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Accepted Version

Tailleux, R. ORCID: https://orcid.org/0000-0001-8998-9107 (2018) Local available energetics of multicomponent compressible stratified fluids. Journal Of Fluid Mechanics, 842. ISSN 0022-1120 doi: 10.1017/jfm.2018.196 Available at https://centaur.reading.ac.uk/75735/

To link to this article DOI: http://dx.doi.org/10.1017/jfm.2018.196

Publisher: Cambridge University Press

Local available energetics of multicomponent compressible stratified fluids

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(Received xx; revised xx; accepted xx)

We extend the local theory of available potential energy (APE) to a general multicomponent compressible stratified fluid, accounting for the effects of diabatic sinks and sources. As for simple compressible fluids, the total potential energy density of a fluid parcel is the sum of its available elastic energy (AEE) and APE density. These respectively represent the adiabatic compression/expansion work needed to bring it from its reference pressure to its actual pressure and the work against buoyancy forces required to move it from its reference state position to its actual position. Our expression for the APE density is new and derived using only elementary manipulations of the equations of motion; it is significantly simpler than existing published expressions, while also being more transparently linked to the relevant form of APE density for the Boussinesq and hydrostatic primitive equations. Our new framework is used to clarify the links between some aspects of the energetics of Boussinesq and real fluids, as well as to shed light on the physical basis underlying the choice of reference state(s) in local APE theory.

1. Introduction

The concept of available potential energy (APE), first developed by Lorenz (1955) and Margules (1903), aims to quantify the part of the total potential energy (PE) of a stratified fluid that is available for (reversible) conversions with kinetic energy (KE). In Lorenz's view, any realisable state of a stratified fluid is viewed as an adiabatic re-arrangement of some notional rest state, whose background potential energy PE_r is 'inert' or 'unavailable' in some sense, thus motivating the partition $PE = APE + PE_r$. From a fundamental viewpoint, the concept of APE provides a natural solution to a number of outstanding problems plaguing the standard PE. Indeed, in contrast to the latter: 1) APE is independent of the arbitrary reference level defining geopotential height and of the unknowable constants generally entering the construction of internal energy; 2) APE is naturally positive definite and at least quadratic in perturbations, thus allowing it to be meaningfully partitioned into mean and eddy components; 3) APE does not suffer from the 'cooling' paradox, which occurs when a given forcing acts as a sink of potential energy (such as high-latitude cooling in the oceans), but nevertheless results in the creation of kinetic energy owing to such a forcing being a source of APE; 4) the APE budget naturally yields a prediction for the thermodynamic efficiency of the system considered that appears to be physically much more meaningful than that predicted by the entropy budget, as discussed by Tailleux (2010). This is particularly important for understanding how to define and quantify the power input due to surface buoyancy fluxes in the ocean for instance. It is no surprise, therefore, that Lorenz APE theory has been and still primarily remains the main tool for discussing the energy cycle of the atmosphere and oceans, e.g., Peixoto & Oort (1992)

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A longstanding difficulty with Lorenz APE theory, however, is that it is only global in nature. This is an important drawback complicating its use in local or regional studies of energetics, which has historically prompted much research effort into identifying a suitable local extension, the main attempts being reviewed in Tailleux (2013*a*). The most successful attempts, and most clearly related to Lorenz APE theory, are the local frameworks proposed by Andrews (1981) for a simple compressible stratified fluid and by Holliday & McIntyre (1981) for a Boussinesq fluid with a linear equation of state; these attempts, along with other related older formulations, were subsequently unified within the context of Hamiltonian theory by Shepherd (1993). Yet, despite having been established over 35 years ago, it is only relatively recently that local theories of APE have started to receive attention in the context of stratified turbulence (Roullet & Klein 2009; Molemaker & McWilliams 2010; Scotti & White 2014; Winters & Barkan 2013), ocean energetics (Scotti *et al.* 2006; Tailleux 2013*b*; Roullet *et al.* 2014; Saenz *et al.* 2015; Zemskova *et al.* 2015; MacCready & Giddings 2016), and atmospheric energetics (Kucharski 1997; Kucharski & Thorpe 2000; Peng *et al.* 2015; Novak & Tailleux 2018).

Despite all recent advances and increased interest, however, a satisfactory generalisation of Andrews (1981)'s local APE theory to a multicomponent compressible stratified fluid accounting for diabatic sources and sinks has remained out of reach. Yet, such a generalisation is essential for the development of any rigorous theoretical understanding of the role played by salinity or humidity, as well as of turbulent mixing, in the oceanic and atmospheric energy cycles. Bannon (2003) attempted such a generalisation by means of Shepherd (1993)'s Hamiltonian framework, but his theory only pertains to an ideal fluid whose constituents are all independent of horizontal position in the reference state, thus excluding the possibility of density compensation (Tailleux *et al.* 2005), a key aspect of real fluids. Recently, Peng *et al.* (2015) have proposed a local APE theory for a moist atmosphere, but it is not exact and limited to the anelastic approximation. How to achieve such a generalisation by means of elementary manipulation of the full Navier-Stokes equations is the main aim of the present paper, and is derived in Section 2. Section 3 discusses the effects of diabatic sources and sinks and the links between the energy conversions taking place in real and Boussinesq fluids. Section 4 reviews and discusses the physical issues underlying the construction of the reference state(s) in local APE theories. Section 5 summarises and discusses the results.

2. Available energetics of a stratified compressible binary fluid

We start by revisiting Andrews (1981)'s construction of potential energy (PE) density so as to generalise it easily to multi-component compressible stratified fluids. As in Andrews (1981) or Bannon (2003), we define the PE density of a stratified fluid as the sum of its Available Elastic Energy (AEE) density and APE density. As showed below, this can be achieved without loss of generality by restricting attention to the case of a binary fluid, that is, one whose equation of state for density depends on composition S (referred to as 'salinity' hereafter) in addition to specific entropy η and pressure p, all quantities being functions of spatial position (x, y, z) and time t. A full set of governing equations in Earth rotating frame is:

$$\rho \frac{D\mathbf{v}}{Dt} + 2\Omega \times \rho \mathbf{v} + \nabla p = -\rho \nabla \Phi + \rho \mathbf{F}, \tag{2.1}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{\upsilon} \frac{D\upsilon}{Dt}, \tag{2.2}$$

$$\frac{D(\eta, S)}{Dt} = (\dot{\eta}, \dot{S}), \tag{2.3}$$

$$\upsilon = \upsilon(\eta, S, p). \tag{2.4}$$

Here, $\mathbf{v} = (u, v, w)$ is the velocity field, $\varOmega$ is Earth rotation vector, $\varPhi(z) = g_0 z$ is the geopotential (g_0 is gravitational acceleration), z is height increasing positively upwards, $\upsilon = 1/\rho$ is the specific volume, ρ is density, $\mathbf{F}$ is frictional force, while $\dot{\eta}$ and $\dot{S}$ represent diabatic sources of entropy and salt.

We first address the case of an ideal fluid, thus setting $\mathbf{F} = \dot{\eta} = \dot{S} = 0$ in (2.1-2.4). In this case, a standard expression for the total energy budget is:

$$\rho \frac{D}{Dt}\left(\frac{\mathbf{v}^2}{2} + \varPhi + e\right) + \nabla \cdot (p\mathbf{v}) = 0, \tag{2.5}$$

where $e = e(\eta, S, p) = h(\eta, S, p) - p\upsilon$ is the specific internal energy and h the specific enthalpy. In order to obtain an expression for the PE density of the fluid considered, one first needs to identify suitably defined reference specific volume $\upsilon_0(z)$ and pressure $p_0(z)$ profiles functions of z only and in mechanical (hydrostatic) equilibrium:

$$\upsilon_0(z)\frac{\partial p_0}{\partial z}(z) = -\frac{\partial \varPhi}{\partial z}(z). \tag{2.6}$$

A discussion of the physical issues pertaining to the construction of the reference state is deferred to Section 4. The next key step is to assign to each fluid parcel a reference position $z_r(\eta, S)$ in that reference state. This can be done most generally by regarding z_r as the particular depth at which the specific volume of the fluid parcel moved adiabatically to that level matches that of the reference state. Mathematically, this is equivalent to defining z_r as a solution of the so-called Level of Neutral Buoyancy (LNB) equation

$$\upsilon(\eta, S, p_r) = \upsilon_0(z_r), \tag{2.7}$$

similarly as in Tailleux (2013*b*), where $p_r(\eta, S) = p_0(z_r(\eta, S))$. Depending on how the reference specific volume $\upsilon_0(z)$ is defined, Eq. (2.7) may have no solution, one solution, or even multiple solutions. In the former case, z_r should be defined either at the top or bottom of the domain depending on the buoyancy of the fluid parcel in the actual state (Tailleux 2013*b*). The latter case is rarely encountered in the ocean (Saenz *et al.* 2015) but is commonplace in a moist atmosphere (Wong *et al.* 2016; Harris & Tailleux 2018). How to select the solution in this case is still poorly understood, however.

Following Andrews (1981), we next use the following identity

$$\nabla \cdot (p_0\mathbf{v}) = \mathbf{v}\cdot\nabla p_0 - \frac{p_0}{\rho}\frac{D\rho}{Dt} = \rho\frac{D}{Dt}\left(\frac{p_0}{\rho}\right), \tag{2.8}$$

to rewrite the standard energy budget (2.5) in terms of the perturbation pressure $p - p_0$ as follows:

$$\rho\frac{D}{Dt}\left(\frac{\mathbf{v}^2}{2} + \mathcal{B}\right) + \nabla\cdot[(p-p_0)\mathbf{v}] = 0, \tag{2.9}$$

where the quantity

$$\mathcal{B}(z, S, \eta, p) = \varPhi(z) + e(\eta, S, p) + \frac{p_0(z)}{\rho(\eta, S, p)} \tag{2.10}$$

is a hybrid function of thermodynamic coordinates and height, reminiscent of the non-kinetic energy part of the classical Bernoulli head. The introduction of $\mathcal{B}$ proves determinant, for it turns out that the quantity

$$\varPi = \mathcal{B} - \mathcal{B}(z_r, S, \eta, p_r) = \varPhi(z) - \varPhi(z_r) + e(\eta, S, p) - e(\eta, S, p_r) + \frac{p_0(z)}{\rho} - \frac{p_r}{\rho_r}, \tag{2.11}$$

obtained as the difference between $\mathcal{B}$ and its value in the reference state naturally happens to be positive definite and our sought-for PE density. The quantity $\Pi + \mathbf{v}^2/2$ is then Shepherd (1993)'s 'pseudo-energy', and obeys the following conservation law:

$$\rho \frac{D}{Dt}\left(\frac{\mathbf{v}^2}{2} + \Pi\right) + \nabla \cdot [(p - p_0)\mathbf{v}] = 0. \tag{2.12}$$

To prove its positive definite character, it is useful to rewrite Π in terms of the enthalpies $h(\eta, S, p) = e(\eta, S, p) + p/\rho$ and $h(\eta, S, p_r) = e(\eta, S, p_r) + p_r/\rho_r$ as follows:

$$\Pi = \Phi(z) - \Phi(z_r) + h(\eta, S, p) - h(\eta, S, p_r) + \frac{p_0(z) - p}{\rho}, \tag{2.13}$$

as this naturally yields the decomposition $\Pi = \Pi_1 + \Pi_2$ where

$$\Pi_1 = h(\eta, S, p) - h(\eta, S, p_0(z)) + \frac{p_0(z) - p}{\rho}, \tag{2.14}$$

$$\Pi_2 = \Phi(z) - \Phi(z_r) - h(\eta, S, p_r) + h(\eta, S, p_0(z)). \tag{2.15}$$

The quantity Π_1 is the so-called Available Elastic Energy (AEE) density previously derived by Andrews (1981) or Bannon (2003), and may be rewritten as follows:

$$\Pi_1 = \int_{p_0(z)}^{p} [\upsilon(\eta, S, p') - \upsilon(\eta, S, p)]\,\mathrm{d}p' = \int_{p_0(z)}^{p}\int_{p'}^{p} \frac{1}{\rho^2 c_s^2}(\eta, S, p'')\,\mathrm{d}p''\mathrm{d}p', \tag{2.16}$$

where the identities $\mathrm{d}h = T\mathrm{d}\eta + \mu \mathrm{d}S + \upsilon \mathrm{d}p$ and $\upsilon_p = -(\rho^2 c_s^2)^{-1}$ have been used. The result $\Pi_1 \geqslant 0$ follows from the ρ and c_s being both strictly positive quantities. For small pressure perturbation $p' = p - p_0$, Π_1 reduces to the well known quadratic approximation:

$$\Pi_1 \approx \frac{(p - p_0(z))^2}{2\rho^2 c_s^2} \tag{2.17}$$

as noted by Andrews (1981) and Shepherd (1993) among others. Physically, Π_1 represents the work required to bring the reference pressure $p_0(z)$ of a fluid parcel to its actual pressure p by means of an adiabatic compression ($p > p_0$) or expansion ($p < p_0$). Andrews (1981) argues that such a term vanishes in the incompressible limit $c_s \to +\infty$, thus justifying its absence in Boussinesq fluids.

The quantity Π_2 is the APE density of the fluid and the only part of the PE density generally retained in incompressible Boussinesq fluids. Physically, it is equivalent to Andrews (1981)'s Π_2, but mathematically, Eq. (2.15) is actually much simpler. Moreover, in contrast to Andrews (1981) or Bannon (2003)'s constructions, it does not require the introduction of purely depth-dependent reference profiles for specific entropy and salinity, which is overly restrictive in real fluids. In order to clarify the link between the APE density of a compressible fluid with that of the Boussinesq or hydrostatic primitive equations, we make use of (2.6) and of the change of variables $p' = p_0(z')$, $\mathrm{d}p' = -\rho_0(z')g_0\mathrm{d}z'$, to rewrite Π_2 in the following mathematically equivalent ways:

$$\Pi_2 = -\int_{z_r}^{z} \upsilon_0(z')\frac{dp_0}{dz}(z')\,\mathrm{d}z' + \int_{p_0(z_r)}^{p_0(z)} \upsilon(\eta, S, p')\,\mathrm{d}p'$$

$$= \int_{p_0(z_r)}^{p_0(z)} [\upsilon(\eta, S, p') - \hat{\upsilon}_0(p')]\,\mathrm{d}p' = g_0 \int_{z_r}^{z} \frac{[\rho(\eta, S, p_0(z')) - \rho_0(z')]}{\rho(\eta, S, p_0(z'))}\,\mathrm{d}z', \tag{2.18}$$

where $\hat{\upsilon}_0(p) = \upsilon_0(Z_0(p))$, $Z_0(p)$ being the inverse function of $p_0(z)$ satisfying $p_0(Z_0(p)) = p$. Eq. (2.18) are classical expressions for the work against buoyancy forces required to move a fluid parcel from its reference position at pressure $p_r = p_0(z_r)$ to its actual position at pressure $p_0(z)$ by means of an adiabatic and isohaline process. It is easily seen that the APE density for a Boussinesq fluid derived by Holliday & McIntyre (1981) or Tailleux (2013*c*) can be recovered: 1) by replacing the denominator $\rho(\eta, S, p)$ in the last term of (2.18) by the constant reference Boussinesq density ρ_{00}; 2) by replacing everywhere the reference pressure $p_0(z)$ by the Boussinesq pressure $p_{00}(z) = -\rho_{00} g_0 z$. As to the APE density for a hydrostatic dry atmosphere discussed by Novak & Tailleux (2018), it is simply recovered from the first term in (2.18) by replacing the upper bound p_0 by the hydrostatic pressure p itself. As a result, Tailleux (2013*b*)'s arguments may be used to prove the positive definite character of Π, as well as its small amplitude approximation $\Pi_2 \approx N_r^2 (z - z_r)^2/2$, with N_r^2 given by

$$N_r^2 = -\frac{g_0}{\rho_r}\left[\frac{d\rho_0}{dz}(z_r) + \frac{\rho_r g_0}{c_s^2(S, \eta, p_r)}\right]. \tag{2.19}$$

Note here that in a binary or multi-component fluid, the possibility of density compensation means that only the reference density and pressure profiles may be assumed to be functions of z alone; all other quantities, including N_r^2, must in general depend on horizontal position as well. To conclude, let us remark that Π is 'local' only in the sense of being definable at any location, since it possesses some degree of non-locality due to being defined relative to a 'globally-defined' reference state. For this reason, Π is best viewed as a sophisticated form of density variance, which — like other standard statistical quantities such as 'anomalies' — is also nonlocal in some sense.

3. Energy conversions in presence of diabatic sinks and sources

We now turn to the issue of how diabatic sinks and sources of (η, S) affect the evolution of the PE density. To that end, we re-introduce the diabatic terms $(\dot{\eta}, \dot{S})$ in the equations for (η, S). Because the reference state may be altered by diabatic effects, we assume from now on that the specific volume and pressure reference profiles $\upsilon_0(z,t)$ and $p_0(z,t)$ also depend on time (Winters *et al.* 1995). The LNB equation (2.7) thus becomes $\upsilon[\eta, S, p_0(z_r, t)] = \upsilon_0(z_r, t)$ and now defines $z_r = z_r(\eta, S, t)$ as a time-dependent material function of (η, S). Evolution equations for Π_1 and Π_2 are obtained by taking the material derivatives of (2.14) and (2.15). Adding an evolution equation for the kinetic energy $E_k = \mathbf{v}^2/2$ (obtained in the usual way) yields the following description of energetics:

$$\rho\frac{DE_k}{Dt} + \nabla\cdot[(p - p_0)\mathbf{v}] = -\nabla\cdot(p_0\mathbf{v}) + \frac{p}{\upsilon}\frac{D\upsilon}{Dt} - \rho\frac{D\Phi}{Dt} + \rho\mathbf{F}\cdot\mathbf{v}, \tag{3.1}$$

$$\rho\frac{D\Pi_1}{Dt} = \nabla\cdot(p_0\mathbf{v}) - \frac{p}{\upsilon}\frac{D\upsilon}{Dt} + \rho\dot{\Pi}_1 + \left(1 - \frac{\rho}{\rho_h}\right)\frac{\partial p_0}{\partial t}(z,t) + \frac{\rho_0}{\rho_h}\rho\frac{D\Phi}{Dt}, \tag{3.2}$$

$$\rho\frac{D\Pi_2}{Dt} = \rho\frac{D\Phi}{Dt} + \rho\dot{\Pi}_2 - \frac{\rho_0}{\rho_h}\rho\frac{D\Phi}{Dt} + \frac{\rho}{\rho_h}\frac{\partial p_0}{\partial t}(z,t) - \frac{\rho}{\rho_r}\frac{\partial p_0}{\partial t}(z_r,t). \tag{3.3}$$

In (3.2) and (3.3), the local diabatic production terms $\dot{\Pi}_1$ and $\dot{\Pi}_2$ are defined by

$$\dot{\Pi}_1 = (T - T_h)\dot{\eta} + (\mu - \mu_h)\dot{S}, \qquad \dot{\Pi}_2 = (T_h - T_r)\dot{\eta} + (\mu_h - \mu_r)\dot{S}, \tag{3.4}$$

where T and μ denote in-situ temperature and relative chemical potential respectively. Moreover, the suffix 'h' indicates a variable estimated at the pressure $p_0(z,t)$, e.g., $T_h = T(\eta, S, p_0(z,t))$, while the suffix 'r' indicates a variable estimated at the pressure $p_r = p_0(z_r,t)$, e.g., $T_r = T(\eta, S, p_r) = T(\eta, S, p_0(z_r,t))$. Adding (3.2) and (3.3) yields the

Energy Conversion	'Standard' view	'Boussinesq' view
$C(\Pi_1, E_k)$	$\frac{p}{\upsilon}\frac{D\upsilon}{Dt} - \nabla \cdot (p_0\mathbf{v})$	$\frac{p}{\upsilon}\frac{D\upsilon}{Dt} - \rho\frac{D\Phi}{Dt} - \nabla \cdot (p_0\mathbf{v})$
$C(E_k, \Pi_2)$	$\rho\frac{D\Phi}{Dt}$	0
$C(\Pi_2, \Pi_1)$	$\frac{\rho_0}{\rho_h}\rho\frac{D\Phi}{Dt}$	$\left(\frac{\rho_0}{\rho_h} - 1\right)\rho\frac{D\Phi}{Dt}$

TABLE 1. The two different possible approaches to defining the energy conversions between E_k, Π_1 and Π_2 in a turbulent multi-component compressible stratified fluid discussed in this paper.

following equation for Π

$$\rho\frac{D\Pi}{Dt} = \underbrace{\nabla \cdot (p_0\mathbf{v}) + \rho\frac{D\Phi}{Dt} - \frac{p}{\upsilon}\frac{D\upsilon}{Dt}}_{C(E_k,\Pi)} + \rho\dot{\Pi} + \frac{\partial p_0}{\partial t}(z,t) - \frac{\rho}{\rho_r}\frac{\partial p_0}{\partial t}(z_r,t), \tag{3.5}$$

where the local diabatic production/destruction term $\dot{\Pi}$ is defined by:

$$\dot{\Pi} = (T - T_r)\dot{\eta} + (\mu - \mu_r)\dot{S}. \tag{3.6}$$

Note that because the diabatic terms $\dot{S}$ and $\dot{\eta}$ couple Π to the background potential energy energy $\mathcal{B}_r = \Phi(z_r) + e(\eta, S, p_r) + p_r/\rho_r$, the pseudo-energy $E_k + \Pi$ is no longer a conservative quantity.

We now discuss the nature of the energy conversions between E_k, Π_1 and Π_2 implied by (3.1-3.3) in order to understand its link with Boussinesq energetics. As is well known, such an exercise is prone to conceptual difficulties, because there is no universally agreed way to define energy conversions, which can often be defined in several plausible mathematically equivalent ways. In the present case, for instance, there appears to be two natural such ways — synthesised in Table 1 — which we refer to as the 'standard' and 'Boussinesq' conversions. The standard conversions are those that naturally follow from the energy budget written as (3.1-3.3), and consists in regarding the work of compression/expansion as a conversion between KE and AEE and the term $\rho g_0 w$ as a conversion between KE and APE. In this standard view, 'large' conversions are permitted to occur between all the different energy reservoirs. The Boussinesq view, on the other hand, exploits the tendency for the standard energy conversions to compensate each other, thus suggesting to recombine the terms of (3.2) and (3.3) as follows:

$$\rho\frac{D\Pi_1}{Dt} = \underbrace{\nabla \cdot (p_0\mathbf{v}) + \rho\frac{D\Phi}{Dt} - \frac{p}{\upsilon}\frac{D\upsilon}{Dt}}_{C(E_k,\Pi_1)} + \rho\dot{\Pi}_1 - \underbrace{\left(1 - \frac{\rho_0}{\rho_h}\right)\rho\frac{D\Phi}{Dt}}_{C(\Pi_1,\Pi_2)} + \text{N.L}, \tag{3.7}$$

$$\rho\frac{D\Pi_2}{Dt} = \underbrace{\left(1 - \frac{\rho_0}{\rho_h}\right)\rho\frac{D\Phi}{Dt}}_{C(\Pi_1,\Pi_2)} + \rho\dot{\Pi}_2 + \text{N.L.}, \tag{3.8}$$

where the acronym *N.L.* denotes the non-local terms proportional to the various $\partial p_0/\partial t$ partial derivatives. The Boussinesq re-organisation has the key advantage of transforming $C(\Pi_1, \Pi_2)$ into a density flux *anomaly*, without actually introducing any form of average. This makes it arguably more relevant/useful to the study of turbulent stratified fluids than the standard view. Physically, (3.7-3.8) no longer allow for any direct conversion between kinetic energy and the APE density Π_2, so that conversion between KE and Π_2 appears to be indirect and mediated via Π_1. As one referee put it, Π_1 becomes a 'pass-through' energy reservoir. The usefulness of this property becomes evident in the incompressible limit $c_s \to +\infty$. Indeed, in this case, Π_1 vanishes but the energy conversions in the right-hand side of (3.7) do not. As a consequence, (3.7) becomes a diagnostic equation imposing the following balance to hold approximately at all times:

$$\underbrace{\nabla \cdot (p_0 \mathbf{v}) + \rho \frac{D\Phi}{Dt} - \frac{p}{\upsilon}\frac{D\upsilon}{Dt}}_{C(E_k,\Pi_1)} + \rho \dot{\Pi}_1 - \underbrace{\left(1 - \frac{\rho_0}{\rho_h}\right) \rho \frac{D\Phi}{Dt}}_{C(\Pi_1,\Pi_2)} + \text{N.L} \approx 0. \tag{3.9}$$

If $\rho\dot{\Pi}_1$ and *N.L.* can be neglected in (3.9), the above balance reduces to $C(\Pi_1,\Pi_2) \approx C(E_k,\Pi_1)$, which formally allows one to regard the density flux anomaly $C(\Pi_1,\Pi_2)$ as a conversion between KE and Π_2, precisely as in Boussinesq energetics. Any kinetic energy converted into Π_1 is instantaneously converted into Π_2 and conversely. To establish that such a view is valid and physically meaningful, one presumably would need to prove that such conversions are primarily achieved by sound waves, as these are the only ones capable of transmitting information at infinite speed in the limit $c_s \to +\infty$, but this is a challenging technical issue that we regard to be beyond the scope of this paper.

4. Issues pertaining to the choice of reference state

How to define and construct the APE reference state has been a continuing source of confusion and difficulty in APE theory since its inception. To date, the issue remains controversial and outstanding. Here, we briefly review and discuss two well known key difficulties in subsections 4.1 and 4.2 below. The first one pertains to the possible use of 'non-resting' reference states. The second one pertains to the apparent arbitrariness of the reference state in local APE theory and whether it implies the possibility to use reference states other than those originally proposed by Lorenz (1955).

4.1. *Resting versus non-resting reference states*

The standard form of APE is commonly understood to originate from horizontal variations in the density field, and therefore naturally defined relative to a 'resting' reference state function of z only (asides possible temporal dependence). A common objection, however, is that in presence of rotation, idealised fluid flow configurations such as steady and stable zonal flows or vortices may have a large fraction of their APE actually locked up by geostrophic or higher order type of balance and therefore seemingly unavailable. Such examples have prompted a few authors to look for a generalisation of APE accounting for momentum constraints, e.g., Codoban & Shepherd (2003); Andrews (2006). Whether such an approach is necessarily the 'right' one is not clear, however. Indeed, the alternative and much simpler approach advocated here considers that the definition of APE should not depend on the presence of rotation and flow stability, and that the potential energy density Π_e defined relative to a non-resting reference state should be regarded as a form of 'eddy' potential energy in a mean/eddy decomposition $\Pi = \Pi_e + \Pi_m + \delta\Pi$ of the total potential energy density Π. Such an approach is easily implemented in the present framework by noting that the construction of Π outlined in Section 2 applies equally well to reference pressure and specific volume $p_m(x,y,z)$ and $\upsilon_m(x,y,z)$ that also depend on horizontal position. As a result, the above components of the partition of Π can be explicitly defined by

$$\Pi_e = \Phi(z) - \Phi(z_{rm}) + e(\eta,S,p) - e(\eta,S,p_{rm}) + \frac{p_m(x,y,z)}{\rho} - \frac{p_{rm}}{\rho_{rm}}, \tag{4.1}$$

$$\Pi_m = \Phi(z_{rm}) - \Phi(z_r) + e(\eta, S, p_{rm}) - e(\eta, S, p_r) + \frac{p_0(z_{rm})}{\rho_{rm}} - \frac{p_r}{\rho_r}, \tag{4.2}$$

$$\delta\Pi = \frac{p_0(z) - p_m(x,y,z)}{\rho} + \frac{p_m(x,y,z_{rm}) - p_0(z_{rm})}{\rho_{rm}}. \tag{4.3}$$

In Eqs. (4.1-4.3), $p_r = p_r(\eta, S)$, $\rho_r = \rho_r(\eta, S)$ and $z_r = z_r(\eta, S)$ have the same meaning as in Section 2 and are functions of the materially conserved variables (η, S) only, whereas z_{rm} is the reference depth solution of the LNB equation for the non-resting state $\upsilon(\eta, S, p(x, y, z_{rm})) = \upsilon_m(x, y, z_{rm})$, with $p_{rm} = p_m(x, y, z_{rm})$. Physically, Π_e represents the 'eddy' potential energy density of a fluid parcel relative to the 'non-resting' mean state, whereas Π_m represents the potential energy density of the fluid parcel in its mean equilibrium position relative to its 'resting' equilibrium position. Note that the above mean/eddy partition of Π does not explicitly depend on the introduction of any particular averaging operator, and is valid for arbitrarily defined 'non-resting' and 'resting' reference state (p_m, υ_m) and (p_0, υ_0); from a mathematical viewpoint, it is the counterpart of the mean/eddy partitioning of the kinetic energy as $\frac{\mathbf{v}^2}{2} = \frac{\mathbf{v}_m^2}{2} + \frac{\mathbf{v}_e^2}{2} + \mathbf{v}_m \cdot \mathbf{v}_e$, which also does not require the introduction of any particular averaging operator. The interaction term $\delta\Pi$ can a priori takes on both signs, and is the counterpart of the interaction $\mathbf{v}_m \cdot \mathbf{v}_e$.

Physically, the expectation is that if (p_m, υ_m) correspond to a steady and stable zonal flow or vortex, rotation should inhibit conversions between Π_m and Π_e. To confirm this, let use derive the evolution equations for $\mathbf{v}^2/2 + \Pi_e$, Π_m and $\delta\Pi$ in the usual manner:

$$\rho\frac{D}{Dt}\left(\frac{\mathbf{v}^2}{2} + \Pi_e\right) + \nabla\cdot\{[p - p_m(x,y,z)]\mathbf{v}\} = -\frac{\rho\mathbf{u}\cdot\nabla_h p_{rm}}{\rho_{rm}}, \tag{4.4}$$

$$\rho\frac{D\Pi_m}{Dt} + \nabla\cdot\left\{\frac{\rho[p_{rm} - p_0(z_{rm})]\mathbf{v}}{\rho_{rm}}\right\} = \frac{\rho\mathbf{u}\cdot\nabla_h p_{rm}}{\rho_{rm}}, \tag{4.5}$$

$$\rho\frac{D(\delta\Pi)}{Dt} = \nabla\cdot\left\{\left[p_0(z) - p_m(x,y,z) + \frac{\rho(p_{rm} - p_0(z_{rm}))}{\rho_{rm}}\right]\mathbf{v}\right\}, \tag{4.6}$$

where ∇_h denotes the horizontal part of the gradient operator. It is easily verified that summing (4.4-4.6) recovers (2.5). Eqs. (4.4-4.5) show that the conversion between Π_m and Π_e is controlled by the product of the horizontal velocity times the horizontal gradient of p_{rm}, which vanishes for a steady geostrophically balance state $\mathbf{u} \propto f_0^{-1}\mathbf{k}\times\nabla_h p_{rm}$, where f_0 denotes the local rotation rate, and $\mathbf{k}$ the unit vector pointing upwards. Therefore, as far as we can judge, the inhibiting effects of rotation can be understood in the context of the mean/eddy decomposition of Π (see Scotti & White (2014) and Novak & Tailleux (2018) for different ways of performing such a decomposition), and does not therefore seem to require a definition of APE accounting for momentum constraints. Obviously, this is a subtle and complicated issue that will warrant extensive discussion in a separate and forthcoming dedicated paper.

4.2. *On the arbitrariness of the reference state*

The apparent arbitrariness of the reference state in local APE theory, first noted by Andrews (1981) and Shepherd (1993), has been exploited by a few authors to use reference states easier or more convenient to compute than Lorenz reference state, such as the horizontally-averaged density field (Tailleux 2013*b*) or an analytic profile (Peng *et al.* 2015). Because the choice of reference state affects the magnitude of the APE reservoir, as well as the rates at which it is created, stored and destroyed (Wong *et al.* 2016), the present state of affairs is obviously unsatisfactory and will remain so until the physical basis for how to specify it is identified. The key role played by $\mathcal{B}$ in the construction of Π suggests that the reference pressure $p_0(z)$ or $p_0(z,t)$ should be chosen so as to approximate the actual pressure field $p(x,y,z,t)$ as accurately as feasible. If least square were the relevant principle, one would have a physical basis for arguing that the reference density field $\rho_0(z,t)$ should be defined as the horizontally averaged density $\overline{\rho}(z,t)$ rather than in terms of Lorenz reference state. Such an approach would also ensure that Π vanishes identically in a state of rest, a desirable property of Π that is not necessarily satisfied with Peng *et al.* (2015)'s analytically specified reference state. Clarifying this issue once and for all therefore remains a key priority for future research.

5. Conclusions

In this paper, we succeeded in extending Andrews (1981)'s local concept of potential energy density Π to a multi-component compressible stratified fluid and in deriving evolution equations for the attendant energy cycle accounting for diabatic sinks and sources. In contrast to the previous approaches by Bannon (2003) and Andrews (1981), our construction of APE density stands out by its simplicity and economy, as well as by its considerably more transparent link to the APE density of the Boussinesq and hydrostatic primitive equations. The key step making our approach so simple is the introduction of the quantity $\mathcal{B}(z,\eta,S,p) = \Phi(z) + e(\eta,S,p) + p_0(z)\upsilon(\eta,S,p)$, which is just the standard potential energy plus the term p_0/ρ. Indeed, all one has to do is to subtract the reference value of $\mathcal{B}$ to obtain $\Pi = \mathcal{B}(z,\eta,S,p) - \mathcal{B}(z_r,\eta,S,p_0(z_r))$, a quantity that is naturally positive definite, in contrast to the integrand of Lorenz (1955) globally defined APE. Such an approach can be done for any arbitrary resting or non-resting reference state, which points to new ways to rigorously partition Π into mean and eddy components or to think about the role played by rotation on occasionally inhibiting a large fraction of the total APE. Nevertheless, the exact physical principle(s) underlying the specification of the reference state(s) remain an outstanding issue in need of further research.

As in Andrews (1981) and Bannon (2003), the total potential energy density of a fluid parcel is the sum of its available elastic energy (AEE) and APE density, of which only the latter subsists in sound-proof systems of equations such as the Boussinesq or hydrostatic primitive equations. This can be formally justified by noting that AEE vanishes in the limit $c_s \to +\infty$, as pointed out by Andrews (1981). However, although AEE vanishes in the incompressible limit, this is not in general the case for the individual terms entering its evolution equation. As a result, the AEE evolution equation becomes a diagnostic equation in the limit $c_s \to +\infty$, which turns out to be key for establishing a formal link between the energetics of Boussinesq and real fluids.

The present results are important, for they provide for the time the tools needed to tackle many outstanding issues in the field, ranging from elucidating the role of salt and humidity in the oceanic and atmospheric energy cycles, to the development of new energetically and thermodynamically consistent mixing parameterisations for use in numerical oceanic and atmospheric models, as we hope to demonstrate in future studies.

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