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# Quasi Kronecker products and a determinant formula

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**Abstract**

We introduce an extension of the Kronecker product for matrices which retains many of the properties of the usual Kronecker product. As an application we study matrices over divisor-closed sets with multiplicative entries, and show how these are quasi Kronecker products over the primes of simpler matrices. In particular this gives a formula for the determinant of such matrices which combines and generalizes a number of previous results on Smith type determinants.

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**Introduction**

In [5], inspired by a result of Codéca and Nair [2], it was shown that matrices with multiplicative entries indexed over a divisor set $D(k) = \{d \in \mathbb{N} : d|k\}$ factorize as Kronecker (or tensor) products; namely, for $f : \mathbb{N}^2 \to \mathbb{C}$ multiplicative (as a function of two variables) and $A = (f(m,n))_{m,n\in D(k)}$,

$$A = \bigotimes_{p|k} A_p$$

where $A_p = (f(p^i, p^j))_{0\le i,j\le r}$ and $p^r||k$.[1] Since Kronecker products satisfy many useful properties, this makes is possible to deduce lots of information about $A$ from the $A_p$ like its eigenvalues, norm and determinant.

It is natural to enquire what we can say more generally about matrices $A_S = (f(m,n))_{m,n\in S}$ for some finite set $S \subset \mathbb{N}$, in particular when $f$ is multiplicative. We find a natural condition on $S$ is that it should be *divisor closed*; i.e. $n \in S$ implies $d \in S$ whenever $d|n$. For example $S = \{1, \ldots, N\}$, which gives the usual $N \times N$ truncation, is divisor closed. Determinants of matrices over divisor-closed sets have been discussed by many authors (see for example, [1], [3]), after the well-known Smith determinant from 1876 [6].

We show in this more general setting that $A_S$ still factorizes over the primes in $S$ as a type of psuedo-Kronecker product. This more general Kronecker product still retains a number of useful properties which we investigate here. In particular, we find linearity, commutativity and associativity are retained, even if multiplicativity fails. Furthermore, a neat formula for the determinant (already found in [4] for $S = \{1, \ldots, N\}$) and results on positive definiteness are obtained. As a consequence, we find a formula for $\det A_S$ whenever $S$ is divisor-closed and $f$ is multiplicative. This generalises a number of earlier results concerning determinants of arithmetical matrices over divisor-closed sets.

**§1. Quasi Kronecker products**

Let $A = (a_{ij})$ be an $n \times n$ matrix, $B = (b_{ij})$ an $m \times m$ matrix, and $l = (l_1, \ldots, l_n) \in \mathbb{N}^n$ where $\max\{l_1, \ldots, l_n\} = m$. Let $B_{rs} = (b_{ij})_{i\le r, j\le s}$ denote the $r \times s$ truncation of $B$. If $r = s$ we simply write $B_r$.Put $L = l_1 + \cdots + l_n$. Define $A \otimes_l B$ to be the $L \times L$ matrix given by the block matrix

$$A \otimes_l B = (a_{ij} B_{l_i l_j})_{i,j\le n}. \tag{0.1}$$

[1] Here $p^r||k$ means, as usual, that $p^r|k$ but $p^{r+1} \not| k$.

**Example 1** With $l = (3, 2)$,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes_l \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c & 2a & 2b \\ d & e & f & 2d & 2e \\ g & h & i & 2g & 2h \\ 3a & 3b & 3c & 4a & 4b \\ 3d & 3e & 3f & 4d & 4e \end{pmatrix}.$$

*Remarks 1*

(a) If the $l_i$ are constant, say $l_i = m$ for all $i$, then $A \otimes_l B$ reduces to $A \otimes B$, the usual Kronecker product.

(b) Note the asymmetrical nature of this generalised product whenever the $l_i$ are not constant. In the above example, the top left corners of $A$ and $B$ feature more prominently than the bottom right.

(c) Since $L = l_1 + \cdots + l_n$, we can call $l = (l_1, \ldots, l_n)$ a *partition* of $L$ (of length $n$). If we let $L_0 = 0$ and $L_i = l_1 + \cdots + l_i$ for $i = 1, \ldots, n$, then

$$(A \otimes_l B)_{uv} = a_{ij} b_{u-L_{i-1}, v-L_{j-1}}, \tag{0.2}$$

where $i, j \leq n$ are the unique positive integers such that $L_{i-1} < u \leq L_i$ and $L_{j-1} < v \leq L_j$.

(d) For $X$ and $Y$ of the same size, let us write $X \cong Y$ to mean $X = P^{-1} Y P$ for some permutation matrix $P$. Note that $A \otimes_l B \cong A \otimes_{l^*} B$ where $l^* = (l_1^*, \ldots, l_n^*)$ is the permutation of $l$ with $l_1^* \geq \cdots \geq l_n^*$.

(e) We can view $A \otimes_l B$ as a 'projection' of $A \otimes B$ onto a smaller matrix. More precisely (and using the same notation as above), there is a $mn \times L$ matrix $P$ such that

$$A \otimes_l B = P^*(A \otimes B)P.$$

Indeed, with $P = (p_{ij})_{i \leq mn, j \leq L}$ we have

$$p_{ij} = 1 \quad \text{if } j = L_{r-1} + s,\ i = m(r-1) - L_{r-1} + s \text{ for } 1 \leq r \leq n \text{ and } 1 \leq s \leq l_r$$

and $p_{ij} = 0$ otherwise.

Equivalently, there exists a (diagonal) orthogonal projection $Q$ such that

$$Q(A \otimes B)Q \cong \left( \begin{array}{c|c} A \otimes_l B & 0 \\ \hline 0 & 0 \end{array} \right).$$

Indeed $Q = \mathrm{diag}(d_1^{(1)}, \ldots, d_m^{(1)}, \ldots, d_1^{(n)}, \ldots, d_m^{(n)})$ where

$$d_r^{(i)} = \begin{cases} 1 & \text{if } r \leq l_i \\ 0 & \text{if } l_i < r \leq m \end{cases} \quad \text{for } i = 1, \ldots, n.$$

For Example 1, we have $Q = \mathrm{diag}(1, 1, 1, 1, 1, 0)$.

### 1.1 Properties of the Kronecker product

The Kronecker product satisfies many properties as we see below (cf. [8]). For $A, B, C$ of appropriate sizes:

(a) $(\lambda A) \otimes B = A \otimes (\lambda B) = \lambda A \otimes B$;

(b) $(A + B) \otimes C = A \otimes C + B \otimes C$; $A \otimes (B + C) = A \otimes B + A \otimes C$;

(c) $A \otimes B = 0$ if and only if $A = 0$ or $B = 0$;

(d) If $B \cong C$, then $A \otimes B \cong A \otimes C$;

(e) $A \otimes B \cong B \otimes A$;

(f) $A \otimes (B \otimes C) = (A \otimes B) \otimes C$;

(g) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$;

(h) $A, B$ symmetric/unitary/normal/positive definite implies $A \otimes B$ symmetric/unitary/normal/positive definite respectively;

(i) $\det(A \otimes B) = (\det A)^m (\det B)^n$ if $A$ is $n \times n$ and $B$ is $m \times m$;

(j) $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \operatorname{tr}(B)$;

(k) $\|A \otimes B\| = \|A\| \|B\|$ and $\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$; [2]

(l) $\sigma(A \otimes B) = \sigma(A)\sigma(B)$, where $\sigma(A)$ is the set of eigenvalues of $A$;

(m) Defining $A \oplus B = A \otimes I + I \otimes B$ to be the *Kronecker sum* of $A$ and $B$, we have $e^A \otimes e^B = e^{A \oplus B}$. Also $\sigma(A \oplus B) = \sigma(A) + \sigma(B)$.

**1.2 Properties of the Quasi Kronecker product**

Here we investigate how the above properties (a) to (m) generalize. We shall find that (a), (b), (e), (f), (i) and (j) all generalize directly or in some suitable sense, while for the other parts, only less information can be salvaged. For example, in (k), equality is replaced by inequality. It would be especially useful if (l) could be generalized in a suitable way.

Trivially, we find (a) and (b) continue to hold with $\otimes$ replaced by $\otimes_l$. Part (c) fails; e.g. with $l = (3, 2)$

$$\begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix} \otimes_l \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Indeed, with the notation of (0.1), $A \otimes_l B = 0$ if and only if $a_{ij} B_{l_i l_j} = 0$ for all $i, j \leq n$.

Part (d) is also false in general whenever $l$ is not constant as can be readily shown. Thus $I_2 \otimes_{(2,1)} I_2 = I_3$ but

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes_{(2,1)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{0.3}$$

which is not permutation similar to $I_3$.

For (e) and (f), we have the following elegant generalizations which shows the role partitions play. Since $L = l_1 + \cdots + l_n$, we can regard $l = (l_1, \ldots, l_n)$ as a partition of $L$ of length $n$. By Remark 1(d), we may assume that the $l_i$ decrease; as such $l_1 = m$. Its *conjugate* partition is $l' = (l'_1, \ldots, l'_m)$ where $l'_r = \#\{j : l_j \geq r\}$. Note that $l'_1 = n$. The conjugate partition is easiest to visualize by a Ferrer's diagram, which has each $l_i$ as a sequence of horizontal dots. Transposing the diagram (or viewing it vertically) gives the conjugate partition. For example, with $l = (3, 2)$, we have $l' = (2, 2, 1)$:

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \end{array} \qquad \text{has dual} \qquad \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \end{array}$$

[2] Here $\|\cdot\|$ is the operator norm, i.e. $\|A\| = \sup_{\|x\|=1} \|Ax\|$ and $\|\cdot\|_2$ is the Hilbert-Schmidt norm: for $A = (a_{ij})_{i,j \leq n}$, we have $\|A\|_2^2 = \sum_{i,j \leq n} |a_{ij}|^2$.

**Theorem 1**
*We have $A \otimes_l B \cong B \otimes_{l'} A$, where $l'$ is the conjugate partition of $l$. In particular, if $l = l'$, then $A \otimes_l B \cong B \otimes_l A$.*

We postpone the proof until after Theorem 5 as it follows from that proof.

Thus in example 1, we have $l' = (2, 2, 1)$ and we find

$$B \otimes_{l'} A = \begin{pmatrix} a & 2a & b & 2b & c \\ 3a & 4a & 3b & 4b & 3c \\ d & 2d & e & 2e & f \\ 3d & 4d & 3e & 4e & 3f \\ g & 2g & h & 2h & i \end{pmatrix}.$$

With $P$ representing the permutation (4235); i.e.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

we have $P^{-1}(A \otimes_l B)P = B \otimes_{l'} A$.

**Theorem 2**
*Given partitions $k, l$ and square matrices $A, B, C$ of appropriate sizes, there exist partitions $\tilde{k}$ and $\tilde{l}$ dependent on $k$ and $l$ only such that*

$$A \otimes_k (B \otimes_l C) = (A \otimes_{\tilde{k}} B) \otimes_{\tilde{l}} C. \tag{1.1}$$

*Proof.* We start with some notation. Let $A$ be $n \times n$ and $B$ be $m \times m$ so that $k = (k_1, \ldots, k_n)$ and $l = (l_1, \ldots, l_m)$. Put $K_i = k_1 + \cdots + k_i$ and $L_i = l_1 + \cdots + l_i$ with $K = K_n$ and $L = L_m$. Also write $K_0 = L_0 = 0$. The conditions imply that $\max_i k_i = L$ and that the LHS matrix in (1.1) is $K \times K$.

We define $\tilde{k}$ and $\tilde{l}$ in turn. First $\tilde{k} = (\tilde{k}_1, \ldots, \tilde{k}_n)$ where $\tilde{k}_i$ is the unique $k \in \mathbb{N}$ such that $L_{k-1} < k_i \le L_k$; i.e.

$$L_{\tilde{k}_i - 1} < k_i \le L_{\tilde{k}_i}.$$

As for $k$, let $\tilde{K}_i = \tilde{k}_1 + \cdots + \tilde{k}_i$ and $\tilde{K} = \tilde{K}_n$, with $\tilde{K}_0 = 0$. Note that $A \otimes_{\tilde{k}} B$ is of size $\tilde{K}$.

Next we define $\tilde{l} = (\tilde{l}_1, \ldots, \tilde{l}_{\tilde{K}})$ where, for $i = 1, \ldots, n$,

$$\tilde{l}_{\tilde{K}_{i-1}+j} = l_j \quad \text{for } 1 \le j < \tilde{k}_i, \text{ and} \quad \tilde{l}_{\tilde{K}_i} = k_i - L_{\tilde{k}_i - 1}.$$

As for $l$, let $\tilde{L}_i = \tilde{l}_1 + \cdots + \tilde{l}_i$ and $\tilde{L}_0 = 0$.

It follows in particular that

$$\sum_{\tilde{K}_{i-1} < j \le \tilde{K}_i} \tilde{l}_j = \sum_{j=1}^{\tilde{k}_i - 1} l_j + (k_i - L_{\tilde{k}_i - 1}) = k_i,$$

and, summing from $i = 1$ to $r$ gives, for every $r \le n$,

$$\tilde{L}_{\tilde{K}_r} = \sum_{j \le \tilde{K}_r} \tilde{l}_j = K_r. \tag{1.2}$$

Now using (0.2), we find that

$$(A \otimes_k (B \otimes_l C))_{uv} = a_{ij} b_{pq} c_{u - K_{i-1} - L_{p-1}, v - K_{j-1} - L_{q-1}}, \tag{1.3}$$

where $i, j \leq n$ and $p, q \leq m$ are the unique positive integers such that

$$\begin{array}{c} K_{i-1} < u \leq K_i \\ K_{j-1} < v \leq K_j \end{array} \quad \text{and} \quad \begin{array}{c} L_{p-1} < u - K_{i-1} \leq L_p \\ L_{q-1} < v - K_{j-1} \leq L_q \end{array}.$$

In the same way,

$$((A \otimes_{\tilde{k}} B) \otimes_{\tilde{l}} C)_{uv} = a_{ij} b_{r-\tilde{K}_{i-1}, s-\tilde{K}_{j-1}} c_{u-\tilde{L}_{r-1}, v-\tilde{L}_{s-1}},$$

where $r, s \leq \tilde{K}$ and $i, j \leq n$ are the unique positive integers such that

$$\begin{array}{c} \tilde{L}_{r-1} < u \leq \tilde{L}_r \\ \tilde{L}_{s-1} < v \leq \tilde{L}_s \end{array} \quad \text{and} \quad \begin{array}{c} \tilde{K}_{i-1} < r \leq \tilde{K}_i \\ \tilde{K}_{j-1} < s \leq \tilde{K}_j \end{array}.$$

Write $p = r - \tilde{K}_{i-1}$ and $q = s - \tilde{K}_{j-1}$. Note that $1 \leq p \leq \tilde{K}_i - \tilde{K}_{i-1} = \tilde{k}_i \leq m$ and similarly $1 \leq q \leq \tilde{k}_j \leq m$. Then

$$((A \otimes_{\tilde{k}} B) \otimes_{\tilde{l}} C)_{uv} = a_{ij} b_{pq} c_{u-\tilde{L}_{r-1}, v-\tilde{L}_{s-1}}.$$

For this to equal (1.3) the $c$-entries have to match up; i.e. we have to show that

$$\tilde{L}_{r-1} = K_{i-1} + L_{p-1} \quad \text{and} \quad \tilde{L}_{s-1} = K_{j-1} + L_{q-1}.$$

But

$$\begin{aligned} \tilde{L}_{r-1} &= \tilde{L}_{\tilde{K}_{i-1}} + (\tilde{l}_{\tilde{K}_{i-1}+1} + \cdots + \tilde{l}_{r-1}) \\ &= K_{i-1} + (l_1 + \cdots + l_{r-1-\tilde{K}_{i-1}}) && \text{(by (1.2) and since } r - 1 < \tilde{K}_i) \\ &= K_{i-1} + L_{r-1-\tilde{K}_{i-1}} = K_{i-1} + L_{p-1}, \end{aligned}$$

as $p = r - \tilde{K}_{i-1}$. In exactly the same way $\tilde{L}_{s-1} = K_{j-1} + L_{q-1}$. The result follows.

□

**Example 2** First we illustrate how to find $\tilde{k}$ and $\tilde{l}$ from $k$ and $l$.

Let $k = (8, 12, 3, 2, 5)$ and $l = (4, 3, 3, 2)$. Line up the numbers 8,12,3,2,5 from $k$ and see how many of the $l_i$ (i.e. 4,3,3,2) are needed to add up to each $k_i$ with the last term adjusted to give $k_i$ exactly, as follows:

$$\underbrace{\overbrace{431}^{8}}_{3}\underbrace{\overbrace{4332}^{12}}_{4}\underbrace{\overbrace{\;3\;}^{3}}_{1}\underbrace{\overbrace{\;2\;}^{2}}_{1}\underbrace{\overbrace{41}^{5}}_{2}.$$

The numbers needed are given at the bottom and give $\tilde{k}$, while $\tilde{l}$ is found by the middle line. Thus $\tilde{k} = (3, 4, 1, 1, 2)$ and $\tilde{l} = (4, 3, 1, 4, 3, 3, 2, 3, 2, 4, 1)$.

For an illustration of the associative property, we take smaller partitions.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes_{(4,1)} \left( \begin{pmatrix} x & y \\ z & w \end{pmatrix} \otimes_{(2,2)} \begin{pmatrix} \lambda & \mu \\ \sigma & \tau \end{pmatrix} \right) = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes_{(2,1)} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) \otimes_{(2,2,1)} \begin{pmatrix} \lambda & \mu \\ \sigma & \tau \end{pmatrix}$$

with both sides equalling

$$\begin{pmatrix} ax\lambda & ax\mu & ay\lambda & ay\mu & bx\lambda \\ ax\sigma & ax\tau & ay\sigma & ay\tau & bx\sigma \\ az\lambda & az\mu & aw\lambda & aw\mu & bz\lambda \\ az\sigma & az\tau & aw\sigma & aw\tau & bz\sigma \\ cx\lambda & cx\mu & cy\lambda & cy\mu & dx\lambda \end{pmatrix}.$$

For (g), we find it is false. Indeed, $(A \otimes_l I)(B \otimes_l I) \neq AB \otimes_l I$. Take $l = (2, 1)$ and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Then

$$AB \otimes_l I - (A \otimes_l I)(B \otimes_l I) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & bz & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, each of $A \otimes_l B$, $(A \otimes_l I)(I \otimes_l B)$ and $(I \otimes_l B)(A \otimes_l I)$ are all different.

Part (h): trivially we retain the implication $A, B$ symmetric (or Hermitian) implies $A \otimes_l B$ symmetric (or Hermitian) but (h) fails for unitary/normal matrices. Example (0.3) shows that it fails for unitary $A, B$ while the example

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \otimes_{(2,1)} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = C,$$

say, has $C^*C \neq CC^*$ although the matrices on the left are normal. For positive definite $A, B$ we have the following:

**Theorem 3**
*We have $A, B \geq 0 \implies A \otimes_l B \geq 0$ and $A, B > 0 \implies A \otimes_l B > 0$.*

*Proof.* We know that $A, B \geq 0$ implies $A \otimes B \geq 0$. But $A \otimes_l B$ is a principal submatrix of $A \otimes B$. Thus by Theorem 7.2 of [8], $A \otimes_l B \geq 0$. The same argument also holds for strict inequality. □

For part (i), we have the following result, which was proved in [4]:

**Proposition A** ([4], Proposition 4.4)
*Suppose that $l = (l_1, \ldots, l_n)$ with $l_1 \geq \cdots \geq l_n$. Then*

$$\det(A \otimes_l B) = \prod_{r=1}^{n} (\det A_r)^{l_r - l_{r+1}} \det B_{l_r},$$

*where $A_r = (a_{ij})_{i,j \leq r}$ is the $r \times r$ truncation of $A$ and $l_{n+1} = 0$.*

Part (j) generalizes similarly as

$$\mathrm{tr}(A \otimes_l B) = \sum_{r=1}^{n} (\mathrm{tr}(A_r) - \mathrm{tr}(A_{r-1}))\mathrm{tr}(B_{l_r}) = \sum_{r=1}^{n} \mathrm{tr}(A_r)(\mathrm{tr}(B_{l_r}) - \mathrm{tr}(B_{l_{r+1}})),$$

as can be seen by direct calculation. (Here $A_0 = 0$.)

*Remarks 2*

(a) Proposition A shows that $A \otimes_l B$ is invertible if and only if $A_r$ (if $l_r > l_{r+1}$) and $B_{l_r}$ are invertible for each $r$. In particular, this shows that the existence of $A^{-1}$ and $B^{-1}$ is necessary for $A \otimes_l B$ to be invertible but (in general) it is not sufficient (see (0.3)).

(b) Note that in the generalizations of (i) and (j), the determinant and trace of $A \otimes_l B$ can be expressed in terms of the $A_r$ and $B_{l_r}$. This is not true for $\mathrm{tr}((A \otimes_l B)^2)$. For example,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes_{(2,1)} \begin{pmatrix} 0 & x \\ 1 & y \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 1 & y & 1 \\ 0 & x & 0 \end{pmatrix} = C,$$

say. Then $\mathrm{tr}(C^2) = 4x + y^2$, while $\mathrm{tr}(B_1^2) = 0$ and $\mathrm{tr}(B_2^2) = 2x + y^2$ which is not a function of $4x + y^2$.

The above example also shows$\|A\otimes_l B\|_2$ cannot be expressed in terms of the $\|A_r\|_2$ and $\|B_{l_r}\|_2$ as can be readily verified. On the other hand, for this norm and the usual operator norm we do have the following which may be seen as a generalization of (k):

**Theorem 4**
*Let $A$ and $B$ be square matrices of size $n$ and $m$ respectively. Then $\|A\otimes_l B\|_2 \le \|A\|_2\|B\|_2$ and $\|A\otimes_l B\| \le \|A\|\|B\|$. Furthermore, equality occurs in the former if and only if*

$$a_{ij}b_{rs} = 0 \quad \text{for all } i,j \le n \text{ whenever } l_i < r \le m \text{ or } l_j < s \le m, \tag{1.5}$$

*in which case $\|A\otimes_l B\| = \|A\|\|B\|$.*

*Proof.* The first inequality follows by a straightforward computation and using $\|A\otimes B\|_2 = \|A\|_2\|B\|_2$:

$$\|A\otimes_l B\|_2^2 = \sum_{i,j\le n}|a_{ij}|^2\|B_{l_i,l_j}\|_2^2 = \sum_{i,j\le n}|a_{ij}|^2 \sum_{\substack{r\,\le\, l_i\\ s\,\le\, l_j}}|b_{rs}|^2 \le \sum_{i,j\le n}|a_{ij}|^2\sum_{r,s\le m}|b_{rs}|^2 = \|A\otimes B\|_2^2.$$

From this it is immediate that equality holds if and only if (1.5) holds.

For the second inequality, we have, for some orthogonal diagonal projection matrix $Q$,

$$\|A\otimes_l B\| = \left\|\left(\begin{array}{c|c} A\otimes_l B & 0\\ \hline 0 & 0\end{array}\right)\right\| = \|Q(A\otimes B)Q\| \le \|Q\|\|A\otimes B\|\|Q\| = \|A\|\|B\|,$$

since $\|Q\|=1$.

Suppose now (1.5) holds. Consider $(A\otimes B)x = y$ where

$$x = \begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix} \quad \text{and} \quad x_i = \begin{pmatrix} x_i^{(1)}\\ \vdots\\ x_i^{(m)}\end{pmatrix}$$

and similarly for $y$. Then

$$\|(A\otimes B)x\|^2 = \sum_{i=1}^n \|y_i\|^2 = \sum_{i=1}^n\sum_{r=1}^m |y_i^{(r)}|^2.$$

But

$$y_i^{(r)} = \sum_{j=1}^n\sum_{s=1}^m a_{ij}b_{rs}x_j^{(s)} = \sum_{j=1}^n\sum_{s=1}^{l_j} a_{ij}b_{rs}x_j^{(s)}.$$

Thus, without loss of generality, we may take $x_j^{(s)} = 0$ for $s > l_j$. Let

$$\tilde{x} = \begin{pmatrix} \tilde{x_1}\\ \vdots\\ \tilde{x_n}\end{pmatrix} \quad \text{where} \quad \tilde{x_i} = \begin{pmatrix} x_i^{(1)}\\ \vdots\\ x_i^{(l_i)}\end{pmatrix}.$$

As such $\|\tilde{x}\| = \|x\|$ and $\|(A\otimes B)x\| = \|(A\otimes_l B)\tilde{x}\| \le \|A\otimes_l B\|\|\tilde{x}\|$; i.e. $\|A\otimes_l B\| \ge \|A\otimes B\|$ and the result follows.

□

For example with $l=(3,2)$ and $A = \begin{pmatrix} x & y\\ z & w\end{pmatrix}$ and $B = \begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}$ we see that (1.5) holds if $A, B$ are one of the forms

$$\begin{pmatrix} x & y\\ z & w\end{pmatrix}, \begin{pmatrix} a & b & 0\\ d & e & 0\\ 0 & 0 & 0\end{pmatrix}; \quad \begin{pmatrix} x & y\\ 0 & 0\end{pmatrix}, \begin{pmatrix} a & b & 0\\ d & e & 0\\ g & h & 0\end{pmatrix}; \quad \begin{pmatrix} x & 0\\ z & 0\end{pmatrix}, \begin{pmatrix} a & b & c\\ d & e & f\\ 0 & 0 & 0\end{pmatrix}.$$

For part (l) we find there is no obvious relation between $\sigma(A \otimes_l B)$ and the spectra of truncation of $A$ and $B$. This is no doubt due to the failure of the multiplicative property (g).

In the case when $A$ or $B$ is triangular, we can find the spectrum of $A \otimes_l B$ easily. Suppose $A$ is upper-triangular. Then

$$\det(\lambda I_L - A \otimes_l B) = \det \begin{pmatrix} \lambda I_{l_1} - a_{11} B_{l_1} & & * \\ & \ddots & \\ 0 & & \lambda I_{l_n} - a_{nn} B_{l_n} \end{pmatrix} = \prod_{i=1}^{n} \det(\lambda I_{l_i} - a_{ii} B_{l_i}).$$

It follows that $\lambda \in \sigma(A \otimes_l B)$ if and only if $\det(\lambda I_{l_i} - a_{ii} B_{l_i}) = 0$ for some $i$; i.e. $\lambda \in \sigma(a_{ii} B_{l_i})$ for some $i$. Thus, in this case

$$\sigma(A \otimes_l B) = \bigcup_{i=1}^{n} \sigma(a_{ii} B_{l_i}) = \bigcup_{i=1}^{n} \Big(\sigma(A_i) \setminus \sigma(A_{i-1})\Big) \sigma(B_{l_i}). \tag{1.6}$$

But (1.6) is false more generally. For example, $\begin{pmatrix} 1 & \mu \\ \mu & 1 \end{pmatrix}$ has eigenvalues $1 \pm \mu$ and $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ has eigenvalues $0, 2$, but

$$\sigma\Big( \begin{pmatrix} 1 & \mu \\ \mu & 1 \end{pmatrix} \otimes_{(2,1)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Big) = \Big\{ 0, \tfrac{3}{2} \pm \sqrt{2\mu^2 + \tfrac{1}{4}} \Big\}.$$

For (m), we need to generalize the notion of a Kronecker sum. For $A$, $B$ and $l$ as before, define the **Quasi Kronecker sum** by

$$A \oplus_l B = A \otimes_l I_m + I_n \otimes_l B.$$

As a consequence of Theorems 1 and 2, we have $A \oplus_l B \cong B \oplus_{l'} A$ and $A \oplus_k (B \oplus_l C) = (A \oplus_{\tilde{k}} B) \oplus_{\tilde{l}} C$ (using the notation from Theorem 2). Thus for some permutation matrix $P$,

$$\begin{aligned} A \oplus_l B &= A \otimes_l I_m + I_n \otimes_l B = P^{-1}(I_m \otimes_{l'} A)P + P^{-1}(B \otimes_{l'} I_n)P \\ &= P^{-1}(I_m \otimes_{l'} A + B \otimes_{l'} I_n)P = P^{-1}(B \oplus_{l'} A)P, \end{aligned}$$

while, with $A, B, C$ of size $n, m, r$ respectively, we have

$$\begin{aligned} A \oplus_k (B \oplus_l C) &= A \otimes_k I_L + I_n \otimes_k (B \otimes_l I_r + I_m \otimes_l C) \\ &= A \otimes_k (I_m \otimes_l I_r) + I_n \otimes_k (B \otimes_l I_r) + I_n \otimes_k (I_m \otimes_l C) && \text{(by Remark 1(e))} \\ &= (A \otimes_{\tilde{k}} I_m) \otimes_{\tilde{l}} I_r + (I_n \otimes_{\tilde{k}} B) \otimes_{\tilde{l}} I_r + (I_n \otimes_{\tilde{k}} I_m) \otimes_{\tilde{l}} C && \text{(by Theorem 2)} \\ &= (A \oplus_{\tilde{k}} B) \otimes_{\tilde{l}} I_r + I_{\tilde{K}} \otimes_{\tilde{l}} C && \text{(where } \tilde{K} = \tilde{k}_1 + \cdots + \tilde{k}_m) \\ &= (A \oplus_{\tilde{k}} B) \oplus_{\tilde{l}} C. \end{aligned}$$

We find that part (m) is false; i.e.

$$e^{A \oplus_l B} \neq e^A \otimes_l e^B$$

in general. For we can find invertible $C$ and $D$ such that $C \otimes_l D$ is not invertible. But we may write $C = e^A$ and $D = e^B$ for some $A$ and $B$, while $C \otimes_l D$ is not even an exponential.

For an explicit example, let $a = \frac{i\pi}{2}$ and put $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -a & a \\ a & -a \end{pmatrix}$. Then, with $l = (2, 1)$, we have

$$e^A \otimes_l e^B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{while} \quad e^{A \oplus_l B} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Part (m) is even false for Hermitian $A$ and $B$. Take $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ and $B = \begin{pmatrix} c & d \\ d & c \end{pmatrix}$ where $b, d \neq 0$. We omit the details. Furthermore, with $A, B$ as such and $l = (2,1)$, we have $\sigma(A \oplus_l B) = \{a+c, a+c \pm \sqrt{b^2+d^2}\}$ while $\sigma(A) = \{a \pm b\}$ and $\sigma(B) = \{c \pm d\}$, and there is no obvious relation between these.

## §2. Divisor closed sets

Let $S \subset \mathbb{N}$ be finite and divisor closed; i.e. if $n \in S$ and $d|n$ then $d \in S$. For $p \in S$ ($p$ prime), let $k = k_p$ be the largest power of $p$ in $S$; i.e. $p^k \in S$ but $p^{k+1} \notin S$. We can partition $S$ as follows:

$$S = S_0 \cup pS_1 \cup \cdots \cup p^k S_k, \tag{2.1}$$

where each $S_r$ contains no multiples of $p$. Note that $S_0 = \{n \in S : p \nmid n\}$ and more generally[3] $p^r S_r = \{n \in S : p^r \| n\}$. As such,

(i) each $S_r$ is divisor closed,
(ii) $S_{r'} \subset S_r$ for $r' \geq r$, and
(iii) $|S_0| + \cdots + |S_k| = |S|$.

To see (i) let $n \in S_r$ and $d|n$. Then $p^r n \in S$ and $p \nmid n$. As $S$ is divisor-closed, $p^r d \in S$ also. Since $p \nmid d$ it follows that $d \in S_r$. For (ii), let $n \in S_{r'}$. Then $p^{r'} n \in S$. As $S$ is divisor closed, we must have $p^r n \in S$; i.e. $n \in S_r$. Part (iii) follows from (2.1).

For example, $S = D(m) = \{d \in \mathbb{N} : d|m\}$ is divisor closed. For each $p|m$, say $p^k || m$, we have $S_r = D(m/p^k)$ and so $|S_r| = \tau(m/p^k)$ for each $r = 0, \ldots, k$.

Sometimes we shall write $k = k_p$ and $S_r = S_r^{(p)}$ to highlight the dependence on $p$.

### 2.1 Matrices over divisor closed sets

For a divisor closed set $S$ and $A = (a_{ij})$, we write $A_S = (a_{ij})_{i,j \in S}$. If $p$ prime, we write $\tilde{A}_p = (a_{p^r p^s})_{0 \leq r,s \leq k}$ where $p^k$ is the largest power of $p$ in $S$; i.e. $\tilde{A}_p = A_{T_p}$ where $T_p = \{1, p, \ldots, p^k\}$. From [5], we see that if $S$ is of the form $D(k)$ and $a_{ij}$ is multiplicative (in two variables), then $A_S$ is a Kronecker product over the primes in $S$; namely $A_S = \otimes_{p \in S} \tilde{A}_p$. We generalize this to any divisor closed set using the notion of quasi Kronecker product.

We recall that a function $f : \mathbb{N}^2 \to \mathbb{C}$ is *multiplicative* if $f$ is not identically zero and

$$f(m_1 n_1, m_2 n_2) = f(m_1, m_2) f(n_1, n_2)$$

if $(m_1 m_2, n_1 n_2) = 1$. (See [7] for a survey of multiplicative functions of two or more variables.)

**Theorem 5**
*Let $S \subset \mathbb{N}$ be finite and divisor closed, and let $A = (f(m,n))_{m,n \geq 1}$ where $f$ is multiplicative (of two variables). Let $p \in S$ be prime and define $S_0, \ldots, S_k$ as above. Then*

$$A_S \cong \tilde{A}_p \otimes_l A_{S_0}, \tag{2.2}$$

*where $l = (|S_0|, \ldots, |S_k|)$.*

*Proof.* Order the rows and columns of $A$ along elements of $S_0, \ldots S_k$. The block corresponding to $p^r S_r, p^s S_s$ has $mn^{\text{th}}$-entry (where $m \in S_r, n \in S_s$)

$$f(p^r m, p^s n) = f(p^r, p^s) f(m, n)$$

by multiplicativity of $f$. Hence

$$A_S \cong \Big( f(p^r, p^s)(f(m,n))_{m \in S_r, n \in S_s} \Big)_{0 \leq r,s \leq k} = \tilde{A}_p \otimes_l A_{S_0}.$$

[3] Here, $p^r \| n$ means, as usual, $p^r | n$ but $p^{r+1} \nmid n$.

□

As an immediate consequence of this and Theorem 3, we see that $\tilde{A}_p > 0$ for all $p \in S$ implies $A_S > 0$.

*Remark 3.* Since $S_0$ is again divisor closed, we can apply Theorem 5 to $A_{S_0}$; i.e. for a prime $q \in S_0$, we have $A_{S_0} \cong \tilde{A}_q \otimes_k A_{S_{00}}$ for suitable $k$ and $S_{00}$. But we cannot (in general) conclude that

$$A_S \cong \tilde{A}_p \otimes_l (\tilde{A}_q \otimes_k A_{S_{00}})$$

due to the failure of (d) for quasi Kronecker products.

*Proof of Theorem 1.* We are now in a position to prove Theorem 1. Let $A = (a_{ij})_{i,j \leq n}$, $B = (b_{ij})_{i,j \leq m}$ and $l = (l_1, \ldots, l_n)$ with $l_1 = m$. It is clearly sufficient to prove the result when $a_{11}, b_{11} \neq 0$. By rescaling, we may assume that $a_{11} = b_{11} = 1$. The idea is to identify $A$ and $B$ with some matrices of the form $\tilde{A}_2$ and $\tilde{A}_3$ derived from some matrix $A_S$ where $S$ is a suitable divisor closed subset of $\{2^r 3^s : r, s \geq 0\}$.

We choose $S$ to be the (divisor closed) set

$$S = \{2^{r-1} 3^q : 1 \leq r \leq n, 0 \leq q < l_r\}.$$

Thus $S_{r-1} = \{3^q : 0 \leq q < l_r\}$ and $|S_{r-1}| = l_r$ for $r = 1, \ldots, n$. Let $f : \mathbb{N}^2 \to \mathbb{C}$ be multiplicative and defined at the prime powers by

$$f(2^{i-1}, 2^{j-1}) = a_{ij} \text{ for } 1 \leq i, j \leq n, \quad f(3^{i-1}, 3^{j-1}) = b_{ij} \text{ for } 1 \leq i, j \leq m,$$

and $f(p^i, p^j) = 0$ in all other cases. As such, $A = \tilde{A}_2$ and $B = \tilde{A}_3$. We have the partitions

$$S = \bigcup_{r=0}^{n-1} 2^r S_r = \bigcup_{q=0}^{m-1} 3^q T_q,$$

where $T_q = \{2^{r-1} : r \geq 1 \text{ and } 2^{r-1} 3^q \in S\}$. Theorem 5 with $p = 2$ says

$$A_S \cong A \otimes_l B,$$

where $l = (|S_0|, \ldots, |S_{n-1}|)$. On the other hand, applying Theorem 5 with $p = 3$ gives

$$A_S \cong B \otimes_{l'} A,$$

where $l' = (|T_0|, \ldots, |T_{m-1}|)$. As $|S| = |S_0| + \cdots + |S_{n-1}| = |T_0| + \cdots + |T_{m-1}| = L$ say, both $l$ and $l'$ are partitions of $L$. We need to show they are conjugate. This involves showing that

$$|T_i| = \#\{r \geq 0 : |S_r| \geq i + 1\} \quad \text{for } i = 0, 1, \ldots, m - 1.$$

Since $S$ is divisor closed and its elements are only products of powers of 2 and 3, we see that $|S_r| = \#\{j \geq 0 : 2^r 3^j \in S\}$. Thus $|S_r| \geq i + 1$ if and only if $2^r 3^i \in S$; i.e.

$$\#\{r \geq 0 : |S_r| \geq i + 1\} = \#\{r \geq 0 : 2^r 3^i \in S\}.$$

But this equals $|T_i|$.

□

### 2.2 A determinant formula

Applying Proposition A to Theorem 5 leads to a formula for the determinant of $A_S$. This generalizes both Theorem 1 from [2] where $S = D(N)$ and $f(m, n) = \frac{h((m,n))}{mn}$ with $h$ multiplicative and Theorem 3.1/3.2 from [4] where $S = \{1, \ldots, N\}$ and $f(m, n) = F(\frac{m}{n})$ with $F$ multiplicative on $\mathbb{Q}^+$.

**Theorem 6**
*Let $S \subset \mathbb{N}$ be finite and divisor closed, and let $A_S = (f(m,n))_{m,n\in S}$ where $f$ is multiplicative. Then, with $\tilde{A}_p, S_r^{(p)}$ and $k_p$ as in (2.1) and in Section 2.1, we have*

$$\det A_S = \prod_{p\in S}\prod_{r=1}^{k_p}\Big(\det(\tilde{A}_p)_{r+1}\Big)^{|S_r^{(p)}|-|S_{r+1}^{(p)}|}.$$

(Here $(\tilde{A}_p)_{r+1}$ is the $(r+1)\times(r+1)$ truncation of $\tilde{A}_p$ and $|S_{k_p+1}^{(p)}| = 0$.)

*Proof.* Let $p \in S$. Apply the determinant formula of Proposition A to (2.2). Thus, with $l_r = |S_{r-1}^{(p)}|$,

$$\det A_S = \det(\tilde{A}_p \otimes_l A_{S_0^{(p)}}) = \prod_{r=1}^{k_p+1}(\det(\tilde{A}_p)_r)^{l_r - l_{r+1}} \det(A_{S_0^{(p)}})_{l_r}.$$

Since $(\tilde{A}_p)_1 = (1)$, the $r = 1$ term may be dropped, so

$$\det A_S = \prod_{r=1}^{k_p}(\det(\tilde{A}_p)_{r+1})^{|S_r^{(p)}|-|S_{r+1}^{(p)}|} \cdot \prod_{r=1}^{k_p+1}\det(A_{S_0^{(p)}})_{l_r}.$$

Note that the determinant on the right has no terms with $f(p^r, p^s)$. Since both sides are just polynomials in all these variables, we can factor out the first term on the right above for each $p \in S$. Thus the result follows.

□

With a little extra work we see this result also contains the determinant formula in Theorem 2 from [1]. There, the matrix is of the form $(g([m,n]))_{m,n\in S}$ with $S$ divisor closed and $g$ multiplicative, where $[m,n]$ is the LCM of $m$ and $n$. Writing $f(m,n) = g([m,n])$, we see that $f$ is multiplicative and the matrix is just $A_S$.

Now $(\tilde{A}_p)_{r+1} = (g(p^{\max\{i,j\}}))_{0\le i,j\le r}$ which has determinant[4] $g(p^r)\prod_{m=1}^r(g(p^{m-1}) - g(p^m))$. Applying Theorem 6 and manipulating the formula then leads to the formula in [1].

As a special case of Theorem 6, suppose $S$ has only squarefree elements. Then $k_p = 1$ for all $p \in S$. Hence

$$\det A_S = \prod_{p\in S}(f(p,p) - f(p,1)f(1,p))^{N_p},$$

where $N_p = |S_1^{(p)}| = \#\{n \in S : p|n\}$.

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