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Accepted Version

Perfekt, K.-M. (2019) On the spaces of bounded and compact multiplicative Hankel operators. *New York Journal of Mathematics*, 25. pp. 589-602. Available at <http://centaur.reading.ac.uk/83830/>

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# On the spaces of bounded and compact multiplicative Hankel operators

Karl-Mikael Perfekt

ABSTRACT. A multiplicative Hankel operator is an operator with matrix representation  $M(\alpha) = \{\alpha(nm)\}_{n,m=1}^{\infty}$ , where  $\alpha$  is the generating sequence of  $M(\alpha)$ . Let  $\mathcal{M}$  and  $\mathcal{M}_0$  denote the spaces of bounded and compact multiplicative Hankel operators, respectively. In this note it is shown that the distance from an operator  $M(\alpha) \in \mathcal{M}$  to the compact operators is minimized by a nonunique compact multiplicative Hankel operator  $N(\beta) \in \mathcal{M}_0$ . Intimately connected with this result, it is then proven that the bidual of  $\mathcal{M}_0$  is isometrically isomorphic to  $\mathcal{M}$ ,  $\mathcal{M}_0^{**} \simeq \mathcal{M}$ . It follows that  $\mathcal{M}_0$  is an M-ideal in  $\mathcal{M}$ . The dual space  $\mathcal{M}_0^*$  is isometrically isomorphic to a projective tensor product with respect to Dirichlet convolution. The stated results are also valid for small Hankel operators on the Hardy space  $H^2(\mathbb{D}^d)$  of a finite polydisk.

## CONTENTS

1. Introduction	1
2. Results	5
References	12

## 1. Introduction

Given a sequence  $\alpha: \mathbb{N} \rightarrow \mathbb{C}$ , we consider the corresponding multiplicative Hankel operator  $m = M(\alpha): \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ , defined by

$$\langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})} = \sum_{n,m=1}^{\infty} a(n)\overline{b(m)}\alpha(nm), \quad a, b \in \ell^2(\mathbb{N}).$$

Initially, we consider this equality only for finite sequences  $a$  and  $b$ . It defines a bounded operator  $M(\alpha): \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ , with matrix representation  $\{\alpha(nm)\}_{n,m=1}^{\infty}$  in the standard basis of  $\ell^2(\mathbb{N})$ , if and only if there is a constant  $C > 0$  such that

$$|\langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})}| \leq C\|a\|_{\ell^2(\mathbb{N})}\|b\|_{\ell^2(\mathbb{N})}, \quad a, b \text{ finite sequences.}$$

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2010 *Mathematics Subject Classification.* 46B28, 47B35.

*Key words and phrases.* essential norm, Hankel operator, bidual, M-ideal, weak product space.

Multiplicative Hankel operators are also known as Helson matrices, having been introduced by Helson in [14, 15].

There are two common alternative interpretations. One is in terms of Dirichlet series. Let  $\mathcal{H}^2$  be the Hardy space of Dirichlet series, the Hilbert space with  $(n^{-s})_{n=1}^{\infty}$  as a basis. Elements  $f \in \mathcal{H}^2$  are holomorphic functions in the half-plane  $\{s \in \mathbb{C} : \operatorname{Re} s > 1/2\}$ . If

$$f(s) = \sum_{n=1}^{\infty} a(n)n^{-s}, \quad g(s) = \sum_{n=1}^{\infty} \overline{b(n)}n^{-s}, \quad \rho(s) = \sum_{n=1}^{\infty} \overline{\alpha(n)}n^{-s},$$

then

$$\langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})} = \langle fg, \rho \rangle_{\mathcal{H}^2}.$$

Hence there is an isometric correspondence between Helson matrices and Hankel operators on  $\mathcal{H}^2$ , since the forms associated with the latter are precisely of the type  $(f, g) \mapsto \langle fg, \rho \rangle_{\mathcal{H}^2}$ .

The second interpretation is in terms of the Hardy space of the infinite polytorus  $H^2(\mathbb{T}^{\infty})$ , the Hilbert space with basis  $(z^{\kappa})_{\kappa}$ , where  $z = (z_1, z_2, \dots)$ , and  $\kappa = (\kappa_1, \kappa_2, \dots)$  runs through the countably infinite, but finitely supported, multi-indices. Identify each integer  $n$  with a multi-index  $\kappa$  of this type through the factorization of  $n$  into the primes  $p_1, p_2, \dots$ ,

$$n \longleftrightarrow \kappa \text{ if and only if } n = \prod_{j=1}^{\infty} p_j^{\kappa_j}.$$

Under this equivalence, multiplicative Hankel operators correspond to additive Hankel operators on a countably infinite number of variables,

$$\langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})} = \sum_{\kappa, \kappa'} a(\kappa) \overline{b(\kappa')} \alpha(\kappa + \kappa').$$

Hence the multiplicative Hankel operators correspond isometrically to small Hankel operators on  $H^2(\mathbb{T}^{\infty})$ , since the matrix representations of the latter are of the form  $\{\alpha(\kappa + \kappa')\}_{\kappa, \kappa'}$ . See [14, 15] for details.

In particular, the Helson matrices generalize the small Hankel operators on the Hardy space of any finite polytorus  $H^2(\mathbb{T}^d)$ ,  $d < \infty$ . In fact, the results in this note have analogous statements for small Hankel operators on  $H^2(\mathbb{T}^d)$ ; every proof given remains valid verbatim after restricting the number of prime factors, that is, the number of variables.

The first result is the following. We denote by  $\mathcal{B}(\ell^2(\mathbb{N}))$  and  $\mathcal{K}(\ell^2(\mathbb{N}))$ , respectively, the spaces of bounded and compact operators on  $\ell^2(\mathbb{N})$ .

**Theorem 1.** *Let  $M(\alpha)$  be a bounded multiplicative Hankel operator. Then there exists a compact multiplicative Hankel operator  $N(\beta)$  such that*

$$(1) \quad \|M(\alpha) - N(\beta)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = \inf \{ \|M(\alpha) - K\|_{\mathcal{B}(\ell^2(\mathbb{N}))} : K \in \mathcal{K}(\ell^2(\mathbb{N})) \}.$$

*The minimizer  $N(\beta)$  is never unique, unless  $M(\alpha)$  is compact.*

The quantity on the right-hand side of (1) is known as the essential norm of  $M(\alpha)$ . For classical Hankel operators on  $H^2(\mathbb{T})$ , this result was proven by Axler, Berg, Jewell, and Shields in [6], and can be viewed as a limiting case of the theory of Adamjan, Arov, and Krein [1]. The demonstration of Theorem 1 requires only a minor modification of the arguments in [6], the main point being that a characterization of the class of bounded multiplicative Hankel operators is not necessary for the proof.

On  $H^2(\mathbb{T})$ , Nehari's theorem [21] states that the class of bounded Hankel operators can be isometrically identified with  $L^\infty(\mathbb{T})/H^\infty(\mathbb{T})$ , where  $L^\infty(\mathbb{T})$  and  $H^\infty(\mathbb{T})$  denote the spaces of bounded and bounded analytic functions on  $\mathbb{T}$ , respectively. By Hartman's theorem [13], the class of compact Hankel operators is isometrically isomorphic to  $(H^\infty(\mathbb{T}) + C(\mathbb{T}))/H^\infty(\mathbb{T})$ , where  $C(\mathbb{T})$  denotes the space of continuous functions on  $\mathbb{T}$ . Note that the spaces  $L^\infty$ ,  $H^\infty$ , and  $H^\infty + C$  are all algebras, as proven by Sarason [26].

Luecking [20] observed, through a very illustrative argument relying on function algebra techniques, that the compact Hankel operators form an M-ideal in the space of bounded Hankel operators. The concept of an M-ideal will be defined shortly, but let us note for now that M-ideality implies proximality; the distance from a bounded Hankel operator to the compact Hankel operators has a minimizer. Thus Luecking reproved some of the results of [6]. Since

$$((H^\infty + C)/H^\infty)^{**} \simeq L^\infty/H^\infty,$$

it follows that the bidual of the space of compact Hankel operators is isometrically isomorphic to the space of bounded Hankel operators. Spaces which are M-ideals in their biduals are said to be M-embedded.

The multiplicative Hankel operators, on the other hand, have thus far resisted all attempts to characterize their boundedness. It has been shown that a Nehari-type theorem cannot exist [22], and positive results only exist in special cases [14, 24]. In spite of this, the main theorem shows that Luecking's result holds for multiplicative Hankel operators.

Let

$$\mathcal{M}_0 = \{m = M(\alpha) : M(\alpha) : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N}) \text{ compact}\}$$

and

$$\mathcal{M} = \{m = M(\alpha) : M(\alpha) : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N}) \text{ bounded}\}.$$

Equipped with the operator norm,  $\mathcal{M}_0$  and  $\mathcal{M}$  are closed subspaces of  $\mathcal{K}(\ell^2(\mathbb{N}))$  and  $\mathcal{B}(\ell^2(\mathbb{N}))$ , respectively. For a Banach space  $Y$ , we denote by  $\iota_Y$  the canonical embedding  $\iota_Y : Y \rightarrow Y^{**}$ ,

$$\iota_Y y(y^*) = y^*(y), \quad y \in Y, y^* \in Y^*.$$

**Theorem 2.** *There is a unique isometric isomorphism  $U : \mathcal{M}_0^{**} \rightarrow \mathcal{M}$  such that  $U \iota_{\mathcal{M}_0} m = m$  for every  $m \in \mathcal{M}_0$ . Furthermore,  $\mathcal{M}_0$  is an M-ideal in  $\mathcal{M}$ .*

*Remark.* As pointed out earlier, Theorem 2 is also true when stated for small Hankel operators on  $H^2(\mathbb{T}^d)$ ,  $d < \infty$ . The biduality has in this case been demonstrated isomorphically in [18], with an argument based on the non-isometric Nehari-type theorems proven in [10, 17].

The M-ideal property means the following: there is an (onto) projection  $L: \mathcal{M}^* \rightarrow \mathcal{M}_0^\perp$  such that

$$\|m^*\|_{\mathcal{M}^*} = \|Lm^*\|_{\mathcal{M}^*} + \|m^* - Lm^*\|_{\mathcal{M}^*}, \quad m^* \in \mathcal{M}^*,$$

where  $\mathcal{M}_0^\perp$  denotes the space of functionals  $m^* \in \mathcal{M}^*$  which annihilate  $\mathcal{M}_0$ . M-ideals were introduced by Alfsen and Effros [3] as a Banach space analogue of closed two-sided ideals in  $C^*$ -algebras. Very loosely speaking, the fact that  $\mathcal{M}_0$  is an M-ideal in  $\mathcal{M}$  implies that the norm of  $\mathcal{M}$  resembles a maximum norm and, in this analogy, that  $\mathcal{M}_0$  is the subspace of elements vanishing at infinity. The book [12] comprehensively treats M-structure theory and its applications.

We will make use of the following consequences of Theorem 2. Proximality of  $\mathcal{M}_0$  in  $\mathcal{M}$  was already mentioned, but the M-ideal property also implies that the minimizer is never unique [16]. It also ensures that  $\mathcal{M}_0^*$  is a strongly unique predual of  $\mathcal{M}$  [12, Proposition III.2.10]. This means that every isometric isomorphism of  $\mathcal{M}$  onto  $Y^*$ ,  $Y$  a Banach space, is weak\*-weak\* continuous, that is, arises as the adjoint of an isometric isomorphism of  $Y$  onto  $\mathcal{M}_0^*$ . On the other hand,  $\mathcal{M}_0^*$  has infinitely many different preduals [11, Theoreme 27].

The predual of  $\mathcal{M}$  is well known to have an almost tautological characterization as a projective tensor product with respect to Dirichlet convolution,

$$\mathcal{X} = \ell^2(\mathbb{N}) \hat{\star} \ell^2(\mathbb{N}).$$

The space  $\mathcal{X}$  is also referred to as a weak product space. We defer the precise definition to the next section – after establishing the main theorems, we essentially show, following [25], that all reasonable definitions of  $\mathcal{X}$  coincide.

**Theorem 3.** *There is an isometric isomorphism  $L: \mathcal{X} \rightarrow \mathcal{M}_0^*$  such that  $L^*U^{-1}: \mathcal{M} \rightarrow \mathcal{X}^*$  is the canonical isometric isomorphism of  $\mathcal{M}$  onto  $\mathcal{X}^*$ , where  $U: \mathcal{M}_0^{**} \rightarrow \mathcal{M}$  is the isometric isomorphism of Theorem 2.*

Informally stated,  $\mathcal{M}_0^* \simeq \mathcal{X}$  and  $\mathcal{X}^* \simeq \mathcal{M}$ . Theorem 3 follows at once from Theorem 2 and the uniqueness of the predual of  $\mathcal{M}$ , but we also supply a direct proof. While the duality  $\mathcal{X}^* \simeq \mathcal{M}$  is a rephrasing of the definition of  $\mathcal{M}$ , it is difficult to identify a common approach to dualities of the type  $\mathcal{M}_0^* \simeq \mathcal{X}$  in the existing literature. Often, the latter duality is deduced (isomorphically) via a concrete description of  $\mathcal{M}$ . For a small selection of relevant examples, see [4, 8, 12, 18, 19, 23, 28].

The idea behind this note is that the direct view of  $\mathcal{M}$  as a subspace of  $\mathcal{B}(\ell^2(\mathbb{N}))$  already provides sufficient information to prove Theorems 1, 2, and 3. In this direction, Wu [28] worked with an embedding into the space of

bounded operators to deduce duality results for certain Hankel-type forms on Dirichlet spaces.

The proofs of the results only have two main ingredients. The first is a device to approximate elements of  $\mathcal{M}$  by elements of  $\mathcal{M}_0$  (Lemma 4). Such an approximation property is necessary, because if  $\mathcal{M}_0^{**} \simeq \mathcal{M}$ , then the unit ball of  $\mathcal{M}_0$  is weak\* dense in the unit ball of  $\mathcal{M}$ . The second ingredient is an inclusion of  $\mathcal{M}$  into a reflexive space; in our case,  $\ell^2(\mathbb{N})$ . Analogous theorems could be proven for many other linear spaces of bounded and compact operators using the same technique.

## 2. Results

For a sequence  $a$  and  $0 < r < 1$ , let

$$D_r a(n) = r^{\sum_{j=1}^{\infty} j \kappa_j} a(n), \text{ where } n = \prod_{j=1}^{\infty} p_j^{\kappa_j}.$$

Note that

$$\sum_{\kappa} r^{2 \sum_{j=1}^{\infty} j \kappa_j} = \prod_{j=1}^{\infty} \frac{1}{1 - r^{2j}} < \infty.$$

Hence it follows by the dominated convergence theorem that  $D_r: \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$  is a compact operator. Furthermore,  $D_r$  is self-adjoint and contractive,  $\|D_r\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq 1$ . The dominated convergence theorem also implies that  $D_r \rightarrow \text{id}_{\ell^2(\mathbb{N})}$  in the strong operator topology (SOT) as  $r \rightarrow 1$ , that is,  $\lim_{r \rightarrow 1} D_r a = a$  in  $\ell^2(\mathbb{N})$ , for every  $a \in \ell^2(\mathbb{N})$ . A study of the operators  $D_r$  in the context of Hardy spaces of the infinite polytorus can be found in [2].

The Dirichlet convolution of two sequences  $a$  and  $b$  is the new sequence  $a \star b$  given by

$$(a \star b)(n) = \sum_{k|n} a(k) \overline{b(n/k)}, \quad n \in \mathbb{N}.$$

If  $a$  and  $b$  are two finite sequences, then

$$(2) \quad \langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})} = (\alpha, a \star b),$$

where  $(a, b) = \sum_{n=1}^{\infty} a(n)b(n)$  denotes the bilinear pairing between  $a, b \in \ell^2(\mathbb{N})$ . Note also that, for  $0 < r < 1$ ,

$$(3) \quad D_r(a \star b) = D_r a \star D_r b.$$

The following simple lemma is key.

**Lemma 4.** *Let  $M(\alpha)$  be a bounded multiplicative Hankel operator,  $M(\alpha) \in \mathcal{M}$ . For  $0 < r < 1$ , let  $\alpha_r = D_r \alpha$ . Then  $M_{\alpha_r} \in \mathcal{M}_0$ ,*

$$\|M_{\alpha_r}\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq \|M_{\alpha}\|_{\mathcal{B}(\ell^2(\mathbb{N}))},$$

*and  $M_{\alpha_r} \rightarrow M_{\alpha}$  and  $M_{\alpha_r}^* \rightarrow M_{\alpha}^*$  SOT as  $r \rightarrow 1$ .*

**Proof.** By (2) and (3), it holds for finite sequences  $a$  and  $b$  that

$$\langle M(\alpha_r)a, b \rangle_{\ell^2(\mathbb{N})} = \langle M_\alpha D_r a, D_r b \rangle_{\ell^2(\mathbb{N})}.$$

Hence  $M_{\alpha_r} = D_r M_\alpha D_r$ . We conclude that  $M_{\alpha_r}$  is compact,  $\|M_{\alpha_r}\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq \|M_\alpha\|_{\mathcal{B}(\ell^2(\mathbb{N}))}$ , and  $M_{\alpha_r} \rightarrow M_\alpha$  SOT as  $r \rightarrow 1$ . Similarly,  $M_{\alpha_r}^* = M_{\bar{\alpha}_r} \rightarrow M_{\bar{\alpha}} = M_\alpha^*$  SOT as  $r \rightarrow 1$ .  $\square$

The following is a recognizable consequence, cf. [27, Theorem 1]. Note that if  $S_n$  and  $T_n$  are operators such that  $S_n \rightarrow S$  and  $T_n \rightarrow T$  SOT, and if  $C$  is a compact operator, then  $S_n C T_n^* \rightarrow S C T^*$  in operator norm.

**Proposition 5.** *Let  $M(\alpha) \in \mathcal{M}$ . Then  $M(\alpha) \in \mathcal{M}_0$  if and only if*

$$(4) \quad \lim_{r \rightarrow 1} \|M(\alpha_r) - M(\alpha)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = 0.$$

**Proof.** If (4) holds, then  $M(\alpha) \in \mathcal{M}_0$ , since  $M(\alpha_r)$  is compact for every  $0 < r < 1$ . If  $M(\alpha) \in \mathcal{M}_0$ , then (4) holds, since  $M(\alpha_r) = D_r M(\alpha) D_r = D_r M(\alpha) D_r^*$  and  $D_r \rightarrow \text{id}_{\ell^2(\mathbb{N})}$  SOT as  $r \rightarrow 1$ .  $\square$

Recall next the main tool from [6].

**Theorem 6 ([6]).** *Let  $T: \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$  be a non-compact operator and  $(T_n)$  a sequence of compact operators such that  $T_n \rightarrow T$  SOT and  $T_n^* \rightarrow T^*$  SOT. Then there exists a sequence  $(c_n)$  of non-negative real numbers such that  $\sum_n c_n = 1$  for which the compact operator*

$$J = \sum_n c_n T_n$$

satisfies

$$\|T - J\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = \inf \{ \|T - K\|_{\mathcal{B}(\ell^2(\mathbb{N}))} : K \in \mathcal{K}(\ell^2(\mathbb{N})) \}.$$

Lemma 4 and Theorem 6 immediately yield the existence part of Theorem 1.

**Proof of Theorem 1.** Let  $M(\alpha)$  be a bounded multiplicative Hankel operator and let  $(r_k)$  be a sequence such that  $0 < r_k < 1$  and  $r_k \rightarrow 1$ . Then  $M(\alpha)$  has a best compact approximant of the form

$$N = \sum_k c_k M(\alpha_{r_k}).$$

But then  $N = N(\beta)$  is a multiplicative Hankel operator,  $\beta = \sum_k c_k \alpha_{r_k}$ .

The non-uniqueness of  $N(\beta)$  follows immediately once we have established Theorem 2, by general M-ideal results [16]. In fact, if  $M(\alpha) \notin \mathcal{M}_0$ , then the set of minimizers  $N(\beta)$  is so large that it spans  $\mathcal{M}_0$ .  $\square$

Note that

$$\|M(\alpha)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \geq \overline{\lim}_{N \rightarrow \infty} \frac{1}{\|(\alpha(n))_{n=1}^N\|_{\ell^2(\mathbb{N})}} \sum_{n=1}^N |\alpha(n)|^2 = \|\alpha\|_{\ell^2(\mathbb{N})}.$$

Therefore the inclusion  $I: \mathcal{M}_0 \rightarrow \ell^2(\mathbb{N})$  is a contractive operator,  $Im = I(M(\alpha)) = \alpha$ . We can state Theorem 2 slightly more precisely in terms of  $I$ .

**Theorem 2.** *Consider the bitranspose  $U = I^{**}: \mathcal{M}_0^{**} \rightarrow \ell^2(\mathbb{N})$ . Then  $U\mathcal{M}_0^{**} = \mathcal{M}$ , viewing  $\mathcal{M}$  as a (non-closed) subspace of  $\ell^2(\mathbb{N})$ . Furthermore,*

$$U\iota_{\mathcal{M}_0}m = m, \quad m \in \mathcal{M}_0,$$

and

$$\|Um^{**}\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = \|m^{**}\|_{\mathcal{M}_0^{**}}, \quad m^{**} \in \mathcal{M}_0^{**}.$$

If  $V: \mathcal{M}_0^{**} \rightarrow \mathcal{M}$  is another isometric isomorphism such that  $V\iota_{\mathcal{M}_0}m = m$  for all  $m \in \mathcal{M}_0$ , then  $V = U$ . Furthermore,  $\mathcal{M}_0$  is an  $M$ -ideal in  $\mathcal{M}$ .

**Proof.** We identify  $(\ell^2(\mathbb{N}))^* \simeq \ell^2(\mathbb{N})$  linearly through the pairing  $(a, b) = \sum_{n=1}^{\infty} a(n)b(n)$  between  $a, b \in \ell^2(\mathbb{N})$ . With this convention,  $I^*: \ell^2(\mathbb{N}) \rightarrow \mathcal{M}_0^*$  is also contractive, and

$$I^*a(m) = (\alpha, a), \quad a \in \ell^2(\mathbb{N}), \quad m = M(\alpha) \in \mathcal{M}_0.$$

Since  $I$  is injective,  $I^*$  has dense range. In particular,  $\mathcal{M}_0^*$  is separable. Furthermore,  $I^{**}: \mathcal{M}_0^{**} \rightarrow \ell^2(\mathbb{N})$  is injective. By the reflexivity of  $\ell^2(\mathbb{N})$ , we have that  $I^{**}\iota_{\mathcal{M}_0} = I$ , since

$$(I^{**}\iota_{\mathcal{M}_0}m, a) = \iota_{\mathcal{M}_0}m(I^*a) = (\alpha, a) = (Im, a)$$

for every  $m = M(\alpha) \in \mathcal{M}_0$  and  $a \in \ell^2(\mathbb{N})$ . The interpretation, viewing  $\mathcal{M}$  as a non-closed subspace of  $\ell^2(\mathbb{N})$ , is that  $I^{**}\iota_{\mathcal{M}_0}m = m$ , for all  $m \in \mathcal{M}_0$ .

Consider any  $m^{**} \in \mathcal{M}_0^{**}$ , and let  $\alpha = I^{**}m^{**} \in \ell^2(\mathbb{N})$ . Since  $\mathcal{M}_0^*$  is separable, the weak\* topology of the unit ball  $B_{\mathcal{M}_0^{**}}$  of  $\mathcal{M}_0^{**}$  is metrizable. As is the case for every Banach space,  $\iota_{\mathcal{M}_0}(B_{\mathcal{M}_0})$  is weak\* dense in  $B_{\mathcal{M}_0^{**}}$ . Hence there is a sequence  $(m_n)_{n=1}^{\infty}$  in  $\mathcal{M}_0$  such that  $\iota_{\mathcal{M}_0}m_n \rightarrow m^{**}$  weak\* and  $\|m_n\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq \|m^{**}\|_{\mathcal{M}_0^{**}}$ . Suppose that  $m_n = M(\alpha_n)$  and let  $a, b \in \ell^2(\mathbb{N})$  be two finite sequences. Then, since  $\iota_{\mathcal{M}_0}m_n \rightarrow m^{**}$  weak\*,

$$\langle M(\alpha_n)a, b \rangle_{\ell^2(\mathbb{N})} = (\alpha_n, a \star b) = I^*(a \star b)(m_n) \rightarrow m^{**}(I^*(a \star b)) = (\alpha, a \star b),$$

as  $n \rightarrow \infty$ . It follows that

$$\begin{aligned} |\langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})}| &= |(\alpha, a \star b)| \leq \overline{\lim}_{n \rightarrow \infty} \|m_n\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \|a\|_{\ell^2(\mathbb{N})} \|b\|_{\ell^2(\mathbb{N})} \\ &\leq \|m^{**}\|_{\mathcal{M}_0^{**}} \|a\|_{\ell^2(\mathbb{N})} \|b\|_{\ell^2(\mathbb{N})}. \end{aligned}$$

Since  $a, b$  were arbitrary finite sequences, it follows that  $M(\alpha) \in \mathcal{M}$  and

$$\|M(\alpha)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq \|m^{**}\|_{\mathcal{M}_0^{**}}.$$

Since  $\alpha = I^{**}m^{**}$  this proves that  $I^{**}$  maps  $\mathcal{M}_0^{**}$  contractively into  $\mathcal{M}$ .

Conversely, suppose that  $m = M(\alpha) \in \mathcal{M}$ . By Lemma 4, for  $0 < r < 1$ ,  $M(\alpha_r) \in \mathcal{M}_0$ ,  $\|M(\alpha_r)\| \leq \|M(\alpha)\|$ , and  $\alpha_r \rightarrow \alpha$  in  $\ell^2(\mathbb{N})$  as  $r \rightarrow 1$ . Define  $m^{**} \in \mathcal{M}_0^{**}$  by

$$(5) \quad m^{**}(I^*a) := (\alpha, a) = \lim_{r \rightarrow 1} (\alpha_r, a) = \lim_{r \rightarrow 1} I^*a(M(\alpha_r)), \quad a \in \ell^2(\mathbb{N}).$$

This specifies an element  $m^{**} \in \mathcal{M}_0^{**}$  since  $I^*$  has dense range in  $\mathcal{M}_0^*$  and

$$|m^{**}(I^*a)| \leq \overline{\lim}_{r \rightarrow 1} \|M(\alpha_r)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \|I^*a\|_{\mathcal{M}_0^*} \leq \|M(\alpha)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \|I^*a\|_{\mathcal{M}_0^*}.$$

From this inequality we also see that

$$(6) \quad \|m^{**}\|_{\mathcal{M}_0^{**}} \leq \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))}.$$

Furthermore, since

$$(I^{**}m^{**}, a) = m^{**}(I^*a) = (\alpha, a), \quad a \in \ell^2(\mathbb{N}),$$

we have that  $I^{**}m^{**} = \alpha$ . Hence  $I^{**}$  maps  $\mathcal{M}_0^{**}$  bijectively and contractively onto  $\mathcal{M}$ . By (6),  $I^{**}: \mathcal{M}_0^{**} \rightarrow \mathcal{M}$  is also expansive, and hence it is an isometric isomorphism.

Recall that  $\mathcal{K}(\ell^2(\mathbb{N}))$  is an M-ideal in  $\mathcal{B}(\ell^2(\mathbb{N}))$  [9] – indeed,  $\mathcal{K}(\ell^2(\mathbb{N}))$  is a two-sided closed ideal in  $\mathcal{B}(\ell^2(\mathbb{N}))$ . It is well known that there is an isometric isomorphism  $E: \mathcal{K}(\ell^2(\mathbb{N}))^{**} \rightarrow \mathcal{B}(\ell^2(\mathbb{N}))$  such that  $E\iota_{\mathcal{K}(\ell^2(\mathbb{N}))}K = K$  for all  $K \in \mathcal{K}(\ell^2(\mathbb{N}))$ . Thus  $\mathcal{K}(\ell^2(\mathbb{N}))$  is M-embedded. Since  $\mathcal{M}_0$  is a closed subspace of  $\mathcal{K}(\ell^2(\mathbb{N}))$ ,  $\mathcal{M}_0$  is also M-embedded [12, Theorem III.1.6]. Hence, since we have shown that  $I^{**}: \mathcal{M}_0^{**} \rightarrow \mathcal{M}$  is an isometric isomorphism for which  $I^{**}\iota_{\mathcal{M}_0}m = m$  for all  $m \in \mathcal{M}_0$ , it follows that  $\mathcal{M}_0$  is an M-ideal in  $\mathcal{M}$ .

Finally, if  $V: \mathcal{M}_0^{**} \rightarrow \mathcal{M}$  is another isometric isomorphism such that  $V\iota_{\mathcal{M}_0}m = m$ ,  $m \in \mathcal{M}_0$ , then  $F = V^{-1}I^{**}: \mathcal{M}_0^{**} \rightarrow \mathcal{M}_0^{**}$  is an isometric isomorphism such that  $F\iota_{\mathcal{M}_0} = \iota_{\mathcal{M}_0}$ . However, since  $\mathcal{M}_0$  is M-embedded,  $F$  must be obtained as the bitranspose,  $F = G^{**}$ , of an isometric isomorphism  $G: \mathcal{M}_0 \rightarrow \mathcal{M}_0$  [12, Proposition III.2.2]. But then  $G = \text{id}_{\mathcal{M}_0}$ , since

$$m^*(Gm) = G^*m^*(m) = F\iota_{\mathcal{M}_0}m(m^*) = m^*(m), \quad m \in \mathcal{M}_0, m^* \in \mathcal{M}_0^*.$$

Hence  $F = \text{id}_{\mathcal{M}_0^{**}}$  and so  $V = I^{**}$ .  $\square$

The predual of a space of Hankel operators usually has an abstract description as a projective tensor product [5, 7, 10]. In the present context, let

$$X = \left\{ c : c = \sum_{\text{finite}} a_k \star b_k, \ a_k, b_k \text{ finite sequences} \right\},$$

and equip  $X$  with the norm

$$\|c\|_X = \inf \sum_{\text{finite}} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})},$$

where the infimum is taken over all **finite** representations of  $c$ . By writing  $c = c \star (1, 0, 0, \dots)$  it is clear that  $\|c\|_X \leq \|c\|_{\ell^2(\mathbb{N})}$  for  $c \in X$ .

We define the projective tensor product space  $\mathcal{X} = \ell^2(\mathbb{N}) \hat{\star} \ell^2(\mathbb{N})$  with respect to Dirichlet convolution as the Banach space completion of  $X$ . It is essentially definition that  $\mathcal{X}^* \simeq \mathcal{M}$ .

**Lemma 7.** For  $m = M(\alpha) \in \mathcal{M}$ , let

$$Jm(c) = (\alpha, c), \quad c \in X.$$

Then  $Jm$  extends to a bounded functional on  $\mathcal{X}$  for every  $m \in \mathcal{M}$ , and  $J: \mathcal{M} \rightarrow \mathcal{X}^*$  is an isometric isomorphism.

**Proof.** Let  $m \in \mathcal{M}$ . If  $c \in X$  and  $\varepsilon > 0$ , choose a representation  $c = \sum_{k=1}^N a_k \star b_k$ , where  $a_k$  and  $b_k$  are finite sequences for every  $k$ , and

$$\sum_{k=1}^N \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} < \|c\|_X + \varepsilon.$$

Then

$$|Jm(c)| = \left| \sum_{k=1}^N \langle M(\alpha)a_k, b_k \rangle_{\ell^2(\mathbb{N})} \right| \leq \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))} (\|c\|_X + \varepsilon).$$

Hence  $\|Jm\|_{\mathcal{X}^*} \leq \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))}$ . Choosing finite sequences  $a$  and  $b$  such that  $\|a\|_{\ell^2(\mathbb{N})} = \|b\|_{\ell^2(\mathbb{N})} = 1$  and  $\langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})} > \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))} - \varepsilon$ , and letting  $c = a \star b$  gives that

$$\|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))} - \varepsilon < \|Jm\|_{\mathcal{X}^*} \|c\|_X \leq \|Jm\|_{\mathcal{X}^*}.$$

Hence  $J$  is an isometry.

The inclusion of finite sequences into  $X$  extends to a contractive map  $E: \ell^2(\mathbb{N}) \rightarrow \mathcal{X}$ . Let  $\ell \in \mathcal{X}^*$  and let  $c \in X$ . Then  $\ell(c) = (\alpha, c)$ , where  $\alpha = E^*\ell \in \ell^2(\mathbb{N})$ . Then  $m = M(\alpha) \in \mathcal{M}$ , since  $\ell \in \mathcal{X}^*$ . Clearly  $Jm = \ell$  and thus  $J$  is onto.  $\square$

**Theorem 3.** For every  $c \in X$ , let

$$Lc(m) = (\alpha, c), \quad m = M(\alpha) \in \mathcal{M}_0.$$

Then  $L$  extends to an isometric isomorphism  $L: \mathcal{X} \rightarrow \mathcal{M}_0^*$ , and

$$L^*U^{-1} = J: \mathcal{M} \rightarrow \mathcal{X}^*$$

is the isometric isomorphism of Lemma 7. Here  $U: \mathcal{M}_0^{**} \rightarrow \mathcal{M}$  is the isometric isomorphism of Theorem 2.

**Proof.** The quickest proof proceeds by noting that  $\mathcal{M}_0^*$  is a strongly unique predual of  $\mathcal{M}_0^{**}$ , since  $\mathcal{M}_0$  is M-embedded. This implies that the isometric isomorphism  $JU: \mathcal{M}_0^{**} \rightarrow \mathcal{X}^*$  is the adjoint of an isometric isomorphism  $E: \mathcal{X} \rightarrow \mathcal{M}_0^*$ ,  $E^* = JU$ . But then, for  $c \in X$  and  $m = M(\alpha) \in \mathcal{M}_0$ ,

$$\begin{aligned} (7) \quad Ec(m) &= \iota_{\mathcal{M}_0} m(Ec) = E^* \iota_{\mathcal{M}_0} m(c) = JU \iota_{\mathcal{M}_0} m(c) \\ &= Jm(c) = (\alpha, c) = Lc(m). \end{aligned}$$

Hence  $L = E$ , and thus  $L$  is an isometric isomorphism.

Alternatively, the weak\*-weak\* continuity of  $JU$  can be proven by hand.  $L$  clearly extends to a contractive operator  $L: \mathcal{X} \rightarrow \mathcal{M}_0^*$ . The computation (7) shows that  $JU \iota_{\mathcal{M}_0} = L^* \iota_{\mathcal{M}_0}$ . Let  $m^{**} \in \mathcal{M}_0^{**}$  and let  $M(\alpha) = Um^{**}$ .

From (5) we deduce that  $m_r^{**} = \iota_{\mathcal{M}_0} M(\alpha_r) \rightarrow m^{**}$  weak\* in  $\mathcal{M}_0^{**}$ . Hence  $L^* m_r^{**} \rightarrow L^* m^{**}$  weak\* in  $\mathcal{X}^*$ . On the other hand, for  $c \in X$ ,

$$\begin{aligned} JU m^{**}(c) &= (\alpha, c) = \lim_{r \rightarrow 1} (\alpha_r, c) = \lim_{r \rightarrow 1} JU m_r^{**}(c) \\ &= \lim_{r \rightarrow 1} L^* m_r^{**}(c) = L^* m^{**}(c). \end{aligned}$$

This shows that  $JU = L^*$ , and hence  $L$  is an isometric isomorphism.  $\square$

*Remark.* In the notation of Theorem 2,  $I^*c = Lc$  for  $c \in X$ . Theorem 3 hence completes the picture of Theorem 2 by giving an interpretation of the operator  $I^*$ .

Suppose that we had instead defined the projective tensor product space  $\ell^2(\mathbb{N}) \hat{\star} \ell^2(\mathbb{N})$  as the sequence space

$$\mathcal{Y} = \left\{ c : c = \sum_{k=1}^{\infty} a_k \star b_k, a_k, b_k \in \ell^2(\mathbb{N}), \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} < \infty \right\},$$

normed by

$$\|c\|_{\mathcal{Y}} = \inf \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})},$$

where the infimum is taken over all representations of  $c$ . One would like to know that  $\mathcal{Y} = \mathcal{X}$ . Indeed, it is not a priori clear that  $\mathcal{X}$  is a sequence space; or if  $\mathcal{X}$  is identifiable with a space of Dirichlet series, if considering multiplicative Hankel operators in that context. For  $\mathcal{Y}$  these properties are immediate.

**Lemma 8.**  $\mathcal{Y}$  is a Banach space.

**Proof.** Since  $|(a \star b)(n)| \leq \|a\|_{\ell^2(\mathbb{N})} \|b\|_{\ell^2(\mathbb{N})}$  it is clear that

$$e_n(c) = c(n), \quad c \in \mathcal{Y},$$

defines an element  $e_n \in \mathcal{Y}^*$ , for every  $n \in \mathbb{N}$ . It follows that  $\|c\|_{\mathcal{Y}} = 0$  if and only if  $c = 0$ .

Suppose that  $\sum_{k=1}^{\infty} c_k$  is an absolutely convergent series in  $\mathcal{Y}$ . Then there are double sequences  $(a_{k,j})$  and  $(b_{k,j})$  such that  $c_k = \sum_{j=1}^{\infty} a_{k,j} \star b_{k,j}$  for every  $k$  and

$$\sum_{k,j=1}^{\infty} \|a_{k,j}\|_{\ell^2(\mathbb{N})} \|b_{k,j}\|_{\ell^2(\mathbb{N})} < \infty.$$

Then  $c = \sum_{k,j=1}^{\infty} a_{k,j} b_{k,j}$  is an element of  $\mathcal{Y}$  and

$$\|c - \sum_{k=1}^N c_k\|_{\mathcal{Y}} \leq \sum_{k=N+1}^{\infty} \sum_{j=1}^{\infty} \|a_{k,j}\|_{\ell^2(\mathbb{N})} \|b_{k,j}\|_{\ell^2(\mathbb{N})} \rightarrow 0, \quad N \rightarrow \infty.$$

Hence  $\sum_{k=1}^{\infty} c_k$  converges in  $\mathcal{Y}$  to  $c$ . Thus  $\mathcal{Y}$  is complete.  $\square$

We now prove that  $\mathcal{Y} = \mathcal{X}$ . The details are similar to those of [25], where projective tensor products of spaces of holomorphic functions were considered. Note that  $X$  is contractively contained in  $\mathcal{Y}$ .

**Proposition 9.** *The inclusion  $V: X \rightarrow \mathcal{Y}$  extends to an isometric isomorphism  $V: \mathcal{X} \rightarrow \mathcal{Y}$ .*

**Proof.** We make the following preliminary observation. Since for every  $0 < r < 1$ ,

$$D_r(a \star b) = D_r a \star D_r b, \quad \|D_r\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq 1,$$

$D_r$  defines a bounded operator  $D_r: \mathcal{X} \rightarrow \mathcal{X}$ ,

$$\|D_r\|_{\mathcal{B}(\mathcal{X})} \leq 1.$$

Furthermore, since  $D_r \rightarrow \text{id}_{\ell^2(\mathbb{N})}$  SOT on  $\ell^2(\mathbb{N})$  as  $r \rightarrow 1$ , it follows that  $\|D_r c - c\|_X \leq \|D_r c - c\|_{\ell^2(\mathbb{N})} \rightarrow 0$  as  $r \rightarrow 1$  for every  $c \in X$ . Hence  $D_r \rightarrow \text{id}_{\mathcal{X}}$  SOT on  $\mathcal{X}$  as  $r \rightarrow 1$ .

As in Lemma 8, for each  $n \in \mathbb{N}$ ,

$$e_n(c) = c(n), \quad c \in X,$$

extends to a functional  $e_n \in \mathcal{X}^*$  with  $\|e_n\|_{\mathcal{X}^*} \leq 1$ . We show now that  $(e_n)$  is a complete sequence in  $\mathcal{X}^*$  with respect to the weak\* topology. Suppose that  $c \in \mathcal{X}$  and that  $e_n(c) = 0$  for all  $n$ . Pick a sequence  $(c_k)$  in  $X$  such that  $c_k \rightarrow c$  in  $\mathcal{X}$ . Then for fixed  $r < 1$ ,

$$\begin{aligned} \|D_r c\|_{\mathcal{X}} &\leq \overline{\lim}_{k \rightarrow \infty} (\|D_r(c - c_k)\|_{\mathcal{X}} + \|D_r c_k\|_{\mathcal{X}}) \\ &= \overline{\lim}_{k \rightarrow \infty} \|D_r c_k\|_{\mathcal{X}} \leq \overline{\lim}_{k \rightarrow \infty} \|D_r c_k\|_{\ell^2(\mathbb{N})}. \end{aligned}$$

Since  $c_k \rightarrow c$  in  $\mathcal{X}$  and  $e_n \in \mathcal{X}^*$ , we have that  $\lim_{k \rightarrow \infty} c_k(n) = e_n(c) = 0$  for every  $n$ . Furthermore,  $|c_k(n)| \leq \|e_n\|_{\mathcal{X}^*} \|c_k\|_X \leq \|c_k\|_X$  is uniformly bounded in  $k$  and  $n$ . Hence it follows by the dominated convergence theorem that  $\overline{\lim}_{k \rightarrow \infty} \|D_r c_k\|_{\ell^2(\mathbb{N})} = 0$  and thus that  $D_r c = 0$ . Since  $D_r c \rightarrow c$  in  $\mathcal{X}$  as  $r \rightarrow 1$  we conclude that  $c = 0$ . Therefore  $(e_n)$  is complete.

Hence  $\mathcal{X}$  is a space of sequences. More precisely, since every evaluation  $e_n$  is a bounded functional on  $\mathcal{Y}$  as well, the extension  $V: \mathcal{X} \rightarrow \mathcal{Y}$  of the inclusion map is given by

$$(8) \quad Vc = (e_n(c))_{n=1}^{\infty}, \quad c \in \mathcal{X}.$$

The completeness of  $(e_n)$  implies that  $V$  is injective.

We next prove that  $V$  is onto. The argument is precisely as in [25], but we include it for completeness. For a sequence  $a$  and  $m \in \mathbb{N}$ , let  $a^m = (a(1), \dots, a(m), 0, \dots)$ . Given  $a \in \ell^2(\mathbb{N})$  and  $\delta > 0$ , choose a sequence  $(m_1, m_2, \dots)$  such that  $\|a - a^{m_k}\|_{\ell^2(\mathbb{N})} \leq 2^{-k}$ . Let  $a_k = a^{m_{k+1}} - a^{m_k}$ . Then, for sufficiently large  $K$ ,

$$a = a^{m_K} + \sum_{k=K}^{\infty} (a^{m_{k+1}} - a^{m_k}), \quad \sum_{k=K}^{\infty} \|a^{m_{k+1}} - a^{m_k}\|_{\ell^2(\mathbb{N})} < \delta.$$

Hence we can write  $a = \sum_{j=1}^{\infty} a_j$ , where each  $a_j$  is a finite sequence and  $\sum_j \|a_j\|_{\ell^2(\mathbb{N})} < \|a\|_{\ell^2(\mathbb{N})} + \delta$ .

Given  $c \in \mathcal{Y}$  and  $\varepsilon > 0$ , choose  $(a_k)_{k=1}^{\infty}$  and  $(b_k)_{k=1}^{\infty}$  such that

$$c = \sum_{k=1}^{\infty} a_k \star b_k, \quad \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} < \|c\|_{\mathcal{Y}} + \varepsilon.$$

For each  $k$ , write, as in the preceding paragraph,  $a_k = \sum_{j=1}^{\infty} a_{k,j}$ ,  $b_k = \sum_{j=1}^{\infty} b_{k,j}$ , where each  $a_{k,j}$  and  $b_{k,j}$  is a finite sequence and

$$\sum_{j=1}^{\infty} \|a_{k,j}\|_{\ell^2(\mathbb{N})} < \|a_k\|_{\ell^2(\mathbb{N})} + \delta_k, \quad \sum_{j=1}^{\infty} \|b_{k,j}\|_{\ell^2(\mathbb{N})} < \|b_k\|_{\ell^2(\mathbb{N})} + \delta_k.$$

Here the  $\delta_k$  are chosen so that

$$\sum_{k=1}^{\infty} (\|a_k\|_{\ell^2(\mathbb{N})} + \delta_k) (\|b_k\|_{\ell^2(\mathbb{N})} + \delta_k) < \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} + \varepsilon.$$

Then  $c = \sum_{k,j,l=1}^{\infty} a_{k,j} \star b_{k,l}$ , and

$$\sum_{k,j,l=1}^{\infty} \|a_{k,j}\|_{\ell^2(\mathbb{N})} \|b_{k,l}\|_{\ell^2(\mathbb{N})} < \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} + \varepsilon < \|c\|_{\mathcal{Y}} + 2\varepsilon.$$

Relabeling, we have a representation  $c = \sum_{n=1}^{\infty} a_n \star b_n$  where  $a_n$  and  $b_n$  are finite sequences and  $\sum_n \|a_n\|_{\ell^2(\mathbb{N})} \|b_n\|_{\ell^2(\mathbb{N})} < \|c\|_{\mathcal{Y}} + 2\varepsilon$ . Let  $c_N = \sum_{n=1}^N a_n \star b_n$ . Then  $c_N \rightarrow c$  in  $\mathcal{Y}$ , and furthermore  $(c_N)$  is a Cauchy sequence in  $X$ , hence has a limit  $\tilde{c}$  in  $\mathcal{X}$ . By continuity of the functionals  $e_n$  on both  $\mathcal{Y}$  and  $\mathcal{X}$ , we find in view of (8) that  $V\tilde{c} = c$ . Hence  $V$  is onto.

Furthermore, since  $V$  is contractive,

$$\|c\|_{\mathcal{Y}} \leq \|\tilde{c}\|_{\mathcal{X}} = \lim_{N \rightarrow \infty} \|c_N\|_X < \|c\|_{\mathcal{Y}} + 2\varepsilon.$$

We already showed that  $V$  is injective, so that  $\tilde{c}$  is uniquely defined by  $c$ . On the other hand,  $\varepsilon$  is arbitrary. We conclude that  $\|c\|_{\mathcal{Y}} = \|\tilde{c}\|_{\mathcal{X}}$ . It follows that  $V$  is an isometric isomorphism.  $\square$

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