Some Hilbert spaces related with the Dirichlet space

Concr. Oper. 2016; 3: 94–101 **DE GRUYTER** OPEN

Concrete Operators **Open Access**

Research Article

Nicola Arcozzi*, Pavel Mozolyako, Karl-Mikael Perfekt, Stefan Richter, and Giulia Sarfatti

Some Hilbert spaces related with the Dirichlet space

DOI 10.1515/conop-2016-0011
Received December 23, 2015; accepted May 16, 2016.

Abstract: We study the reproducing kernel Hilbert space with kernel k^d, where d is a positive integer and k is the reproducing kernel of the analytic Dirichlet space.

Keywords: Dirichlet space, Complete Nevanlinna Property, Hilbert-Schmidt operators, Carleson measures

MSC: 30H25, 47B35

1 Introduction

Consider the Dirichlet space $\mathcal{D}$ on the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ of the complex plane. It can be defined as the Reproducing Kernel Hilbert Space (RKHS) having kernel

$$k_z(w) = k(w,z) = \frac{1}{\bar{z}w} \log \frac{1}{1-\bar{z}w} = \sum_{n=0}^{\infty} \frac{(\bar{z}w)^n}{n+1}.$$

We are interested in the spaces $\mathcal{D}_d$ having kernel k^d, with $d \in \mathbb{N}$. $\mathcal{D}_d$ can be thought of in terms of function spaces on polydiscs, following ideas of Aronszajn [4]. To explain this point of view, note that the tensor d-power $\mathcal{D}^{\otimes d}$ of the Dirichlet space has reproducing kernel $k_d(z_1, \cdots, z_d; w_1, \ldots, w_d) = \Pi_{j=1}^d k(z_j, w_j)$. Hence, the space of restrictions of functions in $\mathcal{D}^{\otimes d}$ to the diagonal $z_1 = \cdots = z_d$ has the reproducing kernel k^d, and therefore coincides with $\mathcal{D}_d$.

We will provide several equivalent norms for the spaces $\mathcal{D}_d$ and their dual spaces in Theorem 1.1. Then we will discuss the properties of these spaces. More precisely, we will investigate:

– $\mathcal{D}_d$ and its dual space HS_d in connection with Hankel operators of Hilbert-Schmidt class on the Dirichlet space $\mathcal{D}$;
– the complete Nevanlinna-Pick property for $\mathcal{D}_d$;
– the Carleson measures for these spaces.

Concerning the first item, the connection with Hilbert-Schmidt Hankel operators served as our original motivation for studying the spaces $\mathcal{D}_d$.

***Corresponding Author: Nicola Arcozzi:** Università di Bologna, Dipartimento di Matematica, Piazza di Porta S.Donato 5, Bologna, E-mail: nicola.arcozzi@unibo.it
Pavel Mozolyako: Chebyshev Lab at St. Petersburg State University, 14th Line 29B, Vasilyevsky Island, St. Petersburg 199178, Russia, E-mail: pmzlcroak@gmail.com
Karl-Mikael Perfekt: Department of Mathematical Sciences, Norwegian University of Science and Technology (NTNU), NO-7491 Trondheim, Norway, E-mail: karl-mikael.perfekt@math.ntnu.no
Stefan Richter: Department of Mathematics, The University of Tennessee, Knoxville, TN 37996, USA, E-mail: richter@math.utk.edu
Giulia Sarfatti: Istituto Nazionale di Alta Matematica "F. Severi", Città Universitaria, Piazzale Aldo Moro 5, 00185 Roma, and Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, 4, place Jussieu, F-75252 Paris, France, E-mail: giulia.sarfatti@imj-prg.fr

Note that the spaces $\mathcal{D}_d$ live infinitely close to $\mathcal{D}$ in the scale of weighted Dirichlet spaces $\tilde{\mathcal{D}}_s$, defined by the norms

$$\|\varphi\|^2_{\tilde{\mathcal{D}}_s} = \int_{-\pi}^{+\pi} \left|\varphi(e^{it})\right|^2 \frac{dt}{2\pi} + \int_{|z|<1} \left|\varphi'(z)\right|^2 (1-|z|^2)^s \frac{dA(z)}{\pi},\ 0 \le s < 1,$$

where $\frac{dA(z)}{\pi}$ is normalized area measure on the unit disc.

Notation: We use multiindex notation. If $n = (n_1, \dots, n_d)$ belongs to $\mathbb{N}^d$, then $|n| = n_1 + \cdots + n_d$. We write $A \approx B$ if A and B are quantities that depend on a certain family of variables, and there exist independent constants $0 < c < C$ such that $cA \le B \le CA$.

Equivalent norms for the spaces $\mathcal{D}_d$ and their dual spaces HS_d

Theorem 1.1. *Let d be a positive integer and let*

$$a_d(k) = \sum_{|n|=k} \frac{1}{(n_1+1)\dots(n_d+1)}.$$

Then the norm of a function $\varphi(z) = \sum_{k=0}^{\infty} \widehat{\varphi}(k) z^k$ in $\mathcal{D}_d$ is

$$\|\varphi\|_{\mathcal{D}_d} = \left(\sum_{k=0}^{\infty} a_d(k)^{-1} |\widehat{\varphi}(k)|^2\right)^{1/2} \approx [\varphi]_d, \tag{1}$$

where

$$[\varphi]_d = \left(\sum_{k=0}^{\infty} \frac{k+1}{\log^{d-1}(k+2)} |\widehat{\varphi}(k)|^2\right)^{1/2}. \tag{2}$$

An equivalent Hilbert norm $|[\varphi]|_d \approx [\varphi]_d$ for φ in terms of the values of φ is given by

$$|[\varphi]|_d = |\varphi(0)|^2 + \left(\int_{\mathbb{D}} |\varphi'(z)|^2 \frac{1}{\log^{d-1}\left(\frac{1}{1-|z|^2}\right)} \frac{dA(z)}{\pi}\right)^{1/2}. \tag{3}$$

Define now the holomorphic space HS_d by the norm:

$$\|\psi\|_{HS_d} = \left(\sum_{k=0}^{\infty} (k+1)^2 a_d(k) \left|\widehat{\psi}(k)\right|^2\right)^{1/2}. \tag{4}$$

Then, $HS_d \equiv (\mathcal{D}_d)^*$ is the dual space of $\mathcal{D}_d$ under the duality pairing of $\mathcal{D}$. Moreover,

$$\|\psi\|_{HS_d} \approx [\psi]_{HS_d} := \left(\sum_{k=0}^{\infty} (k+1) \log^{d-1}(k+2) \left|\widehat{\psi}(k)\right|^2\right)^{1/2} \approx$$
$$|[\psi]|_{HS_d} := \left(|\psi(0)|^2 + \int_{\mathbb{D}} |\psi'(z)|^2 \log^{d-1}\left(\frac{1}{1-|z|^2}\right) \frac{dA(z)}{\pi}\right)^{1/2}. \tag{5}$$

Furthermore, the norm can be written as

$$\|\psi\|^2_{HS_d} = \sum_{(n_1,\dots,n_d)} |\langle e_{n_1} \dots e_{n_d}, \psi\rangle_{\mathcal{D}}|^2, \tag{6}$$

where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis of $\mathcal{D}$, $e_n(z) = \frac{z^n}{\sqrt{n+1}}$.

The remainder of this section is devoted to the proof of Theorem 1.1. The expression for $\|\varphi\|_{\mathcal{D}_d}$ in (1) follows by expanding $(k_z)^d$ as a power series. The equivalence $\|\varphi\|_{\mathcal{D}_d} \approx [\varphi]_d$, as well as $\|\varphi\|_{HS_d} \approx [\varphi]_{HS_d}$, are consequences of the following lemma. We denote by c, C positive constants which are allowed to depend on d only, whose precise value can change from line to line.

Lemma 1.2. *For each $d \in \mathbb{N}$ there are constants $c, C > 0$ such that for all $k \geq 0$ we have*

$$ca_d(k) \leq \frac{\log^{d-1}(k+2)}{k+1} \leq Ca_d(k).$$

Consequently, if $t \in (0,1)$, then

$$c\left(\frac{1}{t}\log\frac{1}{1-t}\right)^d \leq \sum_{k=0}^{\infty}\frac{\log^{d-1}(k+2)}{k+1}t^k \leq C\left(\frac{1}{t}\log\frac{1}{1-t}\right)^d.$$

Proof of Lemma 1.2. We will prove the Lemma by induction on $d \in \mathbb{N}$. It is obvious for $d = 1$. Thus let $d \geq 2$ and suppose the lemma is true for $d-1$. Also we observe that there is a constant $c > 0$ such that for all $k \geq 0$ and $0 \leq n \leq k$ we have

$$c\log^{d-2}(k+2) \leq \log^{d-2}(n+2) + \log^{d-2}(k-n+2) \leq 2\log^{d-2}(k+2).$$

Then for $k \geq 0$

$$\begin{aligned} a_d(k) &= \sum_{n_1+\dots+n_d=k}\frac{1}{(n_1+1)\dots(n_d+1)} \\ &= \sum_{n=0}^{k}\frac{1}{n+1}\sum_{n_2+\dots+n_d=k-n}\frac{1}{(n_2+1)\dots(n_d+1)} \\ &\approx \sum_{n=0}^{k}\frac{1}{n+1}\frac{\log^{d-2}(k-n+2)}{k-n+1} \quad \text{by the inductive assumption} \\ &= \frac{1}{2}\sum_{n=0}^{k}\frac{\log^{d-2}(n+2)+\log^{d-2}(k-n+2)}{(n+1)(k-n+1)} \\ &\approx \log^{d-2}(k+2)\sum_{n=0}^{k}\frac{1}{(n+1)(k-n+1)} \quad \text{by the earlier observation} \\ &= \frac{\log^{d-2}(k+2)}{k+2}\sum_{n=0}^{k}\frac{1}{n+1}+\frac{1}{k-n+1} \\ &\approx \frac{\log^{d-1}(k+2)}{k+1}. \end{aligned}$$

□

Next, we prove the equivalence $[\varphi]_{HS_d} \approx |[\varphi]|_{HS_d}$ which appears in (5).

Lemma 1.3. *Let $d \in \mathbb{N}$. Then*

$$\int_0^1 t^k\left(\frac{1}{t}\log\frac{1}{1-t}\right)^{d-1}dt \approx \frac{\log^{d-1}(k+2)}{k+1}, \quad k \geq d.$$

Given the Lemma, we expand

$$\begin{aligned} |[\psi]|^2_{HS_d} &= |\widehat{\psi}(0)|^2 + \int_{\mathbb{D}}\left|\sum_{k=1}^{\infty}\widehat{\psi}(k)kz^{k-1}\right|^2\log^{d-1}\frac{1}{1-|z|^2}\frac{dA(z)}{\pi} \\ &= |\widehat{\psi}(0)|^2 + \sum_{k=1}^{\infty}k^2\left|\widehat{\psi}(k)\right|^2\int_0^1\log^{d-1}\frac{1}{1-t}t^{k-1}dt \end{aligned}$$

$$\approx |\widehat{\psi}(0)|^2 + \sum_{k=1}^{\infty} k^2 \left|\widehat{\psi}(k)\right|^2 \frac{\log^{d-1}(k+2)}{k+1}$$
$$\approx [\psi]^2_{HS_d},$$

obtaining the desired conclusion.

Proof of Lemma 1.3. The case $d = 1$ is obvious, leaving us to consider $d \geq 2$. We will also assume that $k \geq 2$. Then by Lemma 1.2 we have

$$\int_0^1 t^k \left(\frac{1}{t}\log\frac{1}{1-t}\right)^{d-1} dt \approx \int_0^1 t^k \sum_{n=0}^{\infty} \frac{\log^{d-2}(n+2)}{n+1} t^n dt = \sum_{n=0}^{\infty} \frac{\log^{d-2}(n+2)}{(n+1)(n+k+1)} = S_1 + S_2,$$

where

$$S_1 = \sum_{n=0}^{k-1} \frac{\log^{d-2}(n+2)}{(n+1)(n+k+1)} \approx \frac{1}{k+1}\sum_{n=0}^{k-1} \frac{\log^{d-2}(n+2)}{n+1} \approx \frac{1}{k+1}\int_1^{k+2} \frac{\log^{d-2}(t)}{t} dt$$
$$= \frac{1}{d-1}\frac{\log^{d-1}(k+2)}{k+1}$$

and

$$S_2 = \sum_{n=k}^{\infty} \frac{\log^{d-2}(n+2)}{(n+1)(n+k+1)} \leq \sum_{n=k+1}^{\infty} \frac{\log^{d-2}(n+1)}{n^2} \leq \sum_{j=1}^{\infty}\sum_{n=k^j}^{k^{j+1}-1} \frac{\log^{d-2}(n+1)}{n^2}$$
$$\leq \sum_{j=1}^{\infty} (j+1)^{d-2} \log^{d-2} k \sum_{n=k^j}^{k^{j+1}-1} \frac{1}{n^2} \leq \log^{d-2}(k+2) \sum_{j=1}^{\infty} (j+1)^{d-2} \int_{k^j-1}^{\infty} \frac{1}{x^2} dx$$
$$= \frac{\log^{d-2}(k+2)}{k+1} \sum_{j=1}^{\infty} (j+1)^{d-2} \frac{k+1}{k^j-1} \leq \frac{\log^{d-2}(k+2)}{k+1} \sum_{j=1}^{\infty} (j+1)^{d-2} \frac{k+1}{(k-1)k^{j-1}}$$
$$\leq \frac{\log^{d-2}(k+2)}{k+1} \sum_{j=1}^{\infty} (j+1)^{d-2} \frac{3}{2^{j-1}} = o\left(\frac{\log^{d-1}(k+2)}{k+1}\right). \qquad \square$$

Now, the duality between $\mathcal{D}_d$ and HS_d under the duality pairing given by the inner product of $\mathcal{D}$ is easily seen by considering $[\cdot]_d$ and $[\cdot]_{HS_d}$. They are weighted ℓ^2 norms and duality is established by means of the Cauchy-Schwarz inequality.

Next we will prove that $[\varphi]_d \approx |[\varphi]|_d$. This is equivalent to proving that the dual space of HS_d, with respect to the Dirichlet inner product $\langle\cdot,\cdot\rangle_{\mathcal{D}}$, is the Hilbert space with the norm $|[\cdot]|_d$.

Let $d \in \mathbb{N}$ and set, for $0 < t < 1$, $w_d(t) = \left(\frac{1}{t}\log\frac{1}{1-t}\right)^d$ and, for $0 < |z| < 1$, $W_d(z) = w_d(|z|^2)$ and $W_d(0) = 1$.

Lemma 1.4. *Let $d \in \mathbb{N}$. Then*

$$\int_{1-\varepsilon}^1 w_d(t)dt \cdot \int_{1-\varepsilon}^1 \frac{1}{w_d(t)} dt \approx \varepsilon^2 \quad as\ \varepsilon \to 0.$$

Proof. Write $\tilde{w}(t) = (\log\frac{1}{1-t})^d$, and note that it suffices to establish the lemma for $\tilde{w}$ in place of w_d. Let $\varepsilon > 0$. Then $\tilde{w}$ is increasing in $(0,1)$ and $\tilde{w}(1-\varepsilon^{k+1}) = (k+1)^d(\log\frac{1}{\varepsilon})^d$, hence

$$\int_{1-\varepsilon}^1 \tilde{w}(t)dt = \sum_{k=1}^{\infty} \int_{1-\varepsilon^k}^{1-\varepsilon^{k+1}} \tilde{w}(t)dt \leq \sum_{k=1}^{\infty} \tilde{w}(1-\varepsilon^{k+1})(\varepsilon^k - \varepsilon^{k+1})$$

$$= \sum_{k=1}^{\infty} (k+1)^d (\log \frac{1}{\varepsilon})^d \varepsilon^k (1-\varepsilon) \approx \varepsilon (\log \frac{1}{\varepsilon})^d \frac{1}{(1-\varepsilon)^d}$$

For $1/\tilde{w}$ we just notice that it is decreasing and hence

$$\int_{1-\varepsilon}^{1} \frac{1}{\tilde{w}(t)} dt \leq \frac{1}{\tilde{w}(1-\varepsilon)} \varepsilon = \frac{\varepsilon}{(\log \frac{1}{\varepsilon})^d}$$

Thus as $\varepsilon \to 0$ we have

$$\varepsilon^2 \leq \int_{1-\varepsilon}^{1} \tilde{w}(t) dt \int_{1-\varepsilon}^{1} \frac{1}{\tilde{w}(t)} dt = O(\varepsilon^2).$$

□

For $0 < h < 1$ and $s \in [-\pi, \pi)$ let $S_h(e^{is})$ be the Carleson square at e^{is}, i.e.

$$S_h(e^{is}) = \{re^{it} : 1-h < r < 1, |t-s| < h\}.$$

A positive function W on the unit disc is said to satisfy the Bekollé-Bonami condition (B2) if there exists $c > 0$ such that

$$\int_{S_h(e^{is})} W dA \cdot \int_{S_h(e^{is})} \frac{1}{W} dA \leq ch^4$$

for every Carleson square $S_h(e^{is})$. If $d \in \mathbb{N}$ and if $W_d(z)$ is defined as above, then

$$\int_{S_h(e^{is})} W_d dA \cdot \int_{S_h(e^{is})} \frac{1}{W_d} dA = h^2 \int_{1-h}^{1} w_d(t) dt \cdot \int_{1-h}^{1} \frac{1}{w_d(t)} dt \approx h^4$$

by Lemma 1.4, at least if $0 < h < 1/2$. Observe that both W_d and $1/W_d$ are positive and integrable in the unit disc, hence it follows that the estimate holds for all $0 < h \leq 1$.

Thus W_d satisfies the condition (B2). Furthermore, note that if $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ is analytic in the open unit disc, then

$$\int_{|z|<1} |f(z)|^2 w_d(|z|^2) \frac{dA(z)}{\pi} = \sum_{k=0}^{\infty} w_k |\hat{f}(k)|^2,$$

where $w_k = \int_0^1 t^k w_d(t) dt \approx \frac{\log^d(k+2)}{k+1}$.

A special case of Theorem 2.1 of Luecking's paper [7] says that if W satisfies the condition (B2) by Bekollé and Bonami [5], then one has a duality between the spaces $L_a^2(W dA)$ and $L_a^2(\frac{1}{W} dA)$ with respect to the pairing given by $\int_{|z|<1} f \overline{g} dA$. Thus, we have

$$\int_{|z|<1} |g(z)|^2 \frac{1}{W_d(z)} dA \approx \sup_{f \neq 0} \frac{\left| \int_{|z|<1} g(z) \overline{f(z)} \frac{dA(z)}{\pi} \right|^2}{\int_{|z|<1} |f(z)|^2 W_d(z) dA} = \sup_{f \neq 0} \frac{\left| \sum_{k=0}^{\infty} \frac{\hat{g}(k)}{(k+1)\sqrt{w_k}} \sqrt{w_k} \overline{\hat{f}(k)} \right|^2}{\sum_{k=0}^{\infty} w_k |\hat{f}(k)|^2}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)^2 w_k} |\hat{g}(k)|^2$$

This finishes the proof of (5). It remains to demonstrate (6). We defer its proof to the next section.

By Theorem 1.1 we have the following chain of inclusions:

$$\ldots \hookrightarrow HS_{d+1} \hookrightarrow HS_d \hookrightarrow \ldots \hookrightarrow HS_2 \hookrightarrow HS_1 = \mathcal{D} = \mathcal{D}_1 \hookrightarrow \mathcal{D}_2 \hookrightarrow \ldots \hookrightarrow \mathcal{D}_d \hookrightarrow \mathcal{D}_{d+1} \hookrightarrow \ldots$$

with duality w.r.t. $\mathcal{D}$ linking spaces with the same index. It might be interesting to compare this sequence with the sequence of Banach spaces related to the Dirichlet spaces studied in [3]. Note that for $d \geq 3$ the reproducing kernel of HS_d is continuous up to the boundary. Hence functions in HS_d extend continuously to the closure of the unit disc, for $d \geq 3$.

Hilbert-Schmidt norms of Hankel-type operators

Let $\{e_n\}$ be the canonical orthonormal basis of $\mathcal{D}$, $e_n(z) = \frac{z^n}{\sqrt{n+1}}$. Equation (6) follows from the computation

$$\sum_{k=0}^{\infty}\sum_{|n|=k} |\langle e_{n_1}...e_{n_d}, \psi\rangle|^2 = \sum_{k=0}^{\infty}\sum_{|n|=k} \frac{1}{(n_1+1)\cdot ...\cdot(n_d+1)}|\langle z^{n_1}...z^{n_d}, \psi\rangle|^2$$
$$= \sum_{k=0}^{\infty}\sum_{|n|=k} \frac{1}{(n_1+1)\cdot ...\cdot(n_d+1)}|\langle z^{k}, \psi\rangle|^2 = \sum_{k=0}^{\infty}\sum_{|n|=k} \frac{(k+1)^2}{(n_1+1)\cdot ...\cdot(n_d+1)}|\hat{\psi}(k)|^2$$
$$= \sum_{k=0}^{\infty}(k+1)a_d(k)|\hat{\psi}(k)|^2 \approx \sum_{k=0}^{\infty}\frac{\log^{d-1}(k+2)}{k+1}|\hat{\psi}(k)|^2.$$

Polarizing this expression for $\|\cdot\|_{HS_d}$, the inner product of HS_d can be written

$$\langle\psi_1,\psi_2\rangle_{HS_d} = \sum_{(n_1,...,n_d)}\langle\psi_1, e_{n_1}\dots e_{n_d}\rangle_{\mathcal{D}}\langle e_{n_1}\dots e_{n_d},\psi_2\rangle_{\mathcal{D}}.$$

Hence, for any $\lambda,\zeta\in\mathbb{D}$,

$$\langle k_\lambda, k_\zeta\rangle_{HS_d} = \sum_{n\in\mathbb{N}^d}\langle k_\lambda, e_{n_1}\dots e_{n_d}\rangle_{\mathcal{D}}\langle e_{n_1}\dots e_{n_d}, k_\zeta\rangle_{\mathcal{D}} = \sum_{n\in\mathbb{N}^d}\overline{e_{n_1}(\lambda)\dots e_{n_d}(\lambda)}e_{n_1}(\zeta)\dots e_{n_d}(\zeta)$$
$$= \left(\sum_{i=0}^{\infty}\overline{e_i(\lambda)}e_i(\zeta)\right)^d = k_\lambda(\zeta)^d = \langle k_\lambda^d, k_\zeta^d\rangle_{\mathcal{D}_d}.$$

That is,

Proposition 1.5. *The map $U: k_\lambda\mapsto k_\lambda^d$ extends to a unitary map $HS_d\to\mathcal{D}_d$.*

When $d=2$, HS_2 contains those functions b for which the Hankel operator $H_b:\mathcal{D}\to\overline{\mathcal{D}}$, defined by $\langle H_b e_j, \overline{e_k}\rangle_{\overline{\mathcal{D}}} = \langle e_j e_k, b\rangle_{\mathcal{D}}$, belongs to the Hilbert-Schmidt class.

Analogous interpretations can be given for $d\geq 3$, but then function spaces on polydiscs are involved. We consider the case $d=3$, which is representative. Consider first the operator $T_b:\mathcal{D}\to\overline{\mathcal{D}}\otimes\overline{\mathcal{D}}$ defined by

$$\left\langle T_b f, \overline{g}\otimes\overline{h}\right\rangle_{\overline{\mathcal{D}}\otimes\overline{\mathcal{D}}} = \langle fgh, b\rangle_{\mathcal{D}}.$$

The formula uniquely defines an operator, whose action is

$$\begin{aligned} T_b f(z,w) &= \langle T_b f, \overline{k}_z\overline{k}_w\rangle_{\overline{\mathcal{D}}\otimes\overline{\mathcal{D}}}\\ &= \langle f k_z k_w, b\rangle_{\mathcal{D}}\\ &= \sum_{n,m,j}\hat{f}(j)\frac{\overline{z}^n}{n+1}\frac{\overline{w}^m}{m+1}\langle\zeta^{n+m+j}, b\rangle_{\mathcal{D}}\\ &= \sum_{n,m,j}\hat{f}(j)\overline{\hat{b}(n+m+j)}\frac{n+m+j+1}{(n+1)(m+1)}\overline{z}^n\overline{w}^m \end{aligned}$$

Then, the Hilbert-Schmidt norm of T_b is:

$$\sum_{l,m,n}\left|\langle T_b e_l, e_m e_n\rangle_{\overline{\mathcal{D}}\otimes\overline{\mathcal{D}}}\right|^2 = \sum_{l,m,n}|\langle e_l e_m e_n, b\rangle_{\mathcal{D}}|^2 = \|b\|_{HS_3}^2.$$

Similarly, we can consider $U_b:\mathcal{D}\otimes\mathcal{D}\to\overline{\mathcal{D}}$ defined by

$$\left\langle U_b(f\otimes g), \overline{h}\right\rangle_{\overline{\mathcal{D}}} = \langle fgh, b\rangle_{\mathcal{D}}.$$

The action of this operator is given by

$$U_b(f\otimes g)(\overline{z}) = \sum_{l,m,n=0}^{\infty} \widehat{f}(l)\widehat{g}(m)\frac{(l+m+n+1)\overline{\widehat{b}(l+m+n)}}{n+1}\overline{z}^n.$$

The Hilbert-Schmidt norm of U_b is still $\|b\|_{HS_3}$.

Carleson measures for the spaces $\mathcal{D}_d$ and HS_d

The (B2) condition allows us to characterize the Carleson measures for the spaces $\mathcal{D}_d$ and HS_d. Recall that a nonnegative Borel measure μ on the open unit disc is Carleson for the Hilbert function space H if the inequality

$$\int_{|z|<1} |f|^2 d\mu \le C(\mu)\|f\|_H^2$$

holds with a constant $C(\mu)$ which is independent of f. The characterization [2] shows that, since the (B2) condition holds, then

Theorem 1.6. *For $d \in \mathbb{N}$, a measure $\mu \ge 0$ on $\{|z|<1\}$ is Carleson for $\mathcal{D}_d$ if and only if for $|a|<1$ we have:*

$$\int_{\tilde{S}(a)} \log^{d-1}\left(\frac{1}{1-|z|^2}\right)(1-|z|^2)\mu(S(z)\cap S(a))^2\frac{dxdy}{(1-|z|^2)^2} \le C_1(\mu)\mu(S(a)),$$

where $S(a) = \{z : 0 < 1-|z| < 1-|a|,\ |\arg(z\overline{a})| < 1-|a|\}$ is the Carleson box with vertex a and $\tilde{S}(a) = \{z : 0 < 1-|z| < 2(1-|a|),\ |\arg(z\overline{a})| < 2(1-|a|)\}$ is its "dilation".

The characterization extends to HS_2, with the weight $\log^{-1}\left(\frac{1}{1-|z|^2}\right)$. Since functions in HS_d are continuous for $d \ge 3$, all finite measures are Carleson measures for these spaces. Once we know the Carleson measures, we can characterize the multipliers for $\mathcal{D}_d$ in a standard way.

The complete Nevanlinna-Pick property for $\mathcal{D}_d$

Next, we prove that the spaces $\mathcal{D}_d$ have the Complete Nevanlinna-Pick (CNP) Property. Much research has been done on kernels with the CNP property in the past twenty years, following seminal work of Sarason and Agler. See the monograph [1] for a comprehensive and very readable introduction to this topic. We give here a definition which is simple to state, although perhaps not the most conceptual. An irreducible kernel $k : X \times X \to \mathbb{C}$ has the CNP property if there is a positive definite function $F : X \to \mathbb{D}$ and a nowhere vanishing function $\delta : X \to \mathbb{C}$ such that:

$$k(x,y) = \frac{\overline{\delta}(x)\delta(y)}{1-F(x,y)}$$

whenever x, y lie in X. The CNP property is a property of the kernel, not of the Hilbert space itself.

Theorem 1.7. *There are norms on $\mathcal{D}_d$ such that the CNP property holds.*

Proof. A kernel $k : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ of the form $k(w,z) = \sum_{k=0}^{\infty} a_k(\overline{z}w)^k$ has the CNP property if $a_0 = 1$ and the sequence $\{a_n\}_{n=0}^{\infty}$ is positive and log-convex:

$$\frac{a_{n-1}}{a_n} \le \frac{a_n}{a_{n+1}}.$$

See [1], Theorem 7.33 and Lemma 7.38. Consider $\eta(x) = \alpha \log\log(x) - \log(x)$, with real α. Then, $\eta''(x) = \frac{\log^2(x)-\alpha\log(x)-\alpha}{x^2\log^2(x)}$, which is positive for $x \geq M_\alpha$, depending on α. Let now

$$a_n = \frac{\log^{d-1}(M_d(n+1))}{\log(M_d)\cdot(n+1)} \approx \frac{1}{n+1} + \frac{\log^{d-1}(n+1)}{n+1} \tag{7}$$

Then, the sequence $\{a_n\}_{n=0}^\infty$ provides the coefficients for a kernel with the CNP property for the space $\mathcal{D}_d$. □

The CNP property has a number of consequences. For instance, we have that the space $\mathcal{D}_d$ and its multiplier algebra $M(\mathcal{D}_d)$ have the same interpolating sequences. Recall that a sequence $Z = \{z_n\}_{n=0}^\infty$ is *interpolating* for a RKHS H with reproducing kernel k^H if the weighted restriction map $R : \varphi \mapsto \left\{\frac{\varphi(z_n)}{k^H(z_n,z_n)^{1/2}}\right\}_{n=0}^\infty$ maps H boundedly onto ℓ^2; while Z is interpolating for the multiplier algebra $M(H)$ if $Q : \psi \mapsto \{\psi(z_n)\}_{n=0}^\infty$ maps $M(H)$ boundedly onto ℓ^∞. The reader is referred to [1] and to the second chapter of [8] for more on this topic.

It is a reasonable guess that the *universal interpolating sequences* for $\mathcal{D}_d$ and for its multiplier space $M(\mathcal{D}_d)$ are characterized by a Carleson condition and a separation condition, as described in [8] (see the Conjecture at p. 33). See also [6], which contains the best known result on interpolation in general RKHS spaces with the CNP property. Unfortunately we do not have an answer for the spaces $\mathcal{D}_d$.

Acknowledgement: This work was supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

The first author was partially supported by GNAMPA of INdAM, the fifth author was partially supported by GNSAGA of INdAM and by FIRB "Differential Geometry and Geometric Function Theory" of the Italian MIUR.

References

[1] J. Agler, J. E. McCarthy, Pick interpolation and Hilbert function spaces, Graduate Studies in Mathematics, 44 (American Mathematical Society, Providence, RI, 2002), xx+308 pp. ISBN: 0-8218-2898-3.

[2] N. Arcozzi, R. Rochberg, E. Sawyer, Carleson measures for analytic Besov spaces, Rev. Mat. Iberoamericana 18 (2002), no. 2, 443-510.

[3] N. Arcozzi, R. Rochberg, E. Sawyer, B. D. Wick, Function spaces related to the Dirichlet space, J. Lond. Math. Soc. (2) 83 (2011), no. 1, 1-18.

[4] N. Aronszajn, Theory of reproducing kernels, Trans. Amer. Math. Soc. 68, (1950) 337-404.

[5] D. Bekollé, A. Bonami, Inégalités à poids pour le noyau de Bergman, C. R. Acad. Sci. Paris Sér. A-B 286 (1978), no. 18, A775-A778 (in French).

[6] B. Boe, An interpolation theorem for Hilbert spaces with Nevanlinna-Pick kernel, Proc. Amer. Math. Soc. 133 (2005), no. 7, 2077-2081.

[7] D. H. Luecking, Representation and duality in weighted spaces of analytic functions, Indiana Univ. Math. J. 34 (1985), no. 2, 319-336.

[8] K. Seip, Interpolation and sampling in spaces of analytic functions, University Lecture Series, 33 (American Mathematical Society, Providence, RI, 2004), xii+139 pp. ISBN: 0-8218-3554-8.