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Essential pseudospectra and essential norms of band-dominated operators

Raffael Hagger[*], Marko Lindner[†], Markus Seidel[‡]

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Abstract

An operator A on an l^p-space is called band-dominated if it can be approximated, in the operator norm, by operators with a banded matrix representation. The coset of A in the Calkin algebra determines, for example, the Fredholmness of A, the Fredholm index, the essential spectrum, the essential norm and the so-called essential pseudospectrum of A. This coset can be identified with the collection of all so-called limit operators of A. It is known that this identification preserves invertibility (hence spectra). We now show that it also preserves norms and in particular resolvent norms (hence pseudospectra). In fact we work with a generalization of the ideal of compact operators, so-called $\mathcal{P}$-compact operators, allowing for a more flexible framework that naturally extends to l^p-spaces with $p \in \{1, \infty\}$ and/or vector-valued l^p-spaces.
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1 Introduction

This first section comes as a rough guide to this paper. Proper definitions and theorems are given in later sections.

We study bounded linear operators on a Banach space $\mathbf{X}$. Most of the time, $\mathbf{X}$ is an l^p sequence space with $1 \le p \le \infty$, index set $\mathbb{Z}^N$ and values in another Banach space X, so that an operator on $\mathbf{X} = l^p(\mathbb{Z}^N, X)$ can be identified, in a natural way, with an infinite matrix (a_{ij}) with indices $i, j \in \mathbb{Z}^N$ and all a_{ij} being operators $X \to X$.

For such an operator A on $\mathbf{X}$, write $A \in \mathcal{K}_0(\mathbf{X}, \mathcal{P})$ if its matrix (a_{ij}) has finite support (i.e. only finitely many nonzero entries), and write $A \in \mathcal{A}_0(\mathbf{X})$ if its matrix is a band matrix (i.e. it has only finitely many nonzero diagonals). Clearly, $\mathcal{A}_0(\mathbf{X})$ is an algebra containing $\mathcal{K}_0(\mathbf{X}, \mathcal{P})$ as a (two-sided) ideal. Denote the closure, in the $\mathbf{X} \to \mathbf{X}$ operator norm, of $\mathcal{A}_0(\mathbf{X})$ by $\mathcal{A}(\mathbf{X})$ and the closure of $\mathcal{K}_0(\mathbf{X}, \mathcal{P})$ by $\mathcal{K}(\mathbf{X}, \mathcal{P})$. Then $\mathcal{A}(\mathbf{X})$ is a Banach algebra containing $\mathcal{K}(\mathbf{X}, \mathcal{P})$ as a closed ideal[1].

[*]Raffael.Hagger@tuhh.de, Hamburg University of Technology (TUHH), Germany
[†]Marko.Lindner@tuhh.de, Hamburg University of Technology (TUHH), Germany (corresponding author)
[‡]Markus.Seidel@fh-zwickau.de, University of Applied Sciences Zwickau, Germany
[1]We will explain the notation $\mathcal{K}(\mathbf{X}, \mathcal{P})$ later and say what $\mathcal{P}$ is.

Operators in $\mathcal{A}(\mathbf{X})$ are called band-dominated operators. The ideal $\mathcal{K}(\mathbf{X},\mathcal{P})$ is a generalization of the set of compact operators: If $\dim X < \infty$ then $\mathcal{K}(\mathbf{X},\mathcal{P})$ coincides with the set $\mathcal{K}(\mathbf{X})$ of all compact operators on $\mathbf{X}$ (except in the somewhat pathological cases $p = 1$ and $p = \infty$); otherwise it does not – as already $\mathcal{K}_0(\mathbf{X},\mathcal{P})$ contains non-compact operators. Recall that $\mathcal{K}(\mathbf{X})$ is a closed ideal in the algebra $\mathcal{L}(\mathbf{X})$ of all bounded linear operators $\mathbf{X} \to \mathbf{X}$.

For $A \in \mathcal{A}(\mathbf{X})$, the coset

$$A + \mathcal{K}(\mathbf{X},\mathcal{P}) \text{ in the quotient algebra } \mathcal{A}(\mathbf{X})/\mathcal{K}(\mathbf{X},\mathcal{P}) \tag{1.1}$$

is of interest. If $\mathcal{K}(\mathbf{X},\mathcal{P}) = \mathcal{K}(\mathbf{X})$ then the quotient norm of (1.1) is the usual essential norm of A, the spectrum of (1.1) is the essential spectrum of A, and the invertibility of (1.1) corresponds to A being a Fredholm operator (i.e. having a finite-dimensional kernel and a finite-codimensional range). In the general case one gets generalized versions of these quantities and properties.

In either case, the coset (1.1) is an interesting but complicated object. Our strategy for its study is a localization technique that replaces this one complicated object by a family of many simpler objects. The key observation is that, by the definition of the ideal $\mathcal{K}(\mathbf{X},\mathcal{P})$, the coset (1.1) depends only (and exactly) on the asymptotic behaviour of the matrix behind A. This asymptotic behaviour is extracted as follows: For every $k \in \mathbb{Z}^N$, let $V_k : \mathbf{X} \to \mathbf{X}$ denote the k-shift operator that maps $(x_i)_{i\in\mathbb{Z}^N}$ to $(y_i)_{i\in\mathbb{Z}^N}$ with $y_{i+k} = x_i$, and then look at the translates $V_{-k}AV_k$ of A. The simpler objects that characterize the coset (1.1) are the partial limits of the family $(V_{-k}AV_k)_{k\in\mathbb{Z}^N}$ of all translates of A with respect to the so-called $\mathcal{P}$-topology, to be described below, that corresponds to entry-wise norm convergence of the matrix. More precisely, if $h = (h_1, h_2, ...)$ is a sequence in $\mathbb{Z}^N$ with $|h_n| \to \infty$ and $V_{-h_n}AV_{h_n}$ converges in that topology then we denote the limit by A_h and call it the limit operator of A with respect to the sequence h. Doing this with all such sequences that produce a limit operator yields the collection

$$\sigma_{\mathrm{op}}(A) := \{A_h \,:\, h = (h_1, h_2, ...),\, h_n \in \mathbb{Z}^N,\, |h_n| \to \infty,\, A_h := \mathcal{P}\text{-}\lim V_{-h_n}AV_{h_n} \text{ exists}\} \tag{1.2}$$

of all limit operators – the so-called operator spectrum of A. We have used sequences h to address our partial limits of $(V_{-k}AV_k)_{k\in\mathbb{Z}^N}$. The same set (1.2) can also be constructed as follows ([31],[40]): Extend the mapping $\varphi_A : k \in \mathbb{Z}^N \mapsto V_{-k}AV_k \in \mathcal{A}(\mathbf{X})$ $\mathcal{P}$-continuously to the (Stone-Čech) boundary $\partial\mathbb{Z}^N$ of $\mathbb{Z}^N$. Then (1.2) exactly collects the values of φ_A on $\partial\mathbb{Z}^N$. Enumerating the set (1.2) via $\partial\mathbb{Z}^N$ (rather than via the set of all sequences h in $\mathbb{Z}^N$ for which A_h exists) has the benefit that the index set $\partial\mathbb{Z}^N$ is independent of A, so that two instances of (1.2) can be added or multiplied elementwise. Under these operations, the map $A \mapsto \varphi_A|_{\partial\mathbb{Z}^N} = (1.2)$ turns out to be an algebra homomorphism. Now the crucial point is that $\mathcal{K}(\mathbf{X},\mathcal{P})$ is exactly the kernel of that homomorphism $A \mapsto (1.2)$, whence $(1.1) \mapsto (1.2)$ is a well-defined algebra isomorphism[2]. In short: The set (1.2) nicely reflects the coset (1.1). Actually, besides $A \in \mathcal{A}(\mathbf{X})$, there is one technical condition to make this identification between the coset (1.1) and the set (1.2) work: To make sure that (1.2) is large enough, we have to assume that $\{V_{-k}AV_k : k \in \mathbb{Z}^N\}$ has a sequential compactness property, namely that every sequence h in $\mathbb{Z}^N$ with $|h_n| \to \infty$ has a subsequence g for which the $\mathcal{P}$-limit A_g exists, in which case we call A a rich operator (in the sense that (1.2) is rich enough to reflect all[3] of (1.1)).

[2] To oversimplify matters, think of continuous functions f on a compact set D. Then the subspace (actually the ideal) $C_0(D)$ of continuous functions with zero boundary values is the kernel of the algebra homomorphism $f \mapsto f|_{\partial D}$, whence the coset of f modulo $C_0(D)$ can be identified with $f|_{\partial D}$, by the fundamental homomorphism theorem.

[3] In fact, the map $A \mapsto \sigma_{\mathrm{op}}(A) = (1.2)$ sends some operators $A \in \mathcal{A}(\mathbf{X})$ to $\varnothing$. For some other $A \in \mathcal{A}(\mathbf{X})$, limit operators exist in one "direction" but not in another. Some of the latter A are not in $\mathcal{K}(\mathbf{X},\mathcal{P})$ but have $\sigma_{\mathrm{op}}(A) = \{0\}$, such as our first example in Remark 3.6. These problems are eliminated by imposing existence of sufficiently many limit operators, i.e. richness of A.

This identification between the objects (1.1) and (1.2) for a rich operator $A \in \mathcal{A}(\mathbf{X})$ is at the core of the limit operator method. Here are some of its consequences:

(i) The main theorem on limit operators [27, 28, 17, 20] says that (1.1) is invertible (so that A is a generalized Fredholm operator) iff every element of (1.2) is invertible.

(ii) Expressing this in the language of spectra, we get that

$$\mathrm{sp}_{\mathrm{ess}}(A) = \bigcup_{A_h \in \sigma_{\mathrm{op}}(A)} \mathrm{sp}(A_h), \tag{1.3}$$

where $\mathrm{sp}_{\mathrm{ess}}(A)$ denotes the spectrum of the coset (1.1), the so-called $\mathcal{P}$-essential spectrum of A, and $\mathrm{sp}(A_h)$ denotes the usual spectrum of A_h as an element of $\mathcal{L}(\mathbf{X})$.

(iii) In addition to (i), the inverse[4] of (1.1) corresponds to the elementwise inverse[4] of (1.2), by [33, Theorem 16].

(iv) In the current paper we show that the norm of (1.1) equals[5] the supremum (in fact maximum) norm of (1.2). We refer to the norm of (1.1) as the essential norm of A.

(v) By a combination of (iii) and (iv), one derives

$$\|[(A - \lambda I) + \mathcal{K}(\mathbf{X}, \mathcal{P})]^{-1}\| = \max_{A_h \in \sigma_{\mathrm{op}}(A)} \|(A_h - \lambda I)^{-1}\| \tag{1.4}$$

for all $\lambda \in \mathbb{C}$, which is an equality between corresponding resolvent norms in (1.1) and (1.2). This in turn proves the following pseudospectral version of (1.3)

$$\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) = \bigcup_{A_h \in \sigma_{\mathrm{op}}(A)} \mathrm{sp}_\varepsilon(A_h), \qquad \varepsilon > 0. \tag{1.5}$$

Here $\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A)$ is the set of all $\lambda \in \mathbb{C}$ for which the left-hand side of (1.4) is greater than $\frac{1}{\varepsilon}$, and $\mathrm{sp}_\varepsilon(A_h)$ is the usual pseudospectrum of A_h that we will discuss in a minute. We will see that $\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A)$ is the pseudospectrum, in the same sense, of the coset (1.1); it will henceforth be referred to as the essential pseudospectrum of A.

Here is an important superset of the spectrum: For an operator $A \in \mathcal{L}(\mathbf{X})$ or, more generally, an element a in a Banach algebra B with unit e, it is sometimes a more sensible question to ask whether the inverse of $a - \lambda e$ is large in norm, possibly non-existent, rather than just to ask for the latter. So one defines the *ε-pseudospectrum* of a by

$$\mathrm{sp}_\varepsilon(a) := \left\{ \lambda \in \mathbb{C} : \|(a - \lambda e)^{-1}\| > \frac{1}{\varepsilon} \right\}, \qquad \varepsilon > 0,$$

where we say $\|b^{-1}\| := \infty$ if b is non-invertible, so that $\mathrm{sp}(a) \subset \mathrm{sp}_\varepsilon(a)$. This defines both $\mathrm{sp}_\varepsilon(A)$ as the pseudospectrum of an operator A in $B = \mathcal{L}(\mathbf{X})$ and $\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A)$ as the pseudospectrum of a

[4] Remarkably, this inverse coset, resp. set of inverses, is again a subset of $\mathcal{A}(\mathbf{X})$, by [28, Propositions 2.1.8 and 2.1.9] (we recall this in Theorem 2.5 below).

[5] Note that in the case $\mathbf{X} = l^2(\mathbb{Z}^N, X)$ with a Hilbert space X, $\mathcal{L}(\mathbf{X})$ and $\mathcal{A}(\mathbf{X})$ are C^*-algebras, so that this equality of norms is a simple consequence of (i) since C^*-homomorphisms that preserve invertibility do also preserve norms. In the general case such elegant arguments are not available anymore.

coset (1.1) in the quotient algebra $B = \mathcal{A}(\mathbf{X})/\mathcal{K}(\mathbf{X}, \mathcal{P})$. For $A \in \mathcal{L}(\mathbf{X})$ there is the remarkable coincidence (see e.g. [4, Theorem 7.4])

$$\mathrm{sp}_\varepsilon(A) = \bigcup_{\|T\|<\varepsilon} \mathrm{sp}(A+T) \tag{1.6}$$

for all $\varepsilon > 0$, showing that $\mathrm{sp}_\varepsilon(A)$ exactly measures the sensitivity of $\mathrm{sp}(A)$ with respect to additive perturbations of A by operators $T \in \mathcal{L}(\mathbf{X})$ of norm less than ε. For normal operators A on Hilbert space, $\mathrm{sp}_\varepsilon(A)$ is exactly the ε-neighbourhood of $\mathrm{sp}(A)$. In general it can be much larger. Pseudospectra are interesting objects by themselves since they carry more information than spectra (e.g. about transient instead of just asymptotic behaviour of dynamical systems). Also, they have better convergence and approximation properties than spectra ($\mathrm{sp}_\varepsilon(A)$ depends continuously on A – unlike $\mathrm{sp}(A)$). Still, the ε-pseudospectra approximate the spectrum as $\varepsilon \to 0$.

On the other hand, there is the ($\mathcal{P}$-)essential spectrum $\mathrm{sp}_{\mathrm{ess}}(A)$. This set is robust under ($\mathcal{P}$-)compact perturbations, enabling its study by means of limit operators via (1.3).

The essential pseudospectrum, $\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A)$, nicely blends these properties of essential and pseudospectra: We have already mentioned that it inherits an ε-version, (1.5), of (1.3). We will also show that there is an essential version of (1.6), that is

$$\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) = \bigcup_{\|T\|<\varepsilon} \mathrm{sp}_{\mathrm{ess}}(A+T) \tag{1.7}$$

for all $\varepsilon > 0$, where the perturbations T come from $\mathcal{A}(\mathbf{X})$. So in this new setting, the different properties (1.3) and (1.6) both generalize and meet in one place.

Besides these aesthetical aspects, our argument for the study of $\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A)$ is as follows: When $\mathrm{sp}_{\mathrm{ess}}(A)$ is of interest, the problem with formula (1.3) is the computation of all limit operators A_h and then of their spectra $\mathrm{sp}(A_h)$. It appears more feasible, from a numerical perspective, to compute the pseudospectra $\mathrm{sp}_\varepsilon(A_h)$ for small values of ε, then derive $\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A)$ by (1.5) and finally use that the closure of $\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A)$ tends to $\mathrm{sp}_{\mathrm{ess}}(A)$ in Hausdorff metric as $\varepsilon \to 0$.

Previous work The story of limit operators probably began in the late 1920's in Favard's paper [9] for studying ODEs with almost-periodic coefficients. It continued in the work of Muhamadiev [22, 23, 24, 25] and was later followed by Lange and Rabinovich [16], who were the first to consider Fredholmness for the generic class of band-dominated operators. In the last 20 years, major work was done by Rabinovich, Roch, Roe and Silbermann [27, 28, 26] with recent contributions by some of the authors and Chandler-Wilde [17, 7, 34, 33, 20]. A detailed review of this history is, for example, in the introduction of [7]. A comprehensive presentation of these results, further achievements and applications e.g. to convolution and pseudo-differential operators, as well as the required tools, can be found in the 2004 book [28] of Rabinovich, Roch and Silbermann. This literature shows that the list of parallels between the items (1.1) and (1.2) is actually longer than our list (i)–(v). For example, in [26] it is shown for the case $\mathbf{X} = l^2(\mathbb{Z}^1, \mathbb{C})$ that the Fredholm index of A can be recovered from two fairly arbitrary elements of (1.2).

Apart from the theory of limit operators, there is of course a vast amount of literature on spectral theory. Particularly related is the work of Trefethen, Embree and others (see [41] and references therein) on pseudospectra. We probably have to mention [1, 2], where essential pseudospectra have been defined.

Summary of contents In Section 2 we summarize the main definitions and previously known results including (i), (ii) and (iii) from above. Section 3 is devoted to the proof of (iv). It is then straightforward to conclude (v). Section 4 introduces and gives basic results about essential pseudospectra. Section 5 turns the attention to the so-called lower norm $\nu(B)$ of an operator B, which is the infimum of $\|Bx\|$ over all x with $\|x\| = 1$. While the norm $\|B^{-1}\|$ of the inverse (if existent), as in the right-hand side of (1.4), can be expressed as $1/\nu(B)$, it is the subject of Section 5 to characterize (or at least bound) also the essential norm of the resolvent on the left-hand side of (1.4) by means of lower norms, without explicit reference to limit operators – thereby giving different approaches to the computation of the essential pseudospectrum (or to upper and lower bounds on it). In Section 6 we discuss an application of our results in the context of approximation methods. For reasons of numerical viability, an operator $A \in \mathcal{A}(\mathbf{X})$ is usually approximated by finite-dimensional operators A_n, hoping that their inverses A_n^{-1} will exist and approximate A^{-1}, provided the latter exists. The key question here is about the stability of the sequence $(A_n)_{n\in\mathbb{N}}$ – and that can be translated into the language of $\mathcal{P}$-Fredholmness of an associated operator. Here our previous results yield new quantitative insights, in particular (partial) limits of the norms $\|A_n^{-1}\|$ and the condition numbers $\kappa(A_n)$.

2 Definitions and known results

In this section we give the relevant definitions and state corresponding theorems about what has been known, including points (i), (ii) and (iii) from the introduction.

Banach spaces and projections Throughout this paper $\mathbf{Y}$ denotes a complex Banach space. On $\mathbf{Y}$ we look at a sequence $\mathcal{P} = (P_n)_{n\in\mathbb{N}}$ of projections P_n with the properties

($\mathcal{P}1$) $P_n = P_nP_{n+1} = P_{n+1}P_n$ and $\|P_n\| = \|Q_n\| = 1$ for all $n \in \mathbb{N}$, where $Q_n := I - P_n$;
($\mathcal{P}2$) $C_{\mathcal{P}} := \sup_{U\subset\mathbb{N}} \|\sum_{n\in U}(P_{n+1} - P_n)\| < \infty$ with the supremum taken over all finite sets $U \subset \mathbb{N}$.

Then $\mathcal{P}$ is a so-called *uniform approximate projection* in the sense of [28, 36, 34]. If additionally

($\mathcal{P}3$) $\sup_{n\in\mathbb{N}} \|P_nx\| \geq \|x\|$ for all $x \in \mathbf{Y}$

then $\mathcal{P}$ is said to be a *uniform approximate identity*. Note that the dual sequence $\mathcal{P}^* = (P_n^*)$ then also has the corresponding properties ($\mathcal{P}1$) and ($\mathcal{P}2$) on the dual space $\mathbf{Y}^*$ but not necessarily ($\mathcal{P}3$).

However, in large parts we consider the more particular situation of a (generalized) sequence space $\mathbf{X} = l^p(\mathbb{Z}^N, X)$ with parameters $p \in \{0\} \cup [1, \infty]$, $N \in \mathbb{N}$ and a complex Banach space X. These (generalized) sequences are of the form $x = (x_i)_{i\in\mathbb{Z}^N}$ with all $x_i \in X$. The spaces are equipped with the usual p-norm. In our notation, $l^0(\mathbb{Z}^N, X)$ stands for the closure in $l^\infty(\mathbb{Z}^N, X)$ of the set of all sequences $(x_i)_{i\in\mathbb{Z}^N}$ with finite support. In the context of these sequence spaces $\mathbf{X}$, $\mathcal{P} = (P_n)$ shall always be the sequence of the canonical projections[6] $P_n := \chi_{\{-n,\dots,n\}^N} I$ which obviously forms a uniform approximate identity on $\mathbf{X}$.

Notice that this variety of spaces $\mathbf{X} = l^p(\mathbb{Z}^N, X)$ in particular covers the spaces $L^p(\mathbb{R}^N)$ by the natural isometric identification of a function in $L^p(\mathbb{R}^N)$ with the sequence of its restrictions to the hypercubes $i + [0, 1]^N$ with $i \in \mathbb{Z}^N$ (saying that $L^p(\mathbb{R}^N) \cong l^p(\mathbb{Z}^N, L^p([0, 1]^N)))$.

[6]Here and in what follows we write $\chi_M : \mathbb{Z}^N \to \{0, 1\}$ for the characteristic function of $M \subset \mathbb{Z}^N$, that is $\chi_M(k) = 1$ if $k \in M$ and $= 0$ otherwise.

Operators and convergence The following definitions and results are taken from e.g. [28, 17, 33]. Starting with a Banach space $\mathbf{Y}$ and a uniform approximate projection $\mathcal{P} = (P_n)$, one says that a bounded linear operator K on $\mathbf{Y}$ is *$\mathcal{P}$-compact* if

$$\|(I - P_n)K\| + \|K(I - P_n)\| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.1}$$

The set[7] of all $\mathcal{P}$-compact operators is denoted by $\mathcal{K}(\mathbf{Y}, \mathcal{P})$. Unlike the set $\mathcal{K}(\mathbf{Y})$ of all compact operators on $\mathbf{Y}$, $\mathcal{K}(\mathbf{Y}, \mathcal{P})$ is in general not an ideal in $\mathcal{L}(\mathbf{Y})$. So we introduce the following subset of $\mathcal{L}(\mathbf{Y})$:

$$\mathcal{L}(\mathbf{Y}, \mathcal{P}) := \{A \in \mathcal{L}(\mathbf{Y}) : AK, KA \in \mathcal{K}(\mathbf{Y}, \mathcal{P}) \text{ for all } K \in \mathcal{K}(\mathbf{Y}, \mathcal{P})\}.$$

Now $\mathcal{L}(\mathbf{Y}, \mathcal{P})$ forms a closed subalgebra of $\mathcal{L}(\mathbf{Y})$ containing $\mathcal{K}(\mathbf{Y}, \mathcal{P})$ as a closed two-sided ideal.

Generalizing usual Fredholmness and the essential spectrum, one now studies invertibility modulo $\mathcal{K}(\mathbf{Y}, \mathcal{P})$: An operator $A \in \mathcal{L}(\mathbf{Y})$ is said to be *invertible at infinity* if there is a so-called *$\mathcal{P}$-regularizer* $B \in \mathcal{L}(\mathbf{Y})$ such that $AB - I$ and $BA - I$ are $\mathcal{P}$-compact. Similarly, $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ is called *$\mathcal{P}$-Fredholm* if the coset $A + \mathcal{K}(\mathbf{Y}, \mathcal{P})$ is invertible in the quotient algebra $\mathcal{L}(\mathbf{Y}, \mathcal{P})/\mathcal{K}(\mathbf{Y}, \mathcal{P})$. For $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$, the *$\mathcal{P}$-essential spectrum* $\operatorname{sp}_{\text{ess}}(A)$ is then the set of all $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is not $\mathcal{P}$-Fredholm.

Theorem 2.1. *([36, Theorem 1.16])*
An operator $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ is $\mathcal{P}$-Fredholm if and only if it is invertible at infinity. In this case every $\mathcal{P}$-regularizer of A belongs to $\mathcal{L}(\mathbf{Y}, \mathcal{P})$. Particularly, $\mathcal{L}(\mathbf{Y}, \mathcal{P})$ is inverse closed in $\mathcal{L}(\mathbf{Y})$.

Finally, if $\mathcal{P}$ is a uniform approximate identity, say that a sequence $(A_n) \subset \mathcal{L}(\mathbf{Y})$ converges *$\mathcal{P}$-strongly* to an operator $A \in \mathcal{L}(\mathbf{Y})$ if

$$\|K(A_n - A)\| + \|(A_n - A)K\| \to 0 \quad \text{as} \quad n \to \infty \tag{2.2}$$

for every $K \in \mathcal{K}(\mathbf{Y}, \mathcal{P})$. We shortly write $A_n \xrightarrow{\mathcal{P}} A$ or $A = \mathcal{P}\text{-}\lim A_n$ in that case. Note that (2.1) and (2.2) immediately imply[8] $P_n \xrightarrow{\mathcal{P}} I$. By [17, Theorem 1.65], $A_n \xrightarrow{\mathcal{P}} A$ is equivalent to the sequence (A_n) being bounded and

$$\|P_m(A_n - A)\| + \|(A_n - A)P_m\| \to 0 \quad \text{as} \quad n \to \infty$$

for every $m \in \mathbb{N}$. Also note [28, Prop. 1.1.17] that for the $\mathcal{P}$-limit A one has

$$A \in \mathcal{L}(\mathbf{Y}, \mathcal{P}) \qquad \text{and} \qquad \|A\| \leq \liminf \|A_n\| \tag{2.3}$$

if all A_n are in $\mathcal{L}(\mathbf{Y}, \mathcal{P})$.

$\mathcal{P}$-compactness determines the notions of $\mathcal{P}$-convergence and $\mathcal{P}$-Fredholmness just like compactness does with strong convergence and the usual Fredholmness [28, Section 1.1].

Remark 2.2. For $\mathbf{Y} = \mathbf{X} = l^p(\mathbb{Z}^N, X)$, one has $\mathcal{K}(\mathbf{X}, \mathcal{P}) \supset \mathcal{K}(\mathbf{X})$ if $p \in (1, \infty)$, whereas $\mathcal{K}(\mathbf{X}, \mathcal{P}) \subset \mathcal{K}(\mathbf{X})$ if $\dim X < \infty$. So for $p \in (1, \infty)$ and $\dim X < \infty$, the $\mathcal{P}$-notions coincide with the classical ones: $\mathcal{K}(\mathbf{X}, \mathcal{P}) = \mathcal{K}(\mathbf{X})$, $\mathcal{L}(\mathbf{X}, \mathcal{P}) = \mathcal{L}(\mathbf{X})$, an operator is $\mathcal{P}$-Fredholm if and only if it is Fredholm, and a sequence (A_n) converges $\mathcal{P}$-strongly to A if and only if $A_n \to A$ and $A_n^* \to A^*$ strongly.

[7] This set is the closure (in $\mathcal{L}(\mathbf{Y})$) of the set $\mathcal{K}_0(\mathbf{Y}, \mathcal{P})$ of all $K \in \mathcal{L}(\mathbf{Y})$ for which $\|(I - P_n)K\| + \|K(I - P_n)\| = 0$ for all sufficiently large n. $\mathcal{K}_0(\mathbf{Y}, \mathcal{P})$ corresponds to matrices with finite support if $\mathbf{Y} = \mathbf{X} = l^p(\mathbb{Z}^N, X)$.

[8] It becomes clear that $\mathcal{K}(\mathbf{Y}, \mathcal{P})$ and $\xrightarrow{\mathcal{P}}$ are actually tailor-made by (2.1) and (2.2) for this purpose.

The reason for the definition of the $\mathcal{P}$-notions is to extend the well-known concepts, tools and connections between them in a way that they still apply to relevant operators and sequences in the cases $p \in \{1, \infty\}$ and/or $\dim X = \infty$. For example, although $P_n \not\to I$ if $p = \infty$ and $P_n^* \not\to I^*$ if $p = 1$, one still has $P_n \xrightarrow{\mathcal{P}} I$ in all cases. Also, each P_n is $\mathcal{P}$-compact, although not compact, in case $\dim X = \infty$.

Anyway, on $\mathbf{X}$, the (classical) Fredholm property nicely fits into the generalized $\mathcal{P}$-setting:

Proposition 2.3. *[33, Corollary 12]*
Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ be Fredholm. Then A is $\mathcal{P}$-Fredholm and has a generalized inverse $B \in \mathcal{L}(\mathbf{X}, \mathcal{P})$, i.e. $A = ABA$ and $B = BAB$. Moreover, A is Fredholm of index zero if and only if there exists an invertible operator $C \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ and an operator $K \in \mathcal{K}(\mathbf{X}, \mathcal{P})$ of finite rank such that $A = C + K$.

Equivalent approximate projections If we fix an approximate projection $\mathcal{P}$ and an operator $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$, we can always find an equivalent approximate projection that is tailored for A. This provides noticeable simplifications in many arguments.

Proposition 2.4. *(extension of [36, Theorem 1.15])*
Let $\mathcal{P}$ be a uniform approximate projection on $\mathbf{Y}$ and $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$.
Then there exists a sequence $\hat{\mathcal{P}} = (F_n)$ of operators that satisfies $(\mathcal{P}1)$ and $(\mathcal{P}2)$ with $C_{\hat{\mathcal{P}}} \le C_{\mathcal{P}}$, and for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $F_n P_m = P_m F_n = F_n$ as well as $P_n F_m = F_m P_n = P_n$, and $\|[A, F_n]\| = \|AF_n - F_n A\| \to 0$ as $n \to \infty$.
If $\mathcal{P}$ is a uniform approximate identity, then $\lim_n \|F_n x\| = \|x\|$ for every $x \in \mathbf{Y}$.

Proof. The existence of (F_n) with $F_n P_m = P_m F_n = F_n$ and $P_n F_m = F_m P_n = P_n$ as announced, and $\|[A, F_n]\| = \|AF_n - F_n A\| \to 0$ as $n \to \infty$ was proved in [36, Theorem 1.15]. Actually, for each $n \in \mathbb{N}$, these F_n are of the form (see [36, Equation (1.4)] and the proof there)

$$F_n = \frac{1}{n} \sum_{k=1}^{n} k P_{U_{n-k}^n} = \frac{1}{n} \left(\sum_{k=1}^{n-1} k (P_{r_{n-k+1}^n} - P_{r_{n-k}^n}) + n P_{r_1^n} \right) = \frac{1}{n} \sum_{k=1}^{n} P_{r_k^n}$$

with certain integers $1 < r_1^n < r_2^n < \ldots < r_n^n$. Thus,

$$1 = \|P_1\| = \|P_1 F_n\| \le \|F_n\| \le \frac{1}{n} \sum_{k=1}^{n} \|P_{r_k^n}\| = \frac{n}{n} = 1.$$

Similarly, for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$1 = \|I - P_m\| = \|(I - P_m)(I - F_n)\| \le \|I - F_n\| \le \frac{1}{n} \sum_{k=1}^{n} \|I - P_{r_k^n}\| = \frac{n}{n} = 1.$$

For $F_n = F_n F_{n+1} = F_{n+1} F_n$ and $(\mathcal{P}2)$ see again [36, Theorem 1.15]. Finally, since $\|F_n x\| = \|F_n P_m x\| \le \|P_m x\|$ and $\|P_n x\| = \|P_n F_m x\| \le \|F_m x\|$ for $m \gg n$ we have

$$\sup_n \|F_n x\| = \lim_n \|F_n x\| = \lim_n \|P_n x\| = \sup_n \|P_n x\|$$

for each $x \in \mathbf{Y}$. Hence if additionally $(\mathcal{P}3)$ is fulfilled then $\lim_n \|F_n x\| = \|x\|$ for every $x \in \mathbf{Y}$. $\square$

Band and band-dominated operators Every sequence $a = (a_n) \in l^\infty(\mathbb{Z}^N, \mathcal{L}(X))$ gives rise to an operator $aI \in \mathcal{L}(\mathbf{X})$, a so-called multiplication operator, via the rule $(ax)_i = a_i x_i$, $i \in \mathbb{Z}^N$. For every $\alpha \in \mathbb{Z}^N$, we define the shift operator $V_\alpha : \mathbf{X} \to \mathbf{X}$, $(x_i) \mapsto (x_{i-\alpha})$.

A band operator is a finite sum of the form $\sum a_\alpha V_\alpha$, where $a_\alpha I$ are multiplication operators. In terms of the generalized matrix-vector multiplication

$$(a_{ij})_{i,j\in\mathbb{Z}^N}\ (x_j)_{j\in\mathbb{Z}^N} = (y_i)_{i\in\mathbb{Z}^N} \quad \text{with} \quad y_i \ = \sum_{j\in\mathbb{Z}^N} a_{ij}x_j,\ i \in \mathbb{Z}^N, \quad \text{where } a_{ij} \in \mathcal{L}(X),$$

band operators A act on $\mathbf{X} = l^p(\mathbb{Z}^N, X)$ via multiplication by band matrices (a_{ij}), that means $a_{ij} = 0$ if $|i - j|$ exceeds the so-called band-width of A. Typical examples are discretizations of differential operators on $\mathbb{R}^N$.

In many physical models, however, interaction a_{ij} between data at locations i and j decreases in a certain way as $|i - j| \to \infty$ rather than suddenly stop at a prescribed distance of i and j. An operator is called *band-dominated* if it is contained in the $\mathcal{L}(\mathbf{X})$-closure, denoted by $\mathcal{A}(\mathbf{X})$, of the set $\mathcal{A}_0(\mathbf{X})$ of all band operators . In contrast to $\mathcal{A}_0(\mathbf{X})$ (which is an algebra but not closed in $\mathcal{L}(\mathbf{X})$), the set $\mathcal{A}(\mathbf{X})$ is a Banach algebra, for which the inclusions

$$\mathcal{K}(\mathbf{X}, \mathcal{P}) \subset \mathcal{A}(\mathbf{X}) \subset \mathcal{L}(\mathbf{X}, \mathcal{P}) \subset \mathcal{L}(\mathbf{X})$$

hold. In particular, $\mathcal{K}(\mathbf{X}, \mathcal{P})$ is a two-sided closed ideal in $\mathcal{A}(\mathbf{X})$.

Theorem 2.5. *[28, Propositions 2.1.7 et seq.]*
Let $A \in \mathcal{A}(\mathbf{X})$ be $\mathcal{P}$-Fredholm. Then every $\mathcal{P}$-regularizer of A is band-dominated as well. In particular, the quotient algebra $\mathcal{A}(\mathbf{X})/\mathcal{K}(\mathbf{X}, \mathcal{P})$ is inverse closed in $\mathcal{L}(\mathbf{X}, \mathcal{P})/\mathcal{K}(\mathbf{X}, \mathcal{P})$, and $\mathcal{A}(\mathbf{X})$ is inverse closed in both $\mathcal{L}(\mathbf{X}, \mathcal{P})$ and $\mathcal{L}(\mathbf{X})$.

So for $A \in \mathcal{A}(\mathbf{X})$, the following are equivalent:

- A is invertible at infinity (i.e. it has a $\mathcal{P}$-regularizer in $\mathcal{L}(\mathbf{X})$),
- A is $\mathcal{P}$-Fredholm (it has a $\mathcal{P}$-regularizer in $\mathcal{L}(\mathbf{X}, \mathcal{P})$),
- the coset (1.1) is invertible (A has a $\mathcal{P}$-regularizer in $\mathcal{A}(\mathbf{X})$).

The first studies of particular subclasses of band-dominated operators and their Fredholm properties were for the case of constant matrix diagonals, that is when the matrix entries a_{ij} only depend on the difference $i - j$, so that A is a convolution operator (a.k.a. Laurent or bi-infinite Toeplitz matrix, the stationary case) [10, 39, 11, 12, 5]. Subsequently, the focus went to more general classes, such as convergent, periodic and almost periodic matrix diagonals, until at the current point arbitrary matrix diagonals can be studied – as long as they are bounded. This possibility is due to the notion of limit operators that enables evaluation of the asymptotic behavior of an operator A even for merely bounded diagonals in the matrix (a_{ij}).

Limit operators Say that a sequence $h = (h_n) \subset \mathbb{Z}^N$ tends to infinity if $|h_n| \to \infty$ as $n \to \infty$. If $h = (h_n) \subset \mathbb{Z}^N$ tends to infinity and $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ then

$$A_h := \mathcal{P}\text{-}\lim_{n\to\infty} V_{-h_n} A V_{h_n},$$

if it exists, is called the *limit operator* of A w.r.t. the sequence h. The set (1.2) of all limit operators of A is its *operator spectrum*, $\sigma_{\mathrm{op}}(A)$.

Proposition 2.6. *[28, Proposition 1.2.2] Let $A, B \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ and $h = (h_n) \subset \mathbb{Z}^N$ tend to infinity such that A_h and B_h exist. Then:*

- *also $(A+B)_h$ and $(AB)_h$ exist, where $(A+B)_h = A_h + B_h$ and $(AB)_h = A_h B_h$;*
- *if $p < \infty$, also $(A^*)_h$ exists and equals $(A_h)^*$;*
- *the inequality $\|A_h\| \le \|A\|$ holds.*

Theorem 2.7. *[33, Theorem 16]*
Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ be $\mathcal{P}$-Fredholm. Then all limit operators of A are invertible and their inverses are uniformly bounded. Moreover, the operator spectrum of every $\mathcal{P}$-regularizer B of A equals

$$\sigma_{\mathrm{op}}(B) = \{A_h^{-1} : A_h \in \sigma_{\mathrm{op}}(A)\}. \tag{2.4}$$

We say that $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ has a rich operator spectrum (or we simply call A a *rich operator*) if every sequence $h \subset \mathbb{Z}^N$ tending to infinity has a subsequence $g \subset h$ such that the limit operator A_g of A w.r.t. g exists. The set of all rich operators $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ is denoted by $\mathcal{L}_\$(\mathbf{X}, \mathcal{P})$ and the set of all rich band-dominated operators by $\mathcal{A}_\$(\mathbf{X})$. Recall from [28, Corollary 2.1.17] that $\mathcal{A}_\$(\mathbf{X}) = \mathcal{A}(\mathbf{X})$ whenever $\dim X < \infty$. For rich operators we know

Theorem 2.8. *[33, Corollary 17]*

- *The set $\mathcal{L}_\$(\mathbf{X}, \mathcal{P})$ forms a closed subalgebra of $\mathcal{L}(\mathbf{X}, \mathcal{P})$ and contains $\mathcal{K}(\mathbf{X}, \mathcal{P})$ as a closed two-sided ideal.*
- *Every $\mathcal{P}$-regularizer of a rich $\mathcal{P}$-Fredholm operator is rich. Thus, $\mathcal{L}_\$(\mathbf{X}, \mathcal{P})/\mathcal{K}(\mathbf{X}, \mathcal{P})$ is inverse closed in $\mathcal{L}(\mathbf{X}, \mathcal{P})/\mathcal{K}(\mathbf{X}, \mathcal{P})$ and $\mathcal{L}_\$(\mathbf{X}, \mathcal{P})$ is inverse closed in both $\mathcal{L}(\mathbf{X}, \mathcal{P})$ and $\mathcal{L}(\mathbf{X})$.*

In the case of rich band-dominated operators, the picture is most complete:

Theorem 2.9. *([27], [7, Theorem 6.28] and [20])*
For an operator $A \in \mathcal{A}_\$(\mathbf{X})$, the following are equivalent:

- *A is $\mathcal{P}$-Fredholm,*
- *all limit operators of A are invertible and their inverses are uniformly bounded,*
- *all limit operators of A are invertible.*

This is result (i) from the introduction. Equality (1.3) from point (ii) follows by replacing A by $A - \lambda I$ in Theorem 2.9, noting that $\sigma_{\mathrm{op}}(A - \lambda I) = \sigma_{\mathrm{op}}(A) - \lambda I$. Furthermore, (iii) is an immediate consequence of Theorem 2.7.

The lower norm For an operator A between two Banach spaces, we call

$$\nu(A) \;:=\; \inf\{\|Ax\| : \|x\| = 1\}$$

the *lower norm* of A. For operators on Hilbert space, $\nu(A)$ is the smallest singular value of A. We call A *bounded below* if $\nu(A) > 0$. The following properties are well known. A proof can be found e.g. in [17, Lemmas 2.32, 2.33 and 2.35].

Lemma 2.10. *Let* $\mathbf{Y}_1$ *and* $\mathbf{Y}_2$ *be Banach spaces and* $A : \mathbf{Y}_1 \to \mathbf{Y}_2$ *be a bounded linear operator between them. Then the following hold:*

- *A is bounded below iff A is injective and its range is closed in $\mathbf{Y}_2$;*
- *A is invertible from the left iff A is injective and its range is complementable in $\mathbf{Y}_2$, in which case $\nu(A) \geq \|A^l\|^{-1}$ for every left inverse A^l;*
- *A is invertible iff A and A^* are bounded below, in which case $\nu(A) = \|A^{-1}\|^{-1}$.*

More generally,[9] $\|A^{-1}\|^{-1} = \min\{\nu(A), \nu(A^*)\}$*, where* $\nu(A) = \nu(A^*)$ *if both are positive.*

3 $\mathcal{P}$-essential norm of band-dominated operators

In this section we prove the first new result, point (iv) from the introduction, about the norm of the coset (1.1). As an immediate consequence of (iii) and (iv) we get the first half of (v). Recall that we abbreviate $I - P_n$ by Q_n.

Proposition 3.1. *Let* $\mathcal{P}$ *be a uniform approximate projection on* $\mathbf{Y}$ *and* $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$*. Then*

$$\|A + \mathcal{K}(\mathbf{Y}, \mathcal{P})\| = \|A^* + \mathcal{K}(\mathbf{Y}^*, \mathcal{P}^*)\| = \lim_{m\to\infty} \|AQ_m\| = \lim_{m\to\infty} \|Q_m A\|.$$

Proof. Let $\varepsilon > 0$ and choose $K \in \mathcal{K}(\mathbf{Y}, \mathcal{P})$ such that $\|A + K\| \leq \|A + \mathcal{K}(\mathbf{Y}, \mathcal{P})\| + \varepsilon$ and $m_0 \in \mathbb{N}$ such that $\|KQ_m\| \leq \varepsilon$ for all $m \geq m_0$. It follows

$$\|AQ_m\| = \|A - AP_m\| \geq \|A + \mathcal{K}(\mathbf{Y}, \mathcal{P})\| \geq \|A + K\| - \varepsilon \geq \|(A + K)Q_m\| - \varepsilon \geq \|AQ_m\| - 2\varepsilon$$

for all $m \geq m_0$ and therefore $\|A+\mathcal{K}(\mathbf{Y}, \mathcal{P})\| = \lim_{m\to\infty} \|AQ_m\|$ since ε was arbitrary. The equality $\|A + \mathcal{K}(\mathbf{Y}, \mathcal{P})\| = \lim_{m\to\infty} \|Q_m A\|$ is similar. Finally, $\|A^* Q_m^*\| = \|Q_m A\|$ finishes the proof. □

Now we switch to sequence spaces $\mathbf{X} = l^p(\mathbb{Z}^N, X)$ and band-dominated operators. Our first main theorem is

Theorem 3.2. *Let* $A \in \mathcal{A}_\$(\mathbf{X})$*. Then*

$$\|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\| = \max_{A_g \in \sigma_{\mathrm{op}}(A)} \|A_g\|. \tag{3.1}$$

Note that if $\mathbf{X}$ is a Hilbert space, C^*-algebra techniques can be used to deduce Theorem 3.2 directly from Theorem 2.9 (cf. [28, Thm 2.2.7]). In the general case we require the following auxiliary notion:

Definition 3.3. The *support* of a sequence $x = (x_n) \in \mathbf{X}$ is the set $\operatorname{supp} x := \{n \in \mathbb{Z}^N : x_n \neq 0\}$. The diameter of a subset $M \subset \mathbb{Z}^N$ is defined as $\operatorname{diam} M := \sup\{|n_i - m_i| : n, m \in M; i = 1, \ldots, N\}$. Moreover, we write $|M|$ for the number of elements of any set M. For $D \in \mathbb{N}$ we now define

$$|||A|||_D := \sup\left\{\frac{\|Ax\|}{\|x\|} : x \in \mathbf{X} \setminus \{0\}, \operatorname{diam}\operatorname{supp} x \leq D\right\}.$$

[9] From now on we set $\infty^{-1} := 0$, so that $\|b^{-1}\|^{-1} = 0$ if b is not invertible.

The first step is to show that the operator norm of $A \in \mathcal{A}_0(\mathbf{X})$ can be localized, up to any desired accuracy, in terms of $|||A|||_D$ (i.e. by looking at sequences $x \in \mathbf{X}$ with support of a certain diameter) in the following sense:

Proposition 3.4. *Let $A \in \mathcal{A}_0(\mathbf{X})$ and $\delta > 0$. Then there is a $D \in \mathbb{N}$ such that*

$$(1-\delta)\|A\chi_F I\| \leq |||A\chi_F I|||_D \leq \|A\chi_F I\| \quad \text{for all} \quad F \subset \mathbb{Z}^N.$$

There is a very similar statement in [20, Prop. 6] for the lower norm $\nu(A)$ that we will address in Section 5. Also the proof is very similar. In [20] there are two different proofs given, here we restrict ourselves to showing one of the two proofs that we know of (the one that uses and generalizes a technique from [6]):

Proof. Clearly $|||\cdot|||_D \leq \|\cdot\|$. So let $A \in \mathcal{A}_0(\mathbf{X})$ and let $w \in \mathbb{N}$ be its band-width, i.e. $\chi_U A \chi_V I = 0$ for all $U, V \subset \mathbb{Z}^N$ with $\operatorname{dist}(U,V) := \inf\{|u-v|_\infty : u \in U, v \in V\} > w$.

For arbitrary $n \in \mathbb{N}$ and $k \in \mathbb{Z}^N$, put $C_n := \{-n, ..., n\}^N$, $C_{n,k} := k + C_n$, $D_n := C_{n+w} \setminus C_{n-w}$, $D_{n,k} := k + D_n$, $c_n := |C_n| = |C_{n,k}| = (2n+1)^N$ and $d_n := |D_n| = |D_{n,k}| = c_{n+w} - c_{n-w} \sim n^{N-1}$. Abbreviate $\chi_{C_{n,k}} I =: P_{n,k}$ and $\chi_{D_{n,k}} I =: \Delta_{n,k}$.

We start with the case $p \in [1,\infty)$. Given such p and our arbitrary $\delta > 0$, we choose $n \in \mathbb{N}$ large enough that $\frac{d_n}{c_n} < (\frac{\delta}{4})^p$. Then $D := 2n+1$ will turn out to satisfy what we claim.

Now fix an arbitrary $F \subset \mathbb{Z}^N$ and note that also $B := A\chi_F I$ is a band operator of the same band-width w. W.l.o.g. we may assume that $B \neq 0$. We note the following facts:

(a) For all finite sets $S \subset \mathbb{Z}^N$ and all $x \in \mathbf{X}$, it holds $\sum_{k \in \mathbb{Z}^N} \|\chi_{k+S}\, x\|^p = |S| \cdot \|x\|^p$.

(b) For the commutator $[P_{n,k}, B] := P_{n,k}B - BP_{n,k}$, one has $[P_{n,k}, B] = [P_{n,k}, B]\Delta_{n,k}$, so that for all $x \in \mathbf{X}$, $\|[P_{n,k}, B]x\| = \|[P_{n,k}, B]\Delta_{n,k}x\| \leq \|[P_{n,k}, B]\|\|\Delta_{n,k}x\| \leq 2\|B\|\|\Delta_{n,k}x\|$ and hence

$$\sum_{k \in \mathbb{Z}^N} \|[P_{n,k}, B]x\|^p \;\leq\; \sum_{k \in \mathbb{Z}^N} 2^p\|B\|^p\|\Delta_{n,k}x\|^p \;\overset{(a)}{=}\; 2^p\|B\|^p d_n \|x\|^p.$$

Fixing $x \in \mathbf{X}$ such that $(1-\frac{\delta}{2})\|B\|\|x\| < \|Bx\|$, we conclude as follows, where (M) refers to Minkowski's inequality in $l^p(\mathbb{Z}^N, \mathbb{C})$:

$$\left(1 - \frac{\delta}{2}\right)\|B\|c_n^{1/p}\|x\| \;<\; c_n^{1/p}\|Bx\| \;\overset{(a)}{=}\; \left(\sum_{k \in \mathbb{Z}^N} \|P_{n,k}Bx\|^p\right)^{1/p}$$

$$\leq \left(\sum_{k \in \mathbb{Z}^N} (\|BP_{n,k}x\| + \|[P_{n,k}, B]x\|)^p\right)^{1/p} \overset{(\mathrm{M})}{\leq} \left(\sum_{k \in \mathbb{Z}^N} \|BP_{n,k}x\|^p\right)^{1/p} + \left(\sum_{k \in \mathbb{Z}^N} \|[P_{n,k}, B]x\|^p\right)^{1/p}$$

$$\overset{(b)}{\leq} \left(\sum_{k \in \mathbb{Z}^N} \|BP_{n,k}x\|^p\right)^{1/p} \;+\; 2\|B\|d_n^{1/p}\|x\|$$

Subtract $2(d_n/c_n)^{1/p}\|B\|c_n^{1/p}\|x\| = 2\|B\|d_n^{1/p}\|x\|$ from the resulting inequality, to get

$$\left(1 - \frac{\delta}{2} - 2\left(\frac{d_n}{c_n}\right)^{1/p}\right)\|B\|c_n^{1/p}\|x\| \;<\; \left(\sum_{k \in \mathbb{Z}^N} \|BP_{n,k}x\|^p\right)^{1/p}.$$

Taking p-th powers, using $2\left(\frac{d_n}{c_n}\right)^{1/p} < \frac{\delta}{2}$ and $\sum_{k\in\mathbb{Z}^N}\|P_{n,k}x\|^p = c_n\|x\|^p$, by (a), we get

$$(1-\delta)^p\|B\|^p\sum_{k\in\mathbb{Z}^N}\|P_{n,k}x\|^p \;<\; \left(1-\frac{\delta}{2}-2\left(\frac{d_n}{c_n}\right)^{1/p}\right)^p\|B\|^p c_n\|x\|^p \;<\; \sum_{k\in\mathbb{Z}^N}\|BP_{n,k}x\|^p.$$

The last inequality shows that there must be some $k\in\mathbb{Z}^N$ for which $P_{n,k}x\neq 0$ and

$$(1-\delta)^p\|B\|^p\|P_{n,k}x\|^p < \|BP_{n,k}x\|^p, \quad \text{i.e.} \quad (1-\delta)\|B\|\|P_{n,k}x\| < \|BP_{n,k}x\| \leq |||B|||_D\|P_{n,k}x\|$$

with $D=2n+1$ as fixed above. This finishes the proof for $p\in[1,\infty)$. Finally, let $p\in\{0,\infty\}$, put $D:=2w+1$, take any $F\subset\mathbb{Z}^N$, $B:=A\chi_F I$, $\varepsilon>0$ and $x\in\mathbf{X}$ with $\|x\|_\infty=1$ and $\|Bx\|_\infty\geq\|B\|-\varepsilon/2$. Then there is a $k\in\mathbb{Z}^N$ with $\|\chi_{\{k\}}Bx\|_\infty\geq\|Bx\|_\infty-\varepsilon/2$, so that

$$\begin{aligned}\|B\|-\varepsilon \;&\leq\; \|Bx\|_\infty-\varepsilon/2 \;\leq\; \|\chi_{\{k\}}Bx\|_\infty \;=\; \|\chi_{\{k\}}BP_{w,k}x\|_\infty \;\leq\; \|BP_{w,k}x\|_\infty\\ &\leq\; |||B|||_D\|P_{w,k}x\|_\infty \;\leq\; |||B|||_D\|x\|_\infty \;=\; |||B|||_D \;\leq\; \|B\|\end{aligned}$$

holds. So in case $p\in\{0,\infty\}$ even equality $\|B\|=|||B|||_D$ follows, where $D=2w+1$. □

A closer look at this proof shows that the size of the support that is required to localize the norm of B to the desired accuracy only depends on the band-width w of B, so that the result carries over in a uniform way to all band operators with band-width not more than w. In short:

$$\forall w\in\mathbb{N},\, c\in(0,1)\;\exists D\in\mathbb{N}: \forall B \text{ with band-width}(B)\leq w: \; |||B|||_D\geq c\|B\|. \tag{ONL}$$

This localizability of the operator norm is no longer a property of a particular operator but rather of the space $\mathbf{X}$. There is recent work by X. Chen, R. Tessera, X. Wang, G. Yu and H. Sako (see [32] and references therein) on metric spaces M with a certain measure such that $\mathbf{X}=l^2(M)$ has the operator norm localization property (ONL). Sako proves in [32] that in case of a discrete metric space M with $\sup_{m\in M}|\{n\in M: d(m,n)\leq R\}|<\infty$ for all radii $R>0$ (which clearly holds in our case, $M=\mathbb{Z}^N$), property (ONL) is equivalent to the so-called Property A that was introduced by G. Yu and is connected with amenability. We also want to mention the very recent paper [40] by Špakula and Willett that generalizes the limit operator results from $\mathbb{Z}^N$ to certain discrete metric spaces. Based on the work of Roe [31], combined with ideas of [20], they prove a version of Theorem 2.9 under the sole assumption that these metric spaces have Yu's Property A.

For the current paper we are not interested in extending Proposition 3.4 to band operators of a certain band-width but rather to the operator spectrum of an operator $A\in\mathcal{A}(\mathbf{X})$:

Corollary 3.5. *Let $A\in\mathcal{A}(\mathbf{X})$ and $\delta>0$. Then there is a $D\in\mathbb{N}$ such that*

$$\|B\chi_F I\|-\delta\leq|||B\chi_F I|||_D\leq\|B\chi_F I\| \quad \textit{for all} \quad F\subset\mathbb{Z}^N \quad \textit{and all} \quad B\in\{A\}\cup\sigma_{\mathrm{op}}(A).$$

Proof. Fix $\delta>0$ and take a band operator $\tilde{A}$ such that $\|A-\tilde{A}\|<\delta/3$. Now choose D by applying the previous proposition to $\tilde{A}$ with $\frac{\delta}{3\|\tilde{A}\|}$ instead of δ. Then, for all $F\subset\mathbb{Z}^N$,

$$\begin{aligned}\|A\chi_F I\|\geq|||A\chi_F I|||_D &\geq \left|\left|\left|\tilde{A}\chi_F I\right|\right|\right|_D - \left|\left|\left|(A-\tilde{A})\chi_F I\right|\right|\right|_D > \left(1-\frac{\delta}{3\|\tilde{A}\|}\right)\|\tilde{A}\chi_F I\|-\frac{\delta}{3}\\ &\geq \|\tilde{A}\chi_F I\|-\frac{2\delta}{3}\geq\|A\chi_F I\|-\|(A-\tilde{A})\chi_F I\|-\frac{2\delta}{3}>\|A\chi_F I\|-\delta.\end{aligned}$$

Now let $A_g \in \sigma_{\mathrm{op}}(A)$. The estimate $|||A_g\chi_F I|||_D \le \|A_g\chi_F I\|$ is clear. Further, for every $\varepsilon > 0$ there is an m such that $\|A_g\chi_F I\| \le \|A_g\chi_F P_m\| + \varepsilon$. For $\|A_g\chi_F P_m\|$ we have the estimate

$$\begin{aligned}\|A_g\chi_F P_m\| &\le \|V_{-g_n}AV_{g_n}\chi_F P_m\| + \|(A_g - V_{-g_n}AV_{g_n})P_m\|\\ &= \|A\chi_{F\cap\{-m,\dots,m\}^N+g_n}I\| + \|(A_g - V_{-g_n}AV_{g_n})P_m\|\\ &\le \left|\left|\left|A\chi_{F\cap\{-m,\dots,m\}^N+g_n}I\right|\right|\right|_D + \delta + \|(A_g - V_{-g_n}AV_{g_n})P_m\|\\ &= |||V_{-g_n}AV_{g_n}\chi_F P_m|||_D + \delta + \|(A_g - V_{-g_n}AV_{g_n})P_m\|.\end{aligned}$$

The last summand goes to zero as $n \to \infty$, whereas the 1st one converges to $|||A_g\chi_F P_m|||_D$. By this we obtain $\|A_g\chi_F I\| - \delta \le |||A_g\chi_F P_m|||_D + \varepsilon \le |||A_g\chi_F I|||_D + \varepsilon$ where $\varepsilon > 0$ is arbitrary. Thus the assertion follows. $\square$

Now we are in a position to prove Theorem 3.2.

Proof of Theorem 3.2. For every $K \in \mathcal{K}(\mathbf{X},\mathcal{P})$ and every $A_g \in \sigma_{\mathrm{op}}(A)$, $\|A+K\| \ge \|(A+K)_g\| = \|A_g\|$ holds. Taking the infimum on the left and the supremum on the right proves the estimate "$\ge$".

Now assume that $\|A + \mathcal{K}(\mathbf{X},\mathcal{P})\| > \sup_{A_g\in\sigma_{\mathrm{op}}(A)} \|A_g\| =: N_A$ holds. Then there is an $\varepsilon > 0$ with $\|A+\mathcal{K}(\mathbf{X},\mathcal{P})\| > N_A+\varepsilon$. We conclude that $\|AQ_m\| = \|A-AP_m\| \ge \|A+\mathcal{K}(\mathbf{X},\mathcal{P})\| > N_A+\varepsilon$ for every $m \in \mathbb{N}$. From Corollary 3.5 we get an $n \in \mathbb{N}$ such that $|||AQ_m|||_{2n+1} > N_A + \varepsilon/2$ for every m. In particular, we get $k_1, k_2, \ldots \in \mathbb{Z}^N$ such that, in the notation $P_{n,k} = V_k P_n V_{-k}$ of the proof of Proposition 3.4, $N_A + \varepsilon/2 < \|(AQ_m)P_{n,k_m}\| \le \|AP_{n,k_m}\|$ for every m. Now pass to a subsequence $g = (g_j)$ of the (unbounded) sequence $(k_1, k_2, \ldots)$ for which the limit operator A_g exists. Then

$$N_A + \varepsilon/2 < \|AP_{n,g_j}\| = \|V_{-g_j}AV_{g_j}P_n\| \to \|A_g P_n\| \le \|A_g\| \le N_A, \qquad j \to \infty$$

is a contradiction.

It remains to show that N_A exists as a maximum. The argument is very similar to that in the proof of [20, Theorem 8], where it is explained in more detail (also see Figure 1 and Remark 9 in [20]). We consider the numbers $\gamma_n := 2^{-n}$ and

$$r_l := \sum_{n=l}^{\infty} \gamma_n = 2^{-l+1}.$$

Then (r_l) is a strictly decreasing sequence of positive numbers which tends to 0. From the above corollary we obtain a sequence $(D_l) \subset \mathbb{N}$ of even numbers such that for every $l \in \mathbb{N}$

$$D_{l+1} > 2D_l \quad \text{and} \quad |||B\chi_F I|||_{D_l} > \|B\chi_F I\| - \gamma_l \quad \text{for every } B \in \{A\} \cup \sigma_{\mathrm{op}}(A) \text{ and every } F \subset \mathbb{Z}^N.$$

Choose a sequence $(B_n) \subset \sigma_{\mathrm{op}}(A)$ such that $\|B_n\| \to \sup\{\|A_g\| : A_g \in \sigma_{\mathrm{op}}(A)\}$ as $n \to \infty$. For each $n \in \mathbb{N}$ we are going to construct a suitably shifted copy $C_n \in \sigma_{\mathrm{op}}(A)$ of B_n as follows:

We start with an $x_n^0 \in \mathbf{X}$, $\|x_n^0\| = 1$, $\operatorname{diam}\operatorname{supp} x_n^0 \le D_n$ such that $\|B_n x_n^0\| \ge \|B_n\| - \gamma_n$. We choose a shift $j_n^0 \in \mathbb{Z}^N$ which centralizes $y_n^0 := V_{j_n^0} x_n^0$ such that $y_n^0 = P_{D_n/2} y_n^0$, and define the copy $C_n^0 := V_{j_n^0} B_n V_{-j_n^0} \in \sigma_{\mathrm{op}}(A)$. Then we have $\|B_n\| \ge \|C_n^0 P_{D_n/2}\| \ge \|B_n\| - \gamma_n$.

Now, for $k = 1, \ldots, n$, we gradually perform a fine tuning by choosing $x_n^k \in \operatorname{im} P_{D_{n-(k-1)}/2}$, $\|x_n^k\| = 1$, $\operatorname{diam}\operatorname{supp} x_n^k \le D_{n-k}$ such that $\|C_n^{k-1} P_{D_{n-(k-1)}/2} x_n^k\| \ge \|C_n^{k-1} P_{D_{n-(k-1)}/2}\| - \gamma_{n-k}$, passing to a centralized $y_n^k := V_{j_n^k} x_n^k$ via a shift $j_n^k \in \{-D_{n-(k-1)}/2, \ldots, D_{n-(k-1)}/2\}^N$ and defining

$C_n^k := V_{j_n^k} C_n^{k-1} V_{-j_n^k} \in \sigma_{\rm op}(A)$. For this we observe $\|C_n^k P_{D_{n-k}/2}\| \geq \|C_n^{k-1} P_{D_{n-(k-1)}/2}\| - \gamma_{n-k}$. In particular, for $n > l \geq 1$, the estimates $\|C_n^{n-l} P_{D_l/2}\| \geq \|B_n\| - \sum_{k=l}^n \gamma_k \geq \|B_n\| - r_l$ hold. Finally, we define $C_n := C_n^n$ and notice that $C_n = V_{j_n^n+\ldots+j_n^{n-l+1}} C_n^{n-l} V_{-(j_n^n+\ldots+j_n^{n-l+1})}$, where $|j_n^n + \ldots + j_n^{n-l+1}| \leq D_l$ by construction, thus $\|C_n P_{2D_l}\| \geq \|C_n^{n-l} P_{D_l/2}\| \geq \|B_n\| - r_l$.

By this construction we have obtained a sequence $(C_n) \subset \sigma_{\rm op}(A)$ of limit operators C_n which have adjusted local norms and such that still $\|C_n\| \to \sup\{\|A_g\| : A_g \in \sigma_{\rm op}(A)\}$ as $n \to \infty$ holds. By [17, Prop. 3.104] we can pass to a subsequence (C_{h_n}) of (C_n) with $\mathcal{P}$-strong limit $C \in \sigma_{\rm op}(A)$. Then

$$\|C\| \geq \|CP_{2D_l}\| = \lim_{n\to\infty} \|C_{h_n} P_{2D_l}\| \geq \lim_{n\to\infty} \|B_{h_n}\| - r_l = N_A - r_l$$

for every l. Since r_l goes to 0 as $l \to \infty$ the assertion follows. □

Remark 3.6. In $\mathcal{L}(\mathbf{X}, \mathcal{P})$ the Equality (3.1) does not hold in general.

- Consider $X := L^p[0, 1]$, and the multiplication operators $a_k I \in \mathcal{L}(X)$ with $a_k(x) := \sin(2\pi kx)$. Then the diagonal operator $A := \operatorname{diag}(\ldots, 0, 0, a_1 I, a_2 I, a_3 I, \ldots)$ on $\mathbf{X}$ has operator spectrum $\{0\}$, but essential norm 1.
- Consider the $n \times n$ matrices

$$B_n := \frac{1}{n} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

and the block diagonal operator $A := \operatorname{diag}(\ldots, 0, 0, B_1, B_2, B_3, \ldots)$ on $l^p(\mathbb{Z}, \mathbb{C})$, $1 < p < \infty$. Then A has operator spectrum $\{0\}$, but essential norm 1 (see [20, Example 14]).

The first example is banded but not rich, whereas the second one is rich but not band-dominated. Note that in the extremal cases, $p \in \{0, 1, \infty\}$, the latter cannot happen since rich operators $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ are automatically band-dominated then, by [20, Theorem 15].

Now we combine Equations (3.1) and (2.4):

Corollary 3.7. *Let $A \in \mathcal{A}_\$(\mathbf{X})$ be $\mathcal{P}$-Fredholm, and B be a $\mathcal{P}$-regularizer. Then*

$$\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\| = \|B + \mathcal{K}(\mathbf{X}, \mathcal{P})\| = \max_{B_h \in \sigma_{\rm op}(B)} \|B_h\| = \max_{A_h \in \sigma_{\rm op}(A)} \|A_h^{-1}\|. \tag{3.2}$$

If $A \in \mathcal{A}_\$(\mathbf{X})$ is not $\mathcal{P}$-Fredholm, then both, the RHS and the LHS of (3.2) are infinite.

Proof. The operator B is band-dominated by Theorem 2.5 and rich by Theorem 2.8. Hence Theorem 3.2 applies and Equation (3.1) together with Equation (2.4) from Theorem 2.7 provide (3.2). The last sentence follows from Theorem 2.9. □

What comes as a simple corollary here is in fact a cornerstone for large parts of the subsequent results. Remember Theorem 2.9 for $A \in \mathcal{A}_\$(\mathbf{X})$. It says that

$$\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\| < \infty \qquad \text{if and only if} \qquad \sup_{A_h \in \sigma_{\rm op}(A)} \|A_h^{-1}\| < \infty.$$

Now Corollary 3.7 goes far beyond: It shows that both quantities are always equal and that the supremum is actually attained as a maximum.

Before we continue to look at Equality (3.2) and its ingredients from different angles, we will prove the following lemma that will be helpful in several places but is also of interest in its own right:

Lemma 3.8. *Let $\mathcal{P}$ be a uniform approximate identity on $\mathbf{Y}$ and $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$. Then*

a) The set $\mathbf{Y}_0 := \{y \in \mathbf{Y} : \|Q_n y\| \to 0$ as $n \to \infty\}$ is a closed subspace of $\mathbf{Y}$. The restriction $A_0 := A|_{\mathbf{Y}_0}$ of A to $\mathbf{Y}_0$ belongs to $\mathcal{L}(\mathbf{Y}_0)$, $\|A_0\| = \|A\|$, and $\nu(A) = \nu(A_0)$.

b) The restriction $(A^*)_0 := A^*|_{(\mathbf{Y}^*)_0}$ of A^* to the (analogously defined) subspace $(\mathbf{Y}^*)_0$ belongs to $\mathcal{L}((\mathbf{Y}^*)_0)$ and $\|(A^*)_0\| = \|A^*\|$.

c) If A is invertible then A_0 is invertible with inverse $(A_0)^{-1} = (A^{-1})_0 \in \mathcal{L}(\mathbf{Y}_0)$ and $\|(A^{-1})_0\| = \|A^{-1}\|$. Further, $(A^*)_0$ is invertible in $\mathcal{L}((\mathbf{Y}^*)_0)$ with inverse $((A^*)_0)^{-1} = ((A^*)^{-1})_0 = ((A^{-1})^*)_0$ and $\|((A^*)^{-1})_0\| = \|(A^*)^{-1}\| = \|A^{-1}\|$.

Proof. *a)* It is easily checked that $\mathbf{Y}_0$ is a closed subspace of $\mathbf{Y}$. $A_0(\mathbf{Y}_0) \subset \mathbf{Y}_0$ is from [28, Lemma 1.1.20] or [36, Proposition 1.18.1] and the formula on the norm is [36, Proposition 1.18.2]. The inequality $\nu(A) \le \nu(A_0)$ is trivial and it remains to prove $\nu(A) \ge \nu(A_0)$. We apply the sequence (F_n) given by Proposition 2.4 to obtain

$$\|Ax\| = \|F_n\|\|Ax\| \ge \|F_n Ax\| \ge \|AF_n x\| - \|[A, F_n]\|\|x\| \ge \nu(A_0)\|F_n x\| - \|[A, F_n]\|\|x\|$$

for every $x \in \mathbf{Y}$ and every $n \in \mathbb{N}$. Sending $n \to \infty$ we get $\|Ax\| \ge \nu(A_0)\|x\|$ for every $x \in \mathbf{Y}$, and taking the infimum over all $\|x\| = 1$ we finally arrive at $\nu(A) \ge \nu(A_0)$.

b) The inclusion $(A^*)_0((\mathbf{Y}^*)_0) \subset (\mathbf{Y}^*)_0$ follows by the same means. Here in this dual setting [36, Proposition 1.18.2] may not be applicable anymore since $\mathcal{P}^*$ is not necessarily subject to ($\mathcal{P}$3). Therefore we need another proof for the formula on the norms.

Let $\varepsilon > 0$ and choose $y \in \mathbf{Y}$, $\|y\| = 1$ such that $\|A\| \le \|Ay\| + \varepsilon$. Since $\mathcal{P}$ is an approximate identity we find a k such that $\|A\| \le \|P_k Ay\| + 2\varepsilon$. Now, by Hahn Banach there is a functional g_0 on $\operatorname{im} P_k$, $\|g_0\| = 1$, with $\|P_k Ay\| = |g_0(P_k Ay)|$. Thus, setting $g := g_0 \circ P_k$ we obtain a functional $g \in \mathbf{Y}^*$, $\|g\| = 1$, such that actually $g \in (\mathbf{Y}^*)_0$ with norm 1, hence

$$\|(A^*)_0\| \le \|A^*\| = \|A\| \le |g(Ay)| + 2\varepsilon \le \|A^* g\| + 2\varepsilon \le \|(A^*)_0\| + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary this shows $\|(A^*)_0\| = \|A^*\|$.

c) Let A be invertible. Then A^* is invertible as well where $(A^*)^{-1} = (A^{-1})^*$. The invertibility of A_0 and $(A^*)_0$ as well as the formulas for their inverses follow from [36, Corollary 1.9, Corollary 1.19]. Since A^{-1} is still in $\mathcal{L}(\mathbf{Y}, \mathcal{P})$ by Theorem 2.1 we can apply the already proved formulas on the norms also to A^{-1}. □

Lemma 3.8 enables us to restrict consideration to elements $x \in \mathbf{Y}_0$ when approximating $\|A\|$ or $\nu(A)$ by $\|Ax\|$ – and similarly for A^*, A^{-1} or $(A^*)^{-1}$ in place of A. In combination with $\mathcal{P}$-convergence this turns out to be a lot more convenient than having to work with $x \in \mathbf{Y}$. The proof of the following proposition shows what we mean by that:

Proposition 3.9. *Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ and $A_h \in \sigma_{\mathrm{op}}(A)$. Then $\nu(A_h) \ge \nu(A)$.*

In a sense, this result is a lower counterpart of the norm inequality from Proposition 2.6. Together they show that $\nu(A) \le \nu(A_h) \le \|A_h\| \le \|A\|$.

Proof. Let $\varepsilon > 0$ and let $h = (h_n)$ be a sequence in $\mathbb{Z}^N$ with $V_{-h_n} A V_{h_n} \xrightarrow{\mathcal{P}} A_h$. By closedness of $\mathcal{L}(\mathbf{X}, \mathcal{P})$ under $\mathcal{P}$-strong convergence (the first part of (2.3)), also $A_h \in \mathcal{L}(\mathbf{X}, \mathcal{P})$. We apply Lemma 3.8 a) to A_h. There is a $x_0 \in \mathbf{X}_0$ with $\|x_0\| = 1$ such that $\nu(A_h) = \nu((A_h)_0) > \|A_h x_0\| - \varepsilon$. Now truncate x_0 and renormalize. Since $x_0 \in \mathbf{X}_0$, one has $\|P_k x_0\|^{-1} P_k x_0 \to x_0$ as $k \to \infty$. So, for sufficiently large $k \in \mathbb{N}$, $x := \|P_k x_0\|^{-1} P_k x_0$ also fulfills $\nu(A_h) > \|A_h x\| - \varepsilon$, where $\|x\| = 1$ and now $x = P_k x$. Choose $n \in \mathbb{N}$ large enough that $\|(A_h - V_{-h_n} A V_{h_n}) P_k\| < \varepsilon$ and conclude that

$$\nu(A_h) \geq \|A_h x\| - \varepsilon = \|A_h P_k x\| - \varepsilon \geq \|V_{-h_n} A \underbrace{V_{h_n} P_k x}_{=: x_n}\| - 2\varepsilon = \|A x_n\| - 2\varepsilon \geq \nu(A)\|x_n\| - 2\varepsilon.$$

But since $\|x_n\| = \|P_k x\| = \|x\| = 1$ and $\varepsilon > 0$ is arbitrary, we are finished. $\square$

4 The $\mathcal{P}$-essential pseudospectrum

With our formula (3.2) it is possible to study resolvent norms in $\mathcal{A}_\$(\mathbf{X})/\mathcal{K}(\mathbf{X}, \mathcal{P})$. To do this replace A by $A - \lambda I$ in (3.2) and recall that $(A - \lambda I)_h = A_h - \lambda I$. Then (3.2) turns into (1.4). This motivates to study the following kind of pseudospectra:

Definition 4.1. For $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ and $\varepsilon > 0$, the *$\mathcal{P}$-essential ε-pseudospectrum* is defined as

$$\operatorname{sp}_{\varepsilon,\mathrm{ess}}(A) := \operatorname{sp}_\varepsilon(A + \mathcal{K}(\mathbf{X}, \mathcal{P})) := \{\lambda \in \mathbb{C} : \|(A - \lambda I + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\| > 1/\varepsilon\}.$$

Recall that, in contrast, the $\mathcal{P}$-essential spectrum of A is

$$\begin{aligned} \operatorname{sp}_{\mathrm{ess}}(A) = \operatorname{sp}(A + \mathcal{K}(\mathbf{X}, \mathcal{P})) &= \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not } \mathcal{P}\text{-Fredholm}\} \\ &= \{\lambda \in \mathbb{C} : \|(A - \lambda I + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\| = \infty\}. \end{aligned}$$

Remark 4.2. Recall that in case $\dim X < \infty$ every $\mathcal{P}$-compact operator is compact, hence every $\mathcal{P}$-Fredholm operator also Fredholm. By Proposition 2.3 every Fredholm operator $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ is $\mathcal{P}$-Fredholm, thus we can conclude that for all $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$, $\mathbf{X} = l^p(\mathbb{Z}^N, X)$ with $\dim X < \infty$, the $\mathcal{P}$-essential spectrum and the (classical) essential spectrum coincide:

$$\operatorname{sp}_{\mathrm{ess}}(A) = \operatorname{sp}(A + \mathcal{K}(\mathbf{X})) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\}.$$

We will address this case again in more detail in Section 5.5.

Now here is our immediate consequence of (3.2):

Theorem 4.3. *Let $A \in \mathcal{A}_\$(\mathbf{X})$ and $\varepsilon > 0$. Then*

$$\operatorname{sp}_{\varepsilon,\mathrm{ess}}(A) = \bigcup_{A_h \in \sigma_{\mathrm{op}}(A)} \operatorname{sp}_\varepsilon(A_h). \tag{4.1}$$

Proof. Equation (1.4) clearly implies (4.1) since $\max_{A_h \in \sigma_{\mathrm{op}}(A)} \|(A_h - \lambda I)^{-1}\| > 1/\varepsilon$ if and only if $\lambda \in \cup_{A_h \in \sigma_{\mathrm{op}}(A)} \operatorname{sp}_\varepsilon(A_h)$. The particular case where $\max_{A_h} \|(A_h - \lambda I)^{-1}\| = \infty$ corresponds to $\lambda \in \cup_{A_h} \operatorname{sp}(A_h)$, see (1.3). $\square$

With (4.1) we have arrived at an ε-version (1.5) of (1.3), the second part of (v) in the introduction. It is known that it may be easier to compute pseudospectra of limit operators than their spectra. So, numerically, computing $\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A)$ via (4.1) aka (1.5) is in general simpler than computing $\mathrm{sp}_{\mathrm{ess}}(A)$ via (1.3). In the end, one is probably interested in $\mathrm{sp}_{\mathrm{ess}}(A)$. The good news is that this can be approximated by $\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A)$ as $\varepsilon \to 0$. It is a standard result that the ε-pseudospectra converge to the spectrum as $\varepsilon \to 0$. For the reader's convenience, we state and prove the result here for our concrete setting of $\mathcal{P}$-essential (pseudo)spectra:

Proposition 4.4. *For every $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$, the sets $\overline{\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A)}$ converge*[10] *to $\mathrm{sp}_{\mathrm{ess}}(A)$ w.r.t. the Hausdorff metric as $\varepsilon \to 0$.*

Proof. Clearly, $\mathrm{sp}_{\mathrm{ess}}(A) \subset \mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) \subset \overline{\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A)} \subset \mathrm{sp}_{\delta,\mathrm{ess}}(A)$ for all $0 < \varepsilon < \delta$. On the other hand, assume that there is a sequence (λ_n) of points $\lambda_n \in \mathrm{sp}_{1/n,\mathrm{ess}}(A)$ which stay bounded away from the $\mathcal{P}$-essential spectrum. By a simple Neumann series argument (λ_n) is bounded, hence it has a convergent subsequence. Without loss of generality let already (λ_n) converge to λ. Since the norms $\|(A - \lambda_n I + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\| > n$ tend to infinity, we find that $A - \lambda I + \mathcal{K}(\mathbf{X}, \mathcal{P})$ cannot be invertible in $\mathcal{L}(\mathbf{X}, \mathcal{P})/\mathcal{K}(\mathbf{X}, \mathcal{P})$, that is $\lambda \in \mathrm{sp}_{\mathrm{ess}}(A)$, a contradiction. $\square$

From Theorem 4.3 and Proposition 4.4 we get the following corollary:

Corollary 4.5. *Let $A \in \mathcal{A}_{\$}(\mathbf{X})$. Then*

$$\mathrm{sp}_{\mathrm{ess}}(A) = \lim_{\varepsilon \to 0} \overline{\bigcup_{A_h \in \sigma_{\mathrm{op}}(A)} \mathrm{sp}_\varepsilon(A_h)} = \bigcap_{\varepsilon > 0} \bigcup_{A_h \in \sigma_{\mathrm{op}}(A)} \mathrm{sp}_\varepsilon(A_h).$$

Remark 4.6. a) Note that Corollary 4.5, although derived via our new Equations (3.2) and (4.1), in fact says nothing more than Theorem 2.9 and Equation (1.3).

b) Several authors define pseudospectra with "$\geq 1/\varepsilon$" instead of "$> 1/\varepsilon$", which leads to compact pseudospectra, but sometimes causes additional difficulties. (For example, the analogue of Proposition 4.7 below is no longer true in arbitrary Banach space Y if "$> 1/\varepsilon$" is replaced by "$\geq 1/\varepsilon$" in the definition of $\mathrm{sp}_\varepsilon(A)$ and if the union below is taken over all $\|K\| \leq \varepsilon$ instead of all $\|K\| < \varepsilon$, cf. [38].) Anyway, our preceding results hold for both definitions.

c) Similar observations are to be expected for (N, ε)-pseudospectra as well.

Another well-known and very useful characterization of pseudospectra of operators A is given as the union of spectra of small perturbations of A.

Proposition 4.7. *(Cf. [4, Section 7.1]) Let Y be a Banach space, $A \in \mathcal{L}(Y)$ and $\varepsilon > 0$. Then*

$$\mathrm{sp}_\varepsilon(A) = \bigcup_{\|K\| < \varepsilon} \mathrm{sp}(A + K) = \bigcup_{\|K\| < \varepsilon,\, \mathrm{rank}\, K \leq 1} \mathrm{sp}(A + K).$$

In the following Proposition we improve this result in case of $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$.

Proposition 4.8. *Let $\mathcal{C} \subset \mathcal{L}(\mathbf{X}, \mathcal{P})$ be an algebra containing all rank-1-operators with only finitely many non-zero entries in the respective matrix representation, let $A \in \mathcal{C}$ and let $\varepsilon > 0$. Then*

$$\mathrm{sp}_\varepsilon(A) = \bigcup_{\|K\| < \varepsilon} \mathrm{sp}(A + K) = \bigcup_{\substack{\|K\| < \varepsilon, \\ K \in \mathcal{C}}} \mathrm{sp}(A + K) = \bigcup_{\substack{\|K\| < \varepsilon, \\ K \in \mathcal{K}(\mathbf{X}, \mathcal{P})}} \mathrm{sp}(A + K) = \bigcup_{\substack{\|K\| < \varepsilon, \\ K \in \mathcal{K}(\mathbf{X}, \mathcal{P}) \cap \mathcal{C}, \\ \mathrm{rank}\, K \leq 1}} \mathrm{sp}(A + K).$$

[10] We consider the closure of $\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A)$ since the Hausdorff metric is defined for compact sets only.

Proof. Abbreviate the sets in this claim from left to right by $S_1, ..., S_5$. $S_1 = S_2$ holds by the previous proposition, $S_2 \supset S_3 \supset S_5$ and $S_2 \supset S_4 \supset S_5$ are obvious. Thus, it remains to prove $S_5 \supset S_1$.

So let $\lambda \in S_1$. Since the case $\lambda \in \operatorname{sp}(A)$ is clear, let $B := A - \lambda I$ be invertible with $\|B^{-1}\| > 1/\varepsilon$. By Lemma 3.8, also $B_0 := B|_{\mathbf{X}_0}$ is invertible and $\|(B_0)^{-1}\| = \|B^{-1}\| > 1/\varepsilon$, so that there exists an $x_0 \in \mathbf{X}_0$, $\|x_0\| = 1$, with $\|Bx_0\| = \|B_0x_0\| < \varepsilon$. As in the proof of Proposition 3.9, take k sufficiently large that also $x := \|P_kx_0\|^{-1}P_kx_0$ fulfills $\|Bx\| < \varepsilon$, where $\|x\| = 1$ and $P_kx = x$. By the Hahn-Banach Theorem there exists a functional φ with $\|\varphi\| = \varphi(x) = 1$ and $\varphi \circ P_k = \varphi$. Now, we define $\tilde{K}u := -\varphi(u)x$ and $Ku := -\varphi(u)Bx$ for every $u \in \mathbf{X}$. Then $\tilde{K}$, K have rank 1 and $\|K\| \le \|\varphi\|\|Bx\| < \varepsilon$. Moreover, both $\tilde{K} = P_k\tilde{K}P_k$ and $K = B\tilde{K}$ belong to $\mathcal{K}(\mathbf{X},\mathcal{P}) \cap \mathcal{C}$. Finally, with $(B + K)x = Bx - \varphi(x)Bx = 0$, we summarize: $\lambda \in \operatorname{sp}(A + K)$, $\|K\| < \varepsilon$, $K \in \mathcal{K}(\mathbf{X},\mathcal{P}) \cap \mathcal{C}$, $\operatorname{rank} K = 1$. □

Also for the $\mathcal{P}$-essential pseudospectra for classes of rich band-dominated operators we can obtain a characterization via perturbations.

Theorem 4.9. *Let $\mathcal{C}$ be one of the algebras of all rich band operators or all rich band-dominated operators*[11] *on $\mathbf{X}$ and let $A \in \mathcal{C}$. For $\varepsilon > 0$*

$$\operatorname{sp}_{\varepsilon,\mathrm{ess}}(A) = \bigcup_{\substack{\|T\|<\varepsilon,\\ T\in\mathcal{L}(\mathbf{X},\mathcal{P})}} \operatorname{sp}_{\mathrm{ess}}(A+T) = \bigcup_{\substack{\|T\|<\varepsilon,\\ T\in\mathcal{C}}} \operatorname{sp}_{\mathrm{ess}}(A+T).$$

Proof. For each $L \in \mathcal{L}(\mathbf{X},\mathcal{P})$, abbreviate the coset $L + \mathcal{K}(\mathbf{X},\mathcal{P}) \in \mathcal{L}(\mathbf{X},\mathcal{P})/\mathcal{K}(\mathbf{X},\mathcal{P})$ by L°. Now let $A \in \mathcal{C}$ and $\lambda \notin \operatorname{sp}_{\varepsilon,\mathrm{ess}}(A)$. With $B := A - \lambda I$, the coset B° is invertible and $\|(B^\circ)^{-1}\| \le 1/\varepsilon$. For arbitrary $T \in \mathcal{L}(\mathbf{X},\mathcal{P})$ with $\|T\| < \varepsilon$, one has $\|(B^\circ)^{-1}T^\circ\| < 1$, so that $I^\circ + (B^\circ)^{-1}T^\circ$ is invertible. Thus, $(B + T)^\circ = B^\circ(I^\circ + (B^\circ)^{-1}T^\circ)$ is invertible, whence $\lambda \notin \operatorname{sp}_{\mathrm{ess}}(A + T)$. Together with Theorem 4.3 we conclude the following inclusions:

$$\bigcup_{\substack{\|T\|<\varepsilon,\\ T\in\mathcal{C}}} \operatorname{sp}_{\mathrm{ess}}(A+T) \subset \bigcup_{\substack{\|T\|<\varepsilon,\\ T\in\mathcal{L}(\mathbf{X},\mathcal{P})}} \operatorname{sp}_{\mathrm{ess}}(A+T) \subset \operatorname{sp}_{\varepsilon,\mathrm{ess}}(A) = \bigcup_{A_h\in\sigma_{\mathrm{op}}(A)} \operatorname{sp}_\varepsilon(A_h).$$

It remains to show that the right-most set is contained in the left-most. So let $A_h \in \sigma_{\mathrm{op}}(A)$ and $\lambda \in \operatorname{sp}_\varepsilon(A_h)$. By Proposition 4.8, $\lambda \in \operatorname{sp}(A_h + K)$ for some $K \in \mathcal{K}(\mathbf{X},\mathcal{P}) \cap \mathcal{C}$ with $\|K\| < \varepsilon$. Now choose a subsequence g of h such that all cubes $g_n + \{-n, ..., n\}^N$ are pairwise disjoint, and define

$$T := \sum_{n\in\mathbb{N}} V_{g_n}P_nKP_nV_{-g_n}. \tag{4.2}$$

T is a well-defined block-diagonal operator[12] belonging to $\mathcal{C}$ with $\|T\| \le \|K\| < \varepsilon$ and $T_g = K$. Since

$$(A - \lambda I + T)_g = A_g - \lambda I + T_g = A_h - \lambda I + K,$$

we find that $\lambda \in \operatorname{sp}(A_h + K) = \operatorname{sp}((A + T)_g)$, whence $\lambda \in \operatorname{sp}_{\mathrm{ess}}(A + T)$ by (1.3). □

Remark 4.10. The above proof that the pseudospectrum is a superset of the union of spectra of perturbations works in every Banach algebra. In C^*-algebras also the converse is true, although it may fail in the general case. For more details see e.g. [13, Page 121].

[11] Actually, one can consider many more subalgebras $\mathcal{C}$ of $\mathcal{A}_\$(\mathbf{X})$ as long as one can define operators of the form (4.2) there. Another example is the set of all rich operators in the Wiener algebra, see e.g. [27] or [19, §3.7.3].

[12] By our assumption on g, the blocks $V_{g_n}P_nKP_nV_{-g_n}$ do not overlap.

5 The $\mathcal{P}$-essential lower norm

Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ be $\mathcal{P}$-Fredholm. By Theorem 2.7 and the last point of Lemma 2.10, we can rewrite the right-hand side of (3.2) in terms of lower norms of the limit operators:

$$\max_{A_h \in \sigma_{\mathrm{op}}(A)} \|A_h^{-1}\| = \left(\min_{A_h \in \sigma_{\mathrm{op}}(A)} \nu(A_h) \right)^{-1}. \tag{5.1}$$

Our aim for this section is to present alternative valuable characterizations of the essential norm $\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\|$ on the left-hand side of (3.2) in terms of lower norms of (perturbations and restrictions of) the operator A directly, which do not count on limit operators.

5.1 1st approach: Lower norms of asymptotic compressions

We start again with the abstract setting of a Banach space $\mathbf{Y}$ with a uniform approximate projection $\mathcal{P} = (P_n)$, and we make the following simple observation:

Lemma 5.1. *For $A \in \mathcal{L}(\mathbf{Y})$, $\lim\limits_{m\to\infty} \nu(A|_{\mathrm{im}\, Q_m}) = \sup\limits_{m\in\mathbb{N}} \nu(A|_{\mathrm{im}\, Q_m})$, where $A|_{\mathrm{im}\, Q_m} : \mathrm{im}\, Q_m \to \mathbf{Y}$.*

Proof. The sequence of compressions is bounded by $\nu(A|_{\mathrm{im}\, Q_m}) \le \|A\|$. Convergence to the supremum follows from the monotonicity $\nu(A|_{\mathrm{im}\, Q_{m+1}}) \ge \nu(A|_{\mathrm{im}\, Q_m})$ since $\mathrm{im}\, Q_{m+1} \subset \mathrm{im}\, Q_m$. □

Definition 5.2. For $A \in \mathcal{L}(\mathbf{Y})$ set

$$\tilde{\mu}(A) := \lim_{m\to\infty} \nu(A|_{\mathrm{im}\, Q_m}), \quad \mu(A) := \min\{\tilde{\mu}(A), \tilde{\mu}(A^*)\}.$$

In Section 5.4 we will see that in the case of appropriate Hilbert spaces $\mathbf{Y}$ this $\mu(A)$ serves as a characterization for the essential norm $\|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|$ for every operator $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ (cf. Theorem 5.18). However, beyond the comfortable Hilbert space case we are still able to prove this observation for all rich band-dominated operators on all $\mathbf{X}$.

Theorem 5.3. *Let $A \in \mathcal{A}_\$(\mathbf{X})$. Then*[13]

$$\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\|^{-1} = \mu(A). \tag{5.2}$$

Before we start with the proof, we want to make the following remark. An equivalent way of saying that $\nu(A) = 0$ is that there exists a so-called Weyl sequence of A, that is a sequence (x_n) of elements $x_n \in \mathbf{Y}$ with $\|x_n\| = 1$ for all $n \in \mathbb{N}$, such that $\|Ax_n\| \to 0$ as $n \to \infty$. So A is invertible iff neither A nor A^* has a Weyl sequence (cf. Lemma 2.10). Moreover, A is not even Fredholm if it has a weak Weyl sequence, where the latter refers to a Weyl sequence (x_n) that weakly converges to zero (see e.g. [8, Lemma 4.3.15]). Similarly, we call a Weyl sequence (x_n) a $\mathcal{P}$-Weyl sequence[14] if additionally (instead of weak convergence) $\|P_m x_n\| \to 0$ as $n \to \infty$ for every fixed $m \in \mathbb{N}$. Then we have the following:

[13] We again use the notation $\|b^{-1}\|^{-1} = 0$ for non-invertible elements b.

[14] In [15] a continuous analogue of this concept is mentioned and denoted as Zhislin sequence.

Lemma 5.4. *Let $A \in \mathcal{L}(\mathbf{Y})$. Then $\tilde{\mu}(A) = 0$ iff A has a $\mathcal{P}$-Weyl sequence.*

Proof. If $\tilde{\mu}(A) = 0$, then there exists a sequence (x_n) of elements $x_n \in \mathbf{Y}$ with $\|x_n\| = 1$ such that $x_n \in \operatorname{im} Q_n$ and $Ax_n \to 0$ as $n \to \infty$. This obviously defines a $\mathcal{P}$-Weyl sequence.

Conversely let (x_n) be a $\mathcal{P}$-Weyl sequence of A. Then for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $\|P_m x_n\| < \frac{1}{m}$ and $\|Ax_n\| < \frac{1}{m}$. This implies

$$\frac{\|AQ_m x_n\|}{\|Q_m x_n\|} = \frac{\|Ax_n - AP_m x_n\|}{\|x_n - P_m x_n\|} < \frac{\frac{1}{m} + \|A\|\frac{1}{m}}{1 - \frac{1}{m}} = \frac{1 + \|A\|}{m-1}.$$

Hence $\nu(A|_{\operatorname{im} Q_m}) \to 0$ as $m \to \infty$. □

Thus $\mu(A) = 0$ iff A or A^* has a $\mathcal{P}$-Weyl sequence. Consequently, Theorem 5.3 and further theorems relating $\mu(A) = 0$ to non-$\mathcal{P}$-Fredholmness of A characterize the latter in terms of $\mathcal{P}$-Weyl sequences. So this is a further instance that generalizes from Fredholmness to $\mathcal{P}$-Fredholmness.

The proof of Theorem 5.3 is a simple consequence of the following lemmas:

Lemma 5.5. *Let $\mathcal{P}$ be a uniform approximate projection on the Banach space $\mathbf{Y}$ and $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$. Then $\|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|^{-1} \le \min\{\tilde{\mu}(A), \tilde{\mu}(A^*)\}$. If A is $\mathcal{P}$-Fredholm, then it even holds that $\|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|^{-1} = \tilde{\mu}(A) = \tilde{\mu}(A^*) = \mu(A)$.*

Proof. There is nothing to prove if A is not $\mathcal{P}$-Fredholm, since the LHS equals zero in this case. If A is $\mathcal{P}$-Fredholm let $\varepsilon > 0$ be arbitrary and choose $B_0 \in (A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}$. Since $B_0 A - I =: K \in \mathcal{K}(\mathbf{Y}, \mathcal{P})$ we get for all sufficiently large m that $\|Q_m B_0 A Q_m - Q_m\| = \|Q_m K Q_m\|$ is small enough that $Q_m B_0 A Q_m = Q_m + Q_m K Q_m$ is invertible in $\mathcal{L}(\operatorname{im} Q_m)$ with

$$\underbrace{Q_m (Q_m B_0 A Q_m)^{-1} Q_m B_0}_{=: B_1 \in \mathcal{L}(\mathbf{Y})} A Q_m = Q_m \quad \text{and} \quad \|Q_m B_0 - B_1\| < \varepsilon,$$

and that $\|Q_m B_0\| \le \|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\| + \varepsilon$, taking Proposition 3.1 into account. By this and Lemma 2.10 we get that $\nu(A|_{\operatorname{im} Q_m}) > 0$, hence the compression $A|_{\operatorname{im} Q_m} : \operatorname{im} Q_m \to \operatorname{im} AQ_m$ is invertible and the compression $B_1|_{\operatorname{im} AQ_m} : \operatorname{im} AQ_m \to \operatorname{im} Q_m$ is its (unique) inverse. We conclude that for sufficiently large m

$$\nu(A|_{\operatorname{im} Q_m})^{-1} = \|B_1|_{\operatorname{im} AQ_m}\| \le \|B_1\| \le \|Q_m B_0\| + \|B_1 - Q_m B_0\| \le \|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\| + 2\varepsilon.$$

On the other hand, AQ_m is $\mathcal{P}$-Fredholm and thus has a $\mathcal{P}$-regularizer C. So $\|(AQ_m C - I)Q_k\| < \delta := \varepsilon/(2\|B_1\|)$ if k is large enough. Moreover, from $B_1 A Q_m = Q_m$ and $Q_m \cong I$ modulo $\mathcal{K}(\mathbf{Y}, \mathcal{P})$ we get that B_1 and hence also $B_1 Q_k$ is inverse to A modulo $\mathcal{K}(\mathbf{Y}, \mathcal{P})$. Consequently,

$$\begin{aligned}
\|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\| &= \|B_1 Q_k + \mathcal{K}(\mathbf{Y}, \mathcal{P})\| \le \|B_1 Q_k\| \le \|B_1 A Q_m C Q_k\| + \|B_1\| \|(AQ_m C - I)Q_k\| \\
&< \|B_1|_{\operatorname{im} AQ_m}\| \|AQ_m C Q_k\| + \|B_1\|\delta < \|B_1|_{\operatorname{im} AQ_m}\|(\|Q_k\| + \delta) + \|B_1\|\delta \\
&\le \|B_1|_{\operatorname{im} AQ_m}\| + 2\|B_1\|\delta = \|B_1|_{\operatorname{im} AQ_m}\| + \varepsilon = \nu(A|_{\operatorname{im} Q_m})^{-1} + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we arrive at $\tilde{\mu}(A) = \|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|^{-1}$, by Lemma 5.1. By the same observation for $A^* \in \mathcal{L}(\mathbf{Y}^*, \mathcal{P}^*)$ we find $\tilde{\mu}(A^*) = \|(A^* + \mathcal{K}(\mathbf{Y}^*, \mathcal{P}^*))^{-1}\|^{-1} = \|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|^{-1}$, where Proposition 3.1 justifies the latter equality. □

Lemma 5.6. *Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$. Then $\mu(A) \leq \inf\{\|A_h^{-1}\|^{-1} : A_h \in \sigma_{\mathrm{op}}(A)\}$. For any invertible $A_h \in \sigma_{\mathrm{op}}(A)$ we even have both $\tilde{\mu}(A), \tilde{\mu}(A^*) \leq \|A_h^{-1}\|^{-1}$.*

Proof. Let $A_g \in \sigma_{\mathrm{op}}(A)$.

1st case: A_g is not invertible. For every $\varepsilon > 0$ there is a $\mathcal{P}$-compact operator T of the norm 1 and such that $\|A_g T\| < \varepsilon$ or $\|T A_g\| < \varepsilon$ (cf. [33, Theorem 11]). Let $m \in \mathbb{N}$. It follows from $(Q_m)_g = I$ that

$$\|V_{-g_n} A Q_m V_{g_n} T\| < 2\varepsilon \quad \text{or} \quad \|T V_{-g_n} Q_m A V_{g_n}\| < 2\varepsilon \quad \text{for all sufficiently large } n.$$

Setting $T_n := V_{g_n} T V_{-g_n}$ we have $\|A Q_m T_n\| < 2\varepsilon$ or $\|T_n Q_m A\| < 2\varepsilon$. Since $\|Q_m T_n\|$ and $\|T_n Q_m\|$ tend to 1 as $n \to \infty$ we conclude

$$\frac{\|A Q_m T_n\|}{\|Q_m T_n\|} < 3\varepsilon \quad \text{or} \quad \frac{\|T_n Q_m A\|}{\|T_n Q_m\|} < 3\varepsilon \quad \text{for large } n.$$

This yields $\nu(A|_{\operatorname{im} Q_m}) < 3\varepsilon$ or $\nu(A^*|_{\operatorname{im} Q_m^*}) < 3\varepsilon$, and since ε and m are arbitrary, we conclude $\mu(A) = 0$.

2nd case: A_g is invertible. Now we proceed similarly to the proof of Proposition 3.9. By Lemma 3.8 the compression $(A_g)_0$ is invertible, $((A_g)_0)^{-1} = (A_g^{-1})_0$ and $\|(A_g^{-1})_0\| = \|A_g^{-1}\|$. Let $\varepsilon > 0$. Then there exists an $x_0 \in \mathbf{X}_0$, $\|x_0\| = 1$, with $\|A_g x_0\| = \|(A_g)_0 x_0\| < \nu((A_g)_0) + \varepsilon = \|((A_g)_0)^{-1}\|^{-1} + \varepsilon = \|A_g^{-1}\|^{-1} + \varepsilon$. For sufficiently large k also $x := \|P_k x_0\|^{-1} P_k x_0$ fulfills $\|A_g x\| < \|A_g^{-1}\|^{-1} + \varepsilon$, where $\|x\| = 1$ and $P_k x = x$. For sufficiently large n, $\|(V_{-g_n} A Q_m V_{g_n} - A_g) P_k\| \leq \varepsilon$ holds and we find

$$\|A Q_m V_{g_n} x\| = \|V_{-g_n} A Q_m V_{g_n} P_k x\| \leq \|A_g P_k x\| + \varepsilon = \|A_g x\| + \varepsilon \leq \|A_g^{-1}\|^{-1} + 2\varepsilon,$$

whence $\nu(A|_{\operatorname{im} Q_m}) \leq \|A_g^{-1}\|^{-1} + 2\varepsilon$ holds for every m. Since $\varepsilon > 0$ is arbitrary, $\mu(A) \leq \tilde{\mu}(A) \leq \|A_g^{-1}\|^{-1}$.

In the dual setting we proceed in exactly the same way to get $\tilde{\mu}(A^*) \leq \|(A_g^*)^{-1}\|^{-1} = \|(A_g)^{-1}\|^{-1}$ by considering the compressions $(A_g^*)_0$. □

Thus, we have for all $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ that

$$\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\|^{-1} \leq \mu(A) \leq \inf\{\|A_h^{-1}\|^{-1} : A_h \in \sigma_{\mathrm{op}}(A)\}. \tag{5.3}$$

For rich band-dominated operators the left-hand side and the right-hand side coincide by Corollary 3.7, hence Theorem 5.3 follows.

5.2 2nd approach: Lower norms of $\mathcal{P}$-compact perturbations

For $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ we define the $\mathcal{P}$-essential lower norm of A by

$$\nu_{\mathrm{ess}}(A) = \sup\{\nu(A + K) : K \in \mathcal{K}(\mathbf{Y}, \mathcal{P})\}$$

and we want to study the relations between $\nu_{\mathrm{ess}}(A)$ and $\tilde{\mu}(A)$.

Proposition 5.7. *Let $\mathcal{P}$ be a uniform approximate projection on $\mathbf{Y}$ and $A \in \mathcal{L}(\mathbf{Y},\mathcal{P})$. Then $\nu_{\text{ess}}(A) \le \tilde{\mu}(A)$ and $\nu_{\text{ess}}(A^*) \le \tilde{\mu}(A^*)$. If $\nu(A) > 0$, then $\nu_{\text{ess}}(A) = \tilde{\mu}(A)$. If $\nu(A^*) > 0$, then $\nu_{\text{ess}}(A^*) = \tilde{\mu}(A^*)$.*

Proof. For $\varepsilon > 0$ let $K \in \mathcal{K}(\mathbf{Y},\mathcal{P})$ be s.t. $\nu(A+K) \ge \nu_{\text{ess}}(A) - \varepsilon$ and m s.t. $\|KQ_m\| \le \varepsilon$. Then

$$\tilde{\mu}(A) \ge \nu(A|_{\operatorname{im} Q_m}) \ge \nu((A+K)|_{\operatorname{im} Q_m}) - \|KQ_m\| \ge \nu(A+K) - \varepsilon \ge \nu_{\text{ess}}(A) - 2\varepsilon.$$

Since ε was arbitrary, the estimate $\nu_{\text{ess}}(A) \le \tilde{\mu}(A)$ is proved.

Now, let A be bounded below and assume that there are constants c, d such that $\nu_{\text{ess}}(A) < c < d < \tilde{\mu}(A)$. By the definition $\nu(Q_kA + \alpha P_kA) \le \nu_{\text{ess}}(A)$ for all $k \in \mathbb{N}$ and all scalars α. In particular, for every $k \in \mathbb{N}$ and $\alpha > 0$ there exists $\|x_{k,\alpha}\| = 1$ such that $\|(Q_kA + \alpha P_kA)x_{k,\alpha}\| < c$. This further implies $\|Q_kAx_{k,\alpha}\| < c$ and $\alpha\|P_kAx_{k,\alpha}\| < c$ by ($\mathcal{P}$1). Now, choose $\varepsilon > 0$ such that $c + \varepsilon + 2\varepsilon\|A\|/\nu(A) < d(1 - 2\varepsilon/\nu(A))$, and $\alpha > 1$ such that $c/\alpha < \varepsilon$.

Fix $n \in \mathbb{N}$, take the sequence (F_n) from Proposition 2.4, and choose $m \in \mathbb{N}$ such that $P_nF_m = P_n$ and $\|[F_m, A]\| < \varepsilon$. Then choose $k \in \mathbb{N}$ such that $F_mP_k = F_m$. From $\alpha\|P_kAx_{k,\alpha}\| < c$ we get $\|F_mP_kAx_{k,\alpha}\| < c/\alpha$ and we conclude that $\|AF_mx_{k,\alpha}\| \le c/\alpha + \|[F_m, A]\| < 2\varepsilon$, thus $\|P_nx_{k,\alpha}\| \le \|F_mx_{k,\alpha}\| < 2\varepsilon/\nu(A)$ and $\|Q_nx_{k,\alpha}\| \ge 1 - 2\varepsilon/\nu(A)$. Now

$$\nu(A|_{\operatorname{im} Q_n}) \le \frac{\|AQ_nx_{k,\alpha}\|}{\|Q_nx_{k,\alpha}\|} \le \frac{\|Q_kAx_{k,\alpha}\| + \|P_kAx_{k,\alpha}\| + \|A\|\|P_nx_{k,\alpha}\|}{\|Q_nx_{k,\alpha}\|} < \frac{c + c/\alpha + 2\varepsilon\|A\|/\nu(A)}{1 - 2\varepsilon/\nu(A)} < d$$

and since $n \in \mathbb{N}$ is arbitrary it follows $\tilde{\mu}(A) = \lim_n \nu(A|_{\operatorname{im} Q_n}) \le d < \tilde{\mu}(A)$, a contradiction.

Finally, applying the already proved assertions to $A^* \in \mathcal{L}(\mathbf{Y}^*, \mathcal{P}^*)$ finishes the proof. □

Since ν_{ess} and $\tilde{\mu}$ are invariant under $\mathcal{P}$-compact perturbations it actually holds

Corollary 5.8. *Let $\mathcal{P}$ be a uniform approximate projection on $\mathbf{Y}$ and $A \in \mathcal{L}(\mathbf{Y},\mathcal{P})$. If $A + \mathcal{K}(\mathbf{Y},\mathcal{P})$ contains an operator being bounded below, then $\nu_{\text{ess}}(B) = \tilde{\mu}(B) > 0$ for all $B \in A + \mathcal{K}(\mathbf{Y},\mathcal{P})$. If $A^* + \mathcal{K}(\mathbf{Y}^*,\mathcal{P}^*)$ contains an operator being bounded below, then $\nu_{\text{ess}}(B) = \tilde{\mu}(B) > 0$ for all $B \in A^* + \mathcal{K}(\mathbf{Y}^*,\mathcal{P}^*)$.*

Corollary 5.9. *Let $\mathcal{P}$ be a uniform approximate projection on $\mathbf{Y}$ and $A \in \mathcal{L}(\mathbf{Y},\mathcal{P})$. Then we have either $\nu_{\text{ess}}(A) = 0$ or $\nu_{\text{ess}}(A) = \tilde{\mu}(A) > 0$. Furthermore we have either $\nu_{\text{ess}}(A^*) = 0$ or $\nu_{\text{ess}}(A^*) = \tilde{\mu}(A^*) > 0$.*

Remark 5.10. Fredholm operators with positive index on $l^2(\mathbb{Z}, \mathbb{C})$ show that $0 = \nu_{\text{ess}}(A) < \tilde{\mu}(A)$ can happen. Similarly, negative index yields $0 = \nu_{\text{ess}}(A^*) < \tilde{\mu}(A^*)$.

Corollary 5.11. *Let $A \in \mathcal{L}(\mathbf{X},\mathcal{P})$ and $A + \mathcal{K}(\mathbf{X},\mathcal{P})$ contain a Fredholm operator. Then*

$$\max\{\nu_{\text{ess}}(A), \nu_{\text{ess}}(A^*)\} = \mu(A) = \|(A + \mathcal{K}(\mathbf{X},\mathcal{P}))^{-1}\|^{-1} > 0.$$

If this Fredholm operator has index 0, then additionally $\nu_{\text{ess}}(A) = \nu_{\text{ess}}(A^*)$.

Proof. W.l.o.g. let A be Fredholm. Firstly, we recall that A is automatically $\mathcal{P}$-Fredholm by Proposition 2.3), and that Lemma 5.5 applies. With the help of shifts and projections one easily constructs a one-sided invertible band operator S with banded one-sided inverse and $\operatorname{ind} S = -\operatorname{ind} A$ (cf. e.g. [33, Lemma 24]). We consider the case $\operatorname{ind} A > 0$. Then SA is Fredholm of index zero and with Proposition 2.3 we find $C \in \mathcal{L}(\mathbf{X},\mathcal{P})$ invertible and $K \in \mathcal{K}(\mathbf{X},\mathcal{P})$ such that $SA = C + K$, hence $S^l C = A - S^l K \in A + \mathcal{K}(\mathbf{X},\mathcal{P})$ is right invertible with the right inverse $C^{-1}S \in \mathcal{L}(\mathbf{X},\mathcal{P})$. In the case $\operatorname{ind} A < 0$ we proceed similarly, and in the case $\operatorname{ind} A = 0$, we simply choose $S = I$ and get $C \in A + \mathcal{K}(\mathbf{X},\mathcal{P})$ invertible. Thus Corollary 5.8 applies to A and we get either $\nu_{\mathrm{ess}}(A) = \tilde{\mu}(A)$ or $\nu_{\mathrm{ess}}(A^*) = \tilde{\mu}(A^*)$ (both if $\operatorname{ind} A = 0$). Since $\nu_{\mathrm{ess}}(A) \le \tilde{\mu}(A)$ and $\nu_{\mathrm{ess}}(A^*) \le \tilde{\mu}(A^*)$ by Proposition 5.7 and $\tilde{\mu}(A) = \tilde{\mu}(A^*) = \|(A + \mathcal{K}(\mathbf{X},\mathcal{P}))^{-1}\|^{-1}$ by Lemma 5.5, we conclude

$$\max\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(A^*)\} = \mu(A) = \|(A + \mathcal{K}(\mathbf{X},\mathcal{P}))^{-1}\|^{-1} > 0.$$

The second assertion follows immediately from the considerations above. □

Remark 5.12. Starting with a non-invertible operator $B \in \mathcal{L}(X)$ that is bounded below, one can define the diagonal operator $A := \operatorname{diag}(\dots, B, B, B, \dots)$ on $\mathbf{X}$ which is bounded below, but not Fredholm or $\mathcal{P}$-Fredholm. Hence, $\nu_{\mathrm{ess}}(A)$ is positive, but $\mu(A) = 0$. Thus some kind of Fredholm condition is necessary. This is why we have $\operatorname{sp}_{\varepsilon,\mathrm{ess}}(A) = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) < \varepsilon\}$ but cannot just write

$$\operatorname{sp}_{\varepsilon,\mathrm{ess}}(A) \stackrel{!}{=} \{\lambda \in \mathbb{C} : \max\{\nu_{\mathrm{ess}}(A - \lambda I), \nu_{\mathrm{ess}}((A - \lambda I)^*)\} < \varepsilon\}.$$

However, we will find solutions to this problem in the next sections. Also note that this can not happen if $A \in \mathcal{A}(l^p(\mathbb{Z}, X))$, $\dim X < \infty$ (see Proposition 5.26 below).

5.3 3rd approach: Symmetrization of the problem

In the two previous approaches we used to look at characteristics of both A and A^* in order to get a complete (symmetric) picture. Now we turn the table in a sense, firstly symmetrize the operator and secondly determine its essential lower norm.

Given a Banach space $\mathbf{Y}$ with a uniform approximate projection $\mathcal{P}$, we write $\mathbf{Y} \oplus \mathbf{Y}^*$ for the Banach space of all pairs $(x, f) \in \mathbf{Y} \times \mathbf{Y}^*$, equipped with the norm $\|(x, f)\| := \max\{\|x\|, \|f\|\}$. For $A \in \mathcal{L}(\mathbf{Y})$, $B \in \mathcal{L}(\mathbf{Y}^*)$, write $A \oplus B$ for the operator $(x, f) \mapsto (Ax, Bf)$ in $\mathcal{L}(\mathbf{Y} \oplus \mathbf{Y}^*)$. The following properties of $A \oplus B$ are easy to check:

$$\|A \oplus B\| = \max\{\|A\|, \|B\|\}, \qquad \nu(A \oplus B) = \min\{\nu(A), \nu(B)\}.$$

To get a similar equality for the essential norm, we have to work a bit more. Note that $\mathcal{P} \oplus \mathcal{P}^* = (P_n \oplus P_n^*)_n$ is again a uniform approximate projection on $\mathbf{Y} \oplus \mathbf{Y}^*$.

Proposition 5.13. *Let $A \oplus B \in \mathcal{L}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)$. Then*

$$\|A \oplus B + \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)\| = \max\{\|A + \mathcal{K}(\mathbf{Y},\mathcal{P})\|, \|B + \mathcal{K}(\mathbf{Y}^*,\mathcal{P}^*)\|\}. \tag{5.4}$$

Proof. By the definition, the left hand side $\|A \oplus B + \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)\|$ of (5.4) is

$$\inf_{K \in \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)} \|A \oplus B + K\| \le \inf_{\substack{L \in \mathcal{K}(\mathbf{Y},\mathcal{P}) \\ M \in \mathcal{K}(\mathbf{Y}^*,\mathcal{P}^*)}} \|A \oplus B + L \oplus M\|$$

where the latter equals

$$\begin{aligned} \inf_{\substack{L \in \mathcal{K}(\mathbf{Y},\mathcal{P}) \\ M \in \mathcal{K}(\mathbf{Y}^*,\mathcal{P}^*)}} \|(A + L) \oplus (B + M)\| &= \inf_{\substack{L \in \mathcal{K}(\mathbf{Y},\mathcal{P}) \\ M \in \mathcal{K}(\mathbf{Y}^*,\mathcal{P}^*)}} \max\{\|A + L\|, \|B + M\|\} \\ &= \max\left\{\inf_{L \in \mathcal{K}(\mathbf{Y},\mathcal{P})} \|A + L\|, \inf_{M \in \mathcal{K}(\mathbf{Y}^*,\mathcal{P}^*)} \|B + M\|\right\} \end{aligned}$$

which is the right hand side of (5.4), hence proves one direction.

Let $P_{(1)} : \mathbf{Y} \oplus \mathbf{Y}^* \to \mathbf{Y} \oplus \{0\}, (x, f) \mapsto (x, 0)$ and $P_{(2)} : \mathbf{Y} \oplus \mathbf{Y}^* \to \{0\} \oplus \mathbf{Y}^*, (x, f) \mapsto (0, f)$ be the canonical projections. Then $\|P_{(1)}\| = \|P_{(2)}\| = 1$ and for all $K \in \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)$ we have

$$\begin{aligned}\|A \oplus B + K\| &\geq \max\left\{\left\|P_{(1)}(A \oplus B + K)P_{(1)}\right\|, \left\|P_{(2)}(A \oplus B + K)P_{(2)}\right\|\right\} \\ &= \max\left\{\left\|A \oplus 0 + P_{(1)}KP_{(1)}\right\|, \left\|0 \oplus B + P_{(2)}KP_{(2)}\right\|\right\} \\ &\geq \max\left\{\|A + \mathcal{K}(\mathbf{Y}, \mathcal{P})\|, \|B + \mathcal{K}(\mathbf{Y}^*, \mathcal{P}^*)\|\right\}.\end{aligned}$$

Taking the infimum over all K, we get the reversed inequality. □

Remark 5.14. Note that the naive guess $\nu_{\mathrm{ess}}(A \oplus B) = \min\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(B)\}$ is wrong in general. For example, let $\mathbf{X} = l^2(\mathbb{Z}^N, \mathbb{C})$, in which case $\mathcal{K}(\mathbf{X}, \mathcal{P}) = \mathcal{K}(\mathbf{X})$, and let A be Fredholm on $\mathbf{X}$ with index 1. Then $A \oplus A^*$ has index 0 and therefore there exists a compact operator K on $\mathbf{X} \oplus \mathbf{X}$ such that $(A \oplus A^*) + K$ is invertible and in particular bounded below. Therefore $\nu_{\mathrm{ess}}(A \oplus A^*) \geq \nu((A \oplus A^*) + K) > 0$. However, $A + L$ has index 1 and therefore a nontrivial kernel for all $L \in \mathcal{K}(\mathbf{X})$, so that $\nu_{\mathrm{ess}}(A) = 0$.

Corollary 5.15. *Let $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$.*
Then $\tilde{\mu}(A \oplus A^*) = \mu(A)$ and either $\nu_{\mathrm{ess}}(A \oplus A^*) = \mu(A) > 0$ or $\nu_{\mathrm{ess}}(A \oplus A^*) = 0$.

Proof. We have

$$\tilde{\mu}(A \oplus A^*) = \lim_{m\to\infty} \nu(A \oplus A^*|_{\operatorname{im}(Q_m \oplus Q_m^*)}) = \lim_{m\to\infty} \min\{\nu(A|_{\operatorname{im} Q_m}), \nu(A^*|_{\operatorname{im} Q_m^*})\} = \mu(A).$$

Corollary 5.9 applied to $A \oplus A^*$ yields the claim on $\nu_{\mathrm{ess}}(A \oplus A^*)$. □

Now, we end up with the third characterization of $\|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|^{-1}$:

Theorem 5.16. *Let $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ be $\mathcal{P}$-Fredholm and $A + \mathcal{K}(\mathbf{Y}, \mathcal{P})$ contain a Fredholm operator. Then*

$$\nu_{\mathrm{ess}}(A \oplus A^*) = \tilde{\mu}(A \oplus A^*) = \mu(A) = \|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\|^{-1}.$$

Notice that in the case $\mathbf{Y} = \mathbf{X}$ these equalities can be complemented (cf. Corollary 5.11) by $\max\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(A^*)\} = \mu(A)$.

Proof. W.l.o.g. A is already Fredholm. By [36, Corollary 1.9] we have $\mathcal{P}$-compact projections P, P' onto $\ker A$ and parallel to the range of A, resp. Then $(P')^*, P^*$ are $\mathcal{P}^*$-compact projections onto $\ker A^*$ and parallel to the range of A^*, resp., hence $R := P \oplus (P')^*$ is a projection onto the kernel of $A \oplus A^*$ whereas $R' := P' \oplus P^*$ is a projection parallel to its range. Since both projections are of the same finite rank there exists an isomorphism $C : \operatorname{im} R \to \operatorname{im} R'$. Then $A \oplus A^* + R'CR = (I - R')(A \oplus A^*)(I - R) + R'CR$ is invertible, where $R'CR \in \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)$ since

$$\lim_{m\to\infty} \|R(Q_m \oplus Q_m^*)\| = \lim_{m\to\infty} \|(Q_m \oplus Q_m^*)R'\| = 0.$$

Corollaries 5.8 and 5.15 yield $\nu_{\mathrm{ess}}(A \oplus A^*) = \tilde{\mu}(A \oplus A^*) = \mu(A)$. □

5.4 The Hilbert space case

On a Hilbert space $\mathbf{Y}$ we consider a sequence of nested orthogonal projections $\mathcal{P} = (P_n)_{n\in\mathbb{N}}$, i.e. $P_n = P_n^* = P_n^2 = P_n P_{n+1} = P_{n+1} P_n$ for all $n \in \mathbb{N}$. With this condition $\mathcal{P}$ satisfies ($\mathcal{P}1$) and ($\mathcal{P}2$). If additionally ($\mathcal{P}3$) is satisfied, we call $\mathcal{P}$ a *H*ermitian *app*roximate *i*dentity (in short: *happi*) and the pairing $(\mathbf{Y}, \mathcal{P})$ a happi space. In this more particular case of $\mathbf{Y}$ being a Hilbert space and under this natural assumption on $\mathcal{P}$, we will find that now our above results already apply to *all* operators $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$.

Lemma 5.17. *An operator $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ is Moore-Penrose invertible (as an element of the C^*-algebra $\mathcal{L}(\mathbf{Y})$) if and only if $\operatorname{im} A$ is closed. In that case $A^+ \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$.*

Proof. The first part is [13, Theorem 2.4]. For the second just notice that $\mathcal{L}(\mathbf{Y}, \mathcal{P})$ is a C^*-subalgebra of $\mathcal{L}(\mathbf{Y})$ and apply [13, Corollary 2.18]. □

Theorem 5.18. *Let $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ on a happi space $(\mathbf{Y}, \mathcal{P})$. Then*

$$\mu(A) = \|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|^{-1}. \tag{5.5}$$

Moreover, if $\mu(A) > 0$, then $\tilde{\mu}(A) = \tilde{\mu}(A^*)$.

Proof. By Lemma 5.5, $\mu(A) \geq \|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|^{-1}$, and the first assertion obviously holds if $\mu(A) = 0$.

So, let $\mu(A) > 0$. Then $\nu(A|_{\operatorname{im} Q_m}) > 0$ for a sufficiently large m, hence $\operatorname{im}(AQ_m)$ is closed (by Lemma 2.10), and we get from Lemma 5.17 that AQ_m is Moore-Penrose invertible with $(AQ_m)^+ \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$. Moreover, the compression $(AQ_m)^+ : \operatorname{im} AQ_m \to \operatorname{im} Q_m$ is the inverse of $AQ_m : \operatorname{im} Q_m \to \operatorname{im} AQ_m$, i.e. $(AQ_m)^+ AQ_m = Q_m$ which yields that $(AQ_m)^+ + \mathcal{K}(\mathbf{Y}, \mathcal{P})$ is a left inverse for the coset $AQ_m + \mathcal{K}(\mathbf{Y}, \mathcal{P}) = A + \mathcal{K}(\mathbf{Y}, \mathcal{P})$. By the same means we get that $(A^*Q_m)^+ + \mathcal{K}(\mathbf{Y}, \mathcal{P})$ is a left inverse for $A^* + \mathcal{K}(\mathbf{Y}, \mathcal{P})$, thus $A + \mathcal{K}(\mathbf{Y}, \mathcal{P})$ is also right invertible. This proves that A is $\mathcal{P}$-Fredholm. Applying Lemma 5.5, we get

$$\mu(A) = \tilde{\mu}(A) = \tilde{\mu}(A^*) = \|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|^{-1}.$$

□

Proposition 5.19. *Let $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ be $\mathcal{P}$-Fredholm on a happi space $(\mathbf{Y}, \mathcal{P})$. Then there is a $K \in \mathcal{K}(\mathbf{Y}, \mathcal{P})$ such that $A + K$ has a one-sided inverse in $\mathcal{L}(\mathbf{Y}, \mathcal{P})$.*

Proof. By Theorem 5.18 we have that $\mu(A) > 0$ and therefore $\operatorname{im}(AQ_m)$ is closed for m large enough. In order to simplify notations, we may assume that $\operatorname{im}(A)$ is closed. Let $A_0 : \ker(A)^\perp \to \operatorname{im}(A)$ be defined by $A_0 x = Ax$ for all $x \in \ker(A)^\perp$. Then A_0 is invertible by Banach's isomorphism theorem. Now choose orthonormal bases $\{\beta_i\}_{i\in I}$ and $\{\gamma_j\}_{j\in J}$ of $\ker(A)$ and $\operatorname{im}(A)^\perp$ respectively. Depending on the cardinalities $|I|$ and $|J|$ there is an injection $\iota : I \to J$ or $\iota : J \to I$ (if $|I| = |J|$, there is even a bijection). Let us assume that $|I| \leq |J|$. Then ι induces an isometry $\Phi : \ker(A) \to \operatorname{im}(A)^\perp$ by $\Phi(\beta_i) = \gamma_{\iota(i)}$ for all $i \in I$. Let R_1 and R_2 be orthogonal projections onto $\ker(A)$ and $\operatorname{im}(A)^\perp$ respectively. Then $A + R_2\Phi R_1 = A_0 \oplus \Phi$ is left invertible. More precisely, the Moore-Penrose inverse of $A + R_2\Phi R_1$, which is contained in $\mathcal{L}(\mathbf{Y}, \mathcal{P})$ by Lemma 5.17, is a left inverse in this case. It remains to show that $R_2\Phi R_1$ is $\mathcal{P}$-compact. Similarly as in the proof of Theorem 5.18 we have that R_1 and R_2 are $\mathcal{P}$-compact since A is $\mathcal{P}$-Fredholm. This implies

$$\|R_2\Phi R_1 Q_m\| \leq \|R_1 Q_m\| \to 0 \quad \text{and} \quad \|Q_m R_2 \Phi R_1\| \leq \|Q_m R_2\| \to 0$$

as $m \to \infty$. □

Combining Proposition 5.7, Theorem 5.18 and Proposition 5.19, this immediately yields

Corollary 5.20. *Let $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ be $\mathcal{P}$-Fredholm on a happi space $(\mathbf{Y}, \mathcal{P})$. Then*

$$\max\{\nu_{\text{ess}}(A), \nu_{\text{ess}}(A^*)\} = \mu(A) = \|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|^{-1}. \tag{5.6}$$

More precisely:

- *If $A + \mathcal{K}(\mathbf{X}, \mathcal{P})$ contains a left invertible operator, then*

$$\nu_{\text{ess}}(A) = \mu(A) = \|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\|^{-1}.$$

  Otherwise, $\nu_{\text{ess}}(A) = 0$.

- *If $A + \mathcal{K}(\mathbf{X}, \mathcal{P})$ contains a right invertible operator, then*

$$\nu_{\text{ess}}(A^*) = \mu(A) = \|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\|^{-1}.$$

  Otherwise, $\nu_{\text{ess}}(A^*) = 0$.

Corollary 5.21. *Let $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ on a happi space $(\mathbf{Y}, \mathcal{P})$. Then*

$$\nu_{\text{ess}}(A \oplus A^*) = \mu(A) = \|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|^{-1}. \tag{5.7}$$

Proof. In case of A being $\mathcal{P}$-Fredholm apply Corollary 5.20 to $A \oplus A^*$ and take the observations $\tilde{\mu}(A \oplus A^*) = \tilde{\mu}(A^* \oplus A) = \mu(A \oplus A^*) = \mu(A)$ and $\nu_{\text{ess}}(A \oplus A^*) = \nu_{\text{ess}}(A^* \oplus A)$ into account. Then (5.6) gives (5.7).

Thus it remains to consider $\nu_{\text{ess}}(A \oplus A^*) > 0$ and to show that A is $\mathcal{P}$-Fredholm in the case. Combining Corollary 5.9 and Theorem 5.18, we get that $A \oplus A^*$ is $\mathcal{P}$-Fredholm (w.r.t. $\mathcal{P} \oplus \mathcal{P}$ in $\mathbf{Y} \oplus \mathbf{Y}$). Restricting a $\mathcal{P}$-regularizer for $A \oplus A^*$ to the first component yields a $\mathcal{P}$-regularizer for A and thus A is $\mathcal{P}$-Fredholm. □

4th approach: Composition of A and A^* In our three previous approaches to characterize $\|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|$, we always combined information on A and A^*. Similar to the formula $\|A^{-1}\|^{-1} = \min(\nu(A), \nu(A^*))$ from Lemma 2.10, one always needs to look at both of them. What we did so far is to consider the following ideas:

- Take the numbers $\tilde{\mu}(A)$ and $\tilde{\mu}(A^*)$ and look at their minimum $\mu(A)$.
- Take the numbers $\nu_{\text{ess}}(A)$ and $\nu_{\text{ess}}(A^*)$ and look at their maximum.
- Combine both operators to $A \oplus A^*$ and look at the number $\nu_{\text{ess}}(A \oplus A^*)$.

These expressions are found, most notably, in Theorems 5.3, 5.18; in Corollary 5.11 and Corollary 5.20; as well as in Theorem 5.16 and Corollary 5.21. A different approach to the same goal, to find $\|(A+\mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|$, could be to couple the operators A and A^* to a new operator via composition.

Corollary 5.22. *Let $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ on a happi space $(\mathbf{Y}, \mathcal{P})$. Then*

$$\min\left\{\sqrt{\nu_{\text{ess}}(AA^*)}, \sqrt{\nu_{\text{ess}}(A^*A)}\right\} = \mu(A) = \|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|^{-1}.$$

Proof. If A is $\mathcal{P}$-Fredholm then, since $\mathcal{L}(\mathbf{Y},\mathcal{P})/\mathcal{K}(\mathbf{Y},\mathcal{P})$ is a C^*-algebra,

$$\|(AA^*+\mathcal{K}(\mathbf{Y},\mathcal{P}))^{-1}\|^{-1} = \|(A+\mathcal{K}(\mathbf{Y},\mathcal{P}))^{-1}\|^{-2} = \|(A^*A+\mathcal{K}(\mathbf{Y},\mathcal{P}))^{-1}\|^{-1}$$

and the assertion follows from Corollary 5.20 applied to the self-adjoint operators AA^* and A^*A. So it remains to show that $\min\left\{\sqrt{\nu_{\mathrm{ess}}(AA^*)}, \sqrt{\nu_{\mathrm{ess}}(A^*A)}\right\} > 0$ implies that A is $\mathcal{P}$-Fredholm. Combining Proposition 5.7 and Theorem 5.18, we get that AA^* and A^*A are both $\mathcal{P}$-Fredholm. Consequently, $A+\mathcal{K}(\mathbf{Y},\mathcal{P})$ has both right and left inverses. Thus A is also $\mathcal{P}$-Fredholm. □

5.5 The case of finite-dimensional entries

Let us now consider the case $\mathbf{X} = l^p(\mathbb{Z}^N, X)$, with $\dim X < \infty$, which we already addressed in Remark 2.2 and Proposition 2.3. Then $\mathcal{K}(\mathbf{X},\mathcal{P}) \subset \mathcal{K}(\mathbf{X})$ holds (since every $\mathcal{P}$-compact operator K is the norm limit of the sequence of finite rank operators KP_n), hence every $\mathcal{P}$-Fredholm operator is Fredholm. Actually, the $\mathcal{P}$-Fredholm property coincides with Fredholmness by Proposition 2.3, and we even have

Proposition 5.23. *Let $\dim X < \infty$ and $A \in \mathcal{L}(\mathbf{X},\mathcal{P})$. Then*

$$\|A+\mathcal{K}(\mathbf{X})\| = \|A+\mathcal{K}(\mathbf{X},\mathcal{P})\| = \|A^*+\mathcal{K}(\mathbf{X}^*)\| = \|A^*+\mathcal{K}(\mathbf{X}^*,\mathcal{P}^*)\|,$$
$$\|(A+\mathcal{K}(\mathbf{X}))^{-1}\|^{-1} = \|(A+\mathcal{K}(\mathbf{X},\mathcal{P}))^{-1}\|^{-1} = \|(A^*+\mathcal{K}(\mathbf{X}^*))^{-1}\|^{-1} = \|(A^*+\mathcal{K}(\mathbf{X}^*,\mathcal{P}^*))^{-1}\|^{-1}.$$

Note that the essential and $\mathcal{P}$-essential norm obviously do not coincide if $\dim X = \infty$, just consider the operators P_1 and $I-P_1$.

Proof. Since (A Fredholm $\Rightarrow$ A $\mathcal{P}$-Fredholm (by Proposition 2.3) $\Rightarrow$ A^* $\mathcal{P}^*$-Fredholm $\Rightarrow$ A^* Fredholm (by $\mathcal{K}(\mathbf{X}^*) \supset \mathcal{K}(\mathbf{X}^*,\mathcal{P}^*)$) $\Rightarrow$ A Fredholm), all terms in the second line are simultaneously zero or non-zero. If they are non-zero, then Proposition 2.3 provides a generalized inverse $B \in \mathcal{L}(\mathbf{X},\mathcal{P})$ for A, and the second asserted line follows from the first one applied to B.

For the first line we recall Proposition 3.1 which shows that the $\mathcal{P}$-essential norm is invariant under passing to the adjoint A^*. In the cases $p \in \{0\} \cup (1,\infty)$, where $\mathcal{K}(\mathbf{X},\mathcal{P}) = \mathcal{K}(\mathbf{X})$ holds, $\|A+\mathcal{K}(\mathbf{X})\| = \|A+\mathcal{K}(\mathbf{X},\mathcal{P})\|$ is obvious as well, and we next prove this equality for the cases $p=1$ and $p=\infty$:

$p=1$: Let $\varepsilon > 0$ and choose $K \in \mathcal{K}(\mathbf{X})$ such that $\|A+K\| \le \|A+\mathcal{K}(\mathbf{X})\| + \varepsilon$ and $m_0 \in \mathbb{N}$ such that $\|Q_mK\| \le \varepsilon$ for all $m \ge m_0$, which is possible because Q_m converges strongly to 0 as $m \to \infty$ and K is compact. Now we can proceed as in Proposition 3.1:

$$\|Q_mA\| = \|A-P_mA\| \ge \|A+\mathcal{K}(\mathbf{X})\| \ge \|A+K\| - \varepsilon \ge \|Q_m(A+K)\| - \varepsilon \ge \|Q_mA\| - 2\varepsilon$$

for all $m \ge m_0$ and therefore $\|A+\mathcal{K}(\mathbf{X})\| = \lim_{m\to\infty}\|Q_mA\| = \|A+\mathcal{K}(\mathbf{X},\mathcal{P})\|$.

$p=\infty$: Let $\varepsilon > 0$ and choose $K \in \mathcal{K}(\mathbf{X})$ such that $\|A+K\| \le \|A+\mathcal{K}(\mathbf{X})\| + \varepsilon$ and $m_0 \in \mathbb{N}$ such that $\|KQ_m|_{\mathbf{X}_0}\| \le \varepsilon$ for all $m \ge m_0$, which is possible because $(Q_m|_{\mathbf{X}_0})^* = Q_m|_{\ell^1(\mathbb{Z}^N,X^*)}$ converges strongly to 0 as $m \to \infty$ and $(K|_{\mathbf{X}_0})^*$ is compact. Now we can proceed as before, using Lemma 3.8 a):

$$\begin{aligned}\|AQ_m\| = \|A-AP_m\| &\ge \|A+\mathcal{K}(\mathbf{X})\| \ge \|A+K\| - \varepsilon \ge \|(A+K)Q_m\| - \varepsilon \\ &\ge \|(A+K)Q_m|_{\mathbf{X}_0}\| - \varepsilon \ge \|AQ_m|_{\mathbf{X}_0}\| - \|KQ_m|_{\mathbf{X}_0}\| - \varepsilon \ge \|AQ_m\| - 2\varepsilon\end{aligned}$$

for all $m \ge m_0$ and therefore $\|A+\mathcal{K}(\mathbf{X})\| = \lim_{m\to\infty}\|AQ_m\| = \|A+\mathcal{K}(\mathbf{X},\mathcal{P})\|$.

Up to now we have $\|A+\mathcal{K}(\mathbf{X})\| = \|A+\mathcal{K}(\mathbf{X},\mathcal{P})\| = \|A^*+\mathcal{K}(\mathbf{X}^*,\mathcal{P}^*)\|$ for all p, and hence the complete first line for all $p<\infty$ by taking a circuit using the natural and well known dualities: $\|A+\mathcal{K}(\mathbf{X})\| = \|A+\mathcal{K}(\mathbf{X},\mathcal{P})\| = \|A^*+\mathcal{K}(\mathbf{X}^*,\mathcal{P}^*)\| = \|A^*+\mathcal{K}(\mathbf{X}^*)\|$. It remains to prove that in the case $p=\infty$ the essential norm of A coincides with the essential norm of A^*. Actually, such a claim this is not true in general Banach spaces, as was shown in [3]. Anyway, for our particular case $\mathbf{X} = l^\infty(\mathbb{Z}^N, X)$ with $\dim X < \infty$ one can utilize a further observation from [3]: $\mathbf{X}$ is the dual of another Banach space, namely $\mathbf{Y} = l^1(\mathbb{Z}^N, X^*)$, and the adjoint of the canonical embedding $E : \mathbf{Y} \to \mathbf{Y}^{**}$ is an operator $E^* : \mathbf{X}^{**} \to \mathbf{X}$ onto $\mathbf{X}$ of the norm 1. For every $K \in \mathcal{K}(\mathbf{X}^{**})$ (with J denoting the canonical embedding $J : \mathbf{X} \to \mathbf{X}^{**}$)

$$\|A^{**}+K\| \geq \|(A^{**}+K)J\| \geq \|E^*(A^{**}+K)J\| = \|A+E^*KJ\| \geq \|A+\mathcal{K}(\mathbf{X})\|.$$

Taking the infimum over all K it follows $\|A^{**}+\mathcal{K}(\mathbf{X}^{**})\| \geq \|A+\mathcal{K}(\mathbf{X})\|$. Since the adjoint of any compact operator is compact the desired equality follows by

$$\|A+\mathcal{K}(\mathbf{X})\| \geq \|A^*+\mathcal{K}(\mathbf{X}^*)\| \geq \|A^{**}+\mathcal{K}(\mathbf{X}^{**})\| \geq \|A+\mathcal{K}(\mathbf{X})\|.$$

□

In analogy to $\nu_{\mathrm{ess}}(A)$ we denote the classical (w.r.t. $\mathcal{K}(\mathbf{X})$) essential lower norm by $\nu_{\mathrm{ess,c}}(A)$:

$$\nu_{\mathrm{ess,c}}(A) := \sup\{\nu(A+K) : K \in \mathcal{K}(\mathbf{X})\}$$

and we obtain the following improvement and completion of the results in the previous sections:

Theorem 5.24. *Let* $\dim X < \infty$ *and* $A \in \mathcal{L}(\mathbf{X},\mathcal{P})$. *Then*

$$\nu_{\mathrm{ess,c}}(A\oplus A^*) = \nu_{\mathrm{ess}}(A\oplus A^*) = \mu(A) = \|(A+\mathcal{K}(\mathbf{X},\mathcal{P}))^{-1}\|^{-1} = \|(A+\mathcal{K}(\mathbf{X}))^{-1}\|^{-1}. \tag{5.8}$$

Moreover, if A is Fredholm of index zero, then

$$\nu_{\mathrm{ess,c}}(A) = \nu_{\mathrm{ess,c}}(A^*) = \nu_{\mathrm{ess}}(A) = \nu_{\mathrm{ess}}(A^*) = \mu(A) = \tilde{\mu}(A) = \tilde{\mu}(A^*) > 0.$$

Conversely, if $\nu_{\mathrm{ess,c}}(A) > 0$ *and* $\nu_{\mathrm{ess,c}}(A^*) > 0$, *then A is Fredholm of index zero.*

Proof. Let B be a Fredholm operator of index zero on a Banach space $\mathbf{Y}$. Then,

$$\begin{aligned}
\nu_{\mathrm{ess,c}}(B) &= \sup\{\nu(B+K) : K \in \mathcal{K}(\mathbf{Y}), B+K \text{ bounded below}\} \\
&= \sup\{\nu(B+K) : K \in \mathcal{K}(\mathbf{Y}), B+K \text{ invertible}\} && (\mathrm{ind}(B+K)=0) \\
&= \sup\{\|(B+K)^{-1}\|^{-1} : K \in \mathcal{K}(\mathbf{Y}), B+K \text{ invertible}\} && (\text{Lemma } 2.10) \\
&= (\inf\{\|(B+K)^{-1}\| : K \in \mathcal{K}(\mathbf{Y}), B+K \text{ invertible}\})^{-1} \\
&= \|(B+\mathcal{K}(\mathbf{Y}))^{-1}\|^{-1}.
\end{aligned}$$

Hence, if A is Fredholm of index zero, then $\nu_{\mathrm{ess,c}}(A^*) = \nu_{\mathrm{ess,c}}(A) = \|(A+\mathcal{K}(\mathbf{X}))^{-1}\|^{-1}$. Together with the obvious estimates $\nu_{\mathrm{ess,c}}(A) \geq \nu_{\mathrm{ess}}(A)$ and $\nu_{\mathrm{ess,c}}(A^*) \geq \nu_{\mathrm{ess}}(A^*)$, Corollary 5.11 and Proposition 5.23 yield $\nu_{\mathrm{ess,c}}(A) = \nu_{\mathrm{ess,c}}(A^*) = \nu_{\mathrm{ess}}(A) = \nu_{\mathrm{ess}}(A^*) = \mu(A) = \|(A+\mathcal{K}(\mathbf{X}))^{-1}\|^{-1}$. Corollary 5.9 completes this equality: $\nu_{\mathrm{ess}}(A) = \tilde{\mu}(A)$, $\nu_{\mathrm{ess}}(A^*) = \tilde{\mu}(A^*)$.

Whenever A is a (general) Fredholm operator then the operator $A \oplus A^*$ is Fredholm of index zero, and by the above it follows that $\nu_{\mathrm{ess,c}}(A\oplus A^*) = \|(A\oplus A^* + \mathcal{K}(\mathbf{X}\oplus\mathbf{X}^*))^{-1}\|^{-1}$. If B

is a regularizer for A then the latter is $\|B \oplus B^* + \mathcal{K}(\mathbf{X} \oplus \mathbf{X}^*)\|^{-1}$. This equals $\|B + \mathcal{K}(\mathbf{X})\|^{-1}$ which is proved in the same way as Proposition 5.13, taking Proposition 5.23 into account. Thus $\nu_{\text{ess,c}}(A \oplus A^*) = \|(A + \mathcal{K}(\mathbf{X}))^{-1}\|^{-1}$. Since

$$\nu_{\text{ess,c}}(A \oplus A^*) \geq \nu_{\text{ess}}(A \oplus A^*) = \mu(A) \geq \|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\|^{-1} = \|(A + \mathcal{K}(\mathbf{X}))^{-1}\|^{-1},$$

(cf. Theorem 5.16, Lemma 5.5, Proposition 5.23) the claim (5.8) easily follows.

If A is not Fredholm then A is not normally solvable or A has infinite dimensional kernel or A^* has infinite dimensional kernel. In the first and second case this remains true for all $A|_{\operatorname{im} Q_m}$, resp., hence all $\nu(A|_{\operatorname{im} Q_m})$ equal zero. In the latter case all $\nu(A^*|_{\operatorname{im} Q_m^*})$ must be zero, and we conclude that $\mu(A) = 0$. Moreover, $A \oplus A^*$ is not normally solvable or has infinite dimensional kernel, hence $0 = \nu_{\text{ess,c}}(A \oplus A^*) \geq \nu_{\text{ess}}(A \oplus A^*)$. Thus (5.8) also holds in this case.

If $\nu_{\text{ess,c}}(A) > 0$, then there exists a $K \in \mathcal{K}(\mathbf{X})$ such that $\nu(A + K) > 0$. This implies that $A + K$ is injective and normally solvable by Lemma 2.10. This implies that A is normally solvable and has finite-dimensional kernel. Similarly, if $\nu_{\text{ess,c}}(A^*) > 0$, then A^* is normally solvable and has finite-dimensional kernel. In particular, A is Fredholm if both $\nu_{\text{ess,c}}(A) > 0$ and $\nu_{\text{ess,c}}(A^*) > 0$ hold. Moreover, $\nu(A + K) > 0$ implies $\operatorname{ind}(A) = \operatorname{ind}(A + K) \leq 0$ whereas $\nu(A^* + L) > 0$ implies $-\operatorname{ind}(A) = \operatorname{ind}(A^*) = \operatorname{ind}(A^* + L) \leq 0$. Thus the index of A has to be zero. This proves the last part. ☐

Actually, there is an even more abstract version of Theorem 5.24 within the $\mathcal{P}$-framework:

Theorem 5.25. *Let $\mathbf{Y}$ be a Banach space with a uniform approximate identity $\mathcal{P} = (P_n)$ consisting of finite rank projections P_n. Then for every $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ which has the $\mathcal{P}$-dichotomy*

$$\nu_{\text{ess}}(A \oplus A^*) = \mu(A) = \|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|^{-1}. \tag{5.9}$$

Proof. Let A be $\mathcal{P}$-Fredholm. Then $A \oplus A^*$ is $\mathcal{P} \oplus \mathcal{P}^*$-Fredholm, Fredholm of index 0, and

$$\begin{aligned}\nu_{\text{ess}}(A \oplus A^*) &= \sup\{\nu(A \oplus A^* + K) : K \in \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*), A \oplus A^* + K \text{ bounded below}\} \\ &= \ldots = \|(A \oplus A^* + \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*))^{-1}\|^{-1}\end{aligned}$$

as above. It follows that $\nu_{\text{ess}}(A \oplus A^*) = \|(A \oplus A^* + \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*))^{-1}\|^{-1} = \|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\|^{-1}$ also by Proposition 5.13 and Proposition 3.1, taking Theorem 2.1 for a regularizer B of A into account. Theorem 5.16 completes (5.9).

If A is not $\mathcal{P}$-Fredholm then A is $\mathcal{P}$-deficient. Thus $A \oplus A^*$ is $\mathcal{P} \oplus \mathcal{P}^*$-deficient from both sides, and it easily follows $\nu_{\text{ess}}(A \oplus A^*) = \mu(A) = 0$. Thus (5.9) also holds in this case. ☐

In the particular case $N = 1$, we also have the following Proposition for band-dominated operators:

Proposition 5.26. *Let $\dim X < \infty$ and $A \in \mathcal{A}(l^p(\mathbb{Z}, X))$. Then*

$$\max\{\nu_{\text{ess}}(A), \nu_{\text{ess}}(A^*)\} = \mu(A) = \|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\|^{-1} = \|(A + \mathcal{K}(\mathbf{X}))^{-1}\|^{-1}.$$

Proof. The equality of $\mu(A)$, $\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\|^{-1}$ and $\|(A + \mathcal{K}(\mathbf{X}))^{-1}\|^{-1}$ holds by Theorem 5.24. If A is Fredholm, then Corollary 5.11 implies the remaining equality. If A is not Fredholm, then A is not even semi-Fredholm by [35, Theorem 4.3], i.e. either A is not normally solvable or both A and A^* have an infinite-dimensional kernel. This also remains true for all $B \in A + \mathcal{K}(\mathbf{X})$. It follows $\nu_{\text{ess}}(A) \leq \nu_{\text{ess,c}}(A) = 0$ and $\nu_{\text{ess}}(A^*) \leq \nu_{\text{ess,c}}(A^*) = 0$ by the definition of $\nu_{\text{ess,c}}$ and Lemma 2.10, hence $\max\{\nu_{\text{ess}}(A), \nu_{\text{ess}}(A^*)\} = 0$. ☐

5th approach: Approximation numbers and singular values In this more comfortable situation $\dim X < \infty$ we can study further characterizations of $\mu(A)$.

Definition 5.27. (cf. [30, 34, 36]) For an operator $A \in \mathcal{L}(\mathbf{X})$ we define the mth lower Bernstein numbers and (one-sided) approximation numbers by

$$\begin{aligned} B_m(A) &:= \sup\{\nu(A|_V) : \dim \mathbf{X}/V < m\}, \\ s_m^r(A) &:= \inf\{\|A - F\| : F \in \mathcal{L}(\mathbf{X}), \dim \ker F \geq m\}, \\ s_m^l(A) &:= \inf\{\|A - F\| : F \in \mathcal{L}(\mathbf{X}), \dim \operatorname{coker} F \geq m\}. \end{aligned}$$

Moreover we introduce the limits

$$\begin{aligned} B(A) &:= \lim_{m\to\infty} \min\{B_m(A), B_m(A^*)\}, \\ S(A) &:= \lim_{m\to\infty} \min\{s_m^r(A), s_m^l(A)\}, \end{aligned}$$

whose existence is proved by monotonicity as in Lemma 5.1.

Theorem 5.28. *Let $\dim X < \infty$ and $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$. Then*

$$\mu(A) = B(A) = S(A). \tag{5.10}$$

Proof. By definition $\nu(A|_{\operatorname{im} Q_m}) \leq B_{\dim \ker Q_m + 1}(A)$, and $\mu(A) \leq B(A)$ easily follows. Furthermore $B(A) \leq S(A)$ holds by [36, Proposition 2.9] or [34, Proposition 1.36]. Also, it is easily seen that all these numbers are zero if A is not Fredholm.

Let A be Fredholm. Assume that there are constants d, e such that $\tilde{\mu}(A) < d < e < S(A)$. Then $\nu(A|_{\operatorname{im} Q_n}) < d$ for all $n \in \mathbb{N}$. This means that there exist $y_n \in \operatorname{im} Q_n$ such that $\|Ay_n\| < d\|y_n\|$, respectively. Recalling the sequence (F_n) from Proposition 2.4 we further conclude that $\|AF_l y_n\| < d\|F_l y_n\|$ for sufficiently large l, since $\|[A, F_l]\|$ tends to zero and $\|F_l y_n\|$ tends to $\|y_n\|$ as $l \to \infty$. Fix such an l (which depends on n) such that also $F_l P_n = P_n F_l = P_n$ holds, and define $z_n := \|F_l y_n\|^{-1} F_l y_n$, respectively. Then $z_n \in \operatorname{im} Q_n$ is still true since $F_l y_n = F_l Q_n y_n = Q_n F_l y_n$.

Next, we fix $m \in \mathbb{N}$ and choose numbers $n_1, \ldots, n_m$ as follows: Set $n_1 := 1$. Given n_i choose l_i such that $P_{l_i} F_{n_i} = F_{n_i} P_{l_i} = F_{n_i}$. Then z_{n_i} is in the range of $P_{l_i} Q_{n_i}$. Furthermore choose $k_i > l_i$ such that $\|Q_{k_i} A P_{l_i}\| < 2^{-i-1}(e - d)$ and $n_{i+1} > k_i$ such that $\|P_{k_i} A Q_{n_{i+1}}\| < 2^{-i-2}(e - d)$. For every i let R_i be a projection of norm 1 onto $\operatorname{span}\{z_{n_i}\}$ and such that $R_i = R_i P_{l_i} Q_{n_i}$, respectively, and define $S_m := \sum_{i=1}^m R_i$. Then S_m is a projection of rank m, $R_i = R_i S_m$ for all $i = 1, \ldots, m$, and $\|S_m\| = 1$. Moreover,

$$\begin{aligned} \|AS_m x\| &= \left\| \sum_{i=1}^m AR_i x \right\| = \left\| \sum_{i=1}^m P_{k_i} Q_{k_{i-1}} AR_i x + \sum_{i=1}^m P_{k_{i-1}} AR_i x + \sum_{i=1}^m Q_{k_i} AR_i x \right\| \\ &\leq \left\| \sum_{i=1}^m P_{k_i} Q_{k_{i-1}} AR_i x \right\| + \left\| \sum_{i=1}^m P_{k_{i-1}} AQ_{n_i} R_i x \right\| + \left\| \sum_{i=1}^m Q_{k_i} AP_{l_i} R_i x \right\| \\ &\leq \left\| \sum_{i=1}^m P_{k_i} Q_{k_{i-1}} AR_i x \right\| + \sum_{i=1}^m 2^{-i-1}(e - d)\|x\| + \sum_{i=1}^m 2^{-i-1}(e - d)\|x\|. \end{aligned}$$

For the first term we have

$$\left\| \sum_{i=1}^{m} P_{k_i} Q_{k_{i-1}} A R_i x \right\|^p = \sum_{i=1}^{m} \|P_{k_i} Q_{k_{i-1}} A R_i x\|^p \le d^p \sum_{i=1}^{m} \|R_i x\|^p = d^p \|S_m x\|^p \le d^p \|x\|^p$$

in the cases $p \in [1, \infty)$, and similarly for $p \in \{0, \infty\}$. Thus $\|AS_m x\| \le e\|x\|$ for all x, and hence

$$s_m^r(A) = \inf\{\|A - F\| : \dim\ker F \ge m\} \le \|A - A(I - S_m)\| = \|AS_m\| \le e < S(A).$$

Sending $m \to \infty$ we arrive at a contradiction. Thus $\tilde{\mu}(A) \ge S(A)$.

Since A is $\mathcal{P}$-Fredholm by Proposition 2.3, we can apply Lemma 5.5 to obtain

$$S(A) \ge B(A) \ge \mu(A) = \tilde{\mu}(A) \ge S(A).$$

$\square$

Remark 5.29. In the case $\mathbf{X}$ being a Hilbert space we even have

$$\mu(A) = \Sigma(A) := \lim_{m \to \infty} \min\{\sigma_m(A), \sigma_m(A^*)\},$$

where $\sigma_m(A)$ denotes the m-th singular value of A (see [36, Corollary 2.12]).

5.6 On the characterization of essential (pseudo)spectra

We have seen that in the following cases there are several characterizations of the essential lower norm

- Band-dominated operators on all sequence spaces $\mathbf{X}$
- $\mathcal{L}(\mathbf{X}, \mathcal{P})$-operators in the case $\dim X < \infty$
- $\mathcal{L}(\mathbf{Y}, \mathcal{P})$-operators on happi spaces $(\mathbf{Y}, \mathcal{P})$,

namely $\mu(A)$, $\nu_{\text{ess}}(A \oplus A^*)$ in all these cases and additionally the essential lower norms of AA^* and A^*A in the happi case. The case $\dim X < \infty$ offers the largest collection of characterizations, including also $B(A)$ and $S(A)$, and the classical (non-$\mathcal{P}$) essential lower norm.

Each of them permits to give an equivalent definition of $\mathcal{P}$-essential spectra and pseudospectra:

Theorem 5.30. *a) Let $A \in \mathcal{A}_{\$}(\mathbf{X})$. Then*

$$\begin{aligned} \operatorname{sp}_{\text{ess}}(A) &= \{\lambda \in \mathbb{C} : \|(A - \lambda I + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\|^{-1} = 0\} = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) = 0\}, \\ \operatorname{sp}_{\varepsilon,\text{ess}}(A) &= \{\lambda \in \mathbb{C} : \|(A - \lambda I + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\|^{-1} < \varepsilon\} = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) < \varepsilon\} \\ &= \operatorname{sp}_{\text{ess}}(A) \cup \{\lambda \in \mathbb{C} : \max\{\nu_{\text{ess}}(A - \lambda I), \nu_{\text{ess}}((A - \lambda I)^*)\} < \varepsilon\}. \end{aligned}$$

If even $\mathbf{X} = l^p(\mathbb{Z}, X)$ with $\dim X < \infty$, then

$$\operatorname{sp}_{\varepsilon,\text{ess}}(A) = \{\lambda \in \mathbb{C} : \max\{\nu_{\text{ess}}(A - \lambda I), \nu_{\text{ess}}((A - \lambda I)^*)\} < \varepsilon\}.$$

b) Let $(\mathbf{Y}, \mathcal{P})$ be a happi space and $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$. Then

$$\begin{aligned}
\mathrm{sp}_{\mathrm{ess}}(A) &= \{\lambda \in \mathbb{C} : \mu(A - \lambda I) = 0\} \\
&= \{\lambda \in \mathbb{C} : \nu_{\mathrm{ess}}((A - \lambda I) \oplus (A - \lambda I)^*) = 0\} \\
&= \{\lambda \in \mathbb{C} : \min\{\sqrt{\nu_{\mathrm{ess}}((A - \lambda I)(A - \lambda I)^*)}, \sqrt{\nu_{\mathrm{ess}}((A - \lambda I)^*(A - \lambda I))}\} = 0\}, \\
\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) &= \{\lambda \in \mathbb{C} : \mu(A - \lambda I) < \varepsilon\} \\
&= \{\lambda \in \mathbb{C} : \nu_{\mathrm{ess}}((A - \lambda I) \oplus (A - \lambda I)^*) < \varepsilon\} \\
&= \{\lambda \in \mathbb{C} : \min\{\sqrt{\nu_{\mathrm{ess}}((A - \lambda I)(A - \lambda I)^*)}, \sqrt{\nu_{\mathrm{ess}}((A - \lambda I)^*(A - \lambda I))}\} < \varepsilon\} \\
&= \mathrm{sp}_{\mathrm{ess}}(A) \cup \{\lambda \in \mathbb{C} : \max\{\nu_{\mathrm{ess}}(A - \lambda I), \nu_{\mathrm{ess}}((A - \lambda I)^*)\} < \varepsilon\}.
\end{aligned}$$

c) Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ and $\dim X < \infty$. Then

$$\begin{aligned}
\mathrm{sp}_{\mathrm{ess}}(A) &= \{\lambda \in \mathbb{C} : \|(A - \lambda I + \mathcal{K}(\mathbf{X}))^{-1}\|^{-1} = 0\} \\
&= \{\lambda \in \mathbb{C} : \mu(A - \lambda I) = 0\} \\
&= \{\lambda \in \mathbb{C} : \nu_{\mathrm{ess}}((A - \lambda I) \oplus (A - \lambda I)^*) = 0\} \\
&= \{\lambda \in \mathbb{C} : B(A - \lambda I) = 0\} \\
&= \{\lambda \in \mathbb{C} : S(A - \lambda I) = 0\}, \\
\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) &= \{\lambda \in \mathbb{C} : \|(A - \lambda I + \mathcal{K}(\mathbf{X}))^{-1}\|^{-1} < \varepsilon\} \\
&= \{\lambda \in \mathbb{C} : \mu(A - \lambda I) < \varepsilon\} \\
&= \{\lambda \in \mathbb{C} : \nu_{\mathrm{ess}}((A - \lambda I) \oplus (A - \lambda I)^*) < \varepsilon\} \\
&= \{\lambda \in \mathbb{C} : B(A - \lambda I) < \varepsilon\} \\
&= \{\lambda \in \mathbb{C} : S(A - \lambda I) < \varepsilon\} \\
&= \mathrm{sp}_{\mathrm{ess}}(A) \cup \{\lambda \in \mathbb{C} : \max\{\nu_{\mathrm{ess}}(A - \lambda I), \nu_{\mathrm{ess}}((A - \lambda I)^*)\} < \varepsilon\}.
\end{aligned}$$

d) If the conditions in b) and c) are both fulfilled, we additionally have

$$\begin{aligned}
\mathrm{sp}_{\mathrm{ess}}(A) &= \{\lambda \in \mathbb{C} : \Sigma(A - \lambda I) = 0\}, \\
\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) &= \{\lambda \in \mathbb{C} : \Sigma(A - \lambda I) < \varepsilon\},
\end{aligned}$$

6 On finite sections

In this section we apply our results, in particular Corollary 3.7, in the context of asymptotic inversion of an operator.

Stability Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$. For the approximate solution of an equation $Ax = b$ or, likewise, for the approximation of the pseudospectrum $\mathrm{sp}_\varepsilon(A)$, one is looking for approximations of the inverse of A (or of $A - \lambda I$, respectively) by operators that can be stored and worked with on a computer.

Assuming invertibility of A, a natural idea is to take a sequence of operators $A_1, A_2, \ldots$ in $\mathcal{L}(\mathbf{X}, \mathcal{P})$ with $A_n \xrightarrow{\mathcal{P}} A$ as $n \to \infty$, and to hope that, for all sufficiently large n, also A_n is invertible and $A_n^{-1} \xrightarrow{\mathcal{P}} A^{-1}$. It turns out (see e.g. [28, Theorem 6.1.3], [17, Corollary 1.77], [34, Propositions 1.22, 1.29 and Corollary 1.28]) that this hope will be fulfilled if and only if the sequence (A_n) is *stable*, meaning that there is an n_0 such that all A_n with $n \geq n_0$ are invertible and $\sup_{n \geq n_0} \|A_n^{-1}\| < \infty$. In short:

$$(A_n) \text{ is stable} \qquad :\Longleftrightarrow \qquad \limsup_{n\to\infty} \|A_n^{-1}\| < \infty \tag{6.1}$$

After a positive answer to this qualitative question about stability, one will ask about quantities:

- (Q1) How large is the lim sup in (6.1)?
- (Q2) Is it possibly a limit?
- (Q3) What is the asymptotics of the condition numbers $\kappa(A_n) = \|A_n\|\|A_n^{-1}\|$?
- (Q4) What is the asymptotic behaviour of the pseudospectra $\mathrm{sp}_\varepsilon(A_n)$?

There are different approaches [27, 29, 18, 34, 37] to deal with these questions. We will discuss one of them and we will focus on questions (Q1) and (Q3). The discussion of (Q2) and (Q4) is postponed to a further paper, [14], as it would overstretch both length and scope of the current paper. Moreover, we will restrict ourselves here to studying sequences (A_n) of so-called finite sections (see below) of an $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ as opposed to [14], where we look at more general elements of an algebra of such sequences.

The stacked operator The idea is to identify the whole sequence (A_n) with one single operator, denoted by $\oplus A_n$, that acts componentwise on a direct sum of infinitely many copies of $\mathbf{X}$. To make this precise, first extend the sequence $(A_n)_{n\in\mathbb{N}}$ to the index set $\mathbb{Z}$, for example by $(A_n)_{n\in\mathbb{Z}} := (\cdots, cI, cI, A_1, A_2, \cdots)$ with some $c \neq 0$, and then, recalling that $\mathbf{X} = l^p(\mathbb{Z}^N, X)$, put

$$\mathbf{X}' := \oplus_{n\in\mathbb{Z}}^{l^p} \mathbf{X} := l^p(\mathbb{Z}, \mathbf{X}) \cong l^p(\mathbb{Z}^{N+1}, X).$$

Now each bounded sequence $(A_n)_{n\in\mathbb{Z}} \subset \mathcal{L}(\mathbf{X})$ acts as a diagonal operator on $\mathbf{X}' = l^p(\mathbb{Z}, \mathbf{X})$. We denote this operator by $\oplus A_n : \mathbf{X}' \to \mathbf{X}'$ and refer to it as the *stacked operator* of the sequence (A_n). Then (see [17, Section 2.4.1] or [19, Section 6.1.3])

$$\| \oplus A_n\| = \sup_n \|A_n\|. \tag{6.2}$$

In order to avoid confusion we denote the approximate identity on $\mathbf{X}' = l^p(\mathbb{Z}^{N+1}, X)$ by $\mathcal{P}' = (P_n')$, where $P_n' = \chi_{\{-n,\ldots,n\}^{N+1}} I$ and $P_n = \chi_{\{-n,\ldots,n\}^N} I$.

Finite sections Given $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$, a natural construction for the approximating sequence (A_n) is to look at the so-called *finite sections*

$$A_n := P_n A P_n, \qquad n \in \mathbb{N}, \tag{6.3}$$

of A. Here A_n is understood as operator $\operatorname{im} P_n \to \operatorname{im} P_n$ and is hence represented by a finite matrix. For completeness, put $P_n := 0$ for $n \in \mathbb{Z} \setminus \mathbb{N}$, so that the same formula (6.3) gives $A_n = 0$ then. From $P_n \xrightarrow{\mathcal{P}} I$ it follows that $A_n \xrightarrow{\mathcal{P}} A$ as $n \to \infty$, where we freely identify A_n with its extension by zero to the whole space $\mathbf{X}$. However, when writing A_n^{-1}, we clearly mean the inverse (or its extension by zero to $\mathbf{X}$) of $A_n : \operatorname{im} P_n \to \operatorname{im} P_n$. For the study of stability of a sequence it is more convenient to have all invertibility problems on the same space. To this end we fix a $c > 0$ and look at the extensions

$$A_{n,c} := P_nAP_n + cQ_n, \qquad n \in \mathbb{Z}, \tag{6.4}$$

of A_n, by c times the identity, to $\mathbf{X}$. Clearly, also $A_{n,c} \xrightarrow{\mathcal{P}} A$ as $n \to \infty$. Now $P_n = 0$ implies $A_{n,c} = cI$ for $n \in \mathbb{Z} \setminus \mathbb{N}$. Note that A_n is invertible on $\operatorname{im} P_n$ if and only if $A_{n,c}$ is invertible on $\mathbf{X}$, and that

$$A_{n,c}^{-1} = A_n^{-1} + c^{-1}Q_n, \quad \text{so that} \quad \|A_{n,c}^{-1}\| = \max(\|A_n^{-1}\|, c^{-1}), \tag{6.5}$$

whence both sequences, (A_n) and $(A_{n,c})$, are stable[15] at the same time. Note that the choice $c \geq \|A\| \geq \|A_n\|$ implies $c^{-1} \leq \|A_n\|^{-1} \leq \|A_n^{-1}\|$, so that $\|A_{n,c}^{-1}\| = \|A_n^{-1}\|$, by (6.5).

Theorem 6.1. *([28, Theorem 6.1.6, Lemma 6.1.7] or [17, Proposition 2.22, Theorem 2.28]) Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$, $c > 0$ and $(A_n), (A_{n,c}) \subset \mathcal{L}(\mathbf{X}, \mathcal{P})$ be the sequences defined in (6.3) and (6.4). Then*

- *The stacked operators $\oplus A_n$ and $\oplus A_{n,c}$ are in $\mathcal{L}(\mathbf{X}', \mathcal{P}')$.*
- *If A is rich then the stacked operators $\oplus A_n$ and $\oplus A_{n,c}$ are rich.*
- *If A is band-dominated then the stacked operators $\oplus A_n$ and $\oplus A_{n,c}$ are band-dominated.*
- *(A_n) is stable if and only if the stacked operator $\oplus A_{n,c}$ is $\mathcal{P}'$-Fredholm.*

Combining Theorems 6.1 and 2.9, we get:

Corollary 6.2. *Let A be a rich band-dominated operator, $c > 0$ and $(A_n), (A_{n,c})$ as defined in (6.3) and (6.4). Then: (A_n) is stable if and only if all limit operators of $\oplus A_{n,c}$ are invertible.*

So we are led to studying the limit operators of $\oplus A_{n,c}$. It is easy to see that each of them is again a stacked operator, say $\oplus B_n$. A detailed analysis of $\oplus A_{n,c}$ and its limit operators (see e.g. [29, 18, 19]) shows that the operators B_n to be considered here are:

(a) the operator A itself,
(b) c times the identity operator on $\mathbf{X}$,
(c) all limit operators of A,
(d) certain truncated limit operators of A, extended to $\mathbf{X}$ by c times the identity, and
(e) shifts of all the operators above.

The invertibility of all limit operators of $\oplus A_{n,c}$ reduces to the invertibility[16] of all B_n under consideration, which is of course handy since it brings us back to the $\mathbf{X} \to \mathbf{X}$ setting of the original operator A. In terms of invertibility of all members B_n, there is a lot of redundancy in the list (a)–(e) since cI is invertible, shifts do not change invertibility, and invertibility (even $\mathcal{P}$-Fredholmness) of A implies that of all its limit operators. So it remains to look at points (a) and (d). Without

[15] We also call a bi-infinite operator sequence $(A_n)_{n\in\mathbb{Z}}$ stable if it is subject to (6.1) (with ∞ referring to $+\infty$), i.e. if its semi-infinite part $(A_n)_{n\in\mathbb{N}}$ is stable. The other part, $(A_n)_{n\in\mathbb{Z}\setminus\mathbb{N}}$, as we defined it, is uncritical anyway.

[16] The uniform boundedness of all inverses B_n^{-1} follows automatically from their existence, as can be seen by a slight modification of our $\oplus A_{n,c}$ construction: Assemble $A_1, A_2, \ldots$ into one diagonal operator $D := \sum_{k\in\mathbb{N}} V_{g_k} A_k V_{-g_k} + c(I - \sum_{k\in\mathbb{N}} V_{g_k} P_k V_{-g_k})$, acting on $\mathbf{X}$ (not $\mathbf{X}'$), where the g_k are chosen such that the sets $g_k + \{-k, \ldots, k\}^N$ are pairwise disjoint, as in (4.2) above. Then (see e.g. [29]) the set of all operators B_n in the limit operators $\oplus B_n$ of $\oplus A_{n,c}$ coincides with the set of all limit operators of D, so that, by Theorem 2.9, the inverses of all B_n are uniformly bounded as soon as they all exist.

going into the details of (d), we will denote this remaining set $\{$(a),(d)$\}$ of operators by $S(A,c)$; in [29, 18, 19] it is called the *stability spectrum of A*. From Corollary 6.2 and the discussion above one gets that

$$(A_n) \text{ is stable} \quad\Longleftrightarrow\quad (A_{n,c}) \text{ is stable} \quad\Longleftrightarrow\quad \text{all elements of } S(A,c) \text{ are invertible.} \tag{6.6}$$

Example 6.3. Let $\mathbf{X} = l^2(\mathbb{Z}^1, \mathbb{C})$, $\mu \in [0,1)$ and consider the operator A induced by the block diagonal matrix

$$A \;=\; \mathrm{diag}(\cdots, B, B, 1, B, B, \cdots), \quad \text{where } B \;=\; \begin{pmatrix} \mu & 1 \\ 1 & \mu \end{pmatrix}$$

and the single 1 entry is at position $(0,0)$ of A. Then A is invertible with

$$A^{-1} = \mathrm{diag}(\cdots, B^{-1}, B^{-1}, 1, B^{-1}, B^{-1}, \cdots), \quad \text{where } B^{-1} = \frac{1}{\mu^2-1}\begin{pmatrix} \mu & -1 \\ -1 & \mu \end{pmatrix}$$

and its finite sections correspond to the finite $(2n+1)\times(2n+1)$ matrices

$$A_n \;=\; \begin{cases} \mathrm{diag}\,(B, \cdots, B, 1, B, \cdots, B) & \text{if } n \geq 2 \text{ is even,} \\ \mathrm{diag}\,(\mu, B, \cdots, B, 1, B, \cdots, B, \mu) & \text{if } n \geq 3 \text{ is odd.} \end{cases}$$

If $\mu = 0$ then the A_n with odd n are singular so that the sequence (A_n) is not stable. If $\mu \in (0,1)$ then all A_n are invertible, with

$$\|A_n^{-1}\| \;=\; \begin{cases} \|B^{-1}\| = (1-\mu)^{-1} & \text{if } n \geq 2 \text{ is even,} \\ \max\big(\|B^{-1}\|, \mu^{-1}\big) = \max\big((1-\mu)^{-1}, \mu^{-1}\big) = (\min(1-\mu,\mu))^{-1} & \text{if } n \geq 3 \text{ is odd.} \end{cases}$$

So for $\mu \in (0,1)$ the sequence (A_n) is stable, where the limsup in (6.1) equals $(\min(1-\mu,\mu))^{-1}$. This limsup is a limit if and only if $(\min(1-\mu,\mu))^{-1} = (1-\mu)^{-1}$, i.e. if $\mu \in [\frac{1}{2}, 1)$.

Fix $c \geq \|A\| = \|B\|$, e.g. $c := 2$. Then $A_{n,c} = \mathrm{diag}(\cdots, c, c, A_n, c, c, \cdots)$ and the stability spectrum $S(A,c)$ consists in this example of five operators. They are A,

$$\begin{aligned} C &= \mathrm{diag}\,(\cdots, c, c, c, B, B, B, \cdots) & D &= \mathrm{diag}\,(\cdots, B, B, B, c, c, c, \cdots)\,, \\ E &= \mathrm{diag}\,(\cdots, c, c, c, \mu, B, B, \cdots)\,, & F &= \mathrm{diag}\,(\cdots, B, B, \mu, c, c, c, \cdots)\,. \end{aligned}$$

In case $\mu = 0$ only A, C and D are invertible. In case $\mu \in (0,1)$, all five operators are invertible, where $\|A^{-1}\| = \|B^{-1}\| = \|C^{-1}\| = \|D^{-1}\| = (1-\mu)^{-1}$ and $\|E^{-1}\| = \|F^{-1}\| = (\min(1-\mu,\mu))^{-1}$.

This example suggests that the set $S(A,c)$ not only determines the stability of (A_n) via the invertibility of all members of $S(A,c)$, by (6.6), but also the answer to question (Q1) via the norms of those inverses. It also shows that the answer to question (Q2) is usually negative. Questions (Q3) and (Q4) are fairly straightforward once (Q1) and (Q2) are settled. As we said, in this paper we restrict ourselves to (Q1) and (Q3). So let us turn back to the general setting.

On question (Q1): What is $\limsup \|A_n^{-1}\|$? We start by noting that the elements of $S(A,c)$ are not just those operators from the list (a)–(e) whose invertibility implies that of all other operators on that list – but they also have the largest inverses among (a)–(e), provided that c is large enough.

Proposition 6.4. *Let $A \in \mathcal{A}_\$(\mathbf{X})$, $c \geq \|A\|$ and $(A_{n,c})$ as in (6.4). Then*

$$\max_{L\in\sigma_{\mathrm{op}}(\oplus A_{n,c})} \|L^{-1}\| = \max_{S\in\{\text{(a)}-\text{(e)}\}} \|S^{-1}\| = \max_{S\in S(A,c)} \|S^{-1}\|.$$

Proof. From Theorem 3.2 we know that the LHS indeed exists as a maximum. As in the discussion following Corollary 6.2, we note that each $L \in \sigma_{\mathrm{op}}(\oplus A_{n,c})$ is of the form $L = \oplus B_n$, so that $L^{-1} = \oplus B_n^{-1}$ and the maximum of all $\|L^{-1}\|$ is the supremum of all $\|B_n^{-1}\|$. As in footnote 16, using Theorem 8 of [20], one can see that also this supremum is attained as a maximum. So the LHS equals the maximum of $\|S^{-1}\|$ with S from the list (a)–(e). It remains to show that this maximum is attained in items (a) or (d).

(a) vs. (b): From $\|A\| \leq c$ we get that $\|A^{-1}\| \geq \|A\|^{-1} \geq c^{-1} = \|(cI)^{-1}\|$.

(a) vs. (c): Let A be invertible and $A_h \in \sigma_{\mathrm{op}}(A)$. Then A_h is invertible and $(A_h)^{-1} = (A^{-1})_h$, by Proposition 2.6. By the same proposition, $\|A^{-1}\| \geq \|(A^{-1})_h\| = \|(A_h)^{-1}\|$.

(e): Clearly, taking shifts does not change the norm of the inverse. □

The maximum of $\|S^{-1}\|$ can be attained by (a), $S = A$, or by (d), a truncated limit operator of A. See Example 6.3 with $\mu \in (0, \frac{1}{2})$ for the latter, and replace A by $A = \mathrm{diag}(\cdots, B, B, \frac{\mu}{2}, B, B, \cdots)$, again with $\mu \in (0, \frac{1}{2})$, for the former. Next we rewrite $\limsup \|B_n\|$ as the $\mathcal{P}'$-essential norm of the stacked operator $\oplus B_n$:

Lemma 6.5. *Consider a bounded sequence $(C_n)_{n\in\mathbb{Z}}$ with $C_n : \operatorname{im} P_n \to \operatorname{im} P_n$ for $n \in \mathbb{N}$ and $C_n = 0$ for $n \in \mathbb{Z} \setminus \mathbb{N}$. Now let $0 \leq d \leq \inf_{n\in\mathbb{N}} \|C_n\|$ and $B_n := C_n + dQ_n$. Then $\oplus B_n \in \mathcal{L}(\mathbf{X}', \mathcal{P}')$ and*

$$\| \oplus B_n + \mathcal{K}(\mathbf{X}', \mathcal{P}')\| = \limsup_{n\to\infty} \|B_n\| = \limsup_{n\to\infty} \|C_n\|.$$

Proof. By the construction of B_n, we have $Q'_k(\oplus B_n)P'_m = 0 = P'_m(\oplus B_n)Q'_k$ for all $m \in \mathbb{N}$ and $k \geq m$, so that $\oplus B_n \in \mathcal{L}(\mathbf{X}', \mathcal{P}')$, by [28, Prop. 1.1.8]. Using $\|B_n\| = \max(\|C_n\|, d) = \|C_n\|$ for all $n \in \mathbb{N}$, we derive the equality

$$\| \oplus B_n + \mathcal{K}(\mathbf{X}', \mathcal{P}')\| = \lim_{m\to\infty} \|Q'_m(\oplus B_n)\| = \lim_{m\to\infty} (\sup_{n>m} \|B_n\|) = \limsup_{n\to\infty} \|B_n\| = \limsup_{n\to\infty} \|C_n\|$$

from Proposition 3.1 and (6.2). □

Now we are ready to answer question (Q1):

Theorem 6.6. *Let $A \in \mathcal{A}_\$(\mathbf{X})$, $c \geq \|A\|$ and let (A_n) from (6.3) be stable. Then*

$$\limsup_{n\to\infty} \|A_n^{-1}\| = \max_{S\in S(A,c)} \|S^{-1}\|.$$

Proof. Fix $n_0 \in \mathbb{N}$ so that all A_n and $A_{n,c}$ with $n \geq n_0$ are invertible. Then $\oplus B_n$ with $B_n = A_{n,c}^{-1}$ for $n \geq n_0$ and $B_n = c^{-1}I$ for $n < n_0$ is a $\mathcal{P}'$-regularizer for $\oplus A_{n,c}$. From $c \geq \|A\|$ we get

$$\begin{aligned}
\limsup_{n\to\infty} \|A_n^{-1}\| &= \limsup_{n\to\infty} \|A_{n,c}^{-1}\| && \text{(by (6.5))}\\
&= \limsup_{n\to\infty} \|B_n\| \ = \ \| \oplus B_n + \mathcal{K}(\mathbf{X}', \mathcal{P}')\| && \text{(by Lemma 6.5)}\\
&= \| \left[\oplus A_{n,c} + \mathcal{K}(\mathbf{X}', \mathcal{P}')\right]^{-1} \| \ = \max_{L\in\sigma_{\mathrm{op}}(\oplus A_{n,c})} \|L^{-1}\| && \text{(by Corollary 3.7)}\\
&= \max_{S\in S(A,c)} \|S^{-1}\| && \text{(by Proposition 6.4)},
\end{aligned}$$

which finishes the proof. □

On question (Q3): The asymptotics of the condition numbers From $A_n \xrightarrow{\mathcal{P}} A$ together with (2.3) and $\|A_n\| = \|P_nAP_n\| \leq \|A\|$ we get $\|A\| \leq \liminf \|A_n\| \leq \limsup \|A_n\| \leq \|A\|$, so that $\lim \|A_n\|$ exists and equals $\|A\|$. So the asymptotics of the condition numbers $\kappa(A_n) = \|A_n\|\|A_n^{-1}\|$ is essentially governed by the asymptotics of $\|A_n^{-1}\|$:

Corollary 6.7. *Let $A \in \mathcal{A}_\$(\mathbf{X})$, $c \geq \|A\|$ and let (A_n) from (6.3) be stable. Then*

$$\limsup_{n\to\infty} \kappa(A_n) = \|A\| \cdot \max_{S\in S(A,c)} \|S^{-1}\|.$$

If $\limsup \|A_n^{-1}\|$ is a limit then also $\limsup \kappa(A_n)$ is a limit, but whether or when this happens is the subject of our question (Q2), which is addressed in [14].

We want to mention that versions of both results, Theorem 6.6 and Corollary 6.7, are already contained in the literature: In the Hilbert space case they follow directly from (6.6) by a C^*-algebra argument (as in footnote 5). [21] gives such results while even exceeding the setting of band-dominated operators considerably. The general case $\mathbf{X} = l^p(\mathbb{Z}^N, X)$ is studied in [37] and in Section 3.2 of [34]. While the results of [34, 37] even apply to sequences (A_n) in an algebra of finite section sequences, they put stronger constraints on the operator A (the higher the dimension N, the stronger are the constraints). Our current approach shows how to avoid these constraints on A, and our separate paper [14] combines the benefits of the two approaches.

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Authors:

Raffael Hagger & Marko Lindner
Hamburg University of Technology (TUHH), Institute of Mathematics, 21073 Hamburg, Germany
Raffael.Hagger@tuhh.de, Marko.Lindner@tuhh.de

Markus Seidel
University of Applied Sciences Zwickau, Dr.-Friedrichs-Ring 2a, 08056 Zwickau, Germany
Markus.Seidel@fh-zwickau.de