Empirical likelihood tests for nonparametric detection of differential expression from RNA-seq data


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1 Empirical likelihood

For $n$ i.i.d one dimensional observations $x_1, \ldots, x_n$ the empirical likelihood (Owen, 1988) can be defined as

$$f(x) = \prod_{i=1}^{n} p_i,$$

(1)

where we assign each observation a weight $p_i$, and constrain these such that $\sum_{i=1}^{n} p_i = 1, \forall i, 0 \leq p_i \leq 1$. Focussing on the empirical likelihood for the mean $\mu$ of our observations $x_i$, we simply require that

$$\sum_{i=1}^{n} p_i x_i = \mu.$$  

(2)

Then we have three constraints, and aim to find the the $p_i$ that maximise the empirical likelihood $f(x)$ under these constraints. Fortunately by using Lagrange multipliers we can find the optimal $p_i$ by solving a one dimensional root finding problem. Defining

$$G = \sum_{i=1}^{n} \log(np_i) - n\lambda \sum_{i=1}^{n} p_i (x_i - \mu) + \gamma \left( \sum_{i=1}^{n} p_i - 1 \right),$$

(3)

and taking the partial derivative with respect to $p_i$, applying the method of Lagrange multipliers (Owen, 2001) we have

$$\frac{\partial G}{\partial p_i} = \frac{1}{p_i} - n\lambda (x_i - \mu) + \gamma = 0,$$

(4)

and we can solve for $\gamma$ by considering

$$\sum_{i=1}^{n} p_i \frac{\partial G}{\partial p_i} = 0$$

(5)

$$\sum_{i=1}^{n} (1 - n\lambda p_i (x_i - \mu) + p_i \gamma) = 0$$

(6)

$$n + \gamma = 0,$$

(7)

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since we know $\sum_{i=1}^{n} p_i(x_i - \mu) = 0$. Then substituting $\gamma = -n$ into equation 4 we have

$$\frac{1}{p_i} - n\lambda(x_i - \mu) - n = 0$$  \hspace{1cm} (8)

$$p_i = \frac{1}{n\lambda(x_i - \mu) + n}. \hspace{1cm} (9)$$

and so $p_i$ depends only on solving equation 4 for $\lambda$. We know that

$$\sum_{i=1}^{n} p_i(x_i - \mu) = 0$$  \hspace{1cm} (10)

$$\sum_{i=1}^{n} \frac{(x_i - \mu)}{n\lambda(x_i - \mu) + n} = 0; \hspace{1cm} (11)$$

and so we can solve for $\lambda$ for a given value of $\mu$ using a univariate root finding algorithm. Then using equation 9 we can find the $p_i$ and calculate the empirical likelihood in equation 1.

### 1.1 Euclidean likelihood

The Euclidean likelihood (Baggerly, 1998) defines the log likelihood as

$$\log f(x|\mu) = -\frac{1}{2} \sum_{i=1}^{n} (np_i - 1)^2,$$  \hspace{1cm} (12)

with the constraints $\sum_{i=1}^{n} p_i = 1$ and $\sum_{i=1}^{n} p_i x_i - \mu = 0$. Again we apply the method of Lagrange multipliers (Owen, 2001)

$$G = -\frac{1}{2} \sum_{i=1}^{n} (np_i - 1)^2 - n\lambda \sum_{i=1}^{n} p_i(x_i - \mu) + \gamma \left( \sum_{i=1}^{n} p_i - 1 \right),$$  \hspace{1cm} (13)

and setting the partial derivative of $G$ with respect to $p_i$ to zero we have

$$\frac{\partial G}{\partial p_i} = n(1 - np_i) - n\lambda(x_i - \mu) + \gamma = 0$$  \hspace{1cm} (14)

$$\frac{1}{n} \sum_{i=1}^{n} (n(1 - np_i) - n\lambda(x_i - \mu) + \gamma) = 0$$  \hspace{1cm} (15)

$$-n\lambda(\bar{x} - \mu) + \gamma = 0.$$  \hspace{1cm} (16)

Substituting $\gamma = n\lambda(\bar{x} - \mu)$ back into equation 14

$$n(1 - np_i) - n\lambda(x_i - \mu) + n\lambda(\bar{x} - \mu) = 0$$  \hspace{1cm} (17)

$$p_i = \frac{1}{n} \left( 1 - \lambda(x_i - \bar{x}) \right).$$  \hspace{1cm} (18)

Given that $\sum_{i=1}^{n} p_i(x_i - \mu) = 0$, we can substitute equation 18 to give
\[
\sum_{i=1}^{n} \frac{(x_i - \mu)}{n} \left(1 - \lambda(x_i - \bar{x})\right) = 0 \quad (19)
\]

\[
\bar{x} - \mu - \sum_{i=1}^{n} \frac{\lambda}{n} (x_i - \mu)(x_i - \bar{x}) = 0 \quad (20)
\]

\[
\bar{x} - \mu - \sum_{i=1}^{n} \frac{\lambda}{n} (x_i - \bar{x})(x_i - \bar{x}) = 0 \quad (21)
\]

\[
\bar{x} - \mu - \lambda s = 0, \quad (22)
\]

where \( s \) is defined as \( s = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x}) \). Substituting \( \lambda \) into equation 18 we have

\[
p_i = \frac{1}{n} \left(1 - \frac{1}{s} (\bar{x} - \mu)(x_i - \bar{x})\right), \quad (23)
\]

and substituting \( p_i \) into equation 12 we arrive at

\[
\log f(x|\mu) = -\sum_{i=1}^{n} \left(\frac{1}{s} (\bar{x} - \mu)(x_i - \bar{x})\right)^2 \quad (24)
\]

\[
= -\frac{1}{s^2} (\bar{x} - \mu)^2 \left(\sum_{i=1}^{n} (x_i - \bar{x})^2\right) \quad (25)
\]

\[
= -\frac{1}{s^2} n(\bar{x} - \mu)^2, \quad (26)
\]

References

