mathcal{L})^{**}\to M(X,\mathcal{L})$ be the isomorphism given by Theorem 2.2. With $Z=M_0(X,\mathcal{L})$, (3) gives

$$M(X,\mathcal{L})^*\simeq M_0(X,\mathcal{L})^{***}=M_0(X,\mathcal{L})^*\oplus M_0(X,\mathcal{L})^{\perp}, \tag{8}$$

and we obtain a second decomposition $m\circ I=(m\circ I)_{\omega^*}+(m\circ I)_s$.

Our first goal is to show that the former decomposition is an extension of the latter. More precisely, we have the following characterization.

Claim 3.5. *$(m\circ I)_{\omega^*}\equiv 0$ if and only if $\overline{m}_{\omega^*}$ annihilates $VM(X,\mathcal{L})$.*

Proof. To prove this, let $x\in M(X,\mathcal{L})$ and let $\hat{x}=I^{-1}x\in M_0(X,\mathcal{L})^{**}$. As in the proof of Theorem 2.2, choose $\{x_n\}_{n=1}^{\infty}\subset M_0(X,\mathcal{L})$ with $\sup_n \|x_n\|_{M_0(X,\mathcal{L})}\le\|\hat{x}\|$ such that $x_n\to\hat{x}$ weak-star. Note that x_n in particular converges to x weakly in X. Hence $Lx_n\to Lx$ weakly in Y for every $L\in\mathcal{L}$. Since also $\sup_{n,L}\|Lx_n\|_Y<\infty$, it follows from [13], Theorem 9, that

$$\int_{\mathcal{L}} Lx\,d\mu(L)=\lim_{n\to\infty}\int_{\mathcal{L}} Lx_n\,d\mu(L). \tag{9}$$

To be more precise, Theorem 9 in [13] allows us to move the limit inside the integral when integrating over a compact G_δ-set $\mathcal{K}\subset\mathcal{L}$. However, as in the proof of Theorem 2.2, we obtain (9) by an obvious approximation argument. We thus have

$$\begin{aligned}\overline{m}_{\omega^*}(Vx)&=\lim_{n\to\infty}\int_{\mathcal{L}} Lx_n\,d\mu(L)=\lim_{n\to\infty}\overline{m}_{\omega^*}(Vx_n)=\lim_{n\to\infty}\overline{m}(Vx_n)\\&=\lim_{n\to\infty}m(x_n)=\lim_{n\to\infty}(m\circ I)(x_n)=\lim_{n\to\infty}(m\circ I)_{\omega^*}(x_n)=(m\circ I)_{\omega^*}(\hat{x}),\end{aligned}$$

so that the claim is proven. $\square$

We can now calculate the distance from $x\in M(X,\mathcal{L})$ to $M_0(X,\mathcal{L})$ using duality,

$$\operatorname{dist}(x,M_0(X,\mathcal{L}))_{M(X,\mathcal{L})}=\sup_{\substack{\|m\|=1\\(m\circ I)_{\omega^*}\equiv 0}}|m(x)|=\sup_{\substack{\|m\|=1\\ \overline{m}_{\omega^*}\perp VM(X,\mathcal{L})}}|\overline{m}_s(Vx)|.$$

Since $\|\overline{m}_s\|\le\|m\|$ we obtain by Lemma 3.3 that

$$\operatorname{dist}(x,M_0(X,\mathcal{L}))_{M(X,\mathcal{L})}\le\overline{\lim_{L\to\infty}}\,\|Lx\|_Y\,.$$

The converse inequality is trivial; for any $x_0 \in M_0(X,\mathcal{L})$ we have

$$\|x-x_0\|_{M(X,\mathcal{L})} \geq \varlimsup_{L\to\infty} \|Lx-Lx_0\|_Y \geq \varlimsup_{L\to\infty} (\|Lx\|_Y - \|Lx_0\|_Y) = \varlimsup_{L\to\infty} \|Lx\|_Y . \quad \square$$

Proof of Corollary 2.5. As in the preceding proof, for $m\in M_0(X,\mathcal{L})^{***}$, write

$$m = m_{\omega^*} + m_s,$$

in accordance with (8). Suppose that $m \notin M_0(X,\mathcal{L})^*$, or equivalently, $m_s \neq 0$. Pick $\hat{x} \in M_0(X,\mathcal{L})^{**}$ such that $m_s(\hat{x}) \neq 0$ and let $\{x_n\}_{n=1}^\infty \subset M_0(X,\mathcal{L})$ converge to $\hat{x}$ weak-star. Then $\{x_n\}_{n=1}^\infty$, as a sequence in $M_0(X,\mathcal{L})^{**}$, is a weak Cauchy sequence since

$$\lim_{n\to\infty} m'(x_n) = m'_{\omega^*}(\hat{x}), \quad m' \in M_0(X,\mathcal{L})^{***}.$$

On the other hand,

$$m(\hat{x}) = m_{\omega^*}(\hat{x}) + m_s(\hat{x}) \neq m_{\omega^*}(\hat{x}),$$

so that

$$m(\hat{x}) \neq \lim_{n\to\infty} m(x_n).$$

We have thus verified the condition of Theorem 3.4 for $Z = M_0(X,\mathcal{L})^*$, proving the corollary. $\square$

4. Examples

Example 4.1. Denoting by $L_a^2 = L^2(\mathbb{D}) \cap \mathrm{Hol}(\mathbb{D})$ the usual Bergman space on the unit disc $\mathbb{D}$, let $X = L_a^2/\mathbb{C}$ be the space of functions $f \in L_a^2$ with $f(0)=0$. Let $Y=\mathbb{C}$,

$$\mathcal{L} = \{L_w : L_w f = (1-|w|^2) f'(w) \text{ and } w \in \mathbb{D}\},$$

and let τ be the topology of $\mathbb{D}$. Then $M(X,\mathcal{L}) = B/\mathbb{C}$ is the Bloch space modulo constants and $M_0(X,\mathcal{L}) = B_0/\mathbb{C}$ is the little Bloch space (up to constants). For $f \in B$ it is clear that the dilations f_r, $f_r(z) = f(rz)$, converge to f in L_a^2 as $r \to 1^-$ and that $\|f_r\|_B \leq \|f\|_B$, verifying the hypothesis of Assumption B. From the theorems we obtain that $(B_0/\mathbb{C})^{**} \simeq B/\mathbb{C}$ isometrically via the L_a^2-pairing, as well as the distance formula

$$\operatorname{dist}(f, B_0/\mathbb{C})_{B/\mathbb{C}} = \varlimsup_{|w|\to 1} (1-|w|^2)|f'(w)|.$$

This improves a result previously obtained in [4] and [27]. Corollary 2.5 says furthermore that the Bloch space has a unique predual, reproducing a result found in [23].

Example 4.2. Much more generally, the main theorems can be applied to Möbius invariant spaces of analytic functions through the following construction. Denote by G the Möbius group, consisting of the conformal disc automorphisms $\phi\colon \mathbb{D}\to\mathbb{D}$. Each function in G is of the form

$$\phi_{a,\lambda}(z)=\lambda\frac{a-z}{1-\bar{a}z}, \quad a\in\mathbb{D},\ \lambda\in\mathbb{T}.$$

G is a topological group with the topology of $\mathbb{D}\times\mathbb{T}$. In particular, $\phi_{a,\lambda}\to\infty$ equivalently means that $|a|\to 1$.

Let X be a Banach space whose members $f\in X$ are functions analytic in $\mathbb{D}$ with $f(0)=0$. We assume that X is continuously contained in $\mathrm{Hol}(\mathbb{D})/\mathbb{C}$, the latter space being equipped with the compact-open topology, and that it satisfies the properties:

(I) X is reflexive;

(II) the holomorphic polynomials p with $p(0)=0$ are contained and dense in X;

(III) for each fixed $f\in X$, the map T_f, $T_f\phi=f\circ\phi-f(\phi(0))$, is a continuous map from G to X;

(IV) $\overline{\lim}_{G\ni\phi\to\infty}\|\phi-\phi(0)\|_X=0$.

We now let $Y=X$ and let $\mathcal{L}$ be the collection of composition operators induced by G,

$$\mathcal{L}=\{L_\phi : L_\phi f=f\circ\phi-f(\phi(0)) \text{ and } \phi\in G\},$$

equipping it with the topology of G.

$M(X,\mathcal{L})$ and $M_0(X,\mathcal{L})$ are then Möbius invariant Banach spaces in the sense that if f is in either space then so is $f\circ\phi-f(\phi(0))$, $\phi\in G$, and furthermore

$$\|f\circ\phi-f(\phi(0))\|_{M(X,\mathcal{L})}=\|f\|_{M(X,\mathcal{L})}.$$

Property (IV) implies that $B_1/\mathbb{C}$ is continuously contained in $M_0(X,\mathcal{L})$, where B_1 denotes the analytic Besov 1-space, the minimal Möbius invariant space (see [3]). In particular $M_0(X,\mathcal{L})$ contains the polynomials. The construction of the space $M(X,\mathcal{L})$ has been considered by Aleman and Simbotin in [1].

General Möbius invariant spaces are studied by Arazy, Fisher and Peetre in [3]. The next proposition, saying that Assumption B holds, is essentially contained there. Arazy–Fisher–Peetre have a stricter definition of what a Möbius invariant space is, however, so its proof is included here for completeness.

Proposition 4.3. *Given $f\in M(X,\mathcal{L})$, $f(z)=\sum_{k=1}^\infty a_kz^k$, let*

$$f_n(z)=\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)a_kz^k.$$

Then $\|f_n\|_{M(X,\mathcal{L})}\le\|f\|_{M(X,\mathcal{L})}$ and $f_n\to f$ weakly in X.

Proof. Denote by $\Phi_n(\theta)=\sum_{k=-n}^{n}(1-|k|/n+1)e^{-ik\theta}$ the Fejér kernels. That

$$\|f_n\|_{M(X,\mathcal{L})} \le \|f\|_{M(X,\mathcal{L})}$$

is immediate from the formula

$$f_n\circ\phi - f_n(\phi(0)) = \frac{1}{2\pi}\int_0^{2\pi}(f(e^{i\theta}\phi(\,\cdot\,)) - f(e^{i\theta}\phi(0)))\Phi_n(\theta)\,d\theta,$$

where the integral is to be understood as an X-valued function of θ integrated against the measure $\Phi_n(\theta)\,d\theta$. That $f_n\to f$ weakly follows from the same formula with $\phi(z)=z$, because if $\ell\in X^*$, then

$$\ell(f_n) = \frac{1}{2\pi}\int_0^{2\pi}\ell(f(e^{i\theta}\,\cdot\,))\Phi_n(\theta)\,d\theta \to \ell(f)$$

as $n\to\infty$, by a standard argument about the Fejér kernels. □

Applying the theorems, we obtain that $M_0(X,\mathcal{L})^{**}\simeq M(X,\mathcal{L})$ isometrically, that $M_0(X,\mathcal{L})^*$ is the unique isometric predual of $M(X,\mathcal{L})$ and that the formula

$$\operatorname{dist}(f, M_0(X,\mathcal{L}))_{M(X,\mathcal{L})} = \varlimsup_{|a|\to 1}\|f\circ\phi_{a,\lambda} - f(\phi_{a,\lambda}(0))\|_X$$

holds. There are many examples of Möbius invariant spaces. Letting $X=L_a^2/\mathbb{C}$ we once again obtain the Bloch space, $M(X,\mathcal{L})=B/\mathbb{C}$ and $M_0(X,\mathcal{L})=B_0/\mathbb{C}$, but with a different norm than in Example 4.1. When $X=H^2/\mathbb{C}$ is the Hardy space modulo constants we get the space of analytic BMO functions with its conformally invariant norm, $M(X,\mathcal{L})=\mathrm{BMOA}/\mathbb{C}$ and $M_0(X,\mathcal{L})=\mathrm{VMOA}/\mathbb{C}$ (see [17]).

The Q_K-spaces provide a wide class of Möbius invariant spaces that includes both B and BMOA. For a non-zero, right-continuous, non-decreasing function $K\colon[0,\infty)\to[0,\infty)$, denote by Q_K the space of all $f\in\mathrm{Hol}(\mathbb{D})$ with $f(0)=0$ such that

$$\|f\|_{Q_K}^2 = \sup_{\phi\in G}\int_{\mathbb{D}}|f'(z)|^2K\left(\log\frac{1}{|\phi|}\right)dA(z) < \infty.$$

See Essén and Wulan [14] for a survey of Q_K-spaces. See also [1]. Clearly, $Q_K= M(X_K,\mathcal{L})$, if X_K is the space of all $f\in\mathrm{Hol}(\mathbb{D})$ with $f(0)=0$ such that

$$\|f\|_{X_K}^2 = \int_{\mathbb{D}}|f'(z)|^2K\left(\log\frac{1}{|z|}\right)dA(z) < \infty.$$

If $K(\log(1/|z|))$ is integrable on $\mathbb{D}$ and $K(\rho)\to 0$ as $\rho\to 0^+$, it is easy to verify that X_K is a Hilbert space for which properties (II) and (IV) hold. Furthermore, if

$K(\log(1/|z|))$ is a normal weight in the sense of Shields and Williams [25], standard arguments show that if $\phi_{a,\lambda} \to \phi$ in G, then

$$\|f \circ \phi_{a,\lambda} - f(\phi_{a,\lambda}(0))\|_{X_K} \to \|f \circ \phi - f(\phi(0))\|_{X_K}, \quad f \in X_K.$$

Since $f \circ \phi_{a,\lambda} - f(\phi_{a,\lambda}(0))$ also tends weakly to $f \circ \phi - f(\phi(0))$, we in fact have norm convergence, verifying (III) under these assumptions. Hence, if we denote by $Q_{K,0} = M_0(X_K, \mathcal{L})$ the space of those functions $f \in Q_K$ such that

$$\overline{\lim_{|a|\to 1}} \int_{\mathbb{D}} |f'(z)|^2 K\left(\log \frac{1}{|\phi_{a,\lambda}|}\right) dA(z) = 0,$$

we have proven that $Q_{K,0}^{**} = Q_K$ and that

$$\operatorname{dist}(f, Q_{K,0})_{Q_K} = \overline{\lim_{|a|\to 1}} \int_{\mathbb{D}} |f'(z)|^2 K\left(\log \frac{1}{|\phi_{a,\lambda}|}\right) dA(z).$$

Example 4.4. For an open subset Ω of $\mathbb{C}$, let v be a strictly positive continuous function on Ω. In this example we shall consider the weighted space $Hv(\Omega)$ of analytic functions on Ω bounded under the weighted supremum norm given by the weight v.

For the purpose of applying our construction, choose an auxiliary strictly positive continuous weight function $w\colon \Omega \to \mathbb{R}_+$ such that w is integrable on Ω. Define $X = L_a^2(v^2 w)$ to be the weighted Bergman space on Ω with weight $v^2 w$, consisting of those $f \in \operatorname{Hol}(\Omega)$ such that

$$\|f\|_{L_a^2(v^2 w)} = \int_\Omega |f(z)|^2 v(z)^2 w(z)\, dA(z) < \infty.$$

One easily verifies that X is a Hilbert space continuously contained in $\operatorname{Hol}(\mathbb{D})$. Furthermore, let $Y = \mathbb{C}$,

$$\mathcal{L} = \{L_z : L_z f = v(z) f(z) \text{ and } z \in \Omega\},$$

and let τ be the usual topology of Ω.

It is then clear that $M(X, \mathcal{L}) = Hv(\Omega)$ is the Banach space of all $f \in \operatorname{Hol}(\Omega)$ such that vf is bounded, and that $M_0(X, \mathcal{L}) = Hv_0(\Omega)$ is the corresponding little space, consisting of those f such that vf vanishes at infinity on Ω.

Assumption A (Assumption B) holds if and only if for each $f \in Hv(\Omega)$ there is a sequence $\{f_n\}_{n=1}^\infty \subset Hv_0(\Omega)$ such that $f_n \to f$ pointwise in Ω and $\sup_n \|f_n\|_{Hv(\Omega)} < \infty$ ($\sup_n \|f_n\|_{Hv(\Omega)} \le \|f\|_{Hv(\Omega)}$). We have hence recovered a result of Bierstedt and Summers [6]; $Hv_0(\Omega)^{**} \simeq Hv(\Omega)$ via the natural isomorphism if and only if this pointwise weighted approximation condition holds. The isomorphism is isometric precisely when Assumption B holds.

When either assumption holds we furthermore obtain the distance formula

$$\operatorname{dist}(f, Hv_0(\Omega))_{Hv(\Omega)} = \varlimsup_{\Omega \ni z \to \infty} v(z)|f(z)|,$$

where the limit is taken with respect to the topology of Ω. Bierstedt and Summers give sufficient conditions for radial weights v which ensure that Assumption B holds. For example, it holds when v is a radial weight vanishing at $\partial\Omega$, where Ω is a balanced domain such that $\overline{\Omega}$ is a compact subset of $\{z \in \mathbb{C} : rz \in \Omega\}$ for every $0<r<1$. When $\Omega = \mathbb{C}$, we may take any radial weight v on $\mathbb{C}$ decreasing rapidly at infinity to obtain a space $Hv(\mathbb{C})$ of entire functions satisfying Assumption B. See [6] for details.

For simplicity the above considerations have not been carried out in their full generality. We could, for example, have considered weighted spaces $M(X, \mathcal{L})$ of harmonic functions, or of functions defined on $\mathbb{C}^n$, $n>1$. However, problems arise when Ω is an open subset of an infinite-dimensional Banach space, since local compactness of Ω is lost. This case has been considered by García, Maestre and Rueda in [16]. In a different direction, the biduality problem has been studied for weighted inductive limits of spaces of analytic functions. See Bierstedt, Bonet and Galbis [7] for results in this context.

Example 4.5. We now turn to rectangular bounded mean oscillation on the 2-torus. The space $\mathrm{BMO}_{\mathrm{Rect}}(\mathbb{T}^2)$ consists of those $f \in L^2(\mathbb{T}^2)/\mathbb{C}$ such that

$$\sup \frac{1}{|I|\,|J|} \int_I \int_J |f(\zeta,\lambda) - f_J(\zeta) - f_I(\lambda) + f_{I\times J}|^2 \, ds(\zeta)\, ds(\lambda) < \infty,$$

where the supremum is taken over all subarcs $I, J \subset \mathbb{T}$, ds is arc length measure, $f_J(\zeta) = \int_J f(\zeta,\lambda)\, ds(\lambda)/|J|$ and $f_I(\lambda) = \int_I f(\zeta,\lambda)\, ds(\zeta)/|I|$ are the averages of $f(\zeta,\cdot)$ and $f(\cdot,\lambda)$ on J and I, respectively, and $f_{I\times J}$ is the average of f on $I\times J$. Rectangular BMO is one of several possible generalizations of $\mathrm{BMO}(\mathbb{T})$ to the two-variable case. We focus on this particular one because it fits naturally into our scheme. A treatment of rectangular BMO can be found in Ferguson and Sadosky [15].

To obtain $\mathrm{BMO}_{\mathrm{Rect}}(\mathbb{T}^2)$ via our construction, let $X = L^2(\mathbb{T}^2)/\mathbb{C}$, $Y = L^2(\mathbb{T}^2)$ and

$$\mathcal{L} = \left\{ L_{I,J} : L_{I,J} f = \chi_{I\times J} \frac{1}{|I|\,|J|} (f - f_J - f_I + f_{I\times J}) \right\},$$

where I and J range over all non-empty arcs. Denoting by τ the quotient topology considered in Example 2.1, we equip $\mathcal{L}$ with the corresponding product topology $\tau\times\tau$, so that $L_{I,J} \to \infty$ means that $\min(|I|, |J|) \to 0$. By construction $M(X, \mathcal{L}) = \mathrm{BMO}_{\mathrm{Rect}}(\mathbb{T}^2)$. Accordingly, $M_0(X, \mathcal{L})$ will be named $\mathrm{VMO}_{\mathrm{Rect}}(\mathbb{T}^2)$. Assumption B is verified exactly as in Example 2.4, letting $f_n = P_{1-1/n}(\zeta) * P_{1-1/n}(\lambda) * f$ be a double Poisson integral.

From Theorem 2.2 we hence obtain that $\mathrm{VMO}_{\mathrm{Rect}}(\mathbb{T}^2)^{**}$ is isometrically isomorphic to $\mathrm{BMO}_{\mathrm{Rect}}(\mathbb{T}^2)$ via the $L^2(\mathbb{T}^2)$-pairing, and Theorem 2.3 gives

$$\operatorname{dist}(f, \mathrm{VMO}_{\mathrm{Rect}})_{\mathrm{BMO}_{\mathrm{Rect}}} = \varlimsup_{\min(|I|,|J|)\to 0} \|L_{I,J} f\|_{L^2(\mathbb{T}^2)}.$$

Another possible generalization of $\mathrm{BMO}(\mathbb{T})$ to several variables is known as product BMO. In [21], Lacey, Terwilleger and Wick explore the corresponding product VMO space. It would be interesting to apply our techniques also to this case, but one meets the difficulty of defining a reasonable topology on the collection of all open subsets of $\mathbb{R}^n$. On the other hand, the predual of $\mathrm{BMO}_{\mathrm{Rect}}(\mathbb{T}^2)$ is given as a space spanned by certain "rectangular atoms" (see [15]) and is as such more difficult to understand than the predual of product BMO, which is the Hardy space $H^1(\mathbb{T}^n)$ of the n-torus.

Example 4.6. Let $0<\alpha\le 1$ and let Ω be a compact subset of $\mathbb{R}^n$. In this example we shall treat the Lipschitz–Hölder space $\mathrm{Lip}_\alpha(\Omega)$. By definition, a real-valued function f on Ω is in $\mathrm{Lip}_\alpha(\Omega)$ if and only if

$$\|f\|_{\mathrm{Lip}_\alpha(\Omega)} = \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty.$$

As usual we identify f and $f+C$, C constant, in order to obtain a norm.

X will be chosen as a quotient space of an appropriate fractional Sobolev space (Besov space) $W^{l,p}(\mathbb{R}^n)$. For $0<l<1$ and $1<p<\infty$, $W^{l,p}$ consists of those $f\in L^p(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{pl+n}}\,dx\,dy < \infty.$$

Choose l and p such that $0<l<\alpha$ and $pl>n$. By a Sobolev type embedding theorem ([22], Proposition 4.2.5) it then holds that $W^{l,p}$ continuously embeds into the space of continuous bounded functions $C_b(\mathbb{R}^n)$. Let

$$A_\Omega = \{f\in W^{l,p} : f(x)=0 \text{ and } x\in\Omega\}.$$

We set $X = W^{l,p}/A_\Omega$.

To obtain $\mathrm{Lip}_\alpha(\Omega)$ through our construction, let $Y=\mathbb{C}$ and let every operator $L_{x,y}\in\mathcal{L}$, $x,y\in\Omega$, $x\neq y$, be of the form

$$L_{x,y} f = \frac{f(x)-f(y)}{|x-y|^\alpha}.$$

We give $\mathcal{L}$ the topology of $\{(x,y)\in\Omega\times\Omega : x\neq y\}$. Then $M(X,\mathcal{L})=\mathrm{Lip}_\alpha(\Omega)$. One inclusion is obvious and to see the other one let $f\in\mathrm{Lip}_\alpha(\Omega)$. As in [10], f can be extended to $\hat{f}\in\mathrm{Lip}_\alpha(\mathbb{R}^n)$, $\hat{f}=f$ on Ω. Letting $\chi\in C_c^\infty(\mathbb{R}^n)$ be a cut-off function such that $\chi(x)=1$ for $x\in\Omega$, it is straightforward to check that $\chi\hat{f}\in M(X,\mathcal{L})$, verifying that $\mathrm{Lip}_\alpha(\Omega)\subset M(X,\mathcal{L})$.

Note that $M_0(X,\mathcal{L})=\mathrm{lip}_\alpha(\Omega)$ is the corresponding little Hölder space, consisting of those $f\in\mathrm{Lip}_\alpha(\Omega)$ such that

$$\varlimsup_{|x-y|\to 0}\frac{|f(x)-f(y)|}{|x-y|^\alpha}=0.$$

When $\alpha=1$, the space $\mathrm{lip}_\alpha(\Omega)$ is trivial in many cases, so that Assumption A may fail. However, for $\alpha<1$ we can verify Assumption B in general. Let P_t, $t>0$, be the n-dimensional Poisson kernel,

$$P_t(x)=c\frac{t}{(t^2+|x|^2)^{(n+1)/2}},\quad x\in\mathbb{R}^n,$$

for the appropriate normalization constant c, and let χ denote the same cut-off as before. For $f\in M(X,\mathcal{L})$ it is straightforward to verify that $\chi(P_t*f)\in M_0(X,\mathcal{L})$,

$$\|\chi(P_t*f)\|_{\mathrm{Lip}_\alpha(\Omega)}\le\|f\|_{\mathrm{Lip}_\alpha(\Omega)},$$

and that $\chi(P_t*f)\to f=\chi f$ weakly in X as $t\to 0^+$, the final statement following from the reflexivity of X and the fact that P_t*f tends to f pointwise almost everywhere.

We conclude that, for $0<\alpha<1$,

$$\mathrm{dist}(f,\mathrm{lip}_\alpha(\Omega))_{\mathrm{Lip}_\alpha(\Omega)}=\varlimsup_{|x-y|\to 0}\frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

In addition, Theorem 2.2 says that $\mathrm{lip}_\alpha(\Omega)^{**}\simeq\mathrm{Lip}_\alpha(\Omega)$ isometrically. In Kalton [20] it is proven that $\mathrm{lip}_\alpha(M)^{**}\simeq\mathrm{Lip}_\alpha(M)$ under very general conditions on M, for example whenever M is a compact metric space. It would be interesting to see if the theorems of this paper can be applied in this situation and, if this is the case, which space X to embed $\mathrm{Lip}_\alpha(M)$ into.

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