# RISK DRIVEN INVESTMENT IN PUBLIC REAL ESTATE

**A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY**

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# DECLARATION:

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Signed: ………………………………………………………………

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# DEDICATION

To my late mum and dad, late grandpa John and my family

# ABSTRACT

The global financial crisis towards the end of the last decade saw an increasing need in the role of risk measurement and management in the mainstream financial investment market. Among other things, the measurement and management of market risk, credit risk, and operational risk have become pronounced than ever before. Different strategies have been employed in dealing with the unpredictable nature of the market. This research focuses on the risk-driven investment in public real estate. The aims of this research are threefold

1. To examine whether the real estate allocation based on risk parity leads to better performance compared to other allocation methods
2. To assess the performance of market risk models, namely value at risk (VaR) and expected shortfall on the real estate market.
3. To investigate the volatility transmission of the UK implied volatility index and UK REITs with traded options

The results for the risk allocation generally show that risk parity does in some instances perform better than other allocation methods. Concerning market risk modelling, VaR offers much simple modelling in comparison to expected shortfall. The challenge in the expected shortfall is in its time-consuming nature but it does address the shortcoming of VaR. With regards to the volatility transmission, the results are significant there showing that there is a volatility spillover (transmission) between the changes in implied volatility of the FTSE 100 volatility index, the REIT companies with traded options and the UK REIT index prices.

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# CHAPTER 1

# INTRODUCTION

The global financial crisis towards the end of the last decade saw an increased need in the role of risk measurement and management in the mainstream financial investment market. Among other things, the measurement and management of market risk, credit risk, and operational risk have become pronounced than ever before. Different strategies have been employed in dealing with the unpredictable nature of the market. Investors have seen the need to spread their risk through diversification. However, the conventional asset allocation methods have been challenged given their performance in the crisis. Despite this, the traditional asset allocation methods, i.e., equally-weighted, minimum-variance and mean-variance optimisation based on modern portfolio theory (MPT) are still used with MPT being the most widely used notwithstanding its inherent estimation error caused by the forecasting of expected returns. Portfolio construction using MPT in real estate presents similar challenges where naïve diversification is predominately used to diversify portfolios; this is especially so for liquid real estate investments also referred to as public real estate. The challenges in the return-driven strategies have given rise to risk parity, an asset allocation method whose aim is equal risk allocation across asset classes in a portfolio. By so doing, there is equal risk contribution in a portfolio. The risk parity portfolio strategy has grown in popularity in the last two decades as it is seen to be an investor's 'all weather' strategy compared to the return based strategies. Risk parity is said to provide equivalent returns but with lower risk than conventional methods. This is because it does not require the formulation of expected return assumption as only the asset classes' variances and covariances are needed, and these are easier to estimate than expected returns.

This research was motivated by this risk-based strategy to see if its application to public real estate results in better performing portfolios. The undertaking of a study in this new allocation method, risk parity exposed the research to concepts of volatility measures. Traditionally, standard deviation is used as a measure of risk, but for market risk, value at risk (VaR) was the method of choice particularly in the banking sector. VaR assesses the maximum possible loss of an investment, given a confidence level and it is widely used for both market and credit risk. This method, however, has come under constant criticism as it only considers the maximum loss for the chosen confidence level and ignores any losses beyond that threshold. The underlying assumption is that of normally distributed returns and the hope that extreme events rarely happen. However, financial returns are prone to 'fat tails' therefore the probability of extreme returns is more than what the normality assumption predicts. Following the financial crisis of 2008/9, VaR has lost credibility as a risk measure, and Expected Shortfall (ES) is replacing it. ES is preferred to VaR because it concentrates on the tail side of risk unlike the latter; however, its primary challenge is that it is not easy to backtest compared to VaR. In real estate, there is limited research in modelling market risk, in particular using expected shortfall. The second study in this research hence investigates market risk modelling in real estate and examines the application of both VaR and ES on public real estate.

The modelling of market risk to public real estate brought the modelling of volatility to light and how volatility changes over time and also the underlying assumption of constant volatility in option pricing. So, rather than assuming that volatility is constant, options can be used to reveal the volatility that the market implies when trading in that option. This volatility that is gleaned from the market is referred to as implied volatility. Over the years, research has been undertaken in studying the influence of implied volatility indices on the performance of REITs. Most of the research undertaken has been at index level and also at sector and individual REIT level in the US. The last paper in this research investigates the influence or transmission effect of the UK implied volatility index (VFTSE) on UK REIT companies that have traded options. In summary, the aims of this research are threefold.

1. To examine whether the real estate allocation based on risk parity leads to better performance compared to other allocation methods
2. To assess the performance of market risk models, namely value at risk (VaR) and expected shortfall on public real estate.
3. To investigate the volatility transmission of the UK implied volatility index (VFTSE) and UK REITs with traded options.

# CHAPTER 2

# THE CASE FOR RISK PARITY AS AN ALTERNATIVE STRATEGY FOR ASSET ALLOCATION IN PUBLIC REAL ESTATE PORTFOLIOS

## 2.1. Introduction

The recent global financial crisis saw a significant rise in correlations among risky assets (Chow and Kritzman, 2009) thus bringing to the fore the [increasing] need for risk management in the investment market. The ongoing Eurozone sovereign crisis has further exacerbated this need. The application of efficient asset allocation strategies is one way accomplishing the management of this risk. Most of these allocation strategies entail the creation of diversified portfolios as opposed to putting all the proverbial eggs in one basket. Asset allocation, therefore, plays a crucial role both in risk management and in enhancing investment performance through the achievement of higher returns and/or lower risk. Traditionally, the asset allocation strategies comprise of equally-weighted; minimum-variance, and mean-variance optimisation based on modern portfolio theory (MPT). The question is which strategy can best achieve the best portfolio performance while managing risk through efficient diversification at the same time?

Owing to the poor performance of investments following the 2008 financial turmoil, investors have raised concerns about the efficacy of the conventional asset allocation methods in both the financial and real estate investment markets. This concern led to investment professionals criticising Markowitz's Modern Portfolio Theory (MPT) with some announcing the death of the Markowitz's model as it no longer served institutional investors (Rancalli, 2014 and DiBasio, 2012). While it is tempting to merely redistribute the assets in an inefficient portfolio in response to a crisis, the execution of a plain reallocation does not necessarily lead to true diversification (Qian, 2009). In this vein, research around portfolio allocation approaches that put more emphasis on risk and diversification as opposed to expected returns has been on the rise. This shift of emphasis has led to a rise in the adoption of a different approach called risk parity whose aims is to equalize risk contributions of constituent assets towards the overall portfolio risk. According to Bruder and Roncalli (2013), several large institutional investors have adopted this asset allocation method, notably since 2011. This is because unlike other asset allocation methods, predominantly those based on optimisation; the risk parity method achieves full diversification because of its ability to allocate risk equally to all the assets in a portfolio. Accordingly, corner solutions which lead to risk maximisation are avoided. Furthermore, risk parity also loosely referred to as risk budgeting is assumed to be more robust because it is risk-based and therefore does not dependent on the forecasting of returns.

Risk parity has predominantly been used in portfolios which mostly consist of equities and bonds of different markets for example equities from American, European, Japanese and Emerging Markets, large cap and small cap, REITs as part of multi-asset portfolios. With regards to bonds these could be European sovereign bonds, inflation-linked, corporate and high yield bonds (Bruder and Roncalli, 2013, Dalio, Prince and Jensen, 2015, Wealthfront Risk Parity Fund, 2018)).

Although the assets managed under risk parity amounts to about 175 billion US dollars, it only about represents about 0.21% of the global assets under management (AuM) and about 0.24% of the global stock market which stood at 84 billion and 73 billion US dollars in 2018 and 2017 respectively (Risk Magazine 2018, The Boston Consulting Group, 2018 and Visual capitalist, 2019). Despite its relative size compared to the AuM and the stock market, 175 billion is a significant amount of money given the nascency of this allocation method compared to the maturity of the stock market.

While research on risk parity is mainly conducted using multi-asset portfolios which include real estate, as far as the researcher is aware only one piece of research (Moss, Clare, Thomas, and Seaton, 2017) currently exists which includes risk parity as one of the asset allocation methods for to REITs. However, Moss, Clare, Thomas, and Seaton (2017) only considered the equally-weighted, minimum-variance, mean-variance, and risk-parity approaches to asset allocation by comparing their performance to diversified benchmarks. This research extends the previous research by considering estimation risk and therefore including asset allocation methods that apply various regularisation methods and also concentrates on international public real estate securities. Furthermore, the research investigates the performance of these asset allocation methods using different rolling periods and also considers transaction costs

This research examines whether the risk parity approach to asset allocation, when applied to real estate securities, is likely to provide the most diversification while producing superior risk-reward tradeoff at the same time. As institutional investors constitute the largest investors in real estate, this research has be done with institutional investors; like pension fund, insurance compenies, banks, and asset managers; in mind.

### 2.2. The Asset allocation methods and regularization

Below is a brief review of the asset allocation methods widely employed in mainstream finance. Firstly, the traditional asset allocation method, mean-variance optimization and its variants, equally-weighted and minimum-variance, are presented. This is followed by other non-conventional allocation methods that attempt to address the shortcomings of the traditional mean-variance optimization. These allocation methods form part of the basis for which the ‘new’ allocation method, risk parity is be compared against. This recap has been provided because these methods will be referred to from time to time in the risk parity discussion.

#### 2.2.1. Mean-variance optimization (MVO)

The mean-variance optimization (MVO) approach also referred to as the Modern Portfolio Theory (MPT) approach was famously developed by Markowitz (1952) typically for constructing equity portfolios. The main aim of mean-variance optimisation portfolios is the achievement of the highest return for a given level of risk or the lowest risk for a given level of return. Speidell, Miller, and Ullman (1989) describe portfolio optimization as a procedure for measuring and controlling portfolio risk and expected returns while taking into consideration correlations [or covariances] between assets. In other words, portfolio optimization entails combining assets with return and price movements that balance one another hence offsetting each other’s idiosyncratic risks and ultimately reducing portfolio risk. This risk reduction is as a result of low covariances of returns of assets in the portfolio. By and large, MVO portfolios aim to maximize [or optimize] the risk-adjusted returns, i.e., the [excess] return per unit risk (also called the Sharpe ratio). The resulting portfolios, therefore, tend to allocate more weight to the asset with higher returns. As a consequence, these portfolios become more correlated with these high return assets.

MPT was the predominant asset allocation method used leading to the subprime crisis. However, this approach could not withstand the crisis because most of the portfolios in the mainstream financial market were overweight in equities which unfortunately achieved negative returns of up to -50% during the crisis (Roncalli, 2014). Accordingly, these portfolios performed very poorly as they were more correlated to equity returns. For this reason, it is argued that MVO portfolios are under-diversified and hence fail to provide the required risk control especially in volatile periods such as in financial crises

(Hewitt EnnisKnupp 2012 and Allen, 2010) when there is an increase in correlations between asset classes. Following the crisis, MPT has inevitably faced a lot of criticisms with some even going as far as declaring its demise.

Michaud and Michaud (2008) outline four categories under which criticisms towards the implementation of MVO are likely to fall:

1. Investor utility: the limitations of representing investor utility and investment objectives with the mean variance of return;
2. Normal distribution: the limitations of representing return with normal distribution parameters
3. Multi-period framework: the limitations of mean-variance efficiency as a single-period framework for investors with long-term investment objectives, such as pension plans and endowment funds
4. Asset- Liability financial planning: claims that an asset-liability simulation is a superior approach for asset allocation.

Michaud and Michaud (2008) however argue that the criticisms above (which they refer to as 'traditional,' do not address what they call the most critical limitations of MVO, namely instability and ambiguity arising from the estimation of input parameters.

The execution of Markowitz 's MVO is not straightforward as it requires the estimation of the vector of expected returns and covariance matrix; which are the input parameters needed in order to compute optimized portfolios. These optimized portfolios so created are, however, sensitive to these inputs. Therefore, MVO portfolios are occasionally referred to as "maximum error" portfolios due to their sensitivity to estimation errors arising from the estimation of these expected returns and covariances (Michaud, 1989) which are normally calibrated from historical returns. Chaves et al. (2011) argue that these estimates are subjective as a result of investor influence arising from behavioural biases. They attribute these potential biases as emanating from over-estimation of expected returns due, for example, to the asset's recent strong historical performance or under-estimation of an asset class risk arising from personal familiarity with it. As a result, this can lead to better performance by equal weighted portfolios in some conditions, for instance, if mean-variance optimisation is unconstrained. Furthermore, Roncalli (2014) observes that the problem of MVO does not necessarily stem from the allocation method per se but rather much of its criticism is more in the manner in which it is used – specifically the input parameters. Yet still, Roncalli (2014) argues that because MPT sets outs to construct portfolios that are based on arbitrage factors established, it is an aggressive model of active management as opposed to passive management. With this in mind, Green and Hollifield (1992) argue that the sensitivity to input parameters should hence be expected in MPT. However, while Kritzman (2006) recognises that MVO results are highly sensitive to input errors, he argues that these input errors have little significance to estimates of exposure to loss and gain. With regards to real estate, Armonat and Pfnuer (2004) note that this asset class possesses different attributes compared to financial markets upon which MPT was originally developed for. They hence caution on the blind application of capital market theory to real estate as small changes in the parameters are also big drawbacks in MVO for real estate portfolio construction.

Due to the problems posed by estimation error, in practice, the MVO portfolios are perceived to be less robust and are used in conjunction with constraints. The employment of constraints is one of the regularisation techniques used to improve the robustness of MVO (Bruder and Roncalli, 2013). Other regularisation techniques include shrinkage and resampling which essentially create variants of MVO portfolio (these techniques will be looked at in more detail later on in the study). Because of the aforementioned sensitivity to input parameters that emanate from the estimation of expected returns and also the complex computation required to perform mean-variance analysis, some investors prefer the minimum-variance and equally-weighted portfolio approaches. These approaches are widely used in practice because they are more heuristic[1] and computationally simple while still presumed to be robust (Maillard, Roncalli and Teiletche (2010).

### 2.2.2. Equally weighted (1/N)

The equally-weighted or equal-weighting is arguably the most heuristic portfolio allocation approach due to its straightforward construction that involves equal (value) weighting across a portfolio's asset classes in an attempt to attain diversification. When equal weighting is assigned to each asset in a portfolio, a portfolio with "N" assets will each have a 1/N allocation. Consequently, it is often referred to as naïve diversification due to its simplicity. In contrast to MVO, expectations of asset characteristics (or moments) such as the return and risk are ignored entirely. Moreover, the equally weighted

[1] "Heuristic" means being able to employ experience based techniques and trial and error methods to find an acceptable solution (Roncalli, 2014).

approach is a non-optimised portfolio allocation method that does not involve any forecasting of expected return, and thus its sensitivity to input parameters is significantly reduced.

There are mixed findings regarding the efficiency of the equally weighted approach when compared to other asset allocation approaches. Lee (2011) identifies two contrasting findings. The first is DeMiguel, Garlappi, and Uppal (2009) and the other is Kritzman, Page and Turkington (2010). Despite its simplicity, the results of the former indicate that this (1/N) approach is not necessarily inferior to the other well-known techniques like mean-variance optimisation and minimum variance because they are not consistently better out-of-sample. They further point out that in instances when N was large, the simple 1/*N* approach was more likely to outperform the optimisation strategy because the diversification potential was much enhanced while the optimisation models saw an increase in the number of estimated parameters and hence a corresponding increase in estimation errors. Similar results have been found in real estate by Cheng and Liang (2000) who suggest that the mean-variance approach was not statistically better than naïve diversification. Kritzman, Page and Turkington (2010) on the other hand argue that optimised portfolios usually outperform equal weighting on longer-term samples. They contend that the apparent superiority of the MVO approach arises not from limitations in optimisation but, rather, from reliance on rolling short-term samples for estimating expected returns (Lee 2011). Kritzman, Page and Turkington (2010) further observe [from their simulations] that 1/N portfolios on average exhibited more risk concentration than optimised portfolios. This said, the equally weighted asset allocation method is said to be mean-variance optimal under restrictive conditions, for instance, when the assets have homogeneous returns, volatilities and equivalent correlations between each other (Qian 2011 and Lee 2011). Nevertheless, most assets do not behave the same and have different returns and volatilities. In this instance, the equally weighted approach will still result in risk concentration and therefore lead to very limited diversification due to the application of the same weight regardless of whether the assets have the highest or lowest risk (Lee, 2011). There is also a fallacy that the equally weighted portfolios are passive portfolios which do not need rebalancing. Contrary to this belief, over time, the portfolio will become imbalanced as some asset classes would have gained (or lost) value, therefore, necessitating the need for rebalancing to ensure that each asset has an allocation of 1/N (Stevenson 2002).

### 2.2.3. Minimum variance (MV)

A minimum variance (MV) method is also an optimisation approach but unlike the MVO, only relies on the use of covariances to produce portfolios with the lowest variances while completely ignoring estimates of expected returns. In this respect, it is somewhat similar to the 1/N approach and therefore also appealing because it avoids the estimation risk inherent in the forecasting of expected returns. For that reason, MV portfolios are said to be more stable and therefore perform better compared with mean-variance portfolios in some instances (Jorion 1985). The increasing emphasis on risk management following the recent financial crisis gives credence to this asset allocation approach. Additionally, research shows that on average low volatility stocks have performed just as well or even outperformed the market (Clarke, de Silva and Thorley, 2011 and Ang et al. 2006), a phenomenon referred to as the volatility anomaly. This notwithstanding, there are concerns about these portfolios turning out to be fairly concentrated in one asset thus increasing the impact of idiosyncratic shock to the portfolio out-of-sample (Northern Trust, 2012; and Maillard, Roncalli and Teiletche, 2010) and possibly leading to poorly diversified portfolios (Stoyanov and Goltz, 2011). According to Northtrust (2012), modifications to the minimum variance approach can be made to address these concerns by including additional portfolio construction constraints. If all the expected returns across all the assets are identical, then the minimum-variance portfolio becomes mean-variance optimal.

Having covered the traditional asset allocation approaches, and introduced the main shortcoming of the MVO approach – the estimation error, the study now turns to techniques that are used to minimise this. These techniques come under the umbrella of regularisation and result in portfolios that are also variants of the MVO.

## 2.3. Regularisation approaches to asset allocation

As stated earlier, the starting point of portfolio analysis is the mean-variance optimisation (MVO) technique based on Markowitz's modern portfolio theory (MPT). The two main input parameters needed to create an MVO portfolio are the [expected] return vector and the covariance matrix. These parameters are unknown and therefore have to be estimated. Wolf (2007) notes that expected returns represent the portfolio manager's ability to forecast future price movements while the covariance matrix typically has to be estimated from ex-post return data. Portfolio analysis based on ex-post data is straightforward since it is based on actual returns achieved in the past. The presumption here is that past returns will repeat themselves in future and hence disregarding any possible structural economic changes, and for that reason, it is referred to as the naïve (or unconditional) approach. Despite looking inadequate, it is the most common way of estimating expected returns in asset management (Roncalli, 2014). Apart from the equal weighted approach to asset allocation, the conventional practical implementation of the theory for efficient portfolio techniques often leads to questionable results

(Disatnik and Benninga, 2007). More often than not this produces portfolios with stability problems in most cases and is a reason for the reticence and disapprovals in the use of optimised portfolios by investment managers. The weakness in this approach arises because optimal portfolios are subject to estimation risk because they require the estimation of future returns, and/or risk usually in the form of the covariance matrix. Inevitably, this leads to the existence of estimation errors because, as mentioned before, the future return distributions are unknown. For this reason, the mean-variance approach to portfolio construction can hence result in a portfolio composition that varies substantially due to this sensitivity to expected returns as input parameters. The other input, the covariance matrix is usually estimated by working out the sample covariance matrix derived from ex-post returns thus also leading to errors in the estimation. Ledoit and Wolf (2003) observe that these errors come about when exceptional values taken from extremely unreliable coefficients are predominantly used by optimisers as a basis of placing bets. The implication is that the resulting portfolio is inaccurate and can lead to what Michaud (1989) refers to as “error maximisation”.

It was previously highlighted that because MVO is an aggressive model of active asset management, its sensitivity to input parameters should be expected as it sets out to find arbitrage factors and construct portfolios that play on them (Roncalli, 2014). This means that implied bets are therefore a function of these input parameters. Accordingly, wrong input parameters are expected to cause the resulting portfolio not being satisfied because it will be based on erroneous arbitrage factors and bets (ibid). In order to reduce the impact of the errors caused by input parameters or to stabilise these portfolios, one needs to regularise these input parameters or the objective function. Several portfolio regularisation techniques are usually applied in practice (Bruder and Roncalli 2013, Tutuncu and Koening, 2004, and Lee and Stevenson, 2000). The underlying idea behind portfolio regularisation is that less dynamic parameter values should be employed by investors who want a more defensive model (i.e., less aggressive active portfolios). Bruder and Roncalli (2013) present the following regularisation techniques applied in practice:

- Regularisation of the programme specification by imposing some weight constraints
- Regularisation of the return vector or covariance matrix
- Regularisation of the objective function by using resampling techniques.

Imposing constraints to the weights is the easiest of the regularisation techniques. This is done because unconstrained mean-variance optimisation can result in portfolios with substantial short sale positions which in turn could lead to performance that is inferior to that of the simple equally weighted portfolio

(Michaud, 1989). However, Roncalli (2014) shows that the introduction of weight constraints has the possibility of giving rise to a covariance matrix that could be very different from the original one. Furthermore, these constraints are typically applied arbitrarily and can often give rise to instability rather than reducing it (Michard and Michard, 2008) thus making it hard to generalise the outcomes. Because of the potential for error maximisation, optimal portfolios are said to have a tendency of excessive concentration in a limited subset of the full set of assets (Maillard, Roncalli and Teiletche, 2011), a condition referred to as "corner solutions" by Black and Litterman (1992).

In order to obtain a more satisfactory solution from a financial point of view, a combination of the techniques above can be applied. For example, the resampling technique is generally used in conjunction with constraints on weights. This, according to Jagannathan and Ma (2003) cited in Bruder and Roncalli (2013) corresponds to shrinking the covariance matrix albeit implicitly (Wolf, 2007).

In a separate study, Disatnik and Bennunga (2007) identify two approaches for dealing with the problematic results obtained from mean-variance optimisation; namely the theoretical and implementation approaches. The former involves the re-examining of portfolio optimisation theoretical aspects and assumptions. The latter involves the estimation of the unknown return vector and the covariance matrix, being the two main parameters needed in Markowitz's mean-variance approach as highlighted earlier. This research focuses on the implementation approach and will consider the regularisation of the return vector and covariance matrix through "shrinkage". In addition to this, the regularisation of the objective function by using the "resampling" technique will also be explored.

### 2.3.1. Shrinkage

"Shrinkage" is a transformation applied to sample statistics so as to take into account some uncertainties. Ledoit and Wolf (2003) observe that a fundamental principle of statistical decision theory is that there exists an interior optimum in the trade-off between bias and estimation error and that this optimal trade-off can be obtained by simply taking a weighted average of the biased and unbiased estimators. In other words, the shrinkage process entails pulling ("shrinking") the unbiased estimator full of estimation error towards a fixed target characterised by the biased estimator (Ledoit and Wolf, 2003). Hence, in the quest of reducing estimation error, extreme values of the sample statistics (unbiased estimator) are dragged in the direction of more central value (biased estimator). While the shrinkage method was introduced to asset management by Jorion (1986), it has been used for linear regression for a long time (Roncalli, 2014). In this study, the sample statistics to be "shrunk" are the return vector (Jorion, 1986; Chopra and Ziemba, 1993; Stevenson, 2001) and the covariance matrix (Ledoit and Wolf, 2004; and Vu Anh Tuan, 2013). These shrinkage methods are covered below.

### 2.3.2. Bayes Stein return [vector] shrinkage estimator

A study by Chopra and Ziemba (1993) observes that errors in means (expected returns) are about ten times as important as errors in variances and covariances, and errors in variances, in turn, are about twice as important as errors in covariances. This remark is consistent with Bengstssen (2004) who shows that means possess more estimation errors compared to the covariance matrix. The difference in significance is, however, directly proportional to the risk tolerance. For this reason, Michaud (1989) declares that mean-variance optimisation considerably over-weights those securities that have large estimated returns, negative correlations and small variances and vice versa which eventually leads to portfolios with corner solutions. Stevenson (2000) outlines two serious defects of employing modern portfolio theory in portfolio construction as (1) the intertemporal instability of portfolio weights and (2) the sharp deterioration in performance of optimal portfolios out-of-sample. As stated on several occasions, the sensitivity of mean-variance optimisation portfolios to input parameters leads to estimation errors and moreover these portfolios have a tendency to form corner solutions. These solutions can be suboptimal bearing in mind that resulting extreme allocations go against the principle of diversification.

Bayes Stein estimators are used to decrease these estimation errors as well as also reduce the chance of arriving at corner solutions. This is achieved by “shrinking” extreme input parameters towards a global mean. By implication, the magnitude of the shrinkage increases with the distance from the global mean. Therefore it is a function of the distance from the global mean and asset variability, and it also decreases with the number of historical observations (Michaud and Michaud, 2008). The outcome is that the differences between extreme values are reduced thereby decreasing the sensitivity of these parameters which in turn helps lower the estimation errors (Jorion, 1985, 1986). The general form of the estimators is defined as:

$$E(r_i) = w\bar{r}_g + (1 - w)\bar{r}_i \tag{2.1}$$

Where $E(r_i)$ is the adjusted, mean, $\bar{r_i}$ is the original asset mean, $\bar{r_g}$ is the global mean and $w$ is the shrinkage factor (Lee and Stevenson, 2005, and Stevenson, 2000, 2001). The shrinkage factor can be estimated from a suitable prior using Jorion (1985, 1986)[2].

$$\widehat{w} = \frac{\hat{\lambda}}{(T + \hat{\lambda})} \tag{2.2}$$

$$\hat{\lambda} = \frac{(N+2)(T-1)}{(r - r_0 1)' S^{-1} (\bar{r} - r_0 1)(T - N - 2)} \tag{2.3}$$

Where S is the sample covariance matrix; T the number of observations, 1 (in the denominator) a vector of ones. The sample covariance matrix is used in this instance because it is said to be more stable over time as previously stated. The above formulae show that the Bayes- Steins estimator shrinks the sample mean $\bar{r_i}$ to the global mean, $\bar{r_g}$, while using the sample covariance S.

### 2.3.3. Shrinking the covariance matrix

Unlike the return vector shrinkage, the covariance matrix shrinkage is more challenging and needs more explanation. The covariance matrix measures how assets in a portfolio vary with each other. These assets are paired together and identified as the $i^{th}$ and $j^{th}$ assets thus generating $(i, j)$ elements of the matrix. Michaud and Michaud (2008) describe the covariance matrix as being square with n rows and columns equal to the number of assets where the $i^{th}$ diagonal element is equal to the variance of the $i^{th}$asset. The off-diagonal elements represent all pairwise covariances. For this reason, the covariance matrix can be reduced to a correlation matrix if each variable is normalized to take a unit variance. As observed in the Bayes-Stein return [vector] shrinkage estimator, the covariance matrix is gleaned from historical data and is not subject to shrinkage because it is considered not to be as important as the expected return estimate in terms of portfolio optimization estimation risk. In view of that, it is assumed to have little effect on optimized portfolios. However, the covariance matrix is mainly exposed to either the sampling error or specification error. Disatnik and Benninga (2007) describe sampling error as that which comes about when a sample consists of more estimated parameters compared to the number of observations, that is, there are not enough degrees of freedom per estimated parameter. This implies that significantly more data is required for covariance estimation than is typically available otherwise portfolio optimisation is not feasible because the covariance matrix

[2] Michaud and Michaud (2008) define a prior as either a reasonable guess at the answer or an assumption that imposes exogenous structure on potential solutions. Imposing structure on forecasts lowers the estimation error in sample statistics hence reducing dependence on pure statistically estimated data

becomes singular (Michaud and Michaud, 2008). A signular covarinace matrix does not have an inverse matrix. Specification error is that which arises when some form of structure is imposed on the model used in the estimation process. In this case, the estimator becomes too specific in comparison with reality (Disatnik and Benninga, 2007). They further argue that the development of a better estimator requires the reduction of the huge sampling error of the covariance matrix without the creation of too much specification error. This is because of the existence of a trade-off between sampling error and specification error.

### Why shrink the covariance matrix?

Shrinking the covariance matrix approach is the most common approach of portfolio regularisation (Roncalli, 2014). The precise forecasting of covariances is a challenging aspect of financial economics as this input tends to lead to extreme allocation and under-diversified portfolio (Wolf, 2007). Seeing that the returns data are captured from historical means, they are said to be unreliable and non-robust (unstable) estimators of expected returns due to the presence of outliers which heavily influence the covariance matrix [and are therefore prone to errors] (Parret-Gentil and Victoria-Feser, 2003; Jorion, 1986; and Ledoit and Wolf, 2004). It was stated earlier that according to some schools of thought (Chopra and Ziemba, 1993; Lee and Stevenson, 2005; and Bengtsson, 2004) the estimation of expected returns is more important than that of variances and covariance matrix. The reason behind this argument is because of the belief that errors in the expected returns are more superior compared to variances and covariances, the higher the levels of risk tolerance. However, Michard and Michard (2008) highlight the importance of the practice of dealing with estimation error in both risk and return for defining effective investment- optimised portfolios.

Similarly, Markowitz and Usmen (2003) also take cognisance of the noisy nature of empirically observed means, variances and covariances when applied to the mean-variance analysis. Ledoit and Wolf (2003) go further and show the critical role played by the covariance matrix in the reduction of portfolio risk. This is owing to the fact that a reduction of risk translates into an increase of expected return due to the tradeoff of risk and return in the mean-variance analysis. They further argue that a good estimator of the covariance matrix is needed in the estimation of more precise excess returns. The analysis is done through the use of optimisers whose required inputs are the expected returns and the covariance matrix usually estimated from the sample covariance matrix.

However, MVO is also likely to be affected by the estimation error that comes about as a result of using the sample covariance matrix to estimate the covariance matrix (Wolf, 2007). The estimation error is predominantly high in instances when the number of assets in the portfolio *(N)* exceeds the number of the historical observations *(T)* used in the period from which sample covariance is estimated (Jobson and Korkie, 1980; Ledoit and Wolf, 2003, 2004; Clarke, de Silva, and Thorley, 2006; Scowcroft and Sefton, 2006; Disatnik and Benninga, 2007; and Michaud and Michaud, 2008). This leads to the resulting estimated covariance matrix becoming 'ill-conditioned' (i.e., small errors in the data are likely to produce large errors in the solution) and has come to be referred by Pafka, Potters and Kondor (2004) as the *"curse of dimensions"*. Kwan (2010) shows that in portfolio analysis a covariance matrix has to be "positive definite[3]" to be acceptable. This assertion is true for what Michaud and Michaud (2008) refer to as ad hoc estimators of which the single-index model of which Sharpe (1963) is the best known. The question is whether the covariance matrix estimation problem as mentioned above is relevant to real estate due to few sectors? Maybe not at direct property sector level but the problem should still arise when individual properties, as well as REITs, are considered. According to Ledoit and Wolf (2004), the problem above leads to a tendency in which the "*most extreme coefficients in the estimated covariance matrix take on extreme value not because this is "the truth", but because they contain an extreme amount of error."* For this reason, they contend that the sample covariance matrix should not be used in portfolio optimisation but a "shrunk" covariance matrix ought to be used instead since it is more stable and less sensitive to estimation error. This can be attributed to its ability to reduce outlier influences and improve the accuracy of target allocation (Ledoit and Wolf, 2003, 2004).

### The Ledoit & Wolf (2004) covariance matrix shrinkage estimator

Ledoit & Wolf (2004) propose an explicit shrinkage estimator for the sample covariance matrix based on the Stein method but distinct to Jagannathan and Ma (2003) who employ some form of shrinkage to the sample covariance matrix by adding constraints to mean-variance optimised portfolios. Ledoit and Wolf (2004) address shortcomings of erstwhile research of Muirhead (1987) and Frost and Savarino (1986) who propose shrinkage estimators that cannot be applied when the number of assets exceeds the historical periods examined (i.e. when $N > T$). Ledoit and Wolf (2003, 2004) resolve this problem by proposing a general technique by defining an optimal shrinkage intensity that minimizes the loss function without engaging the covariance matrix inverse. The loss is a quadratic measure of distance between the true and the estimated covariance matrices. This optimal shrinkage intensity is dependent on the correlation between estimation error on the sample covariance matrix and the shrinkage target.

[3] Positive definite covariance matrices have non-zero determinants, are invertible and therefore always provide positive variances of portfolio returns.

The correlation is inversely related to the benefit of combining the information contained by the estimation error and the sample covariance matrix.

Ledoit and Wolf (2004) observe that using this "correlation term resolves a deep logical inconsistency in earlier empirical Bayesian literature, where the prior is estimated from the sample data, yet at the same time is assumed to be independent of the sample data." They start with the sample covariance matrix $S$, which is the traditional and probably most intuitive estimator of the covariance matrix. On one hand, they say it is easy to compute and it has an expected value that is equal to the true covariance matrix (and therefore is assumed to be unbiased). On the other hand, they observe that the sample covariance matrix has a lot of estimation error when the number of data points is of a comparable or even smaller order than the number of individual assets. As an alternative, they suggest the use of estimators that contain little estimation error such as those with a lot of structure like the single-factor model of Sharpe (1963). They however, observe that such estimators tend to be misspecified and can be exceptionally biased. This said, models that integrate multiple factors (multi-factor models) are the industrial standard because they provide more flexibility and are less biased even though they have higher estimation errors compared to single factors models (Ledoit and Wolf, 2003, 2004).

Ledoit and Wolf (2004) employ a shrinkage technique as used in statistics whose outcome is a 'compromise' estimator that merges two 'extreme' estimators but has a better performance than either extreme. The ingredients for this shrinkage technique consist of an estimator with no structure i.e., sample covariance matrix $S$; a highly structured estimator (also called the shrinkage target) $F$; and a shrinkage constant, $\delta$ which is a number between 0 and 1. In other words, this is the weighted average of the sample covariance matrix with an invertible covariance matrix estimator whose diagonal elements are sample variances (Disatnik and Benninga, 2007). The aim is to 'shrink' the covariance matrix towards the structured estimator by computing a convex linear combination. This is shown in equation (2.4) in its basic form

$$\delta F + (1 - \delta)S \tag{2.4}$$

### The shrinkage target

Ledoit and Wolf (2004) observe that the shrinkage target should fulfil two requirements simultaneously; it must involve only a small number of free parameters (i.e., a lot of structure) as well as reflecting essential characteristics of the unknown quantity being estimated. The single factor matrix of Sharpe (1963) was suggested as the shrinkage target in their previous study (Ledoit and Wolf, 2003).

However, in their 2004 study, they suggest “the constant correlation model” which they assert is easier to implement but has comparable performance. In the model all the (pairwise) correlations are therefore identical. The use of constant correlation model is appropriate to this study as assets considered (Real Estate Securities) are from the same asset class, i.e. real estate; however, this may not be so when the assets are from different asset classes (Ledoit and Wolf, 2004). The estimator of the common constant correlation is the average of all the sample correlations. They use this number in conjunction with the vector of sample variances as the shrinkage target, $F$.

### Shrinkage constant

The statistical problem is to estimate the optimal value of $\delta$ i.e., the shrinkage constant/intensity (Roncalli, 2014). The idea for the shrinkage constant is a number between 0 and 1 that gives a compromise between $S$ and $F$ however, there are infinite possibilities. Ledoit and Wolf (2004) suggest an ‘optimal’ shrinkage constant i.e., one that minimizes the expected distance between the shrinkage estimator $F$ and the true covariance matrix $\sum$ thus arriving at the quadratic loss function:

$$L(\delta)\left\|\delta F + (1-\delta)S - \widehat{\sum}\right\|^2 \quad (2.5)$$

As earlier stated, this loss function is not dependent on the inverse of the covariance matrix hence overcoming the typical inverse matrix problem faced when the number of assets (N) is more than the observations (T). In comparison to the typically large off-diagonal elements of the sample covariance matrix, the off-diagonal elements of the shrinkage estimator are shrunk (Disatnik and Benninga, 2007) while the variance elements in the diagonal are left untouched. This shrinkage constant has been denoted by $\delta^*$and therefore the shrinkage estimator for the covariance matrix $\sum$ is:

$$\widehat{\sum}_{Shrink} = \delta^* F + (1-\delta^*)S \quad (2.6)$$

Please see Ledoit and Wolf (2004), and Vu Anh Tuan (2013). for more explanation on the mathematical derivation of $\delta^*$.

Having covered the Stein based shrinkage methods the last regularization technique – resampling is explored next.

### 2.3.4. Resampling

Resampling is another regularisation method for overcoming the earlier mentioned shortcomings of the mean-variance optimization approach. Wolf (2007) observes that resampling is theoretically distinct from shrinkage estimation because it creates artificial return data by resampling from the observed data rather than depending on the improved estimator of either the return vector or the covariance matrix discussed previously. Resampling, sometimes referred to as bootstrapping, is not necessarily a new method. Although it gained much traction in asset allocation following Michaud (1989) famous paper, it was notably applied before by Jobson and Korkie (1981) in the form of Monte Carlo simulation method used on unbounded MVO. Like many others, Michaud and Michaud (2008) observe that though not appreciated by the investment industry, estimation error is critical as ignoring it leads to counterproductive and suboptimal investment practices. They attribute the disregard of the MVO by many equity investment practitioners to their lack of statistical understanding of the MVO process and therefore tend to overuse statistically estimated information which amplifies the impact of estimation errors. Michaud and Michaud (2008) contend that a well understood statistical view of MVO could boost investment value at the same time offering a more intuitive framework for asset management because it leads to new procedures that remove the most severe shortcomings, i.e., instability and ambiguity that lead to portfolio optimality that is not well defined. As earlier alluded to, this is because minor changes in input assumptions frequently result in significant changes in the optimised portfolios leading to estimation errors. Michaud and Michaud (2008) go further and note that “it is not simply a matter of garbage in, garbage out, but rather a molehill of garbage in, a mountain of garbage out.” Consequently, the outcome is the famous “error maximising” portfolios as coined by Michaud (1989).

Resampling entails sampling (taking samples) from a sample hence the name “re-sampling”. After the data has been resampled, the sample covariance matrix is then calculated and utilised into the mean-variance (or minimum-variance depending on the portfolio strategy) optimisation process. This iteration is carried out several times, and the resulting optimal resampled portfolios are averaged. Michaud and Michaud (2008) and Scherer (2002) argue that the subsequent average portfolio is more diversified and is likely to improve out of sample performance compared to that produced from the sample covariance matrix obtained from observed data only.

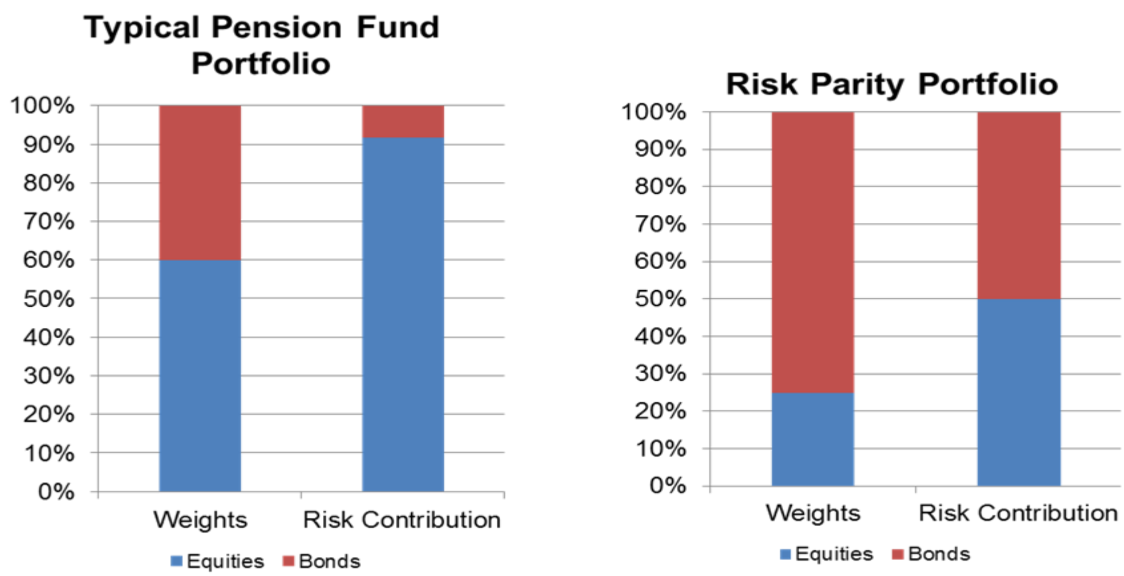
Michard and Michard (2008) have come up with a portfolio allocation strategy based on resampling which they call Resampled Efficient Frontier™ (REF) and have patented it. REF optimisation seem to produce inferior portfolios because by definition, it average returns of resampled portfolios and as expected this produces lower returns and is said to have a more restricted range of risk compared to traditional MVO portfolios. Michard and Michard (2008) argue that this perceptible inadequacy of REF highlights the limitations of in-sample MVO efficiency in portfolio analysis as it is an unreliable and often misleading framework. Due to the instability and sensitivity to input parameters of the MVO, Michard and Michard (2008) suggest that REF optimisation is the paradigm of choice in an environment of information uncertainty as it is less dependent on particular characteristics of these inputs. REF optimised portfolios are likely to perform better out of sample because they are said to present less extreme portfolio weights because they are a product of averages taken from numerous possible outcomes in sharp contrast with the MVO which only depends on one possible outcome. As a consequence, the REF optimisation approach, on the whole, tends to produce safer and more reliable portfolios with better risk-adjusted performance. This is attested by the result of an experiment carried out by Markowitz and Usmen (2003) which showed better performance of the REF optimisation compared to the MVO in all of the independent simulation tests carried out.

Up to this point the study has considered allocation methods that are widely known namely; mean-variance optimization, equal-weighting and minimum-variance. The study has also reviewed regularization techniques that are used to counter some shortcomings of the mean-variance approach and result in variants of the mean-variance optimization approach. These techniques are the Bayes Stein return vector shrinkage; Ledoit and Wolf covariance matrix shrinkage; and resampling. The next section considers a relatively new approach to asset allocation called “Risk Parity”.

## 2.4. Enter Risk Parity

Classically, the investment allocation for most pension funds predominately consists of equities and bonds with other asset classes only accounting for very little. As a result, equities and bonds account for the most contribution to portfolio risk compared to other alternative asset classes. In portfolios consisting of only equity and bonds, pension funds typically take a 60/40 portfolio variant approach to asset allocation. Dalio (2004) however, argues that this allocation is under-diversified in terms of risk exposure because most of the returns are earned from exposures to equity risk and little from bonds.

Due to the usually higher volatility in the equity market compared to that of the bond market and the increase in the realized equity correlation following the advent of the financial crisis, equities contribute more than 90% of the risk in traditional 60/40 equity/bond portfolios (Thiagarajan and Schachter, 2011 and Qian, 2011). This is illustrated in Exhibit 2.1(a). A risk parity approach, on the other hand, will allocate the same risk contribution towards the total portfolio risk resulting in a portfolio that allocates 25% and 75% to equities and bonds respectively as shown in Exhibit 2.1(b).




(a): Typical Equity/Bond pension fund (b): Risk parity Equity/Bond portfolio

Exhibit 2.1: Asset allocation example

For the preceding reason, following the 2007 - 2008 financial crisis, risk management has become more important than performance management (Roncalli, 2014). Accordingly, there has been a growing adoption of 'risk parity' (RP), a strategy that requires less discretionary inputs (Roncalli, 2014), which until recently, was used to manage global multi-asset portfolios. Risk parity was established way back in 1996 by Bridgewater under the name of the "All weather" asset allocation mix (Dalio, 2004). Though this approach has been used under different names the term "risk parity" was first devised by Qian (2005).

Risk parity sometimes referred to as "risk budgeting", "weight budgeting" and "performance budgeting" falls under the umbrella of a suit of budgeting methods in asset allocation (Roncalli, 2014).

Risk budgeting, often confused with "risk attribution" entails risk measurement, risk attribution, and risk allocation when viewed in a risk management setting (Berkelaar, Kobor and Tsumagari, 2006). The weight budgeting approach entails specifying the weights in a portfolio, for example, a typical 60/40 equity and bond policy, would allocate 60% to equities and 40% to bonds. The risk budget approach chooses the risk budgets of the assets in the portfolio. If say 20% risk is required for a portfolio, then 12% and 8% will be the risk budgets for equity and bonds respectively. Subsequently, the weights have to be calculated that will correspond to these risk budgets. Lastly, performance budgeting requires the determination of weights consistent with achieving a set level of portfolio performance. For example, if the level of performance needed is 15%, then, the contribution to the performance of the portfolio will be 9% for equities and 6% for bonds. The preceding budgeting definitions are illustrated in Exhibit 2.2. As part of the risk budgeting strategy, risk parity is an alternative heuristic approach which sets out to attain both superior diversification and higher total returns through an allocation method that enables equal risk contribution to the portfolio's total risk across the asset classes within this portfolio. This risk contribution approach is unique since it offers the flexibility of stipulating the preferred targeted risk contribution profile and hence presenting an additional degree of freedom (Qian, 2011).

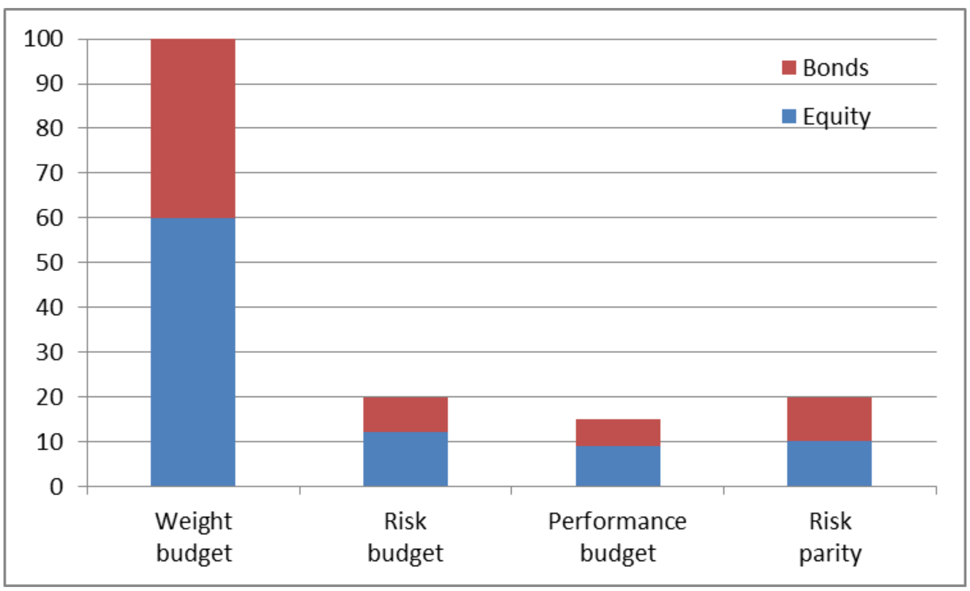


Exhibit 2.2: Risk budgeting methods – adapted from Roncalli (2014)

This study defines risk parity to represent equally weighted risk contribution (or ERC) portfolios. As previously stated this can also be looked at as a risk budget that distributes equally a portfolio's total risk to all the constituent assets. The study will use the terms risk parity and risk budgeting interchangeably.

### 2.4.1. Risk contribution principle

As earlier highlighted, the sensitivity of MVO portfolios to input parameters leads to the use of regularisation methods in the form of resampling, shrinkage or enforcing weight constraints. This introduces discretionary decisions on the portfolio solution (Roncalli, 2014) (and as will be seen later these are not usually easy to implement). He argues that the choice of the regularisation method affects the performance of a portfolio and it is, therefore, difficult to attribute this performance to the allocation method as this could be as a result of the regularisation method, for example, the constraints. The risk parity approach, however, solves this problem. Roncalli (2014) notes that the main difference between a risk budget portfolio and an optimised portfolio are twofold: Firstly unlike the optimised portfolio, a risk budgeting portfolio is not based on the maximisation of a utility function. Secondly, while optimised portfolios hinge on the expected portfolio performance which has to be estimated, risk budget portfolios do not need to explicitly estimate this performance (expected returns).

The interest in risk parity portfolios lies in their ability to mimic the diversification effect of equally weighted portfolios while taking into account single and joint risk contributions of the assets (Maillard, Roncalli and Teiletche, 2010). This means that no asset in the portfolio contributes to the total portfolio risk more than any other. However, if the performance contribution of all the assets are the same, i.e. if they have the same Sharpe ratios; and uncorrelated returns, this then becomes a mean-variance optimal portfolio (Qian 2005, Bruder and Roncalli, 2013).

The portfolio orientated risk management does not end with the measuring of risk. In order to have a more in-depth comprehension of diversification, it is essential to go beyond the risk measure. This is because this risk measure is usually represented by a single number only, that is, either the standard deviation, value at risk or expected shortfall[4] (Roncalli, 2014). As this part of the research focuses on asset allocation based on risk contribution, the principle of “risk contribution” has to come to the forefront. Risk contribution involves the decomposition of the portfolio risk into the proportion that each constituent asset (or sub-portfolio) in the portfolio is responsible for (Tasche, 2008). In other words, the contribution of each asset to the total portfolio risk has to be established as it is fundamental in identifying the concentration and understanding the risk profile of the portfolio (Roncalli, 2014 and Tasche, 2008). Formally, Maillard, Roncalli and Teiletche (2010) define the risk contribution of an asset

[4] It should be explicitly stated that the purpose of this research is not to ascertain the “risk measure” but portfolio allocation. The chapter will therefore not go into the debate as to which risk measure is better but it will use the standard deviation as the risk measure.

$i$ as the share of total portfolio risk attributed to that asset. Roncalli (2014) argues that there are a various methods of achieving this however, the Euler principle is the most used and accepted which he summarizes as shown below based on Tasche (2008).

Let $\prod$ be the Profit and Loss (P & L) of the portfolio. This can be decomposed as the sum of the $n$ asset P & L:

$$\prod = \sum_{i=1}^{n} \prod_i \tag{2.7}$$

If $R(\prod)$ is the risk measure associated with the P& L, then the risk-adjusted performance measure (RAPM) can be defined as

$$\mathrm{RAPM}(\prod) = \frac{\mathrm{E}[\prod]}{\mathrm{R}(\prod)} \tag{2.8}$$

The portfolio-related RAPM of the $i^{th}$ asset is defined by

$$\mathrm{RAPM}(\prod_i|\prod) = \frac{\mathrm{E}[\prod_i]}{\mathrm{R}(\prod_i|\prod)} \tag{2.9}$$

Based on the notion of RAPM, two properties of risk contributions that are desirable from an economic point of view are stated by Tasche (2008):

1. Risk contributions $\mathrm{R}(\prod_i|\prod)$ to portfolio – wide risk $\mathrm{R}(\prod)$ satisfy the full allocation property if:

$$\sum_{i=1}^{n} \mathrm{R}(\prod_i|\prod) = \mathrm{R}(\prod) \tag{2.10}$$

This implies that the sum of all risk contributions of the constituent assets in a portfolio is equal to the portfolio risk.

2. Risk contributions $\mathrm{R}(\prod_i|\prod)$ are RAPM compatible if there are some $\varepsilon_i > 0$ such that:

$$\mathrm{RAPM}(\prod_i|\prod) > \mathrm{RAPM}(\prod) \Rightarrow \mathrm{RAPM}(\prod + \mathrm{h}\prod_i) > \mathrm{RAPM}(\prod) \tag{2.11}$$

for all $0 < \mathrm{h} < \varepsilon_i$

It is shown that if there are risk contributions that are RAPM compatible in the sense of the two previous properties (1) and (2) then $\mathrm{RAPM}(\prod_i|\prod)$ is uniquely determined as:

$$\text{RAPM}(\prod_i|\prod) = \frac{d}{dh}R(\prod + \text{h}\prod_i)|_{h=0} \tag{2.12}$$

and the risk measure is homogeneous of degree 1. In the case of a sub-additive risk measure, it can be shown that:

$$\text{R}(\prod_i|\prod) \leq R(\prod_i) \tag{2.13}$$

This implies that the risk contribution of asset $i$ is always smaller than its stand-alone risk measure. The difference is related to the risk diversification.

The above risk measure becomes $R(x)$ when defined in terms of weights. In this context, the risk contribution is computed as the product of the allocation to asset $i$ and its marginal risk contribution. The marginal contribution equals the change in the total risk of the portfolio induced by an infinitesimal increase in holdings of asset $i$ (Maillard, Roncalli and Teiletche, 2010). This implies that the risk contribution of asset $i$ can be defined, using Bruder and Roncalli (2013), Maillard, Roncalli and Teiletche (2010), Roncalli (2014), and Davies and Menchero (2010), as follows:

Consider a portfolio of $n$ assets. $x_i$ is defined as the exposure (weights) of the $i^{th}$asset and $R(x_1, \ldots., x_n)$ as a risk measure for the portfolio $x$ = $(x_1, \ldots., x_n)$, $\sum$= covariance matrix, $\sigma_{ij}$= covariance between assets $i$ $and$ $j$:

$$R(x_1, \ldots., x_n) = R(x) = \sigma(x) = \sqrt{x^T \textstyle\sum x} \ = \sqrt{\sum_i x_i^2\sigma_i^2 + \sum_i \sum_{i \neq j} x_i x_j \sigma_{ij}} \tag{2.14}$$

$$RC_i = x_i \frac{\partial R(x)}{\partial x_i} \tag{2.15}$$

where $x_i = \text{allocation of asset } i \; and$

$\frac{\partial R(x)}{\partial x_i} = \text{marginal risk contribution}$

The marginal contribution to risk is a partial derivative that represents the change in the portfolio's volatility resulting from an infinitesimal increase in the weight of one asset in the portfolio. From the above equation (2.15), the risk contribution of asset $i$ is equivalent to the product of the assist's weight and its marginal risk contribution (Lee, 2011). This concept is expanded more below.

This risk measure above is said to satisfy the Euler decomposition:

$$R(x) = \sum_{i=1}^{n} x_i \frac{\partial R(x)}{\partial x_i} = \sum_{i=1}^{n} RC_i \qquad (2.16)$$

This relationship is also called the *Euler allocation principle,* and it is also referred to as the risk attribution formula by Davies and Menchero (2010). Euler allocation principle expresses the relationship between the marginal risk contributions and the portfolio risk and is at the core of risk parity portfolios which is used intensively by practitioners (Roncalli, 2014).

### 2.4.2. Computing the risk contributions

Although the value-at-risk and expected shortfall are also volatility measures of returns; this research concentrates on standard deviation as the measure of risk. For a two asset portfolio, risk is defined as:

$$\sigma(x) = \sqrt{x_1^2\sigma_1^2 + 2x_1x_2\rho\sigma_1\sigma_2 + x_2^2\sigma_2^2} \tag{2.17}$$

With this, Roncalli (2014) shows that the marginal risk of the first asset is, therefore:

$$\frac{\partial R(x)}{\partial x_i} = \frac{2x_1\sigma_1^2 + 2x_2\rho\sigma_1\sigma_2}{2\sqrt{x_1^2\sigma_1^2 + 2x_1x_2\rho\sigma_1\sigma_2 + x_2^2\sigma_2^2}}$$

$$= \frac{x_1\sigma_1^2 + 2x_2\rho\sigma_1\sigma_2}{\sqrt{x_1^2\sigma_1^2 + 2x_1x_2\rho\sigma_1\sigma_2 + x_2^2\sigma_2^2}}$$

$$= \frac{cov(R_1, R(x))}{var(R(x))} \tag{2.18}$$

It can be deduced that the risk contribution of the first asset is:

$$RC_i = x_i\frac{\partial R(x)}{\partial x_i}$$

$$= \frac{x_1^2\sigma_1^2 + x_1x_2\rho\sigma_1\sigma_2}{\sqrt{x_1^2\sigma_1^2 + 2x_1x_2\rho\sigma_1\sigma_2 + x_2^2\sigma_2^2}} \tag{2.19}$$

It follows that the sum of two risk contributions is equal to the portfolio's volatility. This is verified below:

$$RC_i + RC_2 = \frac{x_1^2\sigma_1^2 + x_1x_2\rho\sigma_1\sigma_2}{\sqrt{x_1^2\sigma_1^2 + 2x_1x_2\rho\sigma_1\sigma_2 + x_2^2\sigma_2^2}} +$$

$$= \frac{x_2 x_1 \rho \sigma_1 \sigma_2 + x_2^2 \sigma_1^2}{\sqrt{x_1^2 \sigma_1^2 + 2x_1 x_2 \rho \sigma_1 \sigma_2 + x_2^2 \sigma_2^2}}$$

$$= \frac{x_1^2 \sigma_1^2 + 2x_1 x_2 \rho \sigma_1 \sigma_2 + x_2^2 \sigma_2^2}{\sqrt{x_1^2 \sigma_1^2 + 2x_1 x_2 \rho \sigma_1 \sigma_2 + x_2^2 \sigma_2^2}}$$

$$= \sqrt{x_1^2 \sigma_1^2 + 2x_1 x_2 \rho \sigma_1 \sigma_2 + x_2^2 \sigma_2^2}$$

$$= \sigma(x)$$

These preceding formulas can be extended to the case n > 2. Since $\sigma(x) = \sqrt{x^T \sum x}$, it follows that the vector of marginal volatilities is:

$$\frac{\partial R(x)}{\partial x_i} = \frac{1}{2}(x^T \sum x)^{-1}(2 \sum x)$$

$$= \frac{\sum x}{\sqrt{x^T \sum x}} \tag{2.20}$$

The risk contribution of the $i^{th}$ asset is therefore:

$$RC_i = x_i . \frac{(\sum x)_i}{\sqrt{x^T \sum x}} \tag{2.21}$$

Similar to the two-asset case, it can be verified that the full allocation property is:

$$\sum_{i=1}^{n} RC_i = \sum_{i=1}^{n} x_i . \frac{(\sum x)_i}{\sqrt{x^T \sum x}}$$

$$= x^T \frac{\sum x}{\sqrt{x^T \sum x}}$$

$$= \sqrt{x^T \sum x}$$

$$= \sigma(x)$$

Since the aim of risk parity is to attain equal risk contribution to each particular asset in the portfolio, risk parity portfolios are also called risk contribution portfolios. Research has shown similarities between the diversification effect of portfolios with equal weightings and that of out-of-sample mean-variance characteristics of risk parity portfolios. However, the latter also takes single and joint risk contributions of assets into consideration. Maillard, Roncalli and Teiletche (2010) argue that mean-variance also equalise risk contribution albeit only on a marginal basis; meaning that a small increase in allocation of an asset leads to the same increase in the total risk of the portfolio. They prove that when assets in bivariate portfolios have the same volatility, the 1/n, minimum-variance, and risk parity portfolios are identical. However, for portfolios with more than two assets with different correlations; the risk parity portfolio coincides with the 1/n portfolio when the volatilities are the same. Furthermore, they also show that risk parity portfolios correspond to the minimum-variance portfolio when the correlation matrix reaches its lowest possible value (i.e. when cross-diversification is the highest). This suggests that risk parity approach yields portfolios with robust risk-balanced properties.

The question that arises is what weight should be assigned to each asset in order to obtain an equally weighted risk contribution portfolio. Roncalli (2014); Maillard, Roncalli and Teiletche (2010); and Bruder and Roncalli (2013) show how this can be done analytically by first showing a bivariate case and then the general case when there are more than two assets in a portfolio.

The two-asset case (n = 2)

The following terms are first defined. $\sigma_i$ represents the volatility of the $i^{th}$ asset, $\rho$ the correlation and $x = (w, 1 - w)$ the portfolio weights. The marginal risk contributions are:

$$\frac{\partial \sigma(x)}{\partial x} = \frac{\sum x}{\sqrt{x^T \sum x}} = \frac{1}{\sigma(x)} \begin{pmatrix} w\sigma_1^2 + (1-w)\rho\sigma_1\sigma_2 \\ (1-w)^2\sigma_2^2 + w\rho\sigma_1\sigma_2 \end{pmatrix} \quad (2.22$$

Then ERC portfolio satisfies:

$$w.\frac{(w\sigma_1^2 + (1-w)\rho\sigma_1\sigma_2)}{\sigma(x)} = (1-w).\frac{(1-w)\sigma_2^2 + w\rho\sigma_1\sigma_2}{\sigma(x)}$$

They then deduce that:

$$w^2\sigma_1^2 = (1-w)^2\sigma_2^2$$

Because $0 \leq w \leq 1$, the unique solution is:

$$w^* = \frac{1}{\sigma_1} / \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right)$$

$$= \frac{\sigma_2}{\sigma_1 + \sigma_2} \qquad (2.23)$$

This means that the weights of the two asset ERC portfolio are inversely proportional to the volatilities. However, they are independent of the correlation, $\rho$.

Roncalli (2014) compares this portfolio with the equally weighted (or EW) portfolio and the minimum variance (or MV) portfolio. To do this he uses the following parameterization $\sigma_1 = \sigma$ and $\sigma_2 = k\sigma$ with $k \geq 0$.

The volatility for the ERC portfolio thus

$$\sigma(x_{erc}) = \sqrt{2(1-\rho)}\frac{k}{1+k}\sigma$$

While the volatility of an equally weighted portfolio is:

$$\sigma(x_{ew}) = \frac{1}{2}\sigma\sqrt{1 + k^2 + 2\rho k}$$

These expressions, therefore, show that $\sigma(x_{ew}) \geq \sigma(x_{erc})$. When $\sigma(x_{ew}) = \sigma(x_{erc})$ obtainable when $k = 1$.

Roncalli (2014) shows that the minimum variance portfolio for a two asset portfolio has

$$x_{mv} = \frac{1}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\begin{pmatrix}\sigma_2^2 - \rho\sigma_1\sigma_2 \\ \sigma_1^2 + \rho\sigma_1\sigma_2\end{pmatrix}$$

He, therefore, shows that the volatility of the minimum variance portfolio is thus:

$$\sigma(x_{mv}) = k\sigma\sqrt{\frac{1-\rho^2}{1+k^2-2\rho k}}$$

<u>The general case (n > 2)</u>

In instances where a portfolio has more than two assets, it is not possible to find an analytical solution (Roncalli (2014), Bruder and Roncalli (2013) and Maillard, Roncalli and Teiletche (2010). However, the same authors show some closed-form solutions for specific cases, which can be referred to as to the naïve risk parity because either the correlation or the volatility is assumed to be identical for the different assets in the portfolio.

Assuming a constant correlation matrix with $\rho_{i,j} = \rho$ for all $i, j$, it follows that the weights of the ERC portfolios satisfy the following relationship

$$x_i\sigma_i\left((1-p)x_i\sigma_i + \rho\sum_{k=1}^{n} x_k\sigma_k\right) = x_j\sigma_j\left((1-p)x_j\sigma_j + \rho\sum_{k=1}^{n} x_k\sigma_k\right)$$

If follows that $x_i\sigma_i = x_j\sigma_j$. Because $\sum_{i=1}^{n} x_i = 1$, the expression for the risk contribution can be deduced a:

$$x_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^{n}\sigma_j^{-1}} \qquad (2.24)$$

The weight allocated to the asset $i$ is inversely proportional to its volatility. In other words, it is the harmonic average of the volatilities of the assets in the portfolio. Similar to the two asset scenario, this does not depend on the value of the correlation or the covariance. Roncalli (2014) further shows that this solution is connected to the minimum-variance portfolio. Let $\sum = \sigma\sigma^T \circ C_n(\rho)$ with $\Gamma_{i,j} = \sigma_i^{-1}\sigma_j^{-1}$ and:

$$C_n^{-1}(\rho) = \frac{\rho 1 1^T - ((n-1)\rho + 1)I_n}{(n-1)\rho^2 - (n-2)\rho - 1}$$

Because the minimum variance portfolio is:

$$x = \frac{(\Sigma^{-1}1)}{1^T\Sigma^{-1}1}$$

He deduces that the expression for the minimum variance weights is, therefore:

$$x_i = \frac{-\big((n-1)\rho + 1\big)\sigma_i^{-2} + \sigma \sum_{j=1}^{n}(\sigma_i\sigma_j)^{-1}}{\sum_{k=1}^{n}(-((n-1)\rho + 1)\,\sigma_k^{-2} + \sigma \sum_{j=1}^{n}(\sigma_k\sigma_j)^{-1}}$$

Since the lower bound of $C_n(\rho)$ is achieved for $\rho = -(n-1)^{-1}$. The solution becomes:

$$x_i = \frac{\sum_{j=1}^{n}\big(\sigma_i\sigma_j\big)^{-1}}{\sum_{k=1}^{n}\sum_{j=1}^{n}\big(\sigma_k\sigma_j\big)^{-1}} = \frac{\sigma_i^{-1}}{\sum_{k=1}^{n}(\sigma_k)^{-1}}$$

When the correlation is at its lowest possible value, the ERC portfolio is equal to the minimum variance portfolio.

If it is assumed that all volatilities are equal, i.e. $\sigma_i = \sigma$ for all $i$, the risk contribution then becomes

$$RC_i = \frac{\big(\sum_{k=1}^{n} x_i x_k \rho_{i,k}\big)\sigma^2}{\sigma_{(x)}}$$

The equally-weighted risk contribution (ERC) portfolio verifies then:

$$x_i\left(\sum_{k-1}^{n} x_k\rho_{i,k}\right) = x_j\left(\sum_{k-1}^{n} x_k\rho_{j,k}\right)$$

This is then deduced as:

$$x_i = \frac{\left(\sum_{k=1}^{n} x_k\rho_{i,k}\right)^{-1}}{\sum_{k=1}^{n}\left(\sum_{k=1}^{n} x_k\rho_{i,k}\right)^{-1}} \tag{2.25}$$

The weight attributed to asset $i$ is equal to the ratio between the inverse of the weighted average of the correlations of asset $i$ with other assets and the same average across all the assets (Maillard, Roncalli and Teiletche, 2010). Put another way, the weight of asset $i$ is inversely proportional to the weighted average of correlations of asset $i$. However, unlike the previous bivariate case and the case with constant correlations, for higher-order problems, the solution is endogenous owing to the fact that $x_i$, is a function of itself both directly and through the constraint that $\sum_i x_i = 1$. This endogeneity phenomenon unsurprisingly also occurs in the general case where both the volatilities and the correlations are unique.

Maillard, Roncalli and Teiletche, (2010) show the following deduction, by starting from the definition of the covariance of the returns of the asset $i$ with the returns of the aggregated portfolio:

$$\sigma_{ix} = cov(r_i, \sum_j x_j r_j = \sum_j x_j \sigma_{ij}$$

This is equivalent to

$$\sigma_i(x) = \frac{x_i\sigma_{ix}}{\sigma(x)}$$

The beta[5], $\beta_i$, of asset $i$ is now introduced where

$$\beta_i = \frac{\sigma_{ix}}{\sigma^2(x)} \tag{2.26}$$

and

[5] Beta measures the sensitivity of returns of an asset to movements in the markets' returns. In this case, however, the portfolio is used as a proxy for the market. So in essence what is being measured is the sensitivity of the returns asset (in a portfolio) to movements in the portfolio's returns.

$$\sigma_i(x) = x_i\beta_i\,\sigma(x)$$

By definition the ERC portfolio for all $i, j$ is:

$$\sigma_i(x) = \sigma_j(x) = \frac{\sigma(x)}{n}$$

it can, therefore, be deduced that:

$$x_i = \frac{\beta_i^{-1}}{\sum_{j=1}^{n} \beta_j^{-1}}$$

Because $\sum_{i=1}^{n} x_i\beta_i = 1$, this translates to

$$x_i = \frac{1}{n\beta_i} = \frac{\beta_i^{-1}}{n} \tag{2.27}$$

As highlighted earlier, this solution is also endogenous since $x_i$ is a function of the asset's beta, $\beta_i$ which, by definition, is dependant on the portfolio $x$, which inturn is also a function of the sum of portfolio weights $x_i$. The weight attributed to asset $i$ is inversely proportional to its beta. The higher the beta, the lower the weight and vice versa. This implies that the assets with high volatility or high correlations with other assets will be penalized (Maillard, Roncalli and Teiletche, 2010).

### 2.4.3. Comparing ERC with EW and MV portfolios

As earlier discussed, the equally weighted (EW) and minimum-variance (MV) are heuristic approaches that are used in practice as alternatives to the MVO due to their simplicity. A comparison of these two approaches with ERC portfolio by Maillard, Roncalli and Teiletche (2010) reveal that the ERC is located between the EW and the MV portfolios. They show that in the bivariate case, the EW, MV and ERC are identical when the two assets have the same volatility. However, in the general *n-assets* case, with unique correlations, the ERC portfolio coincides with the EW portfolio when all the volatilities are identical. Furthermore, they also show that the ERC portfolio corresponds to the MV portfolio when cross-diversification is the highest, i.e. when the correlation matrix reaches its lowest possible value.

This they argue suggests that the ERC strategy produces portfolios with robust risk-balanced properties. Maillard, Roncalli and Teiletche (2010) demonstrate mathematically, the general case regarding the EW, MV and ERC as follows

$$x_i = x_j \tag{EW}$$

$$\partial x_i \sigma(x) = \partial x_j \sigma(x) \tag{MV}$$

$$x_i \partial x_i \sigma(x) = x_j \partial x_j \sigma(x) \tag{ERC}$$

The previous mathematical annotations show that EW portfolio is represented by equalising the asset weights, while the MV portfolio matches the marginal risk contributions in the portfolio and lastly the ERC equalises the risk contribution of each asset in the portfolio. Ultimately Maillard, Roncalli and Teiletche (2010) and Roncalli 2014) demonstrate that the risk for the ERC portfolio lies in between the MV and EW portfolios. Therefore, the MV portfolio is the least volatile while the EW is the most volatile and in between, is the ERC. This is summarized using their famous inequality:

$$\sigma(x_{mv}) \leq \sigma(x_{erc}) \leq \sigma(x_{ew}) \tag{2.28}$$

### 2.4.4. Diversification benefits of risk parity

As stated previously, while most of the conventional asset allocation approaches entail explicit asset return forecasting with a view to use mean-variance optimisation, risk parity hinges upon risk diversification. Therefore, there is no estimation risk emanating from the estimation of expected returns because only variances and covariances are needed and these are said to be a tenth as important as estimates of means (Chopra and Ziemba, 1993). In this approach, each asset class contributes approximately the same expected fluctuation in the [Dollar] value of the portfolio (Chaves et al., 2011) thus reducing the portfolio’s overall risk which comes about as a result of a reduction in the proportion of risk explained by higher volatility assets like equities. Unlike minimum-variance and mean-variance optimised portfolios that are susceptible corner solutions; risk parity portfolios ensure that each asset has a non-zero weight allocation. Furthermore, in contrast with equally weighted portfolios, correlations of these assets’ returns have an effect on these weight allocations with higher correlated assets getting lower allocations and vice versa. This approach, therefore, limits the risk of overexposure to any individual asset class, while providing sufficient exposure to all of them at the same time (Qian, 2005).

Risk parity can, hence, be seen as an equally weighted portfolio but in terms of risk as opposed to weights. Risk parity thus presents a pure method of risk management since assumptions about expected returns are not required. Consequently, a risk parity portfolio comprising of only equities and bonds will have more funds allocated to bonds (as illustrated earlier in Exhibit 2.1(b) and therefore resulting in reduced volatility. Correspondingly, the effect on the overall portfolio of the losses arising from the constituent assets is constrained. Because of the positive relationship between risk and expected return, the reduced volatility in a risk parity portfolio also leads to a reduction in return possibly below the level that might be required by the investor. To remedy this, leverage can be employed in order to achieve a given level of expected return [where there are no constraints for investors to use leverage]. The investor hence has to decide how much risk should be taken on as well as how this risk is distributed amongst the portfolio's constituent assets.

Getting back to the equity and bond scenario which are held by pension policies, equities are seen as growth assets, and frequently bonds are added to portfolios due to their risk reduction characteristics, primarily as a way to balance this growth. A study by Qian (2011) that compares a 60/40 portfolio with a risk parity portfolio both only consisting of equities and bonds is used to illustrate the diversification benefits of a risk parity approach. Quin's study reveals that the 60/40 balanced portfolio has very high and very low correlations to equities and bonds respectively whereas the risk parity portfolio had the same correlations with both equities and bonds. For this reason, the 60/40 portfolios are said not to offer true risk diversification due to their high dependence on equities. From the risk allocation standpoint, the 60/40 portfolios are highly concentrated albeit, appearing balanced with regards to capital allocation. The risk parity approach, on the other hand, is said to offer true risk diversification and hence improving portfolio diversification and are hence likely to achieve higher Sharpe ratios. In the same study Qian (2011) compared the beta of equity and bonds against both the 60/40 and risk parity portfolios and demonstrated that the equities and the bonds had lower and higher beta respectively in the risk parity portfolio. The reverse was true for the 60/40 portfolio indicating that diversification is attained in the risk parity portfolio because equities (with high volatility) have lower beta and bonds (with low volatility) have higher beta.

The quest for risk parity to allocate equal risk contribution of assets to a portfolio can help the investor determine how much risk to invest in each asset class as well as to ascribe the amount of expected performance to the asset classes. Additionally, because downside risk is a major concern for risk averse investors, it is also essential to consider loss contribution of each asset class in a portfolio to establish when substantial losses are expected to occur (Qian, 2011). Loss contribution is assessed in the same way as risk contribution since it is an accurate indicator of risk loss (Qian, 2005). As a result, in a 60/40 portfolio, a significant loss in equities will result in a loss of similar magnitude for the entire portfolio because equities contribute more than 90% of the portfolio (see Exhibit 2.1a). Because risk parity equates the risk contribution of equities and bonds (i.e. 50 – 50), it limits the downside of the overall portfolio against significant losses by reducing the loss contribution of the high-risk assets like equities in a 60/40 portfolio.

Hewitt (2012) sees risk parity as an active approach that entails the skill to analyse risk and market conditions. He observes that this asset allocation approach is evolving into traditional active management as it employs aspects of return forecasting, tactical allocation and even security allocation allowing managers to differentiate themselves. Sceptics, however, dispute the benefits of risk parity stating that it is not realistic in its assumption of risk being quantified by standard deviation as it may not necessarily encapsulate the risk of an asset[6]. Inker (2011) demonstrates this by considering the standard deviation of AAA-rated credit card backed Asset-Backed Securities from 2001 to 2010 which shows that the risk for this period was very stable despite the impending US home price bubble. However, in 2008 and 2009, the standard deviation rose by a factor of 200 (Inker 2011) validating the unreliability of the standard deviation as a measure of future risk, despite the unprecedented nature of the financial crisis. Thiagarajan and Schachter (2011) and Bhansali (2011) share this concern and go beyond by observing that risk is more comprehensive than any measure of volatility, more so for geared portfolios. Others also criticize the risk parity approach stating that *"investors want high returns and diversification itself does not pay the bills"* (Qian, 2011). Likewise, Thiagarajan and Schachter (2011) observe that while the risk-reward performance of the risk parity approach seems impressive, it is worth keeping in mind that *"higher risk-rewards ratios do not put money in the bank, but returns do".* However, as stated earlier, the implementation of the risk parity approach is driven by the investor's objectives whose goal is not to maximise risk adjusted performance but to manage risk through risk allocation.

[6] Again, as stated earlier the debate as to what the right risk measure should be is beyond the scope of this study.

While the benefits of risk parity have been shown by practitioners and researchers alike; Thiagarajan and Schachter (2011) nevertheless caution about the danger of the exclusive use of risk in portfolio construction where returns are completely ignored. They observe that this could lead to the discarding of potentially valuable information and therefore suggest for a balanced investment policy in which appropriate consideration to risk is provided albeit not exclusively. Lee (2011) is also of the same opinion and views risk-based asset allocation as a subset of modern portfolio theory paradigm rather than a new paradigm. He argues that regardless of whether expected returns are explicitly used as inputs, a portfolio that consistently outperforms the market by definition has more information on future asset returns than the market portfolio. Lee (2011) is also of the view that risk-based portfolios like risk parity are no exception. He asserts that the mean-variance approach is the best and therefore concludes that modern portfolio theory remains modern.

### 2.4.5. Application of risk parity to real estate

Despite the drive to adopting mean-variance optimisation in real estate, naïve diversification has been predominantly utilized in this industry. As shown earlier there are arguments for and against each approach. Even though the naïve approach of equal weighting can be worked out statistically, its practicality is difficult owing to the inherent characteristics of property (Stevenson 2000a). The question is whether risk parity as an alternative risk allocation approach when applied to real estate can provide better performance in terms of risk-adjusted returns and diversification? The ensuing section addresses this using empirical data from real estate.

## 2.5 Data, methodology and Empirical analysis

### 2.5.1. Data

The data used in this study consists of monthly total returns of public real estate securities from four international markets, namely; Australia, France, the United Kingdon (UK) and the United States of America (US). The data was obtained from Thomson Reuters' Datastream®[7] and spans from January 1990 to December 2017. The selected markets were selected based on data availability from 1990 and an extended period, i.e., twenty-seven years was chosen in order to take into consideration different market conditions. These represent the REIT indices on Datastream for the selected markets. Even though all the selected markets, apart from the US, did not have REITs in the entire period covered by the research, Datastream backdated the REIT indices to cover those periods. Monthly returns were used rather than weekly or daily returns because research has shown that higher frequency observations tend to be noisier and this can affect the quality of the estimated covariance matrix (Kwan, 2011). Direct property and private real estate data were not used because their illiquid requires longer holding periods which were not deemed suitable for this study and also for easy analysis. Furthermore, other asset classes were not included because the focus of the study is on the application of risk parity on real estate securities.

Table 2.1 displays the descriptive statistics for the total returns data for international real estate security indices for the period extending from January 1990 to December 2017.

| | Mean return | Standard deviation | Beta |
|---|---|---|---|
| Australia | 0.9444% | 4.3203% | 0.7021 |
| France | 1.3077% | 5.5662% | 0.9325 |
| UK | 0.7401% | 5.9458% | 1.1534 |
| US | 1.1014% | 6.2933% | 1.2120 |

Table 2.1: Summary statistics for International Real Estate Securities for the overall sample period 1990 -2017,

All the sectors had positive monthly average returns ranging from 0.74% to 1.31% with the UK and French markets achieving the lowest and highest respectively. The total risk, measured by the standard deviation ranges from 4.32% to 6.29%, these being the Australian and the US market respectively. As expected, the markets with lower standard deviations were defensive as they exhibited betas less than

[7] Datastream is a platform that provides global financial and macroeconomic data.

unity (1) while the high standard deviation sectors had betas greater than unity and therefore aggressive. The correlation matrix in Table 2.2 reveals that all the sectors are positively correlated with values ranging from about 0.313 to 0.557 which can be classified as weak to moderate.

| | *Australia* | *France* | *UK* | *US* |
|---|---|---|---|---|
| Australia | 1 | | | |
| France | 0.3133 | 1 | | |
| UK | 0.4629 | 0.4190 | 1 | |
| US | 0.4338 | 0.3959 | 0.5569 | 1 |

Table 2.2: Correlation matrix for International Real Estate Securities for the overall sample period 1990 -2017,

### 2.5.2. Methodology

In order to test the performance of risk parity, a comparison with other portfolio allocation strategies, namely; the equally-weighted, minimum-variance, mean-variance optimisation, Bayes Stein (return vector shrinkage), Ledoit and Wolf (covariance matrix shrinkage) and resampling was undertaken. The comparison was made after creating portfolios using the strategies mentioned above and determining the portfolio weights, risk contributions, returns, risk and Sharpe ratios for different rolling sub-periods. Note that at portfolio level the different markets in the portfolio will also be referred to as assets – therefore "markets" and "assets" will be used interchangeably.

In this study, the equally-weighted portfolio as defined previously will allocate equal weights to the four international real estate securities markets. Frost and Savarino (1998) and Chopra (1993) view this strategy as constraining the influence of input parameters unsteadiness. However, this portfolio is both undesirable and impossible to hold in the direct real estate sector due to the indivisibility of property and the marked differences in lot sizes between different property types (Lee and Stevenson, 2005). While it might be possible to do this when creating a real estate securities portfolio, this approach is possibly neither pragmatic nor appropriate. This said, as previously stated the performance of the equally-weighted portfolio, in other studies, is not necessarily inferior to the mean-variance optimisation and minimum-variance approaches as the latter approaches are not consistently better out of sample.

The minimum-variance portfolio will allocate the assets such that the portfolio attains the lowest standard deviation while the mean-variance optimisation's allocation will be based on the achievement of the maximum Sharpe ratio (MSR). The Bayes Stein portfolio will shrink the return vector using equations (2.1) to (2.3). For this study $i$ is the return of the respective international real estate securitties markets, $\bar{r}_g$(the global mean) the mean of the average return of the four markets, $r_0$ the minimum variance portfolio mean, T is 366 representing the total number of observed months and N is 4 representing the number of markets being examined. The minimum-variance portfolio mean has been used as $r_0$because it is determined purely by variances and covariances and therefore obviates the use of mean estimates due to their instability which lead to the increased chances of estimation error (Stevenson, 2001).

The Ledoit and Wolf (2004) allocation approach will shrink the sample covariance matrix[8]. Because the Resampled Efficient Frontier (REF) is patented and the exact simulation is unknown this study uses a basic process to create a resampled portfolio. This is done employing Monte Carlo simulation where a random sample from each sub-period is created, and from this, the mean return; standard deviation; and covariance are then calculated. A Sharpe ratio maximising portfolio is then produced (for this newly created sample), and this iteration is repeated 100 times for each rolling sub-period[9]. The resampled portfolio is then constructed from the average allocations (weights) of the 100 newly created portfolios after resampling.

A strategy that combines the two shrinkage approaches will also be attempted just to see if there is any benefit – and will be called the Bayes Stein + Ledoit and Wolf (BS + LW) shrinkage. As stated previously, the Bayes Stein shrinkage only shrinks the return vector but using the sample covariance matrix while Ledoit and Wolf shrinkage, on the other hand, utilises the sample return but applies the shrinkage on the covariance matrix. Although there are cases that are put forward with regard to which parameter is most sensitive to estimation error i.e., the return vector or the covariance matrix (Chopra and Ziemba, (1993), Lee and Stevenson (2005) and Bengtsson (2004), Michard and Michard (2008), Markowitz and Usmen (2003), Ledoit and Wolf (2003), and Wolf (2007)) what is not disputable is that both of these parameters are subject to estimation errors as they are obtained from ex-post sample data. Hence, the reason why this research correspondingly attempts to combine these two shrinkage approaches to form a new portfolio (i.e., the Bayes Stein + Ledoit and Wolf combo).

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[8] As shown previously the implementation of the covariance shrinkage is very complicated. This study implemented this in Matlab using the code provided by Ledoit and Wolf. See http://www.econ.uzh.ch/faculty/wolf/publications.html

[9] This is a very time consuming task and Matlab in conjunction with Microsoft Excel macros were used to perform this task.

Finally, the risk parity approach as defined earlier will result in portfolios that equalise the risk contribution of all the four markets in the portfolio[10]. As not all investment mandates permit short selling, a no-short selling constraint was imposed on all the portfolios to avoid increasing estimation error impact (Jorion, 1992 and Stevenson, 1999). This means that none of the assets in a portfolio has a negative allocation and the sum of the allocations need to be 100%. It should be noted that the mean-variance optimisation, Bayes Stein, Ledoit and Wolf and Bayes Steins + Ledoit and Wolf aim to maximise the risk-adjusted return by optimising the weights. This risk-adjusted return is defined by the Sharpe ratio (SR):

$$SR = \frac{\bar{r}_p - r_f}{\sigma_p} \tag{2.29}$$

Where $\bar{r}_p$ is the portfolio expected return, $r_f$ is the risk-free rate of return and $\sigma_p$ is the standard deviation of the portfolio. For simplicity, this study assumes that the risk-free rate of return ($r_f$) is zero.

The descriptive statistics (Tables 2.1) and the variance-covariance matrix (hereinafter only referred to as the covariance matrix) are the basis of the calculation of the risk contributions of the various portfolios shown in Exhibit 2.3. The portfolio risk σ is calculated using equation (2.14) and is a function of the allocated weights ($x_i$) and the covariance matrix. The marginal risk contribution ($\mathrm{MRC_i}$) which measures the additional portfolio risk as a result for every addition of asset i is calculated from equation (2.18). While the allocations for the equally weighted portfolio are the same by definition, the marginal risk contribution for each asset is however different. For the minimum-variance portfolio, it can be observed that the marginal risk contributions are identical for the assets with non-zero allocations. The marginal risk contribution can be transformed into the risk contribution $\mathrm{RC_i}$ by multiplying it by the respective weight $\mathrm{x_{,i}}$ as shown by equation (2.19). Furthermore, like underscored earlier, the sum of the $\mathrm{RC_i}$ should equal the portfolio risk σ. The risk contribution ratio ($\mathrm{RC_i}$%) is the asset's risk contribution as a proportion of the portfolio risk. By definition the risk contribution, $\mathrm{RC_i} = \frac{\sigma}{\mathrm{n}}$ for the risk parity portfolio as it shows equal risk contribution and the risk contribution ratio, $\mathrm{RC_i\%} = \frac{100}{\mathrm{n}}$. It is noteworthy that for this study, the risk contribution refers to the risk contribution ratio ($\mathrm{RC_i\%}$) rather than $\mathrm{RC_i}$.

[10] Apart from the equally weighted portfolio, all the strategies where implemented using the "Solver" tool in Microsoft Excel.

Exhibits 3.2(a) through to 3.2(c) are graphical summaries of the performance of the four markets for real estate securities shown in Exhibit 3.1 and displays the standard deviations, Sharpe ratios, risk, allocations and risk contributions of resulting portfolios for the entire period of investigation, i.e. January 1990 to December 2017. The results show that the highest risk and the lowest return are from the equally weighted portfolio. The risk parity portfolio has only outperformed (albeit slightly) the equally weighted portfolio both in terms of risk and Sharpe ratio. This is in harmony with Maillard, Roncalli and Teiletche (2010) and Roncalli 2014) who reveal that the risk parity portfolio lies somewhere between naïve portfolio and the minimum-variance portfolio. For this period, the risk parity portfolio had the third largest standard deviation just behind that of the Ledoit and Wolf covariance shrinking portfolio. What was not expected was the relatively low standard deviation of the mean-variance portfolio relative to that of the risk parity portfolio. As far as the Sharpe ratio is concerned, the risk parity was the second lowest (after the equally weighted portfolio) while the mean-variance was unsurprisingly the highest, as expected. As expected, the portfolio with the lowest risk is the minimum-variance portfolio.

Apart from the equally weighted and the risk parity portfolio, all the other portfolios have allocations that are concentrated in a few assets with some presenting corner solutions as shown in Exhibit (3.2(b)). For example, the minimum-variance, Bayes Stein and the Bayes- Steins and Ledoit and Wolf combo are overweight in the Australian market followed by the French market and a small proportion to the US. The mean-variance, Ledoit & Wolf and Resampling portfolios, on the other hand, have most of their allocations in France, followed by Australia and a small proportion to the US. The mean-variance, Bayes-Steins and Ledoit & Wolf portfolios have completely no allocations assigned to the UK market resulting in corner solutions whereas the Resampling portfolio has only assigned a meagre 1% to the UK. By definition, the equally-weighted portfolio allocates equal weights to all the assets in the portfolio. The risk parity portfolio has allocations in all the markets in the portfolios, and like the equally weighted portfolio, there are no corner solutions.

Exhibit 2.4(c) shows the risk contributions of the assets to the portfolio's overall risk. The pattern of the risk contributions of the assets in all portfolios, except the equally-weighted and risk parity portfolios, are similar to their weight allocations described above. While the equally-weighted portfolio allocates equal weights to all the markets in the portfolio with the risk contributions are range from about 17.6% to 30.2%. The UK and US markets contribute the most risk to the equally weighted portfolio followed by France while the least risk contribution is from Australia. The UK market did not contribute any risk to the mean-variance, Bayes-Steins and Ledoit & Wolf portfolios because no weight was assigned to this market. As expected, all markets in the risk parity portfolio had equal risk contributions to the portfolio's overall risk.

Table 2.1 indicates that the US and Australian markets have the highest and lowest standard deviations respectively. For this reason, the US contributes the most risk in the equally-weighted portfolio while Australia contributes the least. By definition, the risk parity approach allocates lower weights to higher risk assets and vice versa. This explains why the risk parity portfolio has suggested the lowest and highest allocations to the US and Australia respectively. It can be discerned from the preceding that the allocation by risk parity is inverse proportional to the equally weighted portfolio's risk contribution. Furthermore, it is in line with equation (2.27) which shows that the weight attributed to an asset in risk parity is inversely proportional to its beta[11] (Maillard, Roncalli and Teiletche, 2010). This can also be corroborated by looking at the ranking of risk as measured by both the standard deviation and beta as displayed in Table 2.3.

[11] Beta measures the sensitivity of an asset or portfolio returns to market movements.

**Equally Weighted Portfolio**

| | $x_i$ | $MRC_i$ | $RC_i$ | $RC_i\%$ |
|---|---|---|---|---|
| Australia | 25.000% | 0.0296 | 0.740% | 17.553% |
| France | 25.000% | 0.0393 | 0.983% | 23.312% |
| UK | 25.000% | 0.0486 | 1.215% | 28.836% |
| US | 25.000% | 0.0511 | 1.277% | 30.299% |
| ∑ | 100.000% | | 4.215% | 100.000% |
| E(r ) | 1.023% | | | |
| σ | 4.215% | | | |
| E(r )/σ | 0.2428 | | | |

**Minimum Variance Portfolio**

| | $x_i$ | $MRC_i$ | $RC_i$ | $RC_i\%$ |
|---|---|---|---|---|
| Australia | 59.383% | 0.0384 | 2.279% | 59.383% |
| France | 27.018% | 0.0384 | 1.037% | 27.018% |
| UK | 7.500% | 0.0384 | 0.288% | 7.500% |
| US | 6.099% | 0.0384 | 0.234% | 6.099% |
| ∑ | 100.000% | | 3.838% | 100.000% |
| E(r ) | 1.037% | | | |
| σ | 3.838% | | | |
| E(r )/σ | 0.2701 | | | |

**Mean Variance Portfolio**

| | $x_i$ | $MRC_i$ | $RC_i$ | $RC_i\%$ |
|---|---|---|---|---|
| Australia | 47.139% | 0.0334 | 1.575% | 39.875% |
| France | 43.167% | 0.0463 | 1.997% | 50.562% |
| UK | 0.000% | 0.0345 | 0.000% | 0.000% |
| US | 9.694% | 0.0390 | 0.378% | 9.563% |
| ∑ | 100.000% | | 3.949% | 100.000% |
| E(r ) | 1.116% | | | |
| σ | 3.949% | | | |
| E(r )/σ | 0.2827 | | | |

**Bayes - Stein Portfolio**

| | $x_i$ | $MRC_i$ | $RC_i$ | $RC_i\%$ |
|---|---|---|---|---|
| Australia | 56.976% | 0.0369 | 2.103% | 54.390% |
| France | 33.885% | 0.0415 | 1.408% | 36.413% |
| UK | 0.000% | 0.0346 | 0.000% | 0.000% |
| US | 9.139% | 0.0389 | 0.356% | 9.198% |
| ∑ | 100.000% | | 3.866% | 100.000% |
| E(r ) | 1.044% | | | |
| σ | 3.866% | | | |
| E(r )/σ | 0.2700 | | | |

**Ledoit & Wolf Portfolio**

| | $x_i$ | $MRC_i$ | $RC_i$ | $RC_i\%$ |
|---|---|---|---|---|
| Australia | 44.898% | 0.0347 | 1.557% | 37.897% |
| France | 42.627% | 0.0480 | 2.047% | 49.822% |
| UK | 0.000% | 0.0317 | 0.000% | 0.000% |
| US | 12.475% | 0.0404 | 0.505% | 12.280% |
| ∑ | 100.000% | | 4.109% | 100.000% |
| E(r ) | 1.119% | | | |
| σ | 4.109% | | | |
| E(r )/σ | 0.2723 | | | |

**Bayes - Steins + Ledoit & Wolf Portfolio**

| | $x_i$ | $MRC_i$ | $RC_i$ | $RC_i\%$ |
|---|---|---|---|---|
| Australia | 54.638% | 0.0381 | 2.083% | 52.679% |
| France | 28.509% | 0.0429 | 1.224% | 30.942% |
| UK | 6.237% | 0.0354 | 0.221% | 5.588% |
| US | 10.616% | 0.0402 | 0.427% | 10.791% |
| ∑ | 100.000% | | 3.955% | 100.000% |
| E(r ) | 1.033% | | | |
| σ | 3.955% | | | |
| E(r )/σ | 0.2613 | | | |

**Resampled Portfolio**

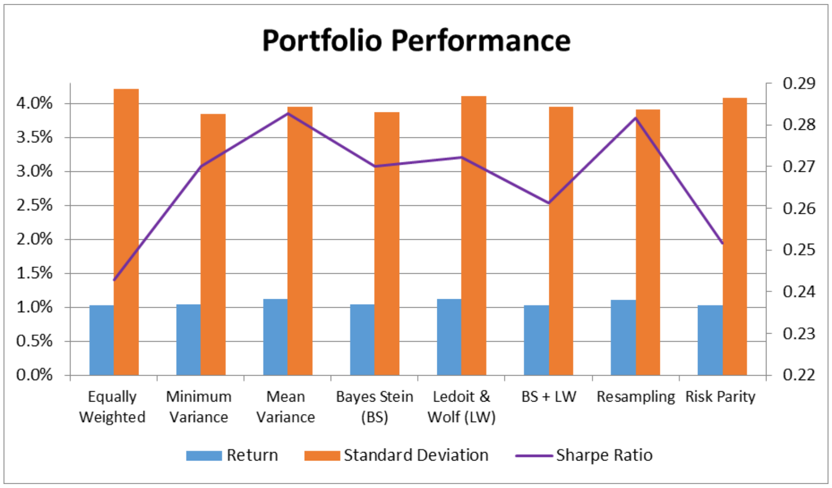
| | $x_i$ | $MRC_i$ | $RC_i$ | $RC_i\%$ |
|---|---|---|---|---|
| Australia | 48.189% | 0.0343 | 1.654% | 42.241% |
| France | 39.144% | 0.0447 | 1.751% | 44.721% |
| UK | 1.017% | 0.0356 | 0.036% | 0.925% |
| US | 11.650% | 0.0407 | 0.474% | 12.112% |
| ∑ | 100.000% | | 3.915% | 100.000% |
| E(r ) | 1.103% | | | |
| σ | 3.915% | | | |
| E(r )/σ | 0.2817 | | | |

**Risk Parity Portfolio**

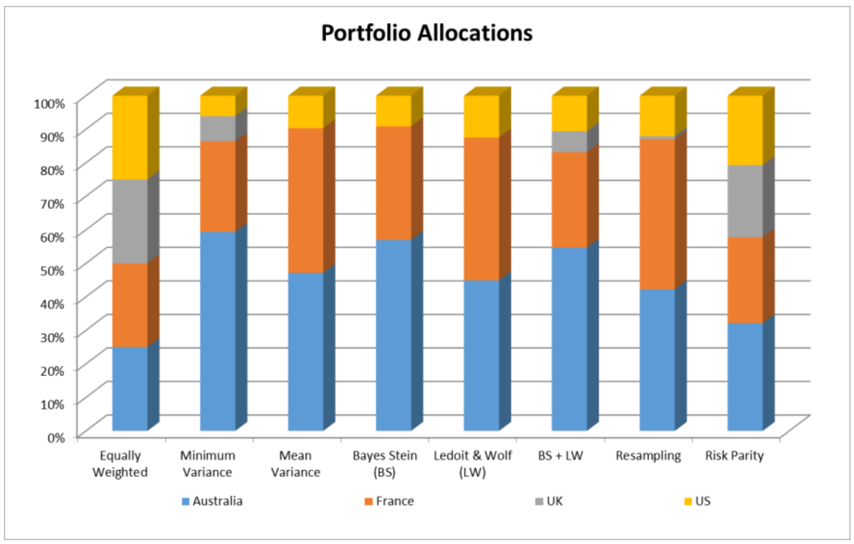
| | $x_i$ | $MRC_i$ | $RC_i$ | $RC_i\%$ |
|---|---|---|---|---|
| Australia | 32.128% | 0.0317 | 1.019% | 25.000% |
| France | 25.617% | 0.0398 | 1.019% | 25.000% |
| UK | 21.523% | 0.0473 | 1.019% | 25.000% |
| US | 20.732% | 0.0492 | 1.019% | 25.000% |
| | 100.000% | | 4.076% | 100.000% |
| E(r ) | 1.026% | | | |
| σ | 4.076% | | | |
| E(r )/σ | 0.2517 | | | |

Exhibit 2.3: Portfolio performances, allocations and risk contributions tables for the overall sample period 1990 - 2017

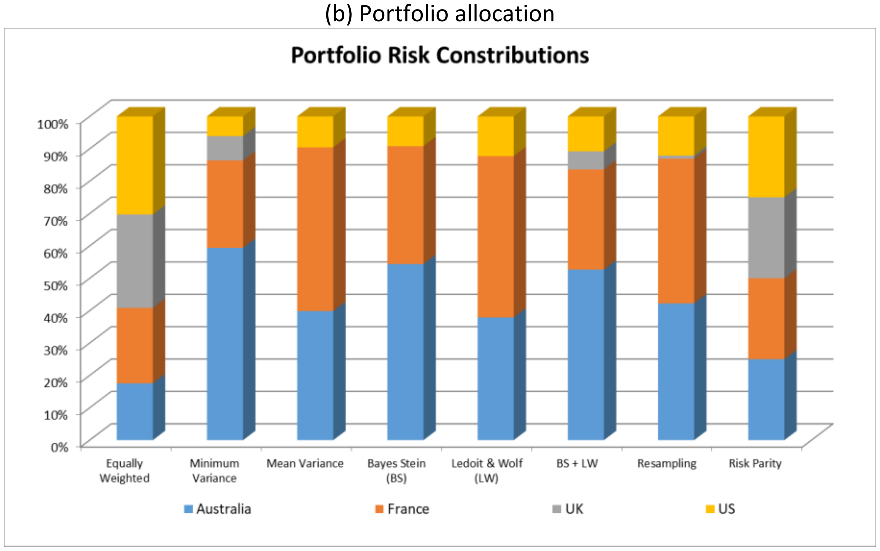
This exhibit shows the weight allocation, marginal risk contribution, risk contribution, and risk contribution percentage for each of the assets in the respective portfolios under investigation. The expected return is calculated from the allocations portfolio allocation while the portfolio standard deviation is calculated by also employing the covariance matrix in addition to the weights.




(a) Portfolio Standard Deviations and Sharpe ratios




(b) Portfolio allocation




(c) Portfolio risk contributions

Exhibit 2.4 International Real Estate Securities Portfolio performance, allocations and risk contributions for the overall sample period 1990 -2017

| Ranking | | | |
|---|---|---|---|
| Std Dev] | Beta | RC (EW) | Weights (RP) |
| 4 | 4 | 4 | 1 |
| 3 | 3 | 3 | 2 |
| 2 | 2 | 2 | 3 |
| 1 | 1 | 1 | 4 |

Table 2.3: Ranking (High to Low) of risk, risk contributions and allocations for EW and RP portfolios
This table shows the ranking of the standard deviation and beta versus that of the risk contribution and weights for the equally-weighted and risk parity portfolios respectively for the overall sample period between 1990 to 2017.

Instead of basing the analysis on the total 28-year period (equivalent to 336 months), the portfolio parameters were rolled and recalculated in accordance to the following sub-periods, i.e. 1-month, 3-months, 6-months, 12-months and 24-months. The data was rolled and rebalanced using the sub-periods above in order to be able to compare both the in-sample and out-of-sample performance. The estimation for the in-sample period was based on a 2-year (24 months) period similar to Lee and Stevenson (2005). For instance, the first estimate for the 1-month rolling periods was done in January 1992 based on a 2 years' worth of data spanning from January 1990 and December 1991. The second estimate was undertaken in February 1992 based on data spanning from February 1990 and January 1992. This rolling was carried out each month until the last estimate which was done in January 2018 which in total results in about 312 worth of monthly estimates. This same methodology was applied by Stevenson (2002) and Lee and Stevenson (2005). The same process was carried out on the other rebalancing periods i.e. 3-months, 6-months, 12-months and 24-months. Table 2.4 summaries how many estimates were performed for each of the rebalancing periods.

| Rolling Period | Number of estimates |
|---|---|
| 1-month | 312 |
| 3-months | 104 |
| 6-months | 53 |
| 12-months | 26 |
| 24-months | 13 |

Table 2.4: Summary of the number of estimates for the different rolling periods

### 2.5.3. Empirical analysis of risk parity

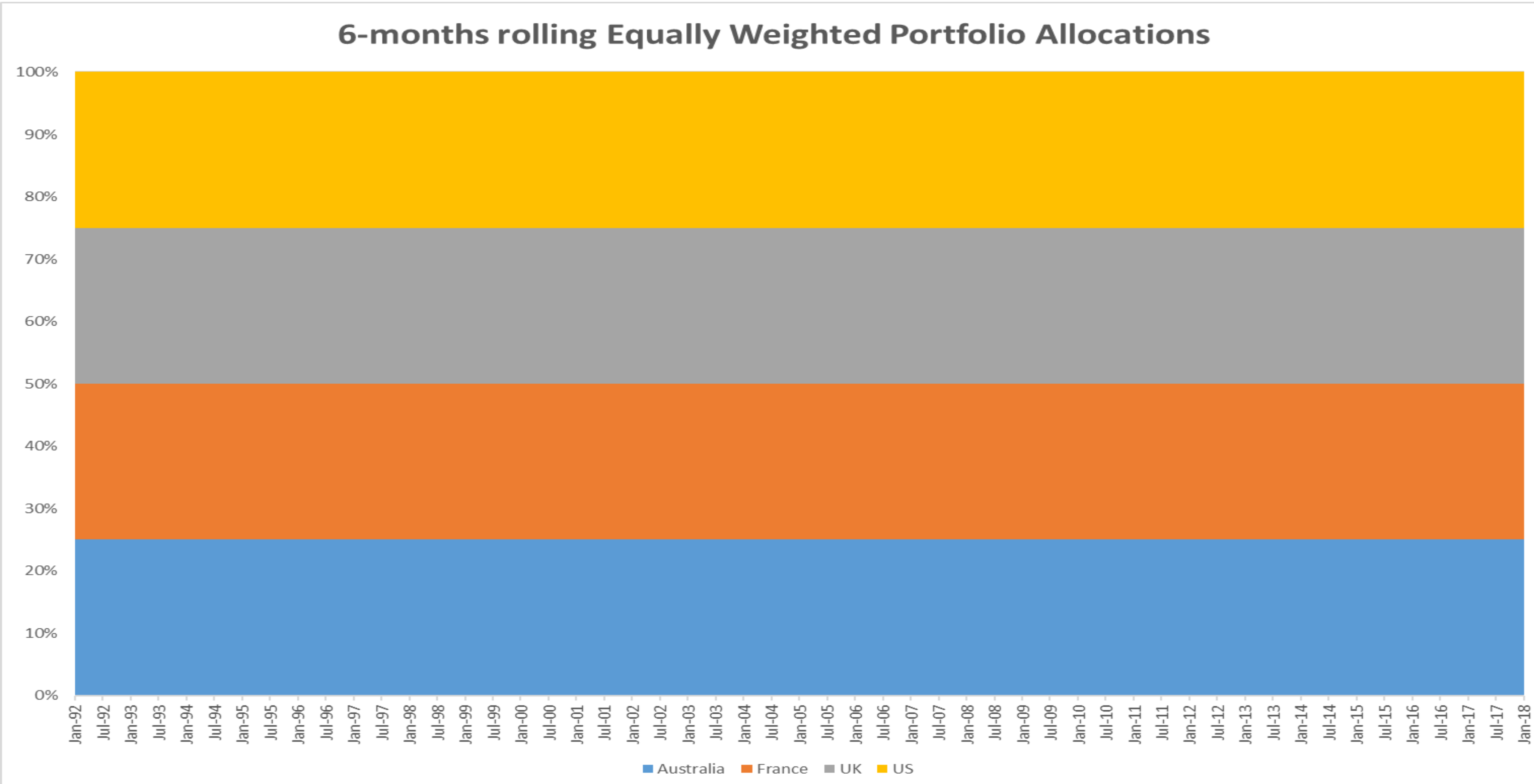
This section tests the robustness of the risk parity approach in the real estate environment using empirical data from the international public real estate securities indices from Australia, France, the UK and the US markets. It compares the conventional approaches, i.e. equally-weighted, minimum-variance, mean-variance; with risk parity. Stein based shrinkage (i.e. of the return vector and the covariance matrix) and resampling regularisation techniques were applied to the portfolio above approaches in order to test their robustness vis--vis risk parity.

In all of the rebalancing periods looked at, the risk parity approach displays more stable allocations in-sample [12]with absolutely no corner solutions in comparison with the other portfolio strategies. Exhibits 2.5 and 2.6 display the allocations and risk contributions suggested by the data based on the 6-months rolling data[13] respectively for all the allocation strategies under investigation. By definition, the equally weighted portfolio displays equal allocations over the rolling periods. However, the resulting risk contributions vary over these periods from a minimum of 6% (Australia) to a maximum of 50% (the UK). Even though risk contributions for the mean-variance, Ledoit and Wolf and the Bayes-Stein are similar, those of the minimum-variance, Bayes-Stein and the Resampling portfolios have marked differences. The weights for all the allocation methods besides the equally weighted all display varying allocations in different time periods showing that there is no stationarity of returns consistent with Stevenson (2001). All these with the exception of risk parity, equally-weighted and resampling portfolios have corner solutions. Compared to the mean-variance portfolio, all the regularised portfolios seem to show slightly less variation in allocations and can arguably be said to be more stable due to shrinkage and resampling. The Bayes-Stein + Ledoit and Wolf combo does seem to provide an improvement, albeit it slight, on the Bayes Stein and Ledoit & Wolf portfolios as far as allocation is concerned. Of the regularised portfolios, the resampling portfolio on many occasions displays more improvement in terms of allocation as it results in far no corner solutions. This said, in about seven instances the allocation to the UK is less than 1%, but despite this, all the assets in this portfolio have allocations assigned to them. By definition risk parity provide uniform (equal) risk contributions for the entire periods which result in the absence of corner and therefore superior stability because there are no significant swings in allocations from period to period The trend revealed in Exhibits 2.5 and 2.6 is consistent for all the portfolios in the other rolling periods investigated (see appendix A for the

[12] In-sample looks at the performance that is based on results of the rolling portfolios from historical data. This provides suggestions (allocations) that can be used ex ante.

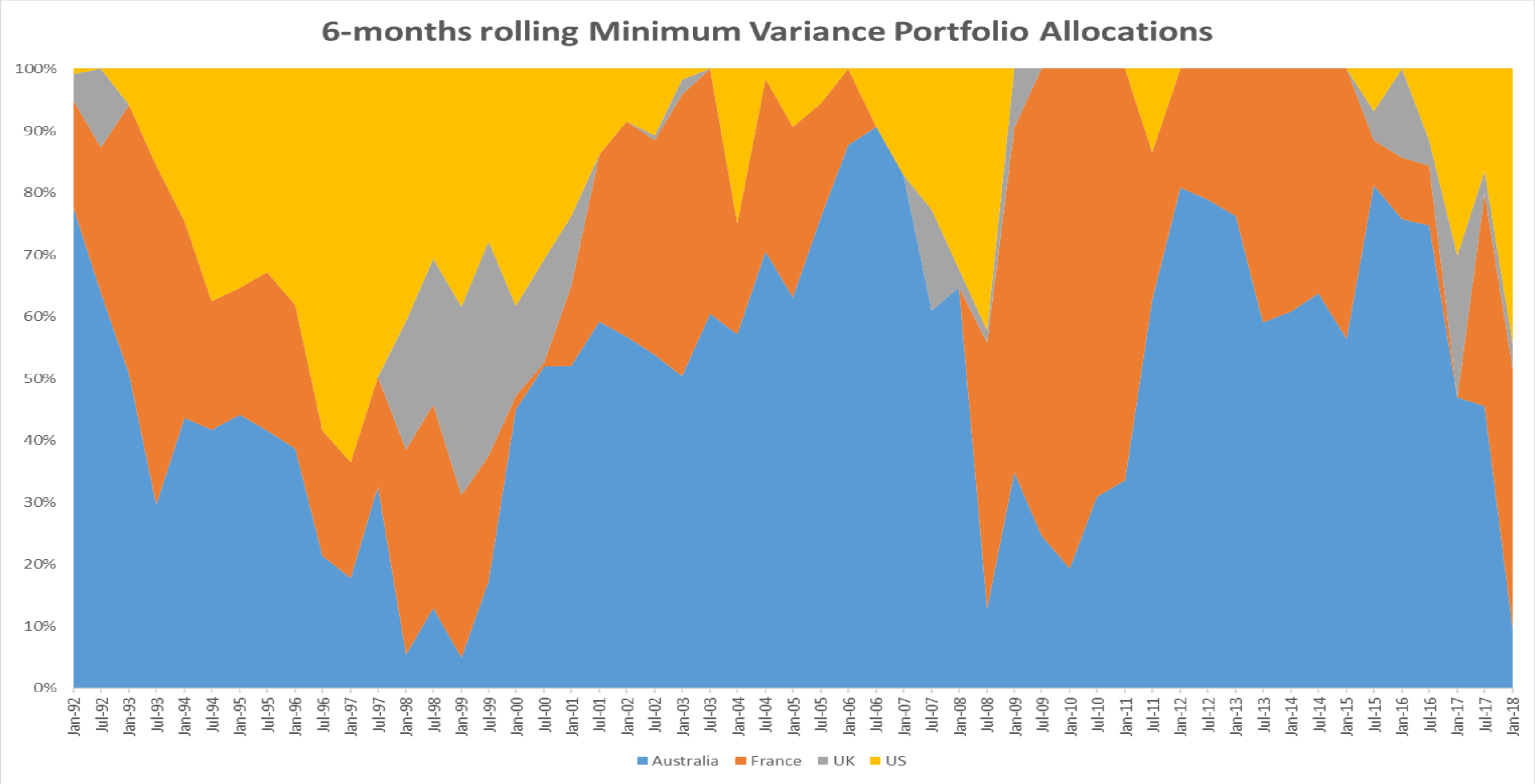
[13] Due to space the allocations and risk contributions for the other rolling periods are shown in appendix A.

allocations of the other the 1-month rolling periods. To save space other rolling periods have not been included but can be made available on request). The question is what in-sample and out of sample performance do these allocations and risk contributions result in? The remainder of the study focuses on this performance.



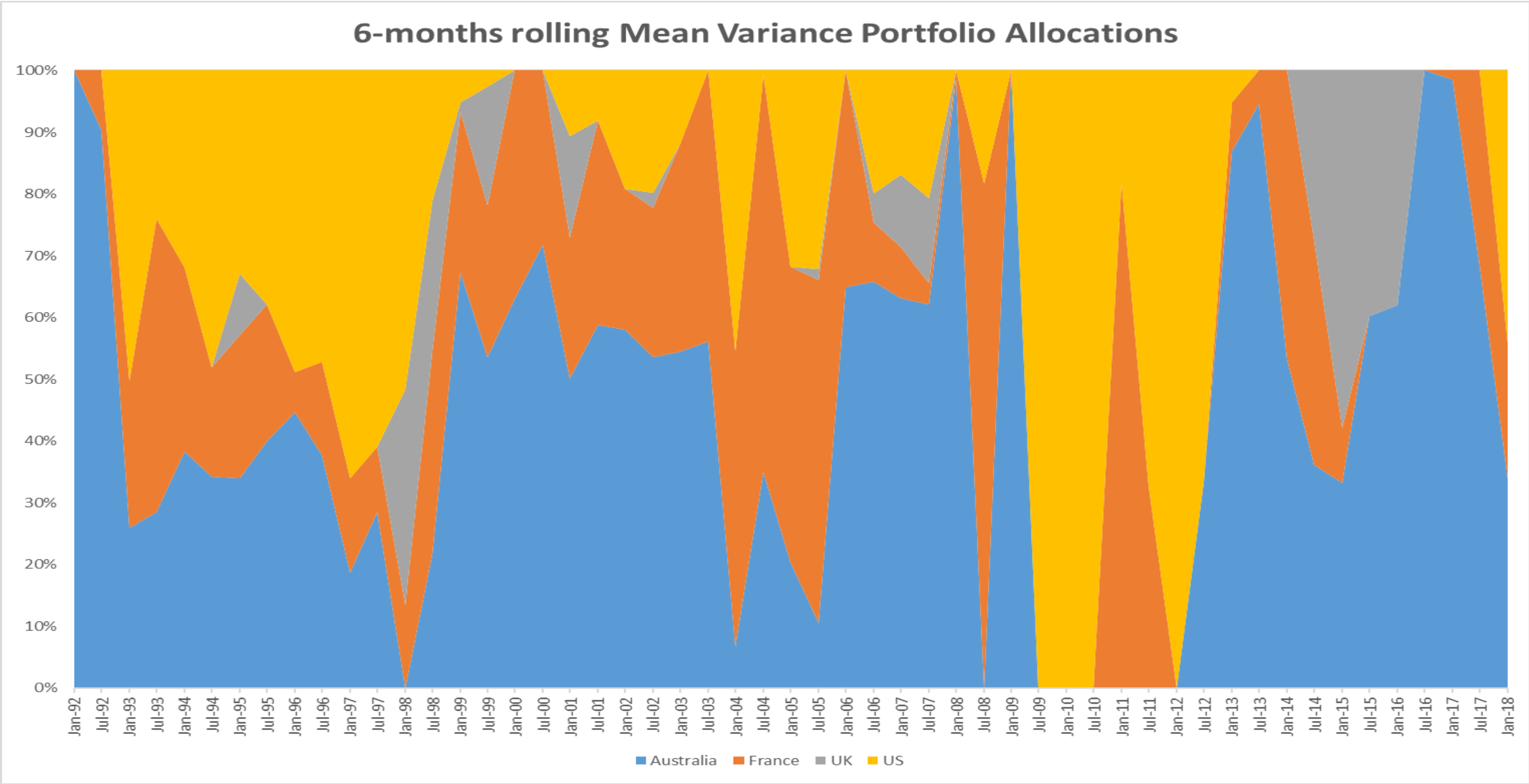

(a)

This chart (a) shows the allocations for the equally-weighted portfolio. As the name suggests, equal weights are assigned to all the markets in the portfolio in all the periods.



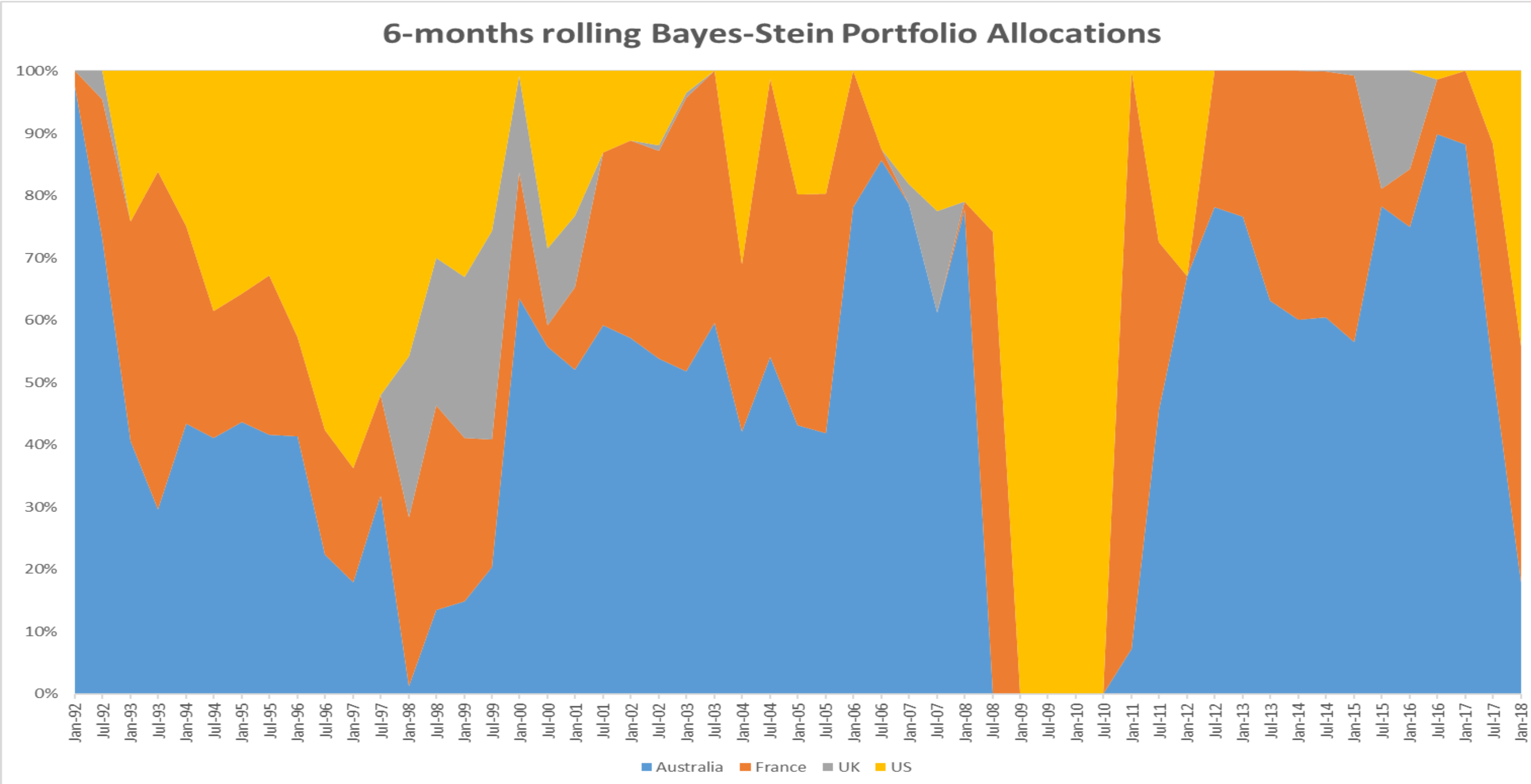

(b)

Chart (b) presents the allocations for the minimum variance portfolio. The allocations result in corner solutions in some instance. Of the allocations to the US market, 100% is allocated to this market 8 per cent of the time. While most of the allocation is assigned to Australia, there is no allocation assigned to France, the UK and the US in 9 per cent, 58 per cent and 30 per cent of the out of the 53 rolling periods.



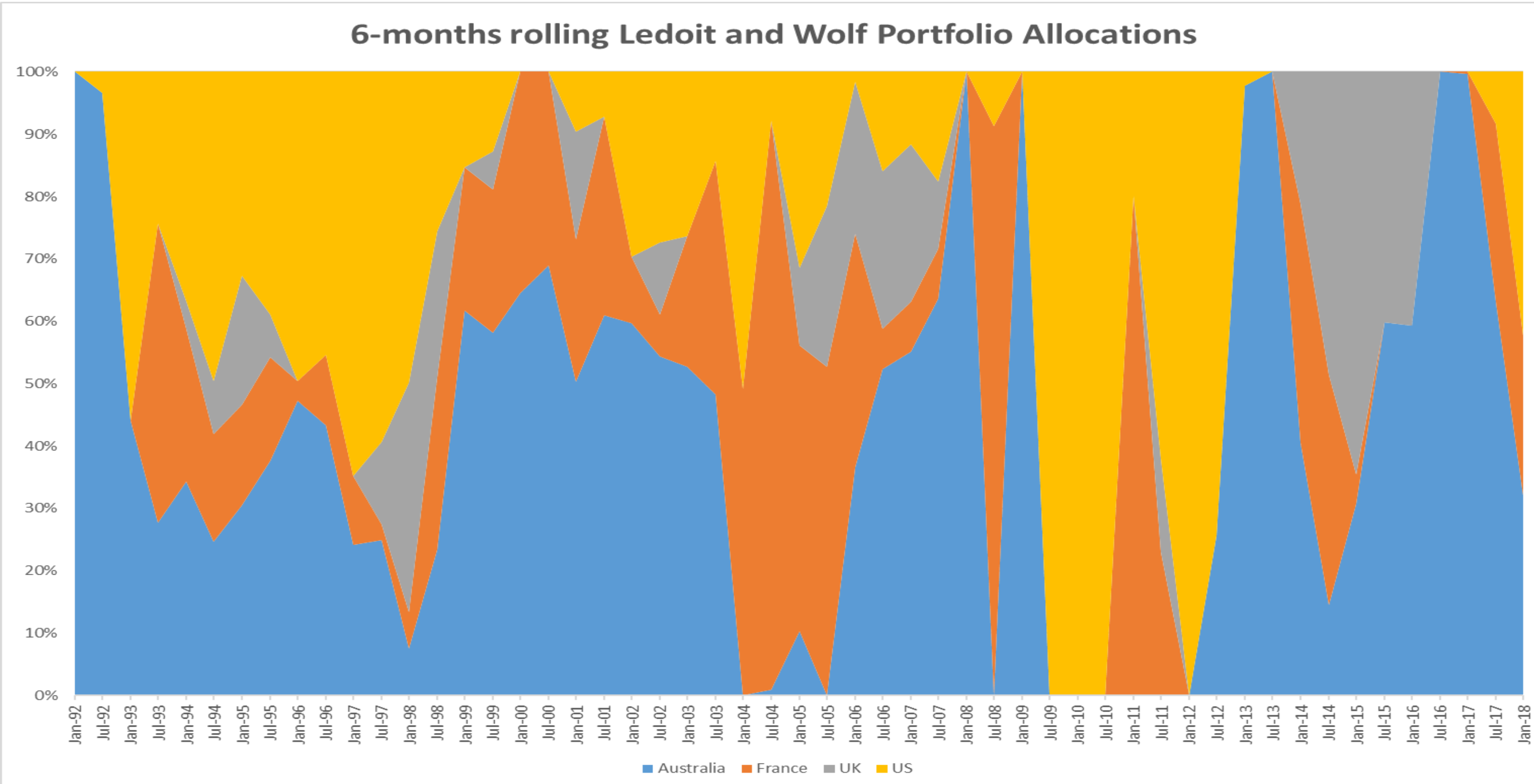

(c)

Chat (c) shows the mean-variance portfolio. Because this portfolio aims to maximise the Sharpe ratio, most of the weights are allocated to high return assets in the portfolio. This frequently results in corner solutions, e.g., in 2 instances 100% is allocated to Australia, and in 4 instances 100% is allocated to the US. In 15%, 19% and 32% of the instances no weights are allocated to Australia, France and the US respectively. While for the UK, in 72% of the time, there is no allocation assigned to this market.



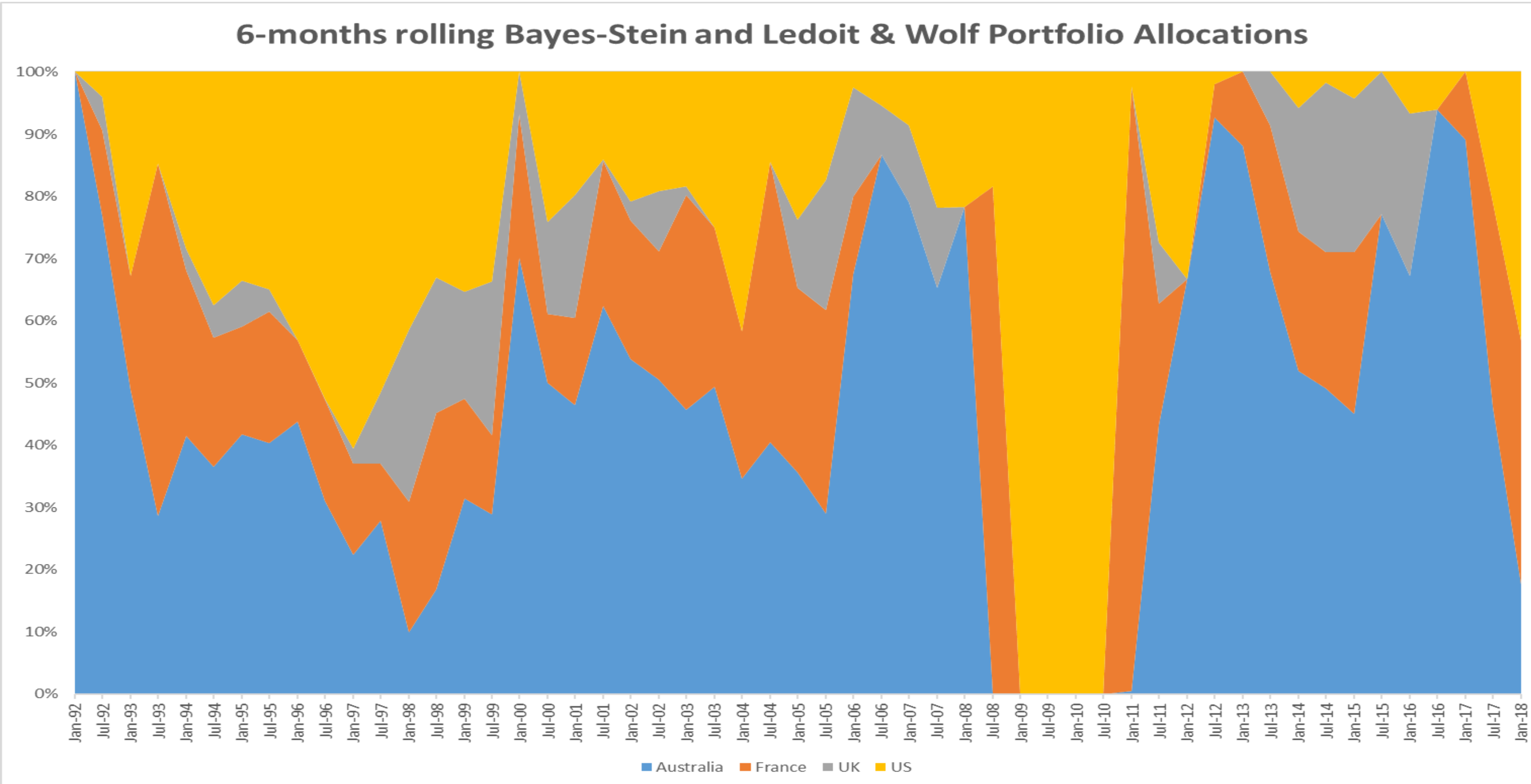

(d)

Chart (d) represents the allocation for the Bayes-Stein (return vector shrinkage) portfolio. Like the minimum-variance and mean-variance portfolios, this results in the allocation of most of the weights in a few of the assets in the portfolio. For example, in 4 instances 100% of the weights are assigned to the US. Australia, France, the UK and the US get no allocations in about 9%, 13%, 70% and 26% of the instances.



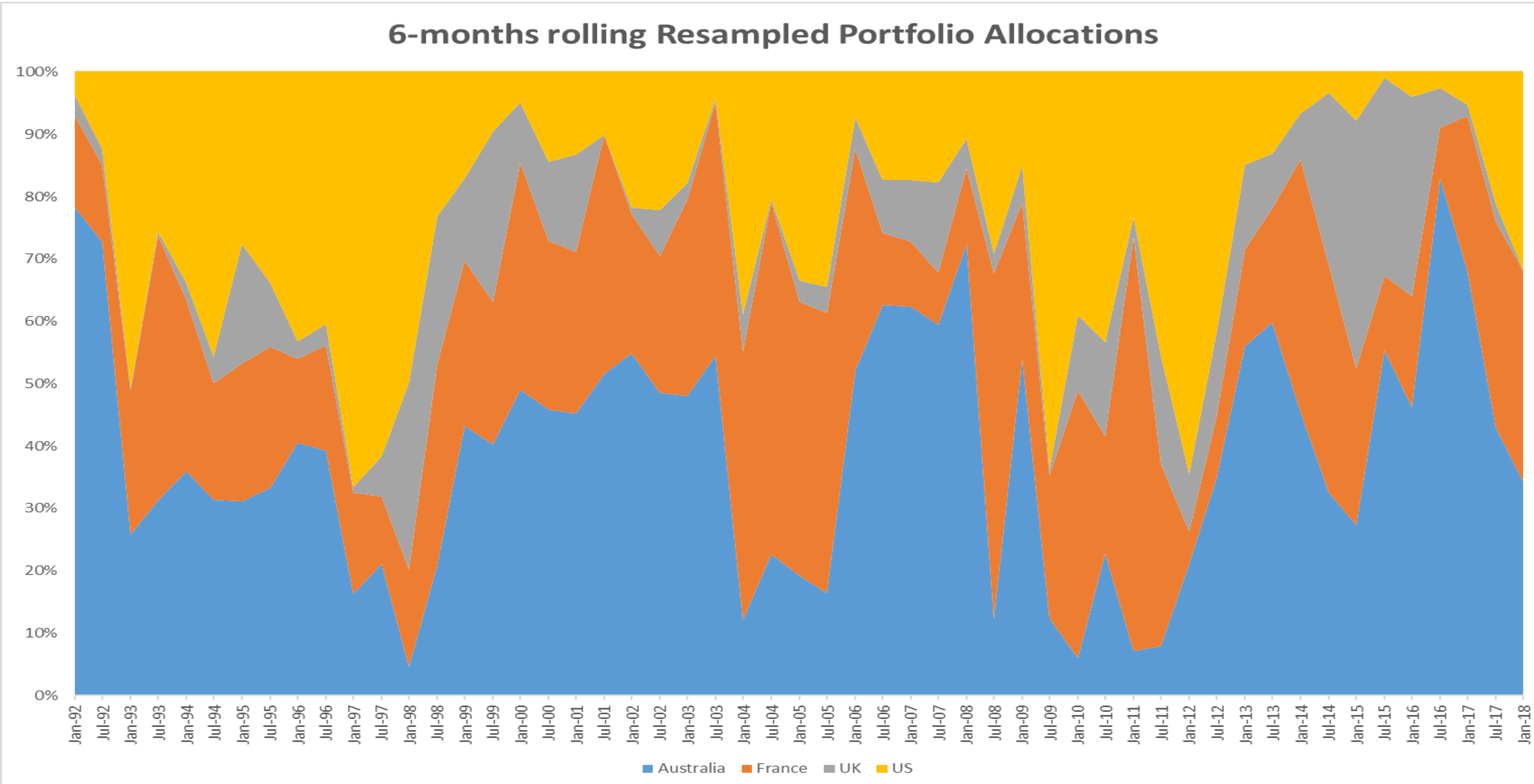

(e)

Chart (e) presents the allocation for the Ledoit and Wolf (covariance shrinkage) portfolio. This portfolio has more instances where 100% is assigned to one asset, i.e. 5 and 4 instances where 100% is assigned to the Australian and US markets respectively. Concerning the proportion of time when no allocation was assigned to an asset this represents 17%, 28%, 58% and 25% for Australia, France, the UK and the US respectively.



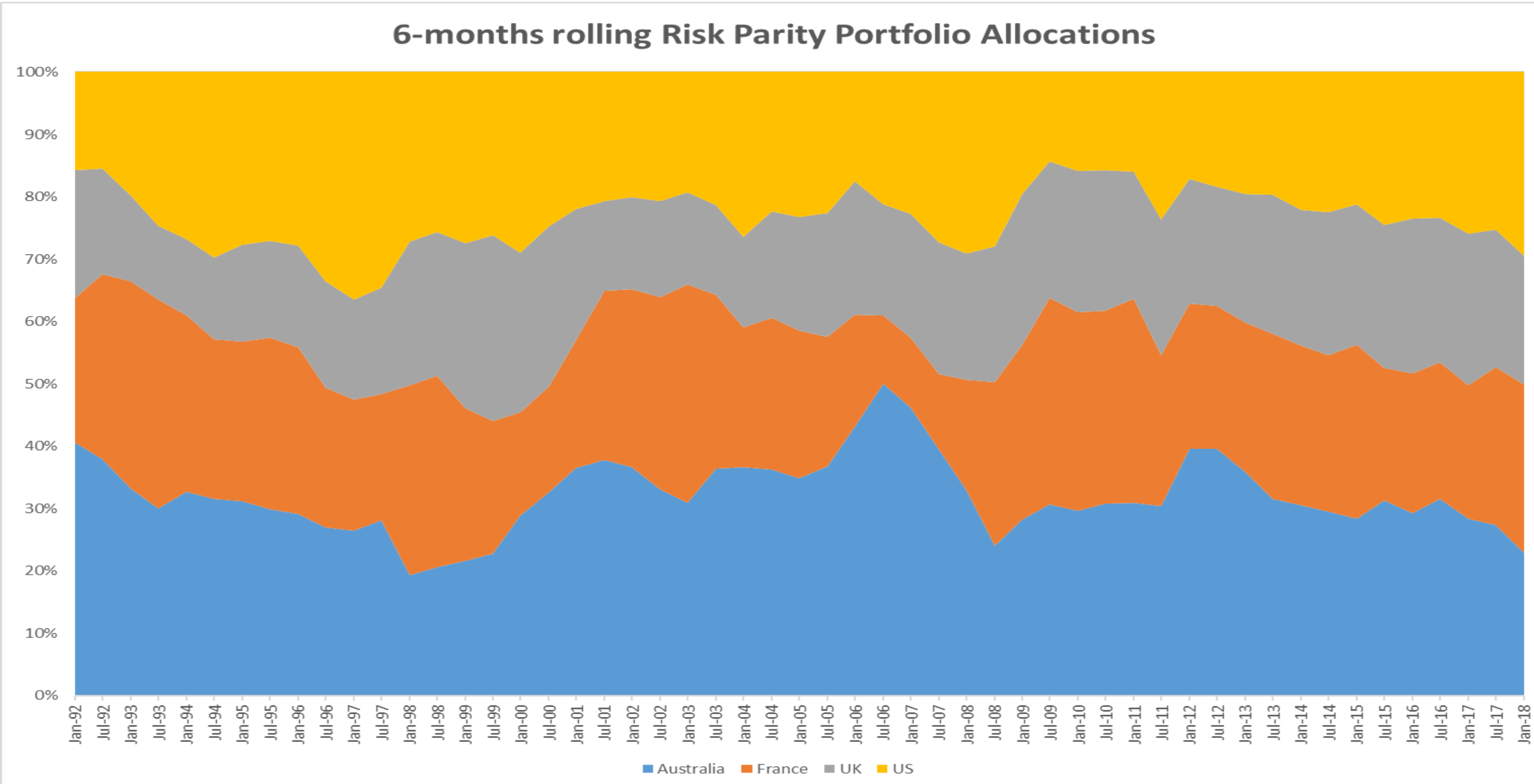

(f)

Chart (f) shows the allocation for the Bayes-Stein and Ledoit & Wolf portfolio. Even if the portfolio also presents some corner solution like the above portfolios, the proportion of those assets with no allocations is less i.e., 9%, 25%, 40% and 11% respectively.




(g)

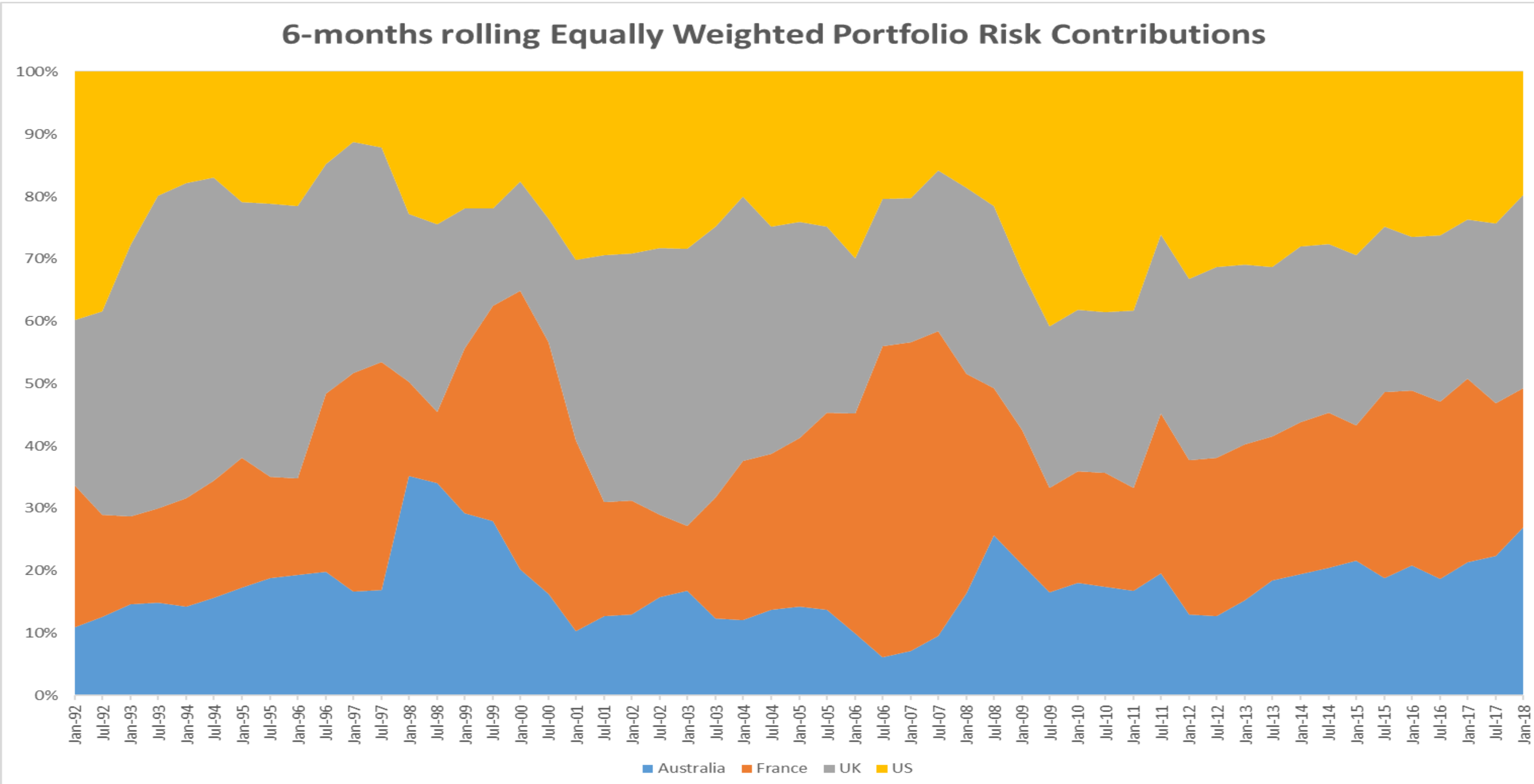
Chart (g) shows the Resampling portfolio allocations. Unlike most of the portfolios, all the assets in the portfolios have weights assigned to them. The weights range from 0.09% to about 83%. This is because the weights are an average of the weights one hundred portfolios based on returns resampled one hundred times.




(h)

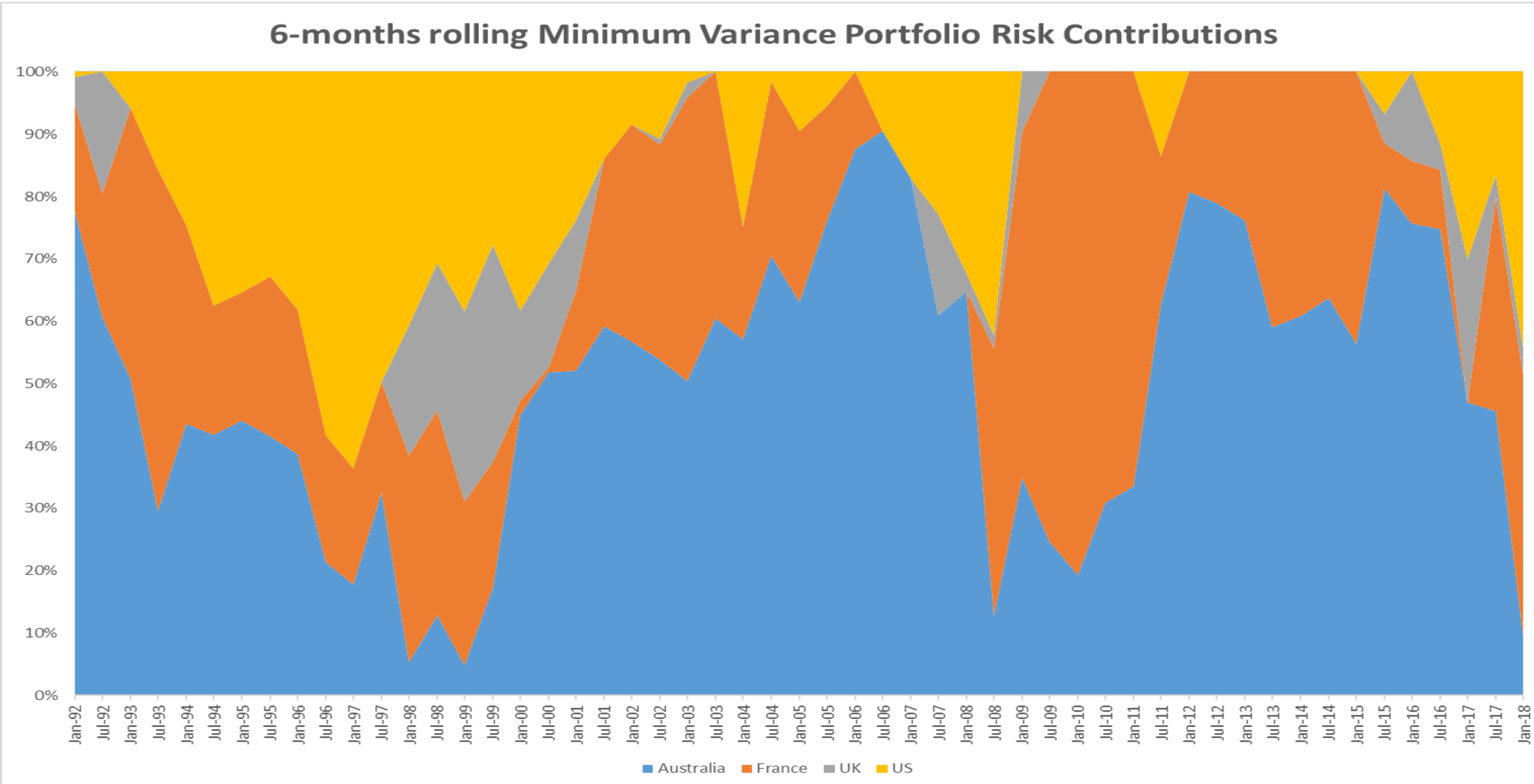
Chart (h) presents the risk parity portfolio allocations. Like the equally-weighted and resampling portfolios, all the assets in the portfolio have assigned weights, and the portfolio presents no corner solutions with weights range from about 11% to 50%. Unlike the other portfolio there rebalanced weights for each of the periods are not significant.

Exhibit 2.5: Charts for 6-months rolling portfolio allocations



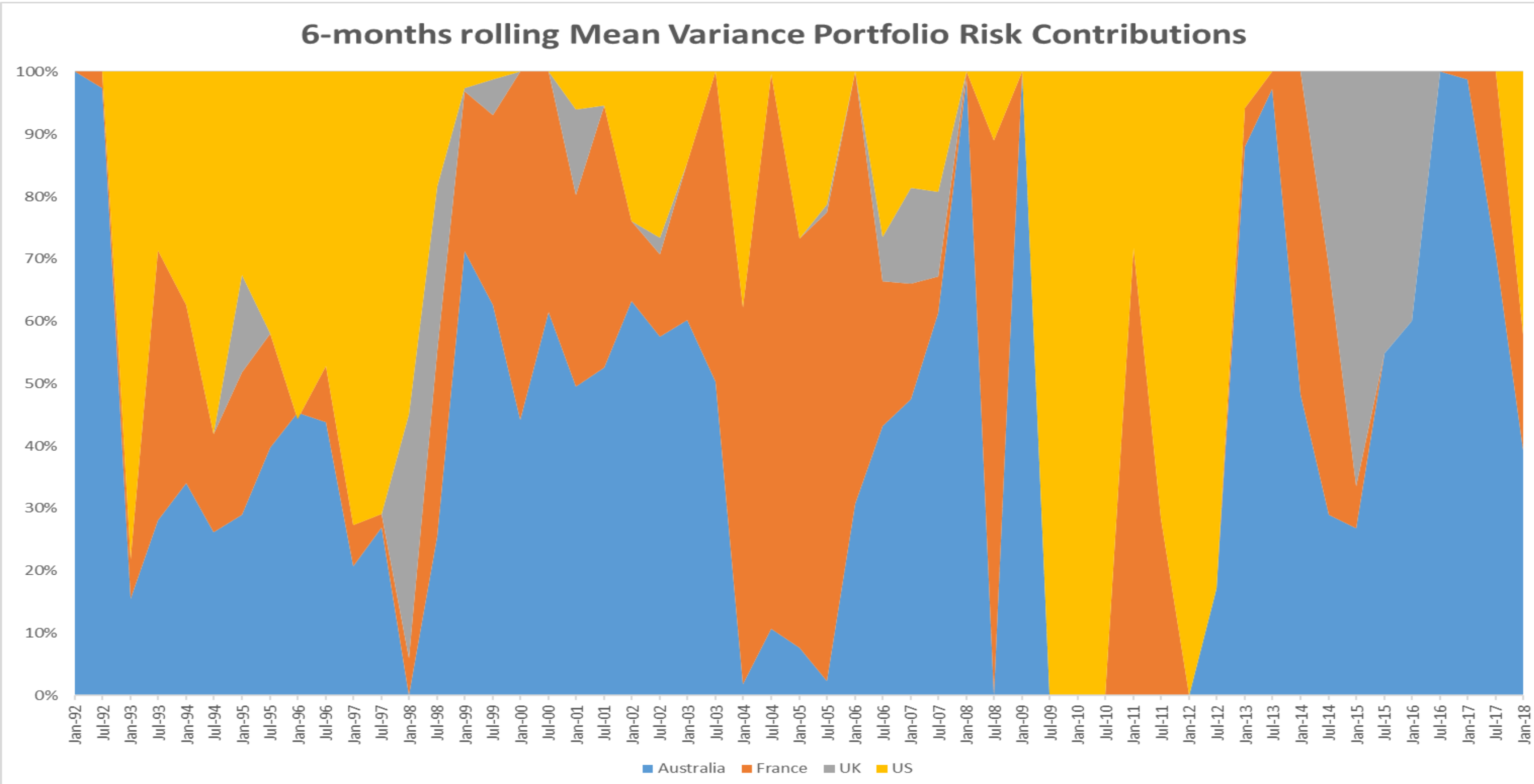

(i)

Chart (i) displays the equally-weighted risk contributions and shows that despite equal weights allocated to the assets in this portfolio, the risk contributions are not equal as they range from about 6% to 50%. On average the UK contributes the most risk to the portfolio whereas Australia has the least contribution.



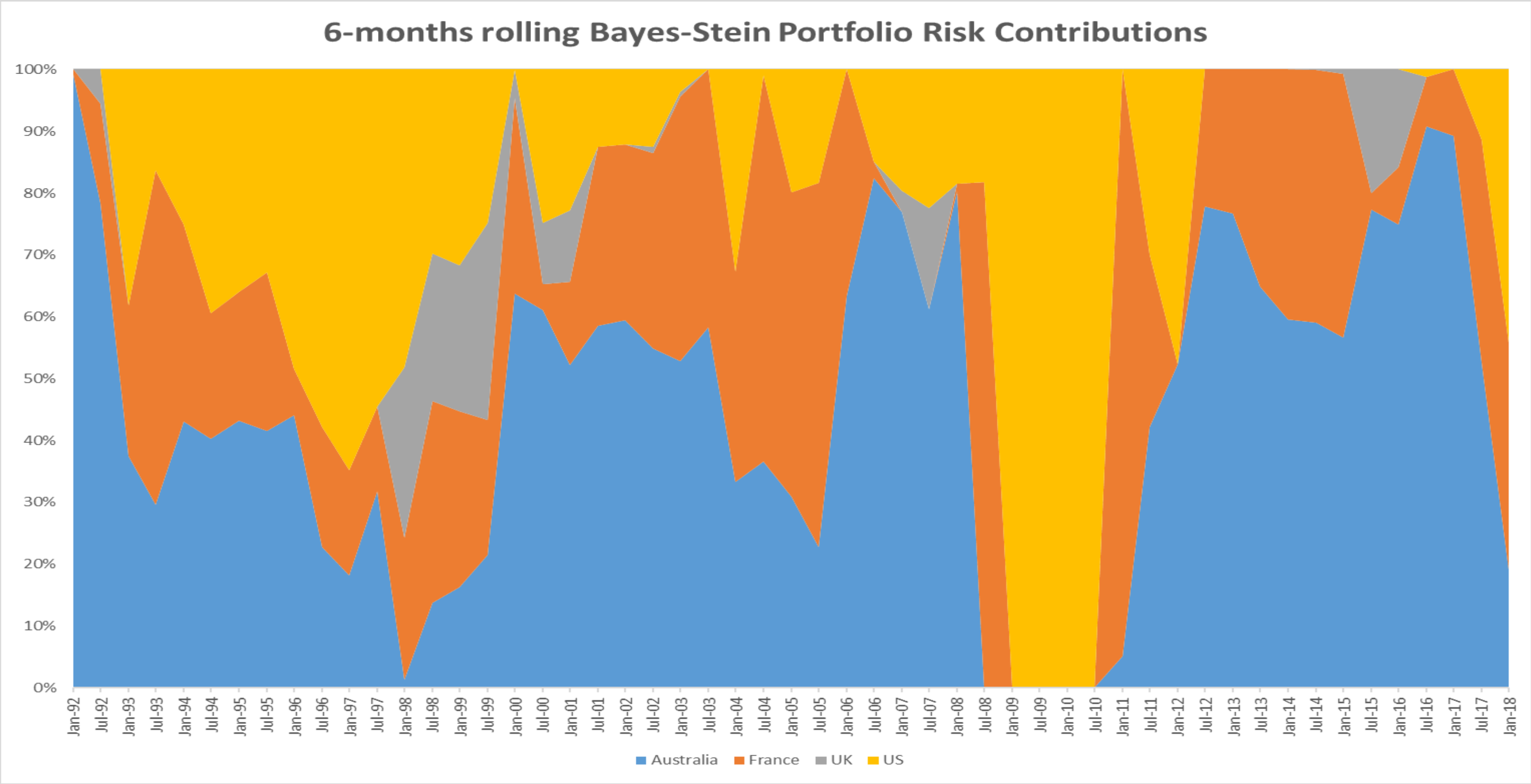

(ii)

Chart (ii) shows that the for minimum variance portfolio ranges from 0% to 91%. Only the assets with weight allocations contribute to the risk of the portfolio, and in some instances, there is risk concentration where most of the risk contribution comes from two or three of the four assets in the portfolio. On average Australia and the UK have the most and least risk contributions respectively.



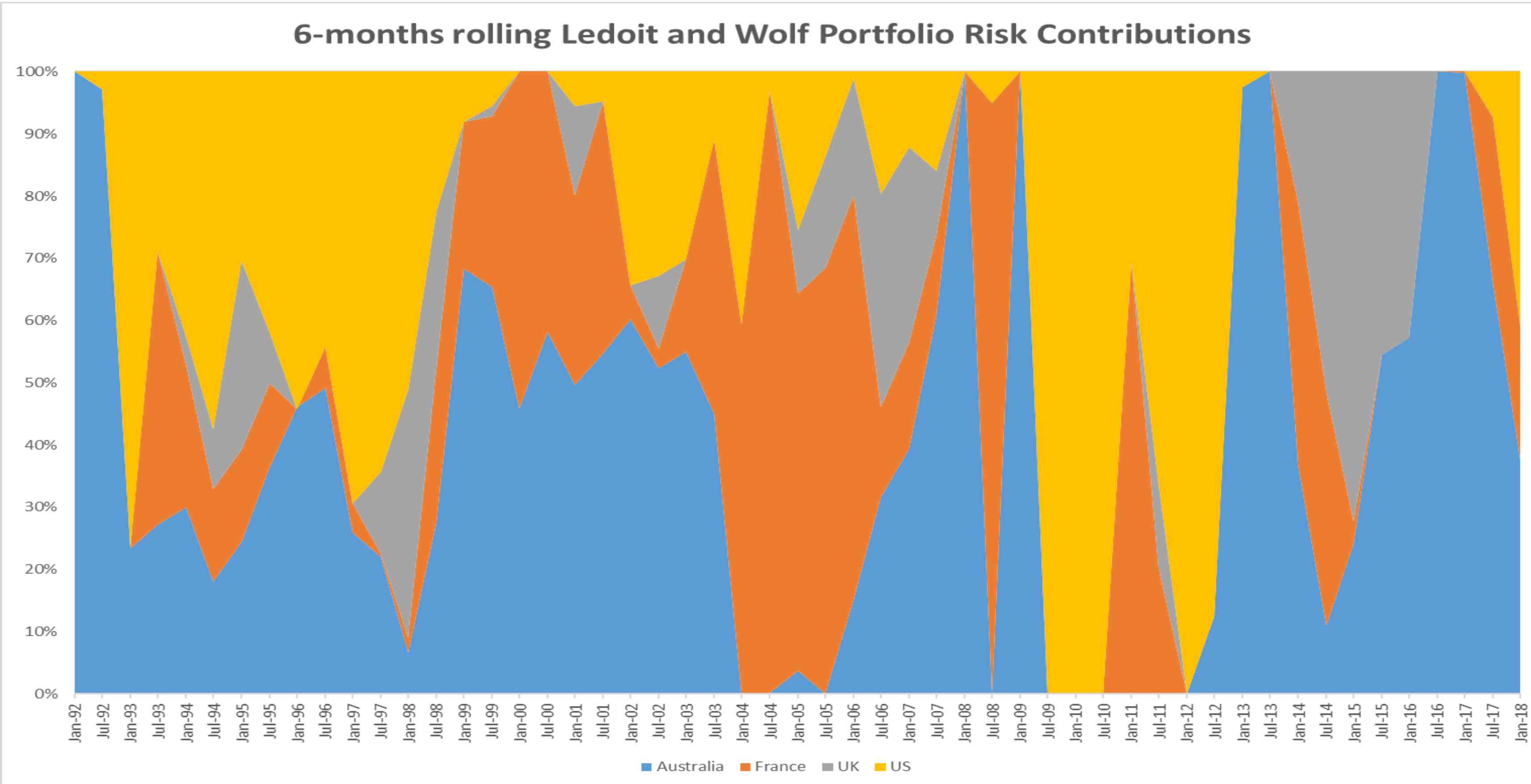

(iii)

Chart (iii) presents the mean-variance risk contributions. Due to corner solutions in the allocations, this portfolio presents some instances where there is risk concentration such that 100% of the risk contribution emanates from one asset in the portfolio namely either Australia or the US market. Similar to the minimum variance portfolio, only assets with weight allocations contribute to the risk of the portfolio.



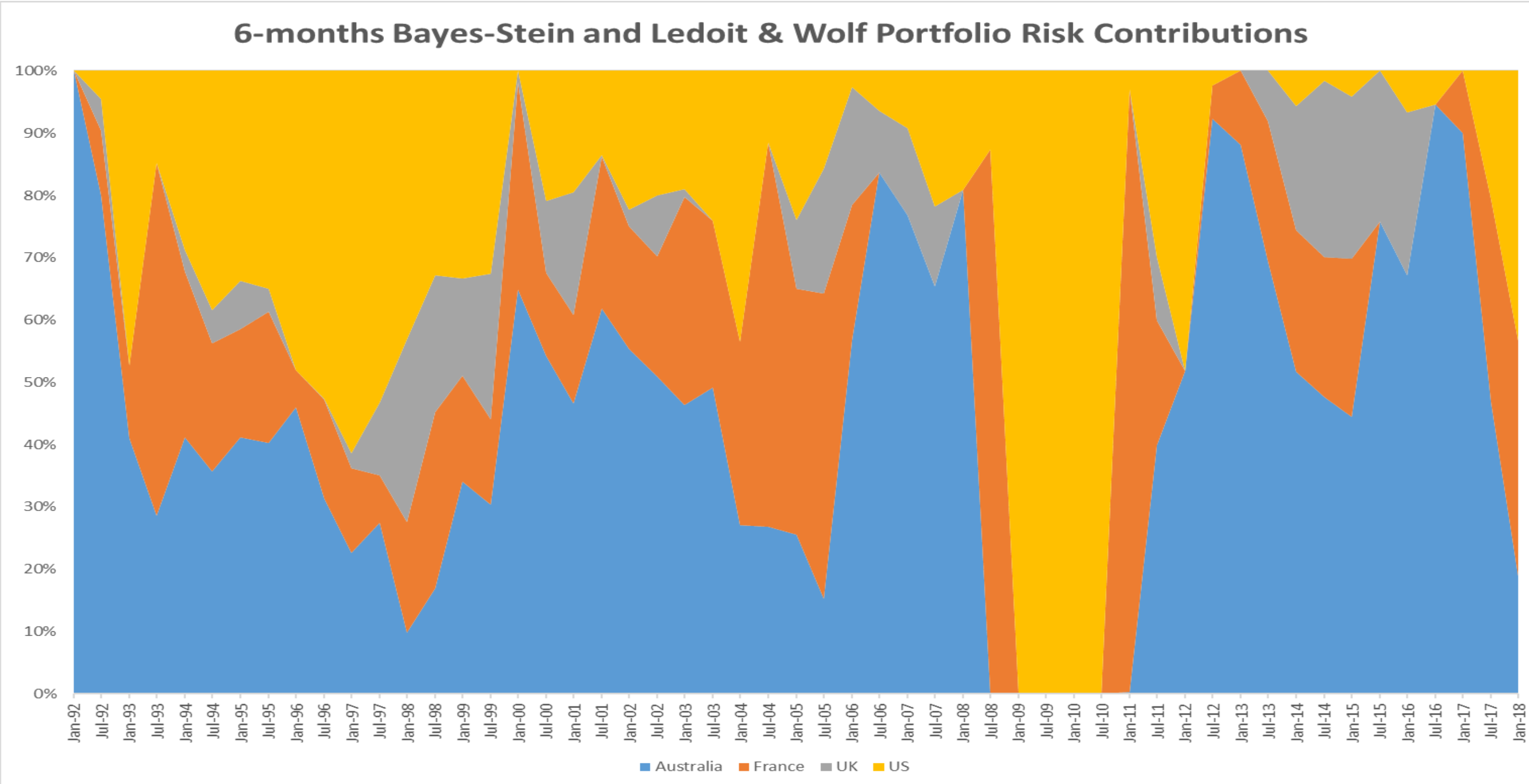

(iv)

Chart (iv) shows the risk contributions for the Bayes-Stein portfolio. For this portfolio also, only the assets with allocations assigned to them contribute to the risk. On average, Australia and the UK account for the most and least risk contributions while France and the US have similar average contributions.



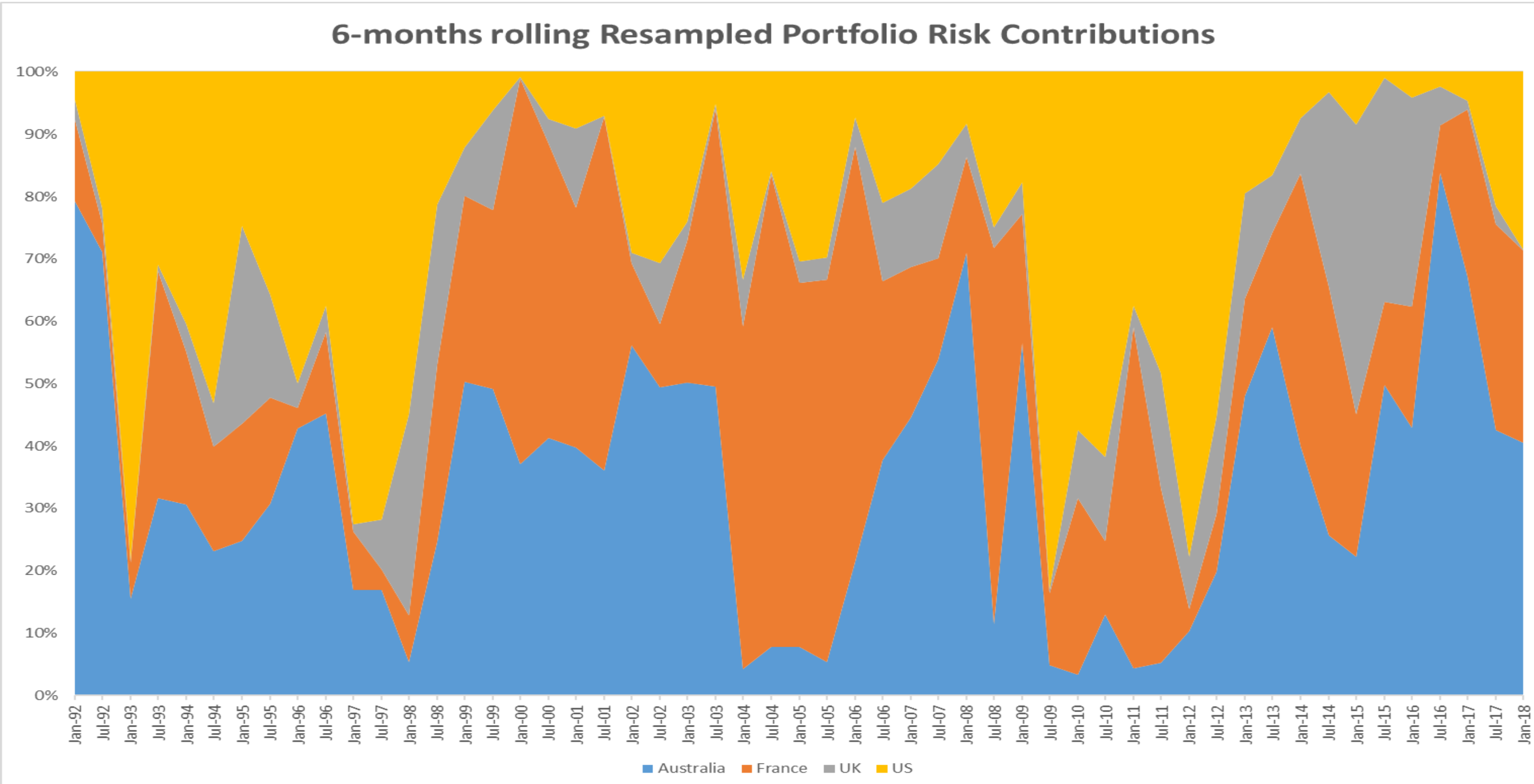

(v)

Chart (v) presents the Ledoit and Wolf portfolio risk contributions. Similar to the mean-variance portfolio, this portfolio also has risk concentration is only one asset as 100% has been assigned to either the Australian or the US markets. On average the distribution of the risk contribution is similar to minimum-variance, mean-variance and the Bayes-Stein portfolios.



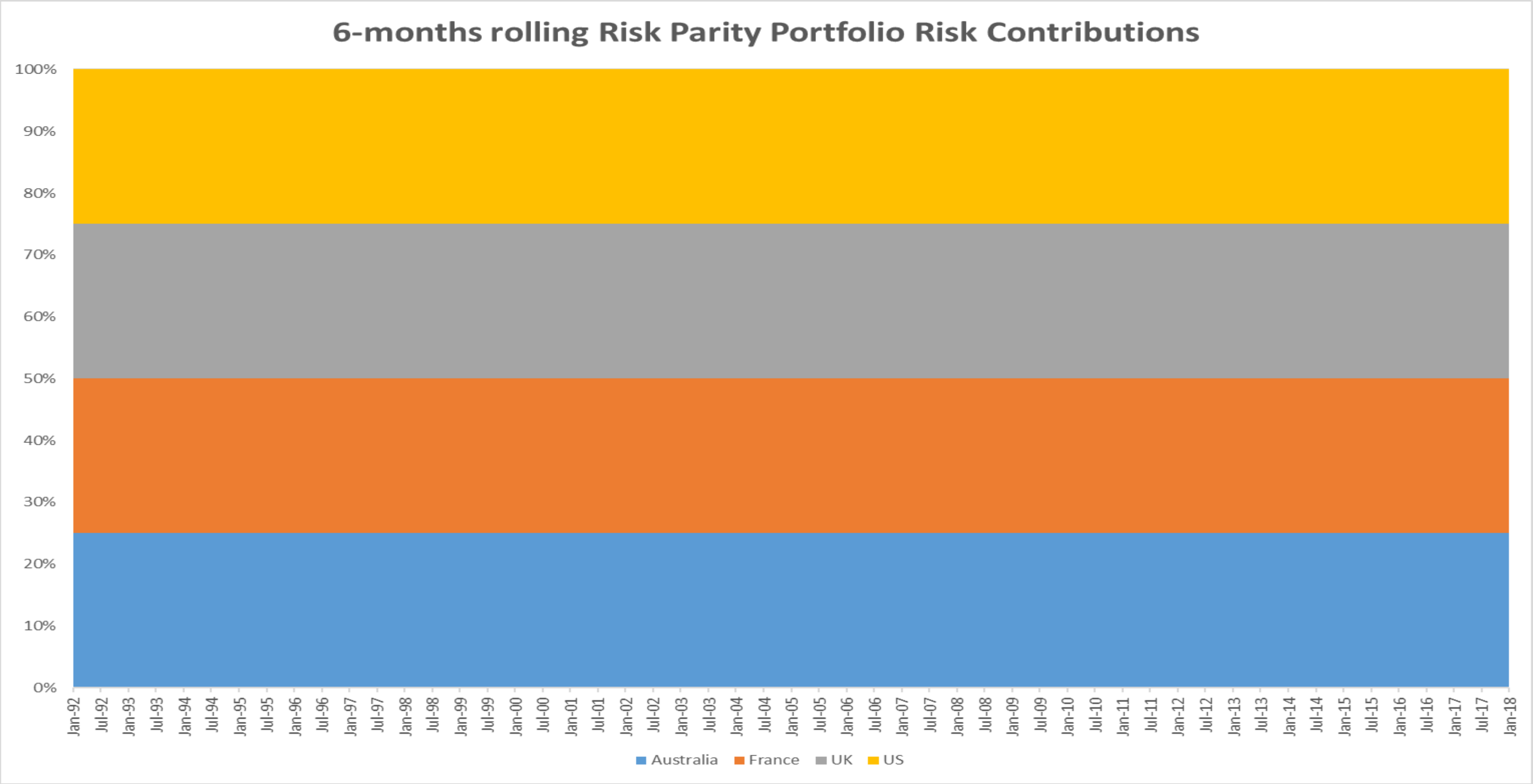

(vi)

Chart (vi) displays the risk contributions for the Bayes-Stein and Ledoit & Wolf portfolio. While this portfolio also has some assets contributing 100% to the portfolio risk, the instances of one asset contributing no risk are less compared to the Bayes-Stein, minimum variance, mean-variance, and Ledoit andWolf portfolios.




(vii)

Chart (vii) shows the resampled portfolio risk contributions. Like the equally-weighted and risk parity portfolios, all the assets have weights allocated to them. However, some of the allocations for this portfolio are very small, in the same token, the risk contributions in some instances are minimal starting from as miniature as 0.8%. These small risk contributions mainly come from the UK and the US markets.



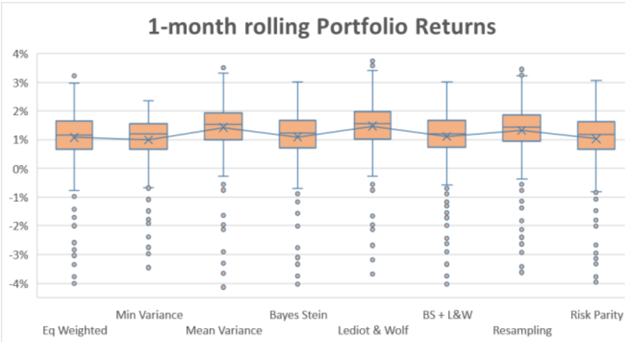

(viii)

Chart (viii) displays the risk contribution for the risk parity portfolio. By definition, this portfolio assigns weights such that each asset contributes equally to the portfolio this. For this reason, the risk parity portfolio has equal risk contributions.

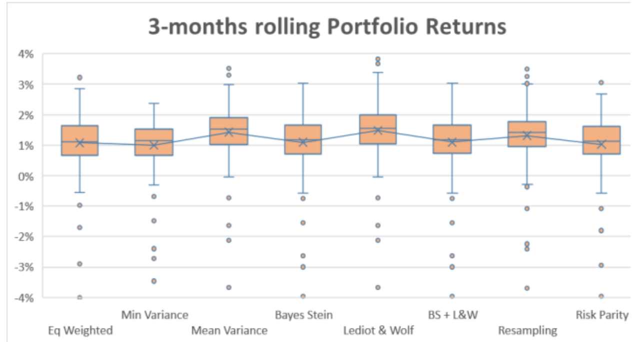
Exhibit 2.6: Charts for the 6-months rolling portfolio risk contributions

### 3.3.1. In-sample and out-of-sample performance

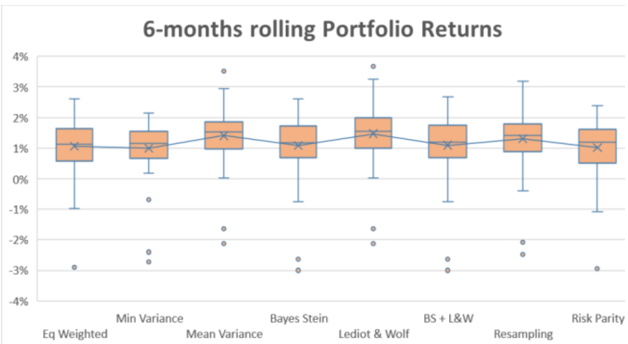
The in-sample performance is based on the estimated portfolio parameters using the past 24 months. So for the 6 months rolling period the historical 24-months' returns are used to determine the ex-ante allocations. These allocations when applied to determine the parameters based on this first 6-months holding period form the in-sample portfolio return, risk and risk contribution. Consistent with Thomson (1997), these allocations (ex-ante) so determined are then applied to the actual performance achieved after the second 6 months holding period (or 1 month, 3 months, 12 months and 24 months depending on what the rolling/holding period is) in order to back-test the performance. The purpose of this is to compare whether this performance would have been better than the actual (ex-post) had those allocations been applied at the beginning of that rolling period (the second period). The out-of-sample performance is, therefore, that which is achieved by applying the ex-ante allocations to the ex-post actual returns. This is done for all the periods within the various rebalancing durations (i.e., 1 month, 3 months, 12 months and 24 months). The problem with the out-of-sample analysis is that weights are assumed to be constant in between the rebalancing periods, but in reality, these will change to reflect changes in prices over the period. So in practice one is not rebalancing from the position that the portfolio was at the start of the period, instead the rebalancing is done from where the portfolio has changed to. The downside with this, especially in short rebalancing periods, is that there is a danger of selling winners and ending up buying losers. This goes against the principles of momentum as the winners are the assets that should be held on to so that they continue to outperform. Exhibits 2.7 through to 2.9 display the in-sample and out-of-sample performance distributions for all the rolling periods.



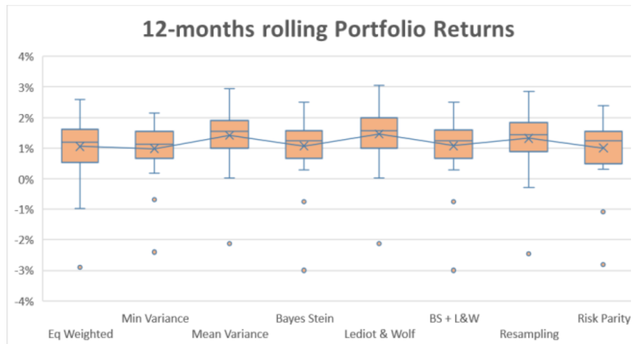

(a)



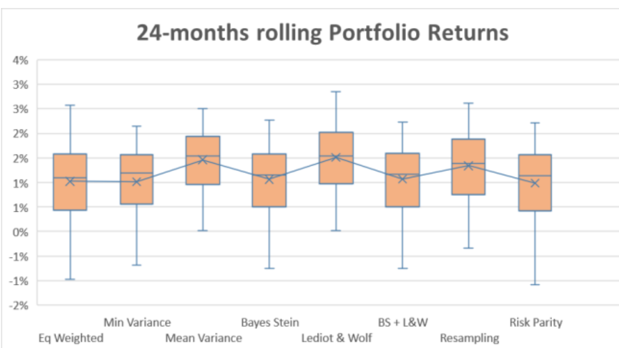

(b)




(c)



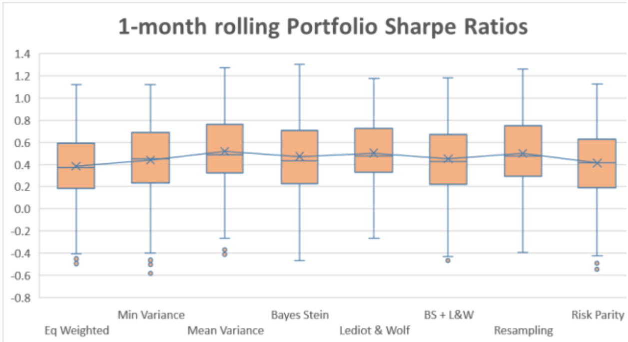

(d)



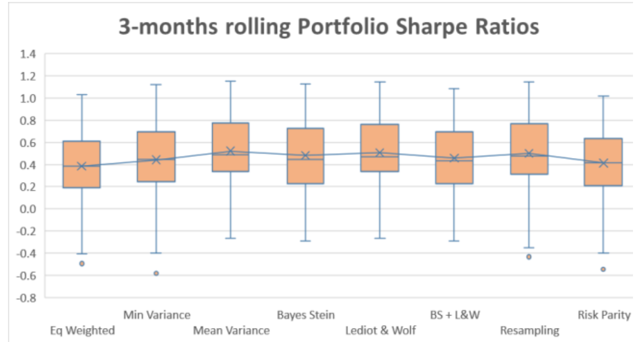

(e)

Exhibit 2.7: Distribution of in-sample returns in different rolling periods

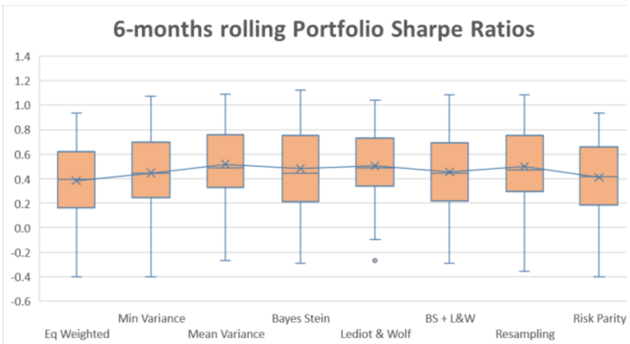
This exhibit shows the in-sample performance distribution for the different portfolios in all the rolling periods under investigation. The returns for all the rolling periods seem to be skewed to the left with the 1-month, 3-months, 6-months and the 12-months rolling periods having similar mean and median returns while the those for the 24-months rolling periods for the mean-variance, Ledoit and Wolf, and the Resampling portfolios are higher.



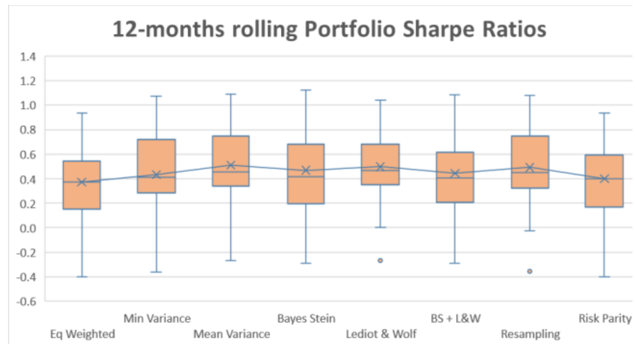

(a)



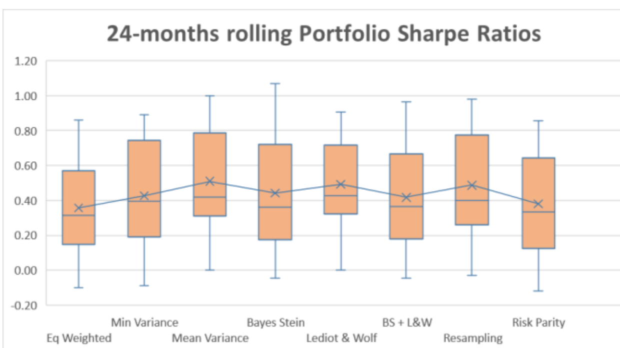

(b)




(c)




(d)




(e)

Exhibit 2.8 Distribution of in-sample Sharpe ratios in different rolling periods

This exhibit presents the distribution for the in-sample Sharpe ratios for all the portfolios in the different rolling periods. With the exception of the 24-months rolling period, the pattern of the distributions of the Sharpe ratios is similar with the equally-weighted, minimum variance and risk parity portfolios having a slight left skew while the other portfolios are skewed to the right. The portfolios in the 24-months rolling period on the hand all have positive skews. Generally, the median values for the Sharpe ratios have dropped in the 24-months rolling period.

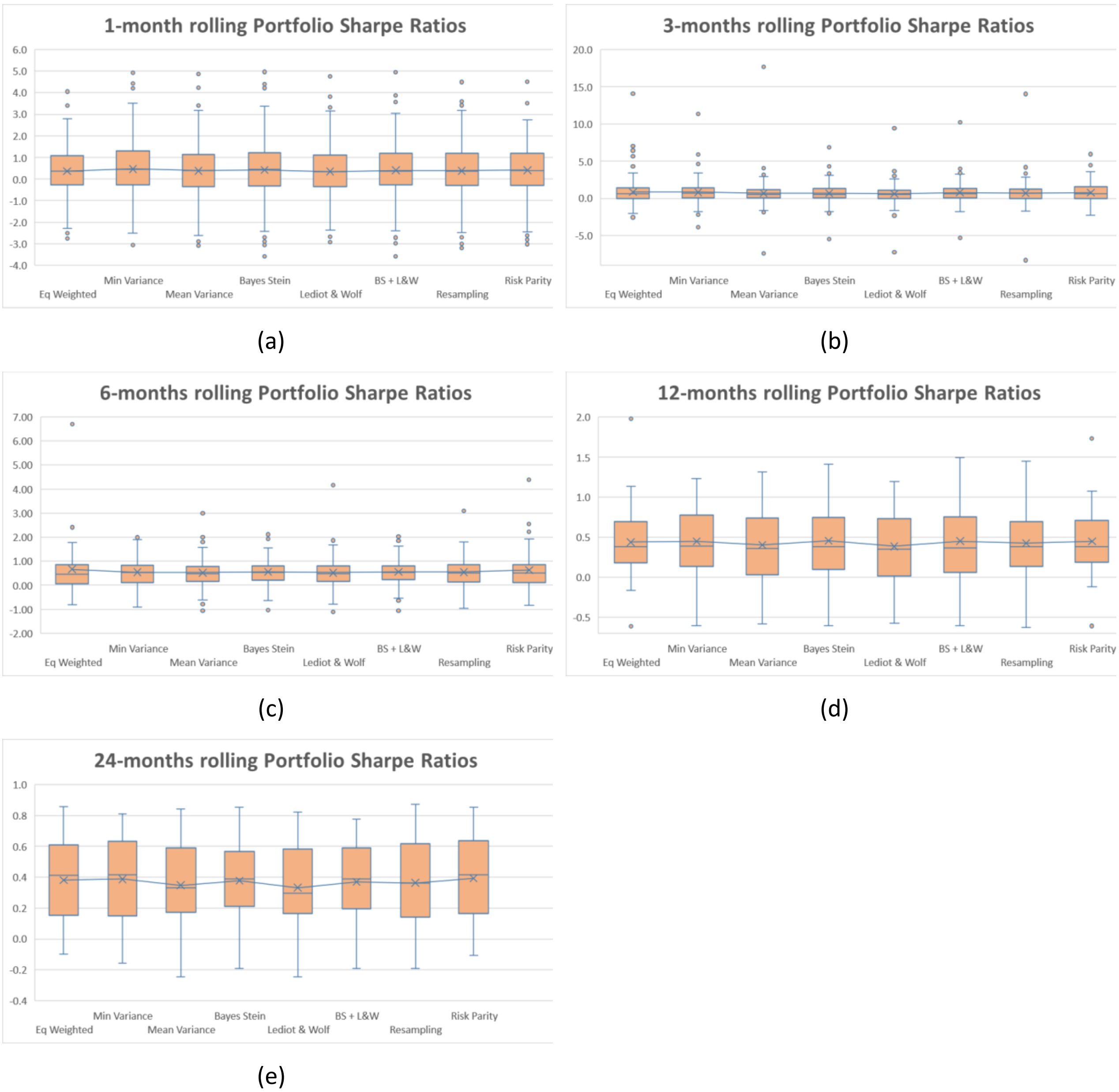



Exhibit 2.9: Distribution of out-of-sample Sharpe Ratios in different rolling periods

Exhibit 2.9 shows the out-of-sample distribution of Sharpe ratios for the different rolling periods of the portfolios under investigations. Overall, there is no marked difference in the distributions amongst the assets in 1-month, 3-months and 6 months rolling periods. However, all the assets in the 12-months rolling period have positive skews while only the mean-variance, Ledoit and Wolf, the resampling portfolio have positive skews in the 24-months rolling periods and the rest have negative skews.

The in-sample performance reveals that the Ledoit and Wolf; mean-variance optimisation and resampling portfolio are consistently the top three achievers in terms of the average returns in all the rolling periods. On the other hand, overall, the risk parity, equally weighted and the minimum variance portfolios are consistently in the 6$^{th}$, 7$^{th}$ and 8$^{th}$ positions respectively in all the rolling periods. This is slightly at variance with the theory which states that the risk parity is expected to fall in between the equally-weighted and minimum-variance portfolio (Maillard, Roncalli and Teiletche, 2010, Roncalli, 2014). However, there is consistency with the theory with regards to the in-sample Sharpe ratios which 4 out 5 times shows the risk parity portfolio lie in between the equally-weighted and minimum-variance portfolio. Again the first three positions in terms of the mean Sharpe ratio are taken by the mean-variance, Ledoit and Wolf and the resampling portfolios; however, surprisingly the Bayes-Steins portfolio reveals the highest positive skew. The performance of the Bayes-Stein and Ledoit & Wolf combo seems to deteriorate the long the rebalancing period.

The Sharpe ratio out-of-sample performance is varied. The performance of the risk parity portfolio in terms of the Sharpe ratio mean value is consistently good compared to its in-sample performance. The risk parity Sharpe ratio mean value is the highest, second and third for the 24-months, 6-months, and 1-month rolling periods respectively while it is fourth place in the 3-months and 12-months rolling periods. The risk parity performance is more stable and is consistent with the literature as observed by Maillard, Roncalli and Teiletche (2010), Roncalli (2014) and Qian (2005, 2011). The equally-weighted portfolio performance reveals surprisingly good performance as it is in the first position for both the 3-months and 6-months rolling periods and third position in the 24-months rolling period. This performance is consistent with DeMiguel, Garlappi, and Uppal (2009) and Cheng and Liang (2000) who argue that the equally-weighted portfolio is not necessarily inferior to the mean-variance and minimum variance portfolios as they are not consistently better out-of-sample. In this research, the equally-weighted portfolio and the risk parity portfolios have consistently outperformed the mean-variance out-of-sample for all the rolling periods. The minimum-variance portfolio's out of sample performance is also very good as it is the highest on two occasions, second also on two occasion and third and sixth on the other occasions. This is because the minimum-variance allocates most of its weight to low volatility assets which outperform the market in some instances as observed by Clarke, de Silva and Thorley (2011) and Ang et al. (2006). Of the regularised portfolio the Bayes-Steins and the Bayes-Steins and Ledoit and Wolf portfolios have performed better out-of-sample followed by the resampling portfolio and with the Ledoit and Wolf portfolio giving the worst performance of all the portfolios under investigation.

Because the purpose of this study is to test the performance of risk parity in public real estate compared to other allocation strategies, the study now evaluates how many times each allocation method has outperformed the others both in-sample and out-of-sample and the results are displayed in Exhibit 2.10[14]. Again only the portfolios for the 6-months rolling period have been shown (see Appendix B for the 1-month rolling period. To save space other rolling periods have not been included but can be made available on request). Even if the risk parity has not done well in terms of average in-sample returns Exhibit 2.10 (a) shows that has outperformed in quite a lot of instances. However, the in-sample Sharpe ratio performance (Exhibit 2.10(b)), results in more outperformance of the risk parity against the equally-weighted portfolio and less that of the minimum variance portfolio. This is consistent with Moss et al (2017) that showed that the minimum-variance outperformed the risk parity portfolio in general. As expected, because the objective of the mean-variance portfolio is to maximise the Sharpe ratio, the risk parity portfolio did not outperform it in-sample. Exhibit 2.10(c) indicates that in comparison to the in-sample performance, the out-of-sample performance for risk parity shows a significant improvement in terms of the number of times it is outperforming other portfolios, with the exception of the equally-weighted portfolio. The other strategies also reveal some outperformance when compared to competitive strategies supporting the earlier findings of DeMiguel, Garlappi and Uppal (2009) of portfolio strategies not being consistently better out of sample. What is evident is that the performance whether in-sample or out-of-sample is different for different rolling periods. The question is whether the performance of the various portfolios is statistically significant. This is explored next through statistical inference by use of hypotheses tests.

| | Risk Parity | Equally Weighted | Minimum Variance | Mean Variance | Bayes Stein (BS) | Ledoit & Wolf (LW) | BS + LW |
|---|---|---|---|---|---|---|---|
| Risk Parity | | | | | | | |
| Equally Weighted | 24 | | | | | | |
| Minimum Variance | 27 | 28 | | | | | |
| Mean Variance | 6 | 10 | 0 | | | | |
| Bayes Stein (BS) | 15 | 20 | 23 | 44 | | | |
| Ledoit & Wolf (LW) | 3 | 7 | 1 | 8 | 7 | | |
| BS + LW | 16 | 18 | 22 | 44 | 11 | 46 | |
| Resampling | 3 | 9 | 4 | 38 | 8 | 49 | 9 |

(a) In sample: Number of times of return outperformance (out of 53)

[14] Note that the portfolios in the columns are compared with those in the rows. For instance the risk parity portfolio in the first column is compared against all the other portfolios and so on.

| | Risk Parity | Equally Weighted | Minimum Variance | Mean Variance | Bayes Stein (BS) | Ledoit & Wolf (LW) | BS + LW |
|---|---|---|---|---|---|---|---|
| Risk Parity | | | | | | | |
| Equally Weighted | 40 | | | | | | |
| Minimum Variance | 17 | 0 | | | | | |
| Mean Variance | 0 | 15 | 0 | | | | |
| Bayes Stein (BS) | 0 | 0 | 0 | 0 | | | |
| Ledoit & Wolf (LW) | 1 | 0 | 21 | 42 | 0 | | |
| BS + LW | 5 | 1 | 11 | 30 | 15 | 0 | |
| Resampling | 1 | 0 | 28 | 49 | 42 | 44 | 0 |

(b) In-sample: Number of times of Sharpe ratio outperformance (out of 53)

| | Risk Parity | Equally Weighted | Minimum Variance | Mean Variance | Bayes Stein (BS) | Ledoit & Wolf (LW) | BS + LW |
|---|---|---|---|---|---|---|---|
| Risk Parity | | | | | | | |
| Equally Weighted | 23 | | | | | | |
| Minimum Variance | 20 | 19 | | | | | |
| Mean Variance | 22 | 23 | 24 | | | | |
| Bayes Stein (BS) | 20 | 19 | 22 | 18 | | | |
| Ledoit & Wolf (LW) | 25 | 24 | 25 | 24 | 25 | | |
| BS + LW | 18 | 17 | 21 | 16 | 19 | 13 | |
| Resampling | 19 | 21 | 23 | 17 | 22 | 17 | 31 |

(c) Out-of-sample: Number of times of returns outperformance (out of 52)

Exhibit 2.10: A comparison of the number of outperformance of portfolios

This exhibit compares the performance of the portfolios in relation to the others. It illustrates the number of times the portfolios in the top row have outperformed the portfolios in the left-most column. For example, the risk parity portfolio in the top is compared to all the other portfolios being examined, i.e. from the equally-weighted to the resampling portfolio. Next, the equally weighted is compared to the remaining portfolios, i.e., minimum-variance to the resampling portfolio and so on.

### 2.5.4. The robustness of the risk parity approach

This section explores how robust the performance of the risk parity approach by undertaking some statistical tests to ascertain the statistical significance of the results.

### Hypothesis testing

The different in-sample and out-of-sample performances have been shown, but this has to be formally compared to see if they are statistically different. In order to do this, two tests have been undertaken both of which test the equality of the Sharpe ratios. Formally stated the hypothesis is

Null Hypothesis, $H_0: Sh_a - Sh_b = 0$

Alternative hypothesis, $H_1: Sh_a - Sh_b \neq 0$

where $Sh$ is the Sharpe ratio.

The first test is the Jobson and Korkie (1981) as corrected by Memmel (2003). The Jobson and Korkie (1981) test has been used several times in real estate studies (Stevenson (2001), Stevenson (2002), and Lee and Stevenson (2005). However, this test was later corrected by Memmel (2003) who found a typographical error that leads to the frequent rejection of the null hypothesis. The corrected test statistic for Jobson and Korkie by Memmel (2003) is, therefore:

$$t = \frac{\sigma_j \mu_i - \sigma_i \mu_j}{\sqrt{\theta}} \qquad (2.30)$$

and

$$\theta = \frac{1}{T}\left[2\sigma_i^2\sigma_j^2 - 2\sigma_i\sigma_j\sigma_{ij} + \frac{1}{2}\mu_i^2\sigma_j^2 + \frac{1}{2}\mu_j^2\sigma_i^2 - \frac{\mu_i\mu_j}{\sigma_i\sigma_j}\sigma_{ij}^2\right]$$

where $\mu_i$ and $\mu_j$ are the mean returns of portfolios $i$ and $j$ respectively; $\sigma_i, \sigma_j$ are their standard deviations and , $\sigma_{ij}$is the covariance between portfolio $i$and $j$. This test assumes normally distributed data (Jobson and Korkie (1981). However, Jobson and Korkie (1981), Jorion (1985) and Stevenson (2002) point out the low power of the Jobson and Korkie test. Ledoit and Wolf (2008) go further and argue that this test is not valid for returns that are not normally distributed or are correlated over time and that it therefore should not be used because these traits are fairly widespread in financial returns. In this light, they proposed another test that also examines the difference of Sharpe ratios but is instead based on a studentised time series bootstrap. It uses resampling from the observed data and they is argue that it is more robust (Ledoit and Wolf, 2008). Formally, they show that the two-sided distribution function of the studentised statistic is approximated via the bootstrap as:

$$L\left(\frac{|\hat{\Delta}-\Delta|}{s(\hat{\Delta})}\right)\approx L\left(\frac{|\hat{\Delta}^{*}-\hat{\Delta}|}{s(\hat{\Delta}^{*})}\right) \tag{2.31}$$

Where $\Delta$ is the true difference between Sharpe ratios, $\hat{\Delta}$ is the estimated difference obtained from the original data, $s(\hat{\Delta})$ is the standard error for $\hat{\Delta}, \hat{\Delta}^{*}$is the estimated difference and $s(\hat{\Delta}^{*})$ is a standard error for $\hat{\Delta}^{*}$ both computed from the boostrap data. $L()$ is the distribution of the random variable $(X)$.

$$s(\hat{\Delta})=\sqrt{\frac{\nabla' f(\hat{v})\hat{\Psi}\nabla f(\hat{v})}{T}}$$

Where $\hat{\Psi}$ is a consistent estimator of $\Psi$, the unknown symmetric positive semi-definite matrix (the covariance matrix). Please see Ledoit and Wolf (2008) for a detailed description of the test. Ledoit and Wolf (2008) admit that the test is complex to implement and offer the corresponding code which can freely be accessed[15].

For this study, this test was implemented by undertaking 5000 iterations to obtain the test statistic. For both the Jobson- Korkie and Ledoit-Wolf tests, the test statistics obtained are examined using the t-test table at 10%, 5%, and 1% significant levels and the results are presented in Exhibits 3.10 and 3.13 . Again, only the results for the 6-months rolling periods are displayed and the results for the the other rolling periods are in Appendix C (To save space other rolling periods have not been included but can be made available on request).

Exhibits 2.11 and 2.12 display the in-sample results of the Jobson-Korkie and Ledoit–Wolf tests respectively for equal Sharpe ratios. For both tests, there is strong evidence of statistical differences in Sharpe ratios where the most significant is at 1% level. The risk parity portfolio does show a significant difference in performance at all the significant levels i.e., 10%, 5% and 1% particularly when the Jobson–Korkie test is used for the rolling periods that are less than 12 months. This implies that $H_o$ should be rejected in most of the instances as there is strong evidence of statistically significant differences in the Sharpe ratios. The exception is for the rolling periods less than 12-months between the risk parity and the equally weighted portfolios. Although the Ledoit-Wolf test suggests that $H_o$ is rejected in most of

[15] Its available at http://www.econ.uzh.ch/faculty/wolf/publications.html

the instances, the number of rejections is far less than that of the Jobson-Korkie test. With regards to the risk parity, there is overwhelming statistical evidence of differences in Sharpe ratios for the rolling periods of 6-months or more. In the 1-month and 3-months rolling periods, however, the evidence is leaning towards failing to reject $H_o$ when risk parity is compared to the equally weighted portfolio for both periods. This is in addition to the minimum-variance for the 1-month rolling period and the mean-variance and resampling portfolios for the 3-months rolling periods (Please see appendix C for the other rolling periods (To save space other rolling periods have not been included but can be made available on request)).

| | Equally Weighted | Minimum Variance | Mean Variance | Bayes Stein (BS) | Ledoit & Wolf (LW ) | BS + LW | Resampling |
|---|---|---|---|---|---|---|---|
| Equally Weighted | | | | | | | |
| Minimum Variance | -1.276 | | | | | | |
| Mean Variance | -4.471*** | -3.178*** | | | | | |
| Bayes Stein (BS) | -2.444** | 0.571 | 4.552*** | | | | |
| Ledoit & Wolf (LW ) | -4.216*** | -2.947*** | 0.361 | -4.204*** | | | |
| BS + LW | -2.389** | 0.477 | 4.601*** | -1.03 | 4.242*** | | |
| Resampling | -4.029*** | -2.287** | 4.075*** | -4.141*** | 3.459*** | -4.213*** | |
| Risk Parity | 0.04 | 1.864* | 4.427*** | 2.408** | 4.139*** | 2.383** | 4.103*** |

Exhibit 2.1: In-sample Jobson - Korkie Hypothesis tests for the 6-months rolling period

| | Equally Weighted | Minimum Variance | Mean Variance | Bayes Stein (BS) | Ledoit & Wolf (LW ) | BS + LW | Resampling |
|---|---|---|---|---|---|---|---|
| Equally Weighted | | | | | | | |
| Minimum Variance | 1.550 | | | | | | |
| Mean Variance | 6.531*** | 1.204 | | | | | |
| Bayes Stein (BS) | 1.079 | 5.704*** | 1.889* | | | | |
| Ledoit & Wolf (LW ) | 3.196*** | 0.813 | 2.994*** | 2.750*** | | | |
| BS + LW | 3.686*** | 6.321*** | 3.029*** | 2.597** | 2.651*** | | |
| Resampling | 13.439*** | 0.770 | 1.841* | 2.761*** | 1.713* | 4.270*** | |
| Risk Parity | 2.818*** | 2.327** | 2.972*** | 1.894* | 3.187*** | 2.062** | 8.526*** |

Exhibit 2.12: In-Sample Ledoit - Wolf Robust Hypothesis tests for the 6-months rolling period

* Significant at the 10% level
** Significant at the 5% level;
*** Significant at the 1% level

Exhibits 2.13 and 2.14 present the results of the portfolios out-of-sample performance using the same two tests. The Jobson-Korkie test shows very little evidence for rejecting $H_o$ for the shorter rolling periods when the risk parity portfolio is compared to other portfolios. However, for the 12 and 24 months rolling period there are more significant differences in the Sharpe ratios in the risk parity portfolio. The trend is similar when the other portfolios' Sharpe ratios are compared with each other. The results for the Ledoit-Wolf Robust test display statistical significance for the risk parity portfolio in relation to the resampling portfolio in the 1-month rolling period, the mean-variance in the 3-months rolling period, and the Ledoit and Wolf and mean-variance portfolios in the 24 months rolling periods (See Appendix C (to save space other rolling periods have not been included but can be made available on request)). For the other portfolios and rolling periods the test fails to reject $H_o$ with regards to the risk parity versus the other portfolios. When it comes to comparing the other portfolios with each other, the equally-weighted and the minimum-variance portfolio to a less extent display significant differences when compared to the others.

| | Equally Weighted | Minimum Variance | Mean Variance | Bayes Stein (BS) | Ledoit & Wolf (LW ) | BS + LW | Resampling |
|---|---|---|---|---|---|---|---|
| Equally Weighted | | | | | | | |
| Minimum Variance | .8921 | | | | | | |
| Mean Variance | .9337** | .4628 | | | | | |
| Bayes Stein (BS) | .0536*** | -.4831 | -.6331 | | | | |
| Ledoit & Wolf (LW ) | .868*** | .367*** | -.4895 | .5781 | | | |
| BS + LW | .0075** | -.5169 | -.6591** | -.3412 | -.6081 | | |
| Resampling | .7879 | -.1735 | -.8498 | .4033 | -.7143 | .4424 | |
| Risk Parity | .1925 | -1.1552 | -.9643 | -.0136** | -.8841 | .0329** | -.8592 |

Exhibit 2.13: Out-of-Sample Jobson - Korkie Hypothesis tests for the 6-months rolling period

| | Equally Weighted | Minimum Variance | Mean Variance | Bayes Stein (BS) | Ledoit & Wolf (LW ) | BS + LW | Resampling |
|---|---|---|---|---|---|---|---|
| Equally Weighted | | | | | | | |
| Minimum Variance | 1.170 | | | | | | |
| Mean Variance | 1.810* | 0.520 | | | | | |
| Bayes Stein (BS) | 0.203 | 1.609 | 1.750* | | | | |
| Ledoit & Wolf (LW ) | 0.128 | 1.004 | 1.884* | 0.031 | | | |
| BS + LW | 1.658 | 0.331 | 0.742 | 1.374 | 1.756* | | |
| Resampling | 0.018 | 1.004 | 1.836* | 0.072 | 0.201 | 1.745* | |
| Risk Parity | 1.364 | 0.165 | 1.392 | 1.492 | 1.017 | 0.945 | 1.216 |

Exhibit 2.14: Out-of-Sample Ledoit - Wolf Robust Hypothesis tests for the 6-months rolling period

* Significant at the 10% level

** Significant at the 5% level;

*** Significant at the 1% level

Further investigation was on the performance is done by considering the effect of incorporating transaction costs of 2% (Stevenson, 2001) for each of the rolling periods used for all the portfolios. Taking the 6 months rolling period, the transaction costs are taken after the portfolio has been rebalanced and was compared to the allocations in the previous 6 months. The 2% transaction cost was then applied to the absolute changes in the allocations, and the out-of-sample returns then adjusted accordingly. This was done for all the estimation periods. This study has also calculated the actual monetary transaction costs by using a notional amount of $50 million. The method is the same as when calculating transaction cost-adjusted returns (described above) but the absolute differences in the allocations and the transaction costs are applied to the notional amount. Exhibits 2.15 and 2.16 display the results of the Jobson-Korkie and Ledoit-Wolf approach when transaction costs are incorporated. The Jobson-Korkie results show that there are significant differences between risk parity and other portfolios for longer rolling periods (above 6-months rolling periods). However, while the

Ledoit-Wolf test also provides strong evidence for the rejection of $H_o$, this is across the different rolling periods apart from the 12 months rolling period which has half of the instances where the null hypothesis cannot be rejected. What is also noticeable is the significant difference in Sharpe ratios of the resampling portfolio in comparison to all the other portfolios. The equally weighted portfolio also reveals a strong case for the rejection of the null hypothesis; however, this may not be the case in practice as explained later. Considering the transaction costs in monetary terms could shed more light on the results of the preceding test and is illustrated in the distributions in Exhibit 2.17.

| | Equally Weighted | Minimum Variance | Mean Variance | Bayes Stein (BS) | Ledoit & Wolf (LW ) | BS + LW | Resampling |
|---|---|---|---|---|---|---|---|
| Equally Weighted | | | | | | | |
| Minimum Variance | 3.6717 | | | | | | |
| Mean Variance | 3.5988** | 1.6922 | | | | | |
| Bayes Stein (BS) | 3.2528*** | .6779 | -1.747 | | | | |
| Ledoit & Wolf (LW ) | 3.7175*** | 1.821*** | .7401 | 1.8313 | | | |
| BS + LW | 3.6412** | 1.3577 | -1.0554** | 2.3972 | -1.2889 | | |
| Resampling | 6.3531 | 3.3568 | .6236 | 2.1666 | .4095 | 1.6271 | |
| Risk Parity | 3.3715 | -3.5037 | -3.2265 | -2.7786** | -3.332 | -3.2403** | -6.064 |

Exhibit 2.15: Out-of-sample Jobson - Korkie Hypothesis tests with transaction costs for 6-months rolling period

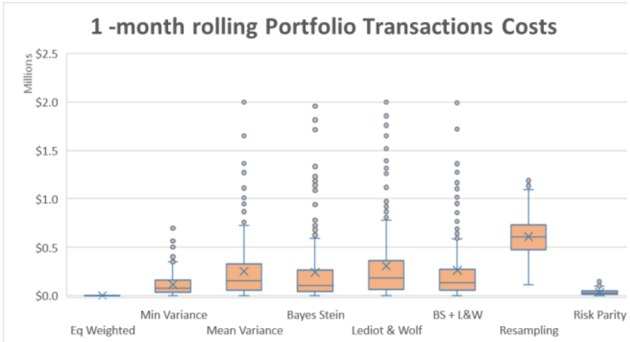
| | Equally Weighted | Minimum Variance | Mean Variance | Bayes Stein (BS) | Ledoit & Wolf (LW ) | BS + LW | Resampling |
|---|---|---|---|---|---|---|---|
| Equally Weighted | | | | | | | |
| Minimum Variance | 2.942*** | | | | | | |
| Mean Variance | 3.477*** | 2.914*** | | | | | |
| Bayes Stein (BS) | 2.769*** | 2.518** | 3.155*** | | | | |
| Ledoit & Wolf (LW ) | 2.815*** | 0.821 | 2.591** | 2.373** | | | |
| BS + LW | 3.327*** | 2.690*** | 1.043 | 3.114*** | 2.557** | | |
| Resampling | 3.047*** | 1.683* | 1.690* | 2.580** | 3.047*** | 1.900* | |
| Risk Parity | 4.067*** | 3.528*** | 0.913 | 3.733*** | 2.297** | 0.575 | 1.890* |

Exhibit 2.16: Out-of-sample Ledoit - Wolf Robust Hypothesis tests with transaction costs for 6-months rolling period

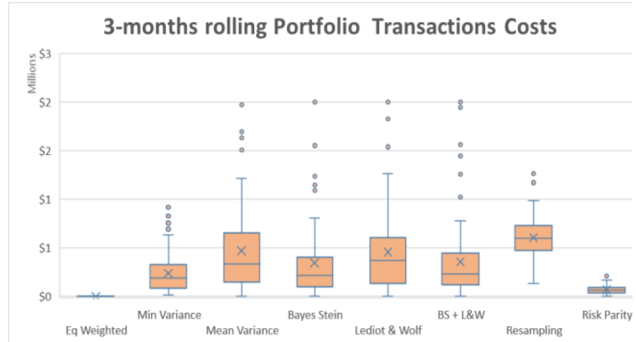
* Significant at the 10% level

** Significant at the 5% level;

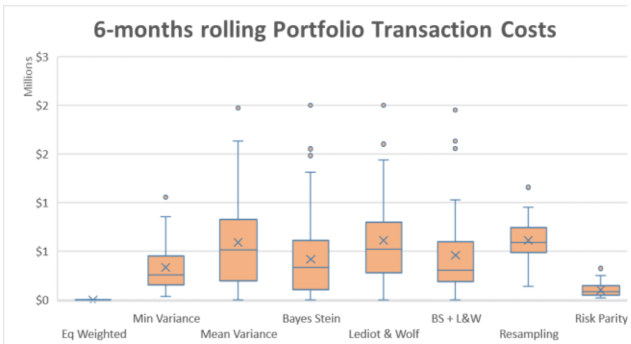
*** Significant at the 1% level



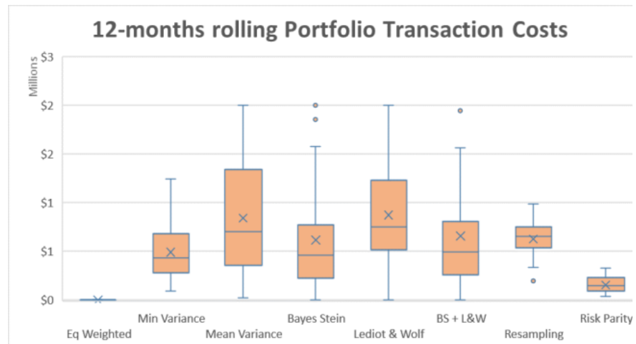

(a)



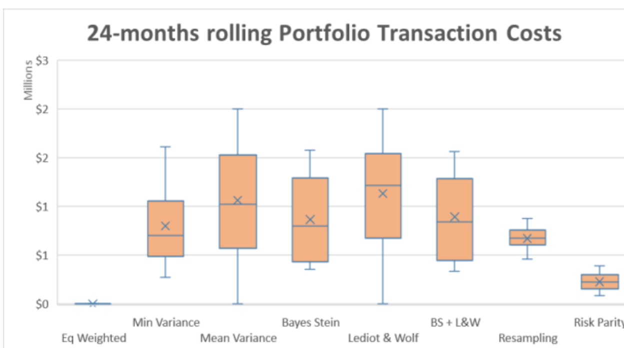

(b)




(c)




(d)




(e)

Exhibit 2.17: Distribution of out-of-sample transaction costs in different rolling periods

This exhibit presents the distribution of transactions costs in US dollar for all the portfolios under investigation in different rolling periods. The transaction costs are based on a notional amount of $50million. Most of the portfolios in all the rolling periods apart from the 24-months rolling period show positively skewed distributions meaning that the number of transaction costs higher than the mean values is more than those that are lower than the mean transaction costs.

Because the transaction costs in this study have been calculated purely using the changes in allocation, the equally-weighted portfolio does not show any transaction costs. This is unlikely to be the case in practice because the equally-weighted portfolio will also be subject to rebalancing as the portfolio value appreciates or depreciates due to the changes in price. This will invariably require rebalancing to maintain equal weights (Stevenson, 2002). What is discernible in the boxplots in Exhibit 2.17 the is that on average the transaction costs for the risk parity portfolio are the consistently the lowest of compared to the all the other portfolios in all the rolling periods. The risk parity transaction costs are clustered together and do not show much variation, unlike the other portfolios.

Furthermore, the low transaction costs for this portfolio is remarkable as even the highest transaction cost for the risk parity is lower than the mean value of each of the other assets and also the median value for all the other assets except for a few portfolios in the 1-month rolling period. The low transaction costs for the risk parity can be attributed to the reasonably stable allocations as shown in Exhibit 2.5 (and Appendix A) compared to the other portfolios especially those with corner solutions. Overall, for shorter rolling periods, i.e. the mean values of the transaction costs are relatively low. This is because, in these periods, the swings in allocations for the other strategies that utilise optimisation (namely mean-variance; Bayes Stein; Ledoit and Wolf; and BS+LW) are minimal since the changes in returns would not have filtered through in the short term. This is despite these optimised portfolios having corner solutions because in these periods the similar allocations are likely to be suggested for some time. Conversely, with longer rolling periods, as the changes in the returns filter through so do the swings in the allocations, and this subsequently leads to an increase in the turnover which accordingly increases the transactions costs as presented in Exhibit 2.17 (c ) through (f). On average, the resampled portfolio, on the other hand, seems to consistently have mean transaction costs that are rather high although these seem to reduce the more extended the rolling period and are finally dwarfed by those of the other portfolios.

## 2.6 Conclusion

This study set out to introduce a relatively new asset allocation method, risk parity, to real estate and examine its performance using the international real estate security markets, namely; Australia, France, the UK and the US. This was done by considering the 'traditional' allocation methods and also more contemporary methods that set out to solve the shortcoming of the well-known mean-variance optimisation based on modern portfolio theory. While most of the allocation approaches are built around the expected returns, risk parity aims to create portfolios whose assets contribute equal risk to the total portfolio risk. The results of the study were mixed. What came out was the diversification superiority of this asset allocation because it resulted in full allocations to all assets thereby helping to constrain a portfolio's overexposure to risk from only a few assets. This is not the case in other portfolio allocation methods examined because corner solutions are prevalent. However, the average in-sample performance was not as good considering that this allocation method underperformed against all the portfolio types based on international real estate securities. However, the out-of-sample performance in terms of the average Sharpe ratio was good for the risk parity as it took the first, third and fourth positions in the rolling periods under investigation. The regularisation based portfolios also performed very well with the Bayes-Stein portfolio leading the pack and followed mainly by the Bayes-Stein and resampled portfolio. Conversely, the performance of the Ledoit and Wolf was not that impressive. A "combo" of the two shrinkage approaches was created to form what was called the Bayes Stein + Ledoit and Wolf portfolio. This produced a significant over the Lediot & Wolf portfolio and also the resampling portfolio and the traditional mean-variance in some instances on an out-of-sample basis.

The hypothesis tests in the main revealed statistically significant differences between risk parity and other portfolios. Again, this can be attributed to the characteristics of this portfolio since it sets out mainly to achieve diversification rather than higher performance.

Transaction costs were also taken into consideration after portfolios are rebalanced and the risk parity portfolio resulted in an impressive performance when absolute transactions costs in dollars were considered as it gave significantly low transactions costs compared to all the other assets for all the rolling period under investigation.

In conclusion, even if the average in-sample performance was not good, the out-of-sample performance was on the whole relatively, and therefore the study has established that the risk parity approach has a place in international real estate equities due to its diversification benefits. The most significant finding is the low transaction cost as a consequence of using the risk parity approach compared to the other asset allocation strategies that were investigated in the study. This is mainly as a result of the relatively stable allocations compared to other methods of asset allocation, particularly those based on optimisation. Arguably, risk parity is also much easier to implement compared to other regularisation portfolios like Bayes-Stein, Ledoit and Wolf, and resampling and can help reduce the estimation risk issue that is prevalent in mean-variance optimisation.

The author takes cognisance that risk parity is not a magic bullet and its implementation, whether within real estate or mainstream finance is dependent on the objectives of the investor, if returns are the primary goal then probably this may not be the allocation method of choice. Nevertheless, if diversification is what is sought then risk parity can be the asset allocation method of choice especially for medium to long term holding periods. Additionally, risk parity can also be of benefit if low transaction costs are important. The implication of the study is that asset allocation methods in real estate should not be restricted to the traditional equally-weighted, minimum-variance and mean-variance but also regularisation methods can be taken into consideration, and risk parity can be applied to international real estate securities investment to help and this can help manage risk through true diversification where there is no risk concentration as allocations are made in such a way that there is equal risk distribution among the assets in a portfolio. The equal distribution of risk in the portfolio circumvents corner solutions and the findings of this study has shown that this can lead to the achievement of good out-of-sample performance and low transaction cost.

It is also recognised that the choice of data may not have brought out the full potential of the different allocation methods examined. This is because the data was restricted to real estate securities as this was the focus of the study and other asset classes were therefore outside the scope of the study. It is possible that including other asset classes could have yielded different results.

The contribution of this study is to apply risk parity to international real estate securities by comparing it to traditional asset allocation methods as well as other methods based on regularisation techniques. Furthermore, the study is undertaken by considering both in-sample and out-of-sample performance in for different rolling periods as well as taking transaction costs into considerations.

The research in risk parity brought about more awareness of volatility in that it can also be measured using Value at Risk and Expected Shortfall. This, therefore, paves the way for the next chapter which investigates market risk modelling.

# CHAPTER 3

# REAL ESTATE MARKET RISK MODELLING

## 3.1. Introduction

The global financial crisis towards the end of the last decade, coupled with the European Sovereign crisis and more recently Brexit has seen an increased interest in the role of risk management in the mainstream financial investment market. Among other things, the measurement and management of market risk, credit risk, and operational risk have become more pronounced than ever before. Value-at-risk (VaR), a tool which assesses the maximum possible loss from an investment, assuming a given confidence level, is widely used in the investment world to measure market and credit risk. This measure has however come under constant criticism as it only considers the maximum loss for that confidence level and ignores any losses beyond that threshold, which could arise from extreme events. Secondly, VaR assumes a normal distribution of returns, and yet this is not the case with most financial returns, which have the added complexity of being susceptible to the phenomenon of 'fat tails'. Thus, the credibility of VaR seems to be losing ground. Though derived from the principles of VaR, expected shortfall (ES) has been forwarded as an alternative proposition due to its ability to overcome some of the shortcomings of VaR, particularly when it comes to dealing with tail risk. To this effect, the ES has been presented as a tool for market risk regulation, replacing VaR in the banking sector as proposed by the Basel Committee on Banking Supervision. This said, the ES has its challenges especially because it cannot be subjected to back-testing due to its non-listable attribute. Furthermore, ES is also said to be quite sensitive to extreme values.

In the real estate market, while VaR has been the subject of most research when it comes to market risk while interest in research on expected shortfall in real estate is also increasing. This study, therefore, investigates market risk modelling for real estate and assess whether and/or the extent to which the expected shortfall model offers a better alternative to VaR in terms of measuring market risk. Public real estate has been chosen as the focus of the study because of the availability of frequent transaction price data which makes it more amenable to the application of VaR compared to private real estate.

The measurement and management of risk have become more pronounced than ever before. The main categories of risk assets and portfolios can be exposed to are market, liquidity and credit risk. These categories of risk are examined below.

### 3.1.1. Market risk

Market risk is associated with the risk emanating from the changes in the level or volatility of market prices. The assessment can be absolute in currency form, by concentrating on the volatility of the total return;or relative by expressing it to a benchmark thus quantifying risk by examining the deviation from the index or tracking error (Jorion, 2007). Market risk is classified as directional and nondirectional. Directional risk relates to exposures to the direction of movements in financial variables, like share prices, interest rates, exchange rates, and commodity prices. The remaining risk is nondirectional, consisting of nonlinear exposures to hedged positions or volatilities (Jorion, 2007).

### 3.1.2. Liquidity risk

Liquidity risk complements market risk by determining the additional loss involved if there is a rapid change in a position. Nonetheless, it is distinct from market risk because it represents a temporary distortion due to transaction pressure (Sheppard 2013). Jorion (2007) observes that liquidity risk is typically treated separately from other risks. It is categorised as asset liquidity risk and funding liquidity risk. The former, also referred to as market/product-liquidity, arises when the size of a position relative to the usual trading lots results in a failure of transactions to be achieved at normal market prices. Prevailing market conditions will have an impact on liquidity risk for different assets over different periods. The bigger the market of an asset the more likely it will be able to be liquidated with ease without substantial price impacts. Examples of such assets are the major currencies, Treasury bonds and large companies (Jorion, 2007 and Jager, 2015). This is unlike smaller companies or exotic 'Over the Counter' (OTC) derivatives contracts where prices are easily affected by any transaction and liquidating them takes a longer time. In VaR and ES, liquidity risk can be factored loosely by a parameter-time horizon H, referred to as the liquidity or holding period (Jager 2013). The other category of liquidity risk, funding liquidity risk also called cash flow risk is concerned with the inability to meet payments obligations, which may force early liquidation and turning "paper" losses into realised losses.

### 3.1.3. Credit risk

Credit risk also referred to as default risk arising due to the inability or unwillingness of counterparties to fulfil their contractual obligations (Jorion 2007) for example repayment of credit, loans or mortgages.

## 3.2 Value at Risk

This research focuses on the uncertainty of future prices of an asset or portfolio and will consequently concentrate on market risk (the other risk classes are out of this study's scope).

Financial tragedies have led to a rise in the need for market risk analysis. This is particularly true following the global financial crisis of 2007/8 coupled with the European Sovereign crisis and more recently Brexit. In order to undertake this analysis, there is a need for the use of some risk measures to assess this market risk. Emma et al. (2013) define risk measures as tools that map loss distributions or random variables to capital amounts. Probability distributions are a way of measuring this risk, but an alternative to this involves the use of a single number related to a capital amount. The common known risk measure of market risk is Value-at-Risk (VaR).

VaR is defined as an estimate of maximum potential loss to be expected over a given period a certain percentage of the time (Beder, 1995). In other words, it is the loss that one is fairly sure, with a fixed probability (normally between 90% and 99%), will not be exceeded if the current asset or portfolio is held over some period of time (Alexander, 2008(d) and Sheppard 2013). Alternatively, significant levels are used, denoting the probability of the losses exceeding the VaR. For example, a 5% daily VaR means that for a given period it is anticipated that 5 per cent of the time the loss will be the VaR or more. The implication is that one day in every twenty days, the asset or portfolio will lose VaR or more. Consequently, in a given period, VaR can also be viewed as the minimum potential loss that a portfolio can suffer in the 5% worst case (Acerbi et al., 2001). An alternative approach is asking "What percentage of the value of the investment is at risk" (Szegö, 2005) given a certain probability (i.e., 5% in this case). As can be seen from the preceding, VaR is concerned with loses, unlike the variance and standard deviation which consider both the upside and downside risk.

From the above definitions, it is discernible that VaR is contingent on a significant level or level of confidence chosen in advance by a user as manifested in the probabilities which it is based on. In spite of this, the selection of the confidence interval or probability is subjective implying that VaR may not be relied on with certainty (Choudhry, 2013). This significant level denoted by ($\alpha$) (or the confidence level denoted by ($1-\alpha$)) is based on the user's attitude to risk. The risk attitude is inversely (directly) proportional to the significant level (confidence level). Therefore, the higher the user's risk aversion, the lower the significance level and the higher the confidence level thus employed. Because VaR measures the volatility of asset prices, which in turn is directly proportional to the probability of loss, a high volatility therefore results in a high VaR figure.

A formal definition of VaR as given by Sheppard (2013) is "the $\alpha$ VaR of a portfolio [or asset] is the largest number such that the probability that the loss in portfolio value over some period of time is greater than the VaR is $\alpha$" i.e.

$$\Pr(R_t < -VaR) = \alpha \quad (3.1)$$

Another way this can be written is:

$$VaR_\alpha(X) = \sup\{x|P[X \geq x] > \alpha\} \quad (3.2)$$

Where $\sup\{x|A\}$ is the upper limit of $x$ given event A, so $\sup\{x|P[X \geq x] > \alpha\}$ indicates the upper $100\alpha$ percentile of loss distribution (Yamai and Yoshiba, 2005). This definition can be applied to both discrete and continuous loss distributions. For returns, this can be simplified below

$$VaR_\alpha = \Phi^{-1}(1-\alpha)\sigma - \mu \quad (3.3)$$

Where $(1-\alpha)$ is the confidence level, $\Phi^{-1}$ is the inverse of the standard normal cumulative distribution and $\mu$ is the mean return.

The time horizon that is used is usually dependant on liquidity and may thus vary from asset to asset. Accordingly, in periods of low liquidity, e.g. downturns, the horizon over which risk is measured should be increased.

Because risk or volatility is at the centre of market risk, it is essential to consider the various ways of capturing it, using volatility models, before delving into the VaR methods. The characteristics of the various volatility models are carried over to the VaR estimation. The research now explores the various models used to estimate volatility.

## 3.3 Market risk volatility models

Volatility and correlation (or covariance) form the underlying building blocks of market risk assessment models, therefore, the apparent changes in these parameters have important outcomes for all types of risk management decisions (Alexander, 2008b). For this reason, it is imperative to model volatility and understand what type of variability of returns models measure.

Before delving in the volatility models, it is vital to define volatility and contextualise it. Alexander (2008d) defines volatility as a measure of the dispersion in a process that is used to model returns. Volatility of returns is measured by standard deviation (the square root of the variance) which is usually expressed as a percentage per annum. The standard deviation is directly proportional to the holding period. It is worth mentioning that the standard deviations for different holding periods are not directly comparable, consequently it necessary to transmute the standard deviations in annual terms (using the square root of time as earlier mentioned).

In contrast to market prices, volatility can only be estimated and forecasted owing to it not being directly observable given that it is not traded on the market in its pure form. Therefore, this makes it challenging to model it. For this reason, Alexander (2008b) argues that estimating volatility can only be done within the context of an assumed statistical model as implied from observing price movements in the market. The statistical model for estimating volatility may give rise to a formula. However this estimate provides volatility that is "realised" by the process in the model for the period under estimation (Alexander 2008d) and hence volatility is referred to as a latent variable (Danielsson, 2011). High fluctuation in prices usually implies high volatility, but it is difficult to ascertain with precision how high this volatility is. According to Danielsson (2011), this is due to the difficulty of distinguishing whether a large shock to prices is temporary or perpetual. Furthermore, the intricacies presented by the existence of features such as non-normalities, volatility clustering and structural breaks makes volatility modelling quite arduous as making strong assumptions and utilising statistical modelling is required.

Alexander (2008b) further observes that only when the normality assumptions are made is volatility a sufficient statistic for dispersion; otherwise, it does not provide a full description of the risks borne by an investment. Thus, understanding more about the distribution of returns in addition to the expected return and volatility is crucial. For this reason, she contends that the best dispersion metric needs basing on the entire distribution function of returns as volatility is concerned about the scale and mean about the location, whereas the dispersion also depends on the shape of the distribution.

The volatility models examined are categorised as equally weighted, exponential weighted moving average and GARCH.

### 3.3.1. Equally weighted

As stated earlier, the typical way of calculating volatility is by using the variance or standard deviation. The standard deviation is obtained by finding the square root of the variance. The variance calculates the average deviation from the mean. The easiest way to forecast volatility is to use historical volatility for a consistent period of time. This results in a moving average as the consistent period is rolled over such that the latest return is added for each day while the oldest drop out. The [consistent] time period or interval over which the average variance is calculated is referred to as the sample size or the look-back period or averaging period (Alexander, 2008a). For this reason, this model is also called the simple moving average as every point within the averaging period is given equal weighting; hence it is eponymously called equally weighted or historical model.

The historical model where equal weights are presumed results in volatility forecast for future periods being the square root of the variance. For time series based on daily data, the mean return is assumed to be zero. This means that the estimate for the equally-weighted variance is simply the average of the squared returns.

The moving average or equally weighted model assumes that an 'independent and identically distributed' (i.i.d.) process with elliptical joint distributions[16], which also implies that the resulting volatility and correlations are constant, drives the returns. As earlier mentioned, the volatility is normally stated in annual terms and the square root of time rule is used to annualise it. So converting a daily volatility to annual terms involves multiplying the daily volatility by the square root of 250 (i.e. assuming there are 250 trading days in a year).

16 Alexander (2008b) defines elliptical distributions as those that have elliptical contours in their bivariate form and these include the normal distribution and the student t distribution.

The historical variance estimated over a period is also referred to as the ‘overall’ or ‘long-term’ average or “unconditional” variance for that period (Alexander, 2008b). It is unconditional in the sense that the variance is constant or in other words, it is not time-varying and can thus be considered to be the long term average variance for the period in question. The estimate of the unconditional parameter only provides an average value of the sample periods’ conditional parameter for VaR. For this reason, the equally-weighted average estimate is said to have limited use in estimating VaR over a short period of time as the current market conditions prevailing may not be manifested. However, the estimation may be useful for long-term estimation of VaR. A longer averaging period (m) results in markedly smooth (stable) risk forecast; in contrast, a shorter period leads to an excessively jagged pattern of forecast variance over time. Nonetheless, even though the more extended period provides more precise estimates, the underlying variation in the volatility could be missed (Jorion, 2007).

Because equal weight is given to all the periods, extreme past events could lead to artificially high volatility levels, which happen for as long as the data range still covers these events. This phenomenon produces what is referred to as ghost effects that lead to volatility clustering. With time this could lead to abrupt changes in volatility not resulting from any significant adjustment in market fundamentals but only that the extreme event has subsequently fallen out of the look-back period T (Brooks, 2014 and Alexander, 2008b)).

According to Danielsson (2011), the moving average should not be used in practice when estimating VaR as it is very sensitive to the choice of the estimation length (which is normally chosen arbitrarily) and normally results in volatility forecasts that jump around and that are generally systematically too high or too low. Furthermore, Alexander (2008b) observes that due to a number of pitfalls historical equally weighted volatility should only be used as an indication of the possible range for long-term volatility as they are of doubtful validity for short term volatility forecasting.

The issues outlined above that the historical or equal moving average volatility models possess are also transferred to the normal linear VaR models (covered later). The ghost effect or the volatility clustering of the equally weighted models result in high linear VaR figures. This is because the linear VaR model assumes constant volatility and this will carry on for T periods. Regardless of the market conditions, this high VaR will become low following the dropping out of the extreme return after T periods. It, therefore, follows that the VaR models that use equally moving average are sensitive to the choice of the sample period T. As highlighted before, this is because the risk sensitivity is inversely proportional to the T, i.e. the larger T is, the less responsive the volatility is to market conditions hence; the less risk-sensitive is the subsequent VaR. It is for this reason that EWMA and GARCH models are preferred as they take volatility clustering into considerations. These models are considered next, in turn.

### 3.3.2. Exponentially Weighted Moving Average (EWMA)

The exponentially weighted moving average (EMWA) addresses the inadequacies presented by the equally-weighted model by giving more weights to recent observations compared to older observations. This is done through the use of λ the 'decay factor' (sometimes called the smoothing constant), which determines how much weight is given to recent versus older observations (Brooks, 2014). When applied to an infinite series of data, EWMA results in equation (5.1) that is recursive.

$$\hat{\sigma}_t^2 = \lambda\, \hat{\sigma}_{t-1}^2 + (1-\lambda) r_{t-1}^2 \tag{3.4}$$

Where $0 < \lambda < 1$; and $\hat{\sigma}_t^2$ is the conditional volatility forecast on day t. Similar with the equally weighted model, the EWMA variance can be converted to EWMA volatility by getting the square root of the variance, and as before, it can be annualised by applying the square root of time rule.

The equation is recursive because today's lambda is a function of the previous day's lambda. Furthermore, the estimated variance is a conditional variance because it is updated with new information from the lagged (previous day's) variance and lagged squared return. In other words the forecasted variance for day t, $\hat{\sigma}_t^2$, that is estimated at the end of day $t-1$ is computed from the variance estimate of the previous day, $\hat{\sigma}_{t-1}^2$ and the previous day's squared return $r_{t-1}^2$. EWMA is therefore a simple updating rule that allows the update of the daily volatility estimate each day based on the most recent daily return (Dowd, 2005). This results in the model being able to capture volatility changes in a way that is broadly consistent with observed returns (Christoffersen, 2012).

$(1-\lambda) r_{t-1}^2$ impacts on how market changes affect the strength of reaction of the volatility to these occurrences. The smaller the λ the bigger the volatility reactions to the market information in the previous day's return. $\lambda\, \hat{\sigma}_{t-1}^2$ controls or defines the persistence (or smoothing) in volatility, the higher the λ, the larger the persistence in volatility following a shock in the market. Put differently a high λ implies a slow decline in the volatility. This means that if previous days' volatility was high then it's likely to remain high today irrespective of what occurs in the market (Alexander, 2008b). A low λ on the other hand implies a quick decline in the volatility.

λ can be estimated using Maximum Likelihood Estimation (MLE) and could vary both across series and over time, resulting in a loss of consistency over different periods.

Furthermore, the EWMA model assumes that the reaction and persistence parameters are not independent, which Alexander (2008b) observes as an unfortunate restriction because it is not generally empirically justified. According to Jorion (2007), the varying values of λ could generate incompatibilities across the covariance terms and may give rise to unreasonable values for correlations. This can occur when volatility is highly reactive but has little persistence or volatility is very persistent but not highly reactive. The cause of such a situation is because often, λ is chosen subjectively (usually ranging between 0.75 and 0.98) rather than determined through statistical methods (Alexander, 2008b). RiskMetrics, established from a large study spanning several asset classes, however, recommends 0.94 and 0.97 for daily and monthly data respectively and these are widely used in practice (Jorion (2007), Brooks (2014), Dowd (2005) and Alexander (2008b)). The recommendations above by RiskMetrics suggest that there are some inconsistencies between the daily and monthly risk forecasts. However, as earlier highlighted, Alexander (2008b) observes that these recommendations by RiskMetrics are ad hoc. Moreover, when VaR is estimated from its conditional volatility, a conditional normality assumption must be brought into play (Boudoukh et al., 1998), which as stated earlier is inconsistent with financial data – as fat tails and skewness properties are challenging to account for within the EWMA method.

The advantage of the EWMA model is that it is easy to use as it only has one unknown parameter, i.e. λ. Despite the elegancy of the EWMA recursive formula, Danielsson (2011) observes that EWMA is not permitted under the Basel Accords to calculate VaR because the exponential weights decline to zero very quickly. He, however, contends that overall, the EWMA model performs well compared to the other more complicated models. Similarly, Jorion (2007) argues that in contrast to other models the use λ as the only parameter to estimate makes EWMA more robust. A separate study by Alexander and Leigh (1997) reveals superiority of EWMA in forecasting the centre return distribution than the equally-weighted and other more complicated model, however, this is not the case in the tails of the distribution.

Despite EWMA overcoming the inadequacies of the equally weighted historical model, it has some shortcomings. The main weakness of the EWMA model is that it is insensitive to market changes because a constant λ is assumed for all periods and identical for all assets (Brooks 2014). This means that it is not optimal for any asset or portfolio. EWMA, therefore, ignores recent dynamics in the data because it predicts that the volatility in the future is likely to level off immediately and remain at this current level. Moreover, the EWMA model is not mean reverting in that it does not tend towards an unconditional or long-run variance. As a result, compared to other more sophisticated models like GARCH, the EWMA model is likely to yield inferior forecasts. Despite this shortcoming and in addition to its ease of implementation it is usually the preferred choice for volatility forecasting model (Daneilsson, 2011).

The shortcomings of EWMA previously highlighted has led to the use of alternative and more superior family of models referred to as GARCH.

### 3.3.3. Generalised Autoregressive Conditional Heteroscedasticity (GARCH)

The moving average models covered earlier are based on the unrealistic assumption that returns are independent and identically distributed (i.i.d.) which implies that the volatility (and correlation) forecast are the same as the current estimates. However, the volatility for financial asset returns is not constant over time and also exhibits volatility clustering whose magnitude depends on the data frequency. The shorter the data frequency the more the volatility clustering therefore, intra-day data is the most prone followed by daily data. Monthly data will exhibit less volatility clustering and annual data will hardly display any. The methods covered so far do not treat volatility clustering in a robust manner. Volatility clustering has important implications for risk management and for pricing and hedging options and therefore should be taken into account (Alexander, 2008b). For this reason, another family of models called GARCH are particularly designed to capture the volatility clustering of returns.

Generalised Autoregressive Conditional Heteroscedasticity (GARCH) models were introduced by Bollerslev (1986) and Taylor (1996) as an extension and generalisation of the Autoregressive Conditional Heteroscedasticity (ARCH) model proposed by Engle (1982). However, ARCH models are not used widely in practice due to a number of inherent limitations (Brooks, 2008). Danielsson (2011) highlights the long lag lengths required to capture the impact of historical returns on current volatility as one of the major problems of ARCH models. Compared to ARCH models, GARCH models are more parsimonious in that they are able to accomplish a desired level of prediction using as few variables as possible. This avoids over-fitting and thus GARCH models are less likely to breach non-negativity constraints (Brooks, 2008 and Jorion, 2007) and allow an infinite number of past square errors to influence the current conditional variance. Consequently, GARCH models are widely used in practice and are in essence a generalised version of the EWMA model because they have the flexibility to accommodate specific aspects of individual assets (Christoffersen, 2012). What is more is that, unlike the EWMA models GARCH models employ optimal exponential weighting of historical returns to obtain volatility forecasts (Danielsson, 2011). Here, the forecasts that result from GARCH models are therefore not the same as the current estimate, but change and converge to the unconditional volatility over time.

The GARCH model, analogous to the EWMA model, has a time series with a form that has a regression on itself and hence is an “autoregressive” approach as the name suggests. The term “conditional heteroscedasticity” stems from the fact that as EWMA, GARCH is conditional in the sense that today’s volatility estimate depends (is conditional) on past [volatilities or] returns and all the information up to that point. Therefore, the distribution of these returns at time $t$ takes into account all past returns up to and including time $t-1$ as being non-stochastic. This is because it is observed and thus non-random so it is conditional on that information set which these returns are a part of (Alexander, 2008). The implication is that markets do not totally follow a random process because yesterday’s events have a bearing on what is likely to happen today (Choudhrey, 2013). However, like EWMA, the relevance of the past data wanes or decays with time as the older data will possess less relevance to today’s events. Accordingly, the data that should be used doesn’t necessarily have to extend too far into the past. Owing to this, the volatility is heteroscedastic in the sense that it changes overtime and is thus not constant. The rationale behind is that, it represents instantaneousness given that it can change from time to time due to its sensitivity to recent events (Alexander, 2008b and Brooks, 2014). Alexander (2008b) observes that due to the forgoing, the GARCH process is not i.i.d. because the second conditional moments at different points in time are related. For that reason, the i.i.d. assumption can be relaxed in the GARCH models and so allowing them to be applied in estimation of VaR particularly in historical VaR and Monte Carlo VaR but she contends that its application to analytic linear VaR is questionable.

### GARCH(1,1)

GARCH is a family of models that consists of numerous GARCH-type models under this umbrella that include AGARCH, EGARCH, JGRGARCH, IGARCH, IGARCH, MARCH, NARCH, QTARCH, SPARCH, SWARCH AND TARCH (Dowd, 2005). GARCH (1,1) is the most popular and simplest GARCH model because it does not involve many parameters and fits the data typically relatively well. It depends on only one lag of squared returns and one lag of the variance, and as such it is most suitable for short-term variance forecasting. In its simplest form, the GARCH(1,1) postulates that today's volatility is contingent on yesterday's volatility and yesterday's squared returns as represented in equation (3.5).

$$\hat{\sigma}_t^2 = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2; \qquad (3.5)$$
$$\omega \geq 0;\ \alpha, \beta \geq 0; \alpha + \beta < 1$$

As seen in equation (5.1) GARCH(1,1) is similar to EWMA, but it has a constant $\omega$ and the sum of $\alpha$ and $\beta$ is less than 1. EWMA can therefore be looked at as a special case of the GARCH(1,1) process that takes place when $\omega = 0$, $\alpha = 1 - \lambda; \beta = \lambda$ and $\alpha + \beta = 1$ (Dowd, 2005 and Christoffersen, 2012). For GARCH (1,1), a high $\beta$ value signifies 'persistent' volatility meaning that it takes a protracted time for the volatility to adjust. As in EWMA, a high persistence means that model will have the tendency to stick to the previous day's variance. A low persistence indicates decay in that there is a tendency to move towards the long-run variance. On the other hand, a high $\alpha$ value suggests a 'spiky' volatility implying a rapid response to market changes (Dowd, 2005). Where there are no market shocks, the GARCH variance ultimately settles down a long run average variance such that $\hat{\sigma}_t^2 = \sigma^2$. The long run average variance is stable and steady i.e. mean-reverting as earlier stated. Generally, in periods following volatile market conditions the average volatility tends to decrease with the forecast horizon, in other words, the term structure forecasts for GARCH converge from above. This is because on average a relatively high volatility will have a tendency to fall over time. Conversely, the GARCH forecasts converge from below in periods following relatively calm market conditions. Likewise, low volatility will have a tendency to rise over time. Alexander (2008b) notes that this characteristic of mean reversion is akin to the implied volatility of option market prices of different maturities that nevertheless have the same strike price[17]. She argues that this is the reason why the GARCH models are popular among practitioners. The variance to which the forecast revert to is the long run variance also known as the unconditional variance (as stated earlier) and is a constant, defined in a similar way to that of the ARCH model as:

$$\sigma^2 = E(\omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2) \qquad (3.6\ )$$

From the above definition, $\hat{\sigma}_t^2 = \hat{\sigma}_{t-1}^2 = \sigma^2$. When this is substituted in the GARCH conditional variance equation( 5.1), it translates to:

[17] Implied volatility is the volatility that is implied by the options market and is covered later in Chapters 8.

$$\sigma^2 = \omega + \alpha\sigma^2 + \beta\sigma^2 \tag{3.7}$$

By rearranging equation 3.7, the unconditional variance is given by

$$\sigma^2 = \frac{\omega}{(1- \alpha - \beta)} \tag{3.8}$$

Taking the annualised square root results in the unconditional volatility which is a result of the positive intercept ω.

Incorporating equation (3.8) into the GARCH equation (3.4) and rearranging it further will result in

$$\hat{\sigma}_t^2 = \sigma^2 + \alpha(r_{t-1}^2 - \sigma^2) + \beta(\sigma_{t-1}^2 - \sigma^2) \tag{3.9}$$

Equation (5.6) indicates that today's variance can be described as the weighted average of the long-run (unconditional) variance, yesterday's squared return and yesterday's variance. In other words, today's variance is the long-run average variance with something added (subtracted) if yesterday's squared return is above (below) its long-run average and something added (subtracted) if yesterday's variance is above (below) its long-run average. Christoffersen (2012) and Dowd (2008) show that variance of the daily return k days ahead can be forecast with the use of only information available at the end of today. Put differently, the GARCH(1,1) forecast can be seen as a weighted average of unconditional variance, the deviation of last period's squared returns from unconditional variance and the deviation of last period's forecast from unconditional variance (Danielsson, 2011).

GARCH is suitable for longer horizons as the volatility is assumed to be time-varying. As before, this is true provided that $\omega, \alpha, \beta > 0$ and $\alpha + \beta < 1$ this is to ensure positive volatility forecast and covariance stationarity respectively (Danielsson, 2011).

If $\alpha + \beta \geq 1$, the unconditional variance of $\sigma^2$ is characterised as 'non-stationarity invariance' as $\sigma^2$ is not defined and is deemed to possess highly undesirable properties (Brooks, 2014). Where $\alpha + \beta = 1$, this is referred to as 'unit root invariance', also known as integrated GARCH or IGARCH (Jorion, 2007). IGARCH is used where the long run variance does not exist i.e. the return is not stationary. In this case the GARCH(1,1) becomes

$$\hat{\sigma}_t^2 = \omega + \beta\sigma_{t-1}^2 + (1-\beta)r_{t-1}^2 \tag{3.10}$$

This will be equivalent to the EWMA model if $\omega = 0$ (see equation 3.3). However, because the EWMA process does not revert to the mean, the longer-period forecasts are significantly distinctive (Jorion, 2007). Alexander (2008b) notes that the GARCH long-term variance is not the same as the EWMA unconditional variance as the latter is based on the i.i.d. return assumptions (hence also called the i.i.d. variance) whilst the former is not based on the same i.i.d. assumption. Moreover, she points out that depending on the GARCH model, this GARCH unconditional variance can vary. Owing to the waning volatility clustering effects in financial asset returns over extended time intervals beyond a week, GARCH models are often estimated on daily or intraday data (Alexander 2008b).

Brooks (2014) illustrations that the general GARCH(1,1) can be extended to a GARCH(p,q) formulation where the current conditional variance $\sigma_t^2$ is parameterised to depend upon q lags of the square errors (or returns) and p lags of the conditional variance:

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i r_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \qquad (3.11)$$

However, he notes that higher order GARCH models are seldom utilised in academic finance literature because the volatility clustering in data can adequate be captured using a GARCH(1,1). This is more so because the parameters of these higher order GARCH models are not easily interpretable (Christofferesen, 2012).

### Estimation of GARCH parameters

In as much as the GARCH models seem to be more superior to the historical simulation and EWMA, they are challenging in that they have several unknown parameters that need estimating. This exigence stems from the fact that the conditional variance must be implicitly estimated because it is not an observable variable (Christoffersen, 2012). The unknown parameters, i.e. $\alpha, \beta$ and $\omega$; can be estimated using the Maximum Likelihood Estimation (MLE), a technique used for finding parameters for both linear and non-linear models. According to Brooks (2014), this is done by finding the most likely values of the parameters that maximise a model, given the actual data[18].

Even though the Microsoft Excel "solver" tool can be used to estimate the parameters of the GARCH models, it only provides an approximation of the optimal estimate. Therefore, the complex nature of

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[18] Please see Brooks (2004) for more details of the use of Maximum Likelihood Estimators is in estimating the parameters for Garch models.

this estimation does not render excel to be the best tool. This is due to the common convergence problem that can render the log likelihood surface in the MLE to be flat (Alexander 2008b). Consequently, the use of econometric software packages that have purpose-built algorithms for maximisation of the GARCH likelihood functions is recommended, particularly Ox[19]. Other programmes like Eviews, MATLAB or R, can also be utilised to achieve in the estimation of these parameters.

In order for the likelihood function to be well defined, a specific minimum amount of data is required. Alexander (2008b) suggests the use of daily data spanning over several years ensures proper convergence of the model; otherwise, the parameter estimates may lack robustness. She further observes that it is typical that the GARCH constant $\omega$ estimates is predominately sensitive to the choice of sample data compared to the changes in the reaction and persistence parameters.

### Leverage effect on GARCH

In the basic ARCH and GARCH models, both negative and positive shocks have the same effect on volatility because they are assumed to be symmetric. However, in some series, such as equities there is a leverage effect in that substantial negative returns have a more significant effect on the volatility as compared to positive returns of the same magnitude. This is unlike commodities where price rises lead to more increases in volatility compared to the price falls of the same scale. Christoffersen (2012) attributes the leverage effect to a drop in the equity value of a company due to the effect of a negative return. This drop-in value results in an increase in a company's gearing level, which in turn increases the financial risk and hence the risk of bankruptcy. A positive return, on the other hand, increases the equity value of a company which in turn reduces the risk of bankruptcy, but usually, this is by a lesser amount (Jorion, 2007). To this end, the basic GARCH models do not accurately model returns because they do not consider the leverage effect. Nevertheless, GARCH models can be adapted into asymmetric models in order to accommodate the leverage effect by using two terms, one for positive shocks and the other for negative shocks, each with its separate alpha coefficient. Therefore, the leverage effect can be captured by defining an indicator variable, $I_{t-1}$ to take on the value 1 if day (t-1)'s return is negative and zero otherwise

$$I_{t-1} = \begin{cases} 1 & if\ r_{t-1} < 0 \\ 0 & if\ r_{t-1} \geq 0 \end{cases}$$

[19] This is the programming language that OxMetrics, an Econometrics software developed by Doornik, J and Hendry, D, is based on.

The variance dynamic can now then be specified as

$$\hat{\sigma}_t^2 = \omega + \alpha r_{t-1}^2 + \alpha\theta I_{t-1} r_{t-1}^2 + \beta\sigma_{t-1}^2 \tag{3.12}$$

Thus, a $\theta$ larger than zero will capture the leverage effect. This is sometimes, referred to as the GJR-GARCH model. The GJR-GARCH is really a modification of the asymmetric GARCH (A-GARCH) model that also accounts for leverage effect (Alexander, 2008b). Another GARCH model that is able to capture the leverage effect is the exponential GARCH (EGARCH), however, it has a major drawback in that future expected variance beyond one period cannot be calculated analytically.

Having covered the volatility models, the study will now explore market risk measures by examining value at risk (VaR) and expected shortfall. However, before delving into these, it is important to examine the stylised facts of financial returns.

### Stylised facts

- Fat tails – Financial returns exhibit a higher probability of observing large losses and large gains than suggested by normal distribution (Chrisoffersen, 2012). In other words, returns for financial assets are not normally distributed but have what is termed as "fat tails" due to extreme events. For risk management, it is important that these fat tails are captured.
- Financial returns exhibit very little serial correlation and, therefore this makes it difficult to predict returns by using historical returns (Christoffersen, 2012). i.e. the financial returns do not fit the assumption of i.i.d. variables. The volatility of financial returns, however, shows a profound serial correlation meaning that volatility is more straightforward to predict than returns.
- Nonlinear dependence – states that multivariate correlation i.e. the correlation between assets appear to be time-varying as they depend on the market situation. For example, bear markets show a high correlation than do bull markets.

With the stylised facts in mind, similar to Jager (2015), in this study, the fat tails that are exhibited due to extreme values will be captured through the application of Extreme Value Theory (EVT). In order to produce an i.i.d. process of random variables a combined AR(1) and GJR GARCH(1,1) process will be utilised, and lastly, the dependence structure of a portfolio will be simulated with copulus. Thus, this approach to modelling financial returns is referred to as the GARCH GJR –EVT- Copula approach.

## 3.4 VaR models

Broadly, VaR models fall into three categories, i.e. parametric, non-parametric and semi-parametric as classified by Manganelli and Engle (2001). The difference in these categories rests in the way in which the distribution is constructed. There are generally three VaR models namely; normal linear VaR, historical simulation and Monte Carlo (Alexander, 2008d, Dowd, 2005, Cheung & Powell, 2012. The volatility models covered earlier can also be categorised as either parametric, non-parametric and semi-parametric. The classification, below of the VaR models and their volatility models is based on Manganelli and Engle (2001), Alexander (2008d) and Dowd (2005).

The VaR models above and some volatility models are discussed below under the three categories.

### 3.4.1 Parametric Models

In parametric models, the risk measure is inferred from the risk estimated by fitting probability curves to the data, and thus these models use additional information contained in the assumed density or distribution (Dowd, 2005).

#### Normal linear VaR

The normal linear VaR is a parametric model used to market risk. It is also referred to as the covariance, variance-covariance, correlation, or analytical VaR. While the covariance matrix is at the centre of this method, the returns on risk factors[20] are assumed to be normally distributed with constant correlations and multivariate normal joint distributions between risk factors (Alexander, 2008d and Choudhry, 2013). Here it is assumed that the risk factor return dependencies are linear and fully encapsulated by one or more correlation matrices. For this reason, all that is required to capture the dependency between the risk factors is the correlation matrix of the risk factor returns.

The normal linear method requires historical data in order to calculate the volatility for each risk factor and the correlations between these risk factors. This is a straightforward way assuming that each return in the time series is equally-weighted. The other method of calculating the volatilities and

20 A risk factor can be described as a characteristic that is measureable, for example exchange rate, interest rate, market price, whose change can affect the value of an asset

correlation is using volatility modelling where the assumption is that the observations in the time series are not equally weighted because more recent observations are given more weight. Despite this difference, both methods assume that future volatilities can be gleaned from historical price movements (Choudhry, 2013).

If VaR is measured over a short period, for example, daily interval, then it can be assumed that the excess return is zero. This, therefore, further simplifies Equation (5.3) resulting in Equation (5.10), which translates as a percentage of the portfolio value, the negative of the standard normal α quantile[21], times the standard deviation of the portfolio returns over the measurement period (Alexander 2008d).

$$VaR_{\alpha} = \Phi^{-1}(1-\alpha)\sigma \tag{3.13}$$

Even though the parametric method is popular due to its simplicity, its drawbacks also stem from its imposition of strong assumptions of constant or stable correlations and hence only estimates linear risk Choudhry (2013) further observes that the overreliance on the normal distribution tends to understate the probability of extreme events occurring despite the presence of "fat tails" (or leptokurtosis - see Mandelbrot (1963) and Fama (1965)) in financial returns. For example, Gibson et al. (2008) argue that the losses incurred in the 2008 financial crisis are extremely rare events, i.e. they are akin to events in excess of "10 standard deviations". They show that according to normal distribution the chance of observing a single 7.5 standard deviation (or greater) one-day return would be in the region of 1 in 33 trillion meaning that the world would have to be roughly ten times older. To help overcome this weakness, extreme value theory (covered later) can be used (Longin, 2000).

There are a number of volatility models that fall under this category, and these are mainly models that estimate conditional volatility such as EWMA (RiskMetrics); GARCH (1,1) and the other asymmetric GARCH models, i.e. EGARCH, GJR GARCH covered earlier.

21 A quantile divides a variable in equal parts. A percentile divides a variable in 100 parts and is therefore a hyponym of a quantile.

### 3.4.2 Non-parametric models

Unlike parametric models, non-parametric models do not make strong assumptions in estimating the risk measures and instead use past data to forecast risk.

#### Historical simulation

The introduction of historical simulation (HS) as a VaR estimation method can be attributed to a series of papers by Boudoukh et al. (1998) and Barone-Adesi et al. (1998, 1999). The historical VaR model is a non-parametric VaR, based purely on past (historical) data from where the volatilities and correlations are calculated. Unlike normal linear VaR, the assumption in historical simulation is that all possible future variations have already been experienced in the past and that the historically simulated distribution is identical to the returns distribution over the forward-looking risk horizon (Alexander 2008d). For this reason, past or empirical percentiles of the historical return distribution are used directly to estimate percentiles for the VaR with skewness, leptokurtosis (fat tails) and other properties already directly accounted for (Boudoukh et al., 1998). In contrast to the parametric model, this method is free of multivariate normality assumptions implying that no conjecture is made that the covariance matrix can capture all the dependencies between risk factors. It should be pointed out, however, that even other models, i.e. normal linear (VaR) and Monte Carlo may if necessitated, also use historical data. Then again, while the other models are capable of incorporating risk factor returns that are skewed and heavy-tailed, the modelling of their multivariate risk factor returns are required to fit a parametric form. The difference, especially with normal linear VaR, is, therefore, the normality assumption.

In performing the historical simulation, it is vital that the number of observations (or sample) is as big as possible to avoid an imprecise VaR that results from having very few data points. This is particularly so in the lower tail of the distribution, particularly at high confidence levels. The implication is that more frequent data (e.g. daily data) over many years is required in order to estimate VaR more precisely for high confidence levels or percentiles (e.g. 99% or 95%) otherwise the VaR estimates are likely to be unstable. This need for big samples sizes to enable the accurate estimation of quantiles is a crucial limitation for historical VaR. This is because large and frequent samples may not always be possible particularly for assets with infrequent return information (e.g. direct real estate). According to

Alexander (2008d), the sample size constraints of historical VaR mean that VaR needs to be estimated at the daily horizon, to begin with then scaled up to longer horizons (h –day horizons).

The big sample size required by historical simulation means that the data may span across several periods or regimes, but this is likely to present another problem. This is because the market risk factors in these regimes may have different behaviour, for example, the financial crisis of 2007/8 where the volatilities and correlations were much higher compared to stable market conditions thus resulting in the earlier mentioned phenomenon referred to as volatility clustering which was referred to earlier. The consequence of volatility clustering is that VaR estimated using historical simulation is likely to be underestimated in periods immediately following stable market conditions while in periods that immediately follow volatile market condition, there is a tendency of overestimating VaR.

Historical simulation, however, does not account for time-varying volatility and volatility clustering (Bollerslev, 1986). This because historical simulation assigns equal weights to all the periods used for the volatility estimate. The no volatility clustering assumption implies that returns have the same likelihood of occurring, which is akin to assuming that they are independently and identically distributed (i.i.d.) (Boudoukh et al., 1998). In other words, the returns are random, so no relationship exists between current returns and past returns.

In summary, the two main problems with historical simulation are that it does not perform well on small samples and secondly, it does not consider time-varying volatility. The first can easily be solved by increasing the sample size. The solution to the second one is the use of EWMA or GARCH models; however; their effectiveness is lost if a longer time frame is used. Accordingly, Boudoukh et al. (1998) observe that the only way to put more weight on recent information within the historical simulation approach is using shorter historical periods or windows. They sum up the problem of historical simulation as follows; *"with long histories, the value of recent information diminishes, while with short histories we encounter estimation problems".* This is also observed by Pritsker (2006) recognises the challenge of choosing the correct sample period because the assumption of constant conditional correlation is violated by too long a period, whereas the accuracy of the model's nonparametric elements is reduced by too short a period.

As earlier indicated, because the historical simulation method is constructed solely from actual historical data it needs minimal analytical capabilities. Its simplicity can, however, lead to substantial distortion particularly when there are options in a portfolio. Nevertheless, its simplicity has led to wide acceptance and use by banks and regulators (Choudhry (2013) and Alexander (2008d)).

### 3.4.3 Semi-parametric models

Semi-parametric models combine both parametric and non-parametric approaches.

#### Filtered Historical Simulation[22]

Filtered Historical Simulation (FHS) is an alternative to the historical simulation that was introduced by Barone-Adesi et al. (1998) and Barone-Adesi et al. (1999). FHS is a semi-parametric model, which is a Monte Carlo, based approach (covered later) that combines parametric modelling of risk factor volatility with nonparametric modelling of the factor innovations (Pritsker, 2006). In essence, FHS overcomes some of the weaknesses of both historical and Monte Carlo simulation at the same time using their advantages. Although making as few assumptions as possible about the distribution of risk factors, FHS also incorporates historical returns' dynamics of past and current volatilities. The advantage over the parametric models is that this ensures consistency of the quantile and therefore the VaR, under weaker conditions because the density of the standardised residuals does not have to be assumed (Sheppard, 2013). Another alternative method is the Cornish-Fisher approximation. This splits the difference between a fully parametric model and a semi-parametric model. Like the FHS, the Cornish-Fisher approximation can be accurate without a parametric assumption; then again, the drawback is that these estimators are said not to be necessarily consistent compared to the HS (Sheppard, 2013).

#### Conditional autoregressive VaR

The Conditional autoregressive VaR (CAViaR) is a quantile regression approach. The quantile regression approach uses the time series of a specified quantile is explicitly modelled using any information considered to be relevant to estimate conditional quantiles. This method avoids the use of any distribution assumptions of the returns (Kuester et al., 2006, 2005 and Engle and Manganelli, 2004).

Volatility models that also deal with extreme VaR such as Extreme Value Theory EVT[23] (covered later) are examples of semi-parametric models. Therefore when a conditional parametric volatility model like GARCH is combined with EVT to make GARCH EVT, it becomes a semi-parametric model.

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[22] The Filtered Historical Simulation model is sometimes categorised as a nonparametric model (Manganelli and Engle (2001). However, Alexander (2008d) and Dowd (2005) point out that this method combines the power of historical simulation (non-parametric) and parametric dynamic flexibility of conditional volatility models such as GARCH.

[23] See chapter 4 for more on EVT

### 3.4.4 Monte Carlo Simulation

Monte Carlo Simulation can be applied to either parametric or non-parametric models. It involves a process which makes use of some given specific user-defined parameters to generate, using a computer programme, a series of an asset's price that is theoretically possible (Choudhry, 2013). These simulated asset prices are then used to estimate the VaR. Similar to the normal linear VaR, Monte Carlo VaR assumes i.i.d. risk factors that have a multivariate normal distribution. The normal distribution assumption implies that the simulated returns are unlikely to be extreme in contrast to financial returns distributions, which tend to have larger densities at the extremes (or fat tails) as earlier indicated. This could potentially lead to underestimation of losses as highlighted before.

Central to Monte Carlo is the assumption of the covariance matrix's ability to capture all possible dependencies between the risk factor returns. Here, a constant covariance matrix is assumed over time. This, however, is not the case in reality as correlations, and therefore covariances tend to increase during stressed market conditions, e.g. the 2007/8 crisis. Nevertheless, compared to the normal linear VaR, Monte Carlo simulation is more flexible and can accommodate many different assumptions about the multivariate distribution of risk factor returns (Alexander 2008d). Furthermore, it is possible for more complex dependency structures to be assumed, and path-dependent behaviour like volatility clustering can be accounted for as well. For instance, this can be done by assuming some form of risk factor evolution, e.g. that a GARCH model captures clustering for both volatility and correlation. This is because both the multivariate normality and the i.i.d. assumptions for risk factors can be generated through simulation while this is done analytically for parametric VaR. Alexander (2008d) further observes on one hand that despite being based on assumptions that are unlikely to hold, the normal linear (parametric) VaR is precise, but on the other hand, Monte Carlo simulation is subject to simulation error. Then again, she contends that if a sufficient number of simulations are used, the VaR estimates from the two models should be similar. This implies that applying Monte Carlo VaR to a linear portfolio has the potential of creating sampling errors not present in the normal linear VaR model. This notwithstanding, Alexander (2008d) is quick to point the usefulness of applying Monte Carlo VaR to a linear portfolio because of its flexibility of being based on practically any multivariate distribution for risk factor returns. This is in contrast to only a few selected distributions that exist in closed form solutions for parametric linear VaR. Because the Monte Carlo approach can also be applied to non-linear portfolios like option portfolios, it is said to be the most flexible of the models due to its versatility in dealing with a great diversity of risk factor return distribution.

The criticism of Monte Carlo Simulation lies in its ‘black box’ nature which may make it impenetrable to some people whereas, in contrast, one can drill down to the data in historical simulation in order to carefully study a given risk measure. The other criticism is that it is very computer intensive and therefore consumes a lot of computer resources.

The three main VaR methodologies (normal linear, historical simulation and Monte Carlo simulation) above assume stationarity of the probability distributions of market price movements. This means that the behaviour of the market over the recent past is a good and unbiased indicator of its near future behaviour (Choudhry, 2013). Thus, this implies that recent historical data can be used in the estimation of future VaR.

### 3.4.5 Regime switching models

The estimation of VaR, based on a long time horizon may not yield the best estimates because this period is likely to cover different market conditions. In other words, models will behave differently in volatile periods compared to periods of relative calm. In order to consider the different market conditions, a regime-switching method can be incorporated in the analysis where the data switches between a stable low-volatility regime and a more unstable high-volatility regime (Hardy, 2001). Hamilton and Susmel (1994) describe the importance of capturing volatility changes because asset prices are a function of volatility and also the correct specification of conditional variance is required in order to make efficient econometric inferences of the conditional mean of a variable. The regime switching was introduced by Hamilton (1989) as the Markov-switching model based on an autoregressive regime-switching process. Regime-switching models have been used in various studies like, Hamilton and Susmel (1994), (Hardy, 2001), Abad and Benito (2013), Kim, Y and Hwang (2018), Herrera, et al (2018), Nyberg (2018), Dias (2013), Chen and Shen (2012), Anderson and Guirguis (2012), Anoruo and Murthy (2017) to capture the dynamic behaviour of volatility in financial.

## 3.5 Performance of VaR models

This section explores the performance of the VaR models. Research by Beder (1995) reveals that the performance of VaR differs depending on what model is used. For example, the results for her research that calculates VaR on three hypothetical portfolios using several methodologies, based on historical and Monte Carlo Simulation models, result in significant discrepancies in the performance of VaR when applied to the same portfolios. This suggests that the set of parameters, data, assumptions, and methodology have a significant bearing on the VaR performance. Though this is the case, the is no set of models for this research that is accepted as correct. For example, while both historical simulation and Monte Carlo methods are based on past data and are used to calculate VaR, they result in different performances due to assumptions and different data needs. This, therefore, becomes a challenge for capital requirements. For this reason, Beder (1995) advocates for uniformity in the methodologies used for VaR. While Barone-Adesi et al. (2002) argue that while historical simulation reflects a more realistic picture of the portfolio's risk, they argue that this results in some disadvantage because the changing risk of assets are not captured and therefore there is the potential of underestimating the risk during high volatile market conditions. This is supported by Pritsker (2006) who demonstrates by investigating the capacity of the VaR estimates to respond to a crush for a portfolio which is long the S&P 500. The results show that the historical simulation VaR estimate has almost no response to the market crash and therefore observes that the performance of any risk measures that rely on historical simulation is affected by its distributional assumptions.

Berkowitz and O'Brien (2002) investigate the accuracy of VaR models at commercial banks by analysing the performance of VaR using bank models of six large U.S. banks. The findings for this study are that the bank model forecasts did not outperform GARCH model forecasts due to their ability to predict changes in the volatility of the P&L. Banks models, on the other hand, exhibited challenges in forecasting changes in the volatility of the P&L.

Another study by Kuester et al. (2006, 2005) explore the univariate VaR prediction using data on the technology index, the NASDAQ, through the comparison of alternative strategies i.e. historical simulation, FHS, GARCH, mixed normal GARCH, EVT, and similar to Engle and Manganeilli (2004) the conditional autoregressive VaR (CAViaR). While the results for this study show VaR underestimation in most of the models, the GARCH based models yield acceptable results with those that combine with EVT performing better together with the FHS methods. However, this performance depends to some extent on the chosen window size. These findings are in line those of Berens et al. (2018), who argue that the accuracy of the risk measure forecast is significantly impacted by the selection of the estimation window strategy. Although Engle and Manganeilli (2004) argue that the CAViaR models can adapt to new risk environments, the performance of CAViaR models in the study by Kuester et al. (2006, 2005 and Bao at al. (2003,2006), is not good. However, the indirect AR(1)-GARCH(1,1) model proposed by Kuester et al. (2006, 2005) is the most promising in the CAViaR class.

Abad and Benito (2013) examines several global stock indexes under two regimes (stable and unstable) to estimate both parametric and non-parametric VaR models; and extreme value theory. The study utilises conditional variance estimated by the EWMA, GARCH and EGARCH volatility models. In terms of the regime performance, all the VaR models perform better in a stable period compared to the unstable period which implies that VaR may not be the best method in volatile periods. Similar to Beder (1995) some results for the VaR methods depend on the volatility model employed, while also the distribution that is used is also important. In this instance, the parametric model estimated using asymmetric (E)GARCH which employed a student's t-distribution, resulted in the best model. This implies that the parametric model, despite its weakness works well with conditional volatility and appropriate return distribution.

Chen and Chen (2013) estimates VaR using the equally-weighted moving average, EWMA, Monte Carlo simulation and historical simulation for a portfolio consisting of securities from the Shanghai stock market. Their study concludes that VaR successfully evaluates financial risk and higher confidence levels resulting in higher levels of VaR albeit being varied for the different approaches. However, at low confidence levels, the different approaches have similar VaR.

Another study undertaken by Dias (2013) investigates the performance of VaR by considering parametric, semi-parametric and non-parametric approaches when estimated for companies with different market capitalisations obtained from different indices. This is done by taking a portfolio consisting of stocks with small to large market capitalisation and bunched into various size sub-portfolios whose VaR is estimated and then aggregated. The results show that the performance of VaR varies with market capitalisation in that the estimates improve when market capitalisation is considered. Also, the study reveals that historical simulation methods provide the best performance in several cases compared to the other methods. Like Abad and Benito (2013), this is done in different states of the market, i.e. crisis and non-crisis periods and provide evidence of the importance of market fundamentals for risk measurement.

In real estate, Lu, Wu and Wu (2009) applies VaR to the US REIT portfolios using five VaR methods namely, equally-weighted moving average, equally weighted moving average method fitted with a t-distribution, EWMA, historical simulation method and the bootstrapping method. These are done at 95% and 99% confidence levels. Like Beder (1995) the results showed that there was a difference in the performance of the different VaR methods. For example, different methods performed differently depending on leverage, i.e. the parametric methods and non-parametric methods performed better in the low leveraging and high leveraging portfolios respectively. While no method dominates the other at different confidence levels, at 95%, the equally-weighted moving average t- distributed method is the worst performer as it significantly overestimates VaR. Even if the two nonparametric methods also overestimating VaR, this is only slightly. The EWMA method, on the other hand, is the worst performer at 99% confidence level. In terms of the expected number of exceptions, in general, the equally weighted moving average has the best performance.

In anther study, Lee and Ou (2010) examine whether VaR provides a better estimate of risk compared to GARCH models for the day-of-the-week effect (DWE) in REITs. The outcome of this study reveals that the VaR estimate that utilises GARCH-DWE results in a more accurate forecast than GARCH and GARCH-DWE models.

The volatility of REITs in turbulent market periods like the 2008/9 financial crisis increases the importance of risk management and therefore research in how this risk can be captured. It is for this reason that Zhou (2012) undertook examines the alternative methods for measuring extreme risk for REITs. Zhou (2012) exploring the prediction of VaR and expected shortfall risk measures by utilising parametric, non-parametric and semi-parametric methods applied to US Equity REITs. The results indicate that on the one hand, two parametric models, the normal GARCH and the RiskMetrics provide the worst performance. On the other hand, the other parametric models E-GARCH -skewed t, and GARCH-t the best performers together with the semi-parametric model, GARCH-EVT whereas, FHS[24] falls somewhere in between. A similar study to Zhou (2012) is undertaken by Zhou and Anderson (2012) which also investigated extreme risk measures in REITs but at international level instead. It examines REITs in nine major global markets and estimates the risk measures (VaR and expected shortfall) under the three categories of non-parametric, parametric and semi-parametric models. The results show that while no particular method in the study is universally adequate to measure extreme risk across the global REIT markets, the FHS (nonparametric model) is the best method overall, unlike the results Zhou (2012) where this method falls somewhere in between the best and the worst performers. The study also compares the performance to non-REIT equities and finds that even if the financial crisis increased the extreme risk to both markets, the extreme risk for the REIT market was higher.

[24] In this study, FHS was categorised under the non-parametric model similar to Zhou and Anderson (2012).

The preceding literature shows varying levels of performance of VaR as an estimate of risk for both non-REITs and REITs, with the performance varying with the model employed. Overall, it would seem that the parametric and semiparametric models perform better compared to nonparametric models. As can be seen from the preceding, the use of VaR as a risk measure is ubiquitous. Even though there is some good performance in some intances, the question is whether VaR is a good risk measure. In order to answer this, one has to ascertain whether the risk measure in question is coherent.

### 3.5.1. Coherent risk measures

A risk measure should have specific characteristics in order to be considered good. Artzner et al. (1999) outline the axioms that should be satisfied by a 'good' risk measure. These axions are monotonicity, subadditivity, positive homogeneity, and translation invariance.

To satisfy the monotonicity property, an investment that has a worse result than another, for every event or state of the world, ought to have a higher risk measure (Hull 2015). Monotonicity is also referred to as stochastic dominance. However, Alexander (2008d) observes that this property is not always preserved by some risk-adjusted measures like the Sharpe ratio as it would in some instances result in a dominated asset exhibiting a higher Sharpe ratio than an asset that dominates it. Subadditivity stipulates that the risk measure for a portfolio of two investments should be less or equal to the weighted average of their respective risk measures (Hull, 2015 and Yamai and Yoshiba, 2005). This axiom accounts for diversification whose aim is to reduce risk without which there would be no incentive to hold portfolios. A risk measure that does not satisfy the subadditivity axiom would be used, in some instances, as a motivation to split up a large firm into two smaller ones (Tasche, 2002). Furthermore, the lack of subadditivity can render a specific risk measure unusable for risk budgeting.

Positive homogeneity is satisfied in an instance whereby the size of an investment changing by a factor h, results in the risk measure being multiplied by h (Hull, 2015). Relative sizes of the constituent assets should be held constant if this is a portfolio. The implication is that doubling the size of the investment or the portfolio without adjusting its relative composition would produce double the risk. Likewise, this would, in turn, entail doubling the capital requirement to cover the losses. Translational invariance suggests that adding (respectively, subtracting) the initial amount *n* to the initial position and investing it in the reference instrument X simply decreases (increases) the risk measure by *n* (Artzner et al., 1999). The reason for this is that this decreases (increases) the required capital cover by that amount (n).

Translational invariance is also referred to as the risk-free axiom (Danielson et al., 2005). Please see Artzner et al. (1999) for the mathematical proof and expansion of coherent risk measures.

Coherent risk measures need to satisfy all the above-desired axioms (Artzner et al., 1999). In general, risk measures that can aggregate risks in ways that take account for the diversification effects are favoured. For this reason, subadditivity is the most important property that a risk measure is expected to satisfy because it is key to diversification in line with Markowitz's modern portfolio theory(Dowd, 2005). A risk measure that is not subadditive does not guarantee a convex efficient frontier; hence, the implication is that it does not allow risk to be aggregated appropriately and priced (Kondor, 2014). Subadditivity in the case of internal risk management indicates that the overall risk of a financial firm is equal or less than the sum of the risks of individual departments of the firm (Danielson et al., 2003), a situation which if not upheld may lead to regulatory arbitrage (Kondor, 2014).

### 3.5.2 Problems with VaR

Acerbi et al. (2001) highlight the strength of VaR as being dependent on its capacity to apply to any financial instrument and its ability to be expressed in the same unit of measure, i.e. the money lost. They also point out that VaR includes an estimate of future events and permits the conversion of the risk of a portfolio into a single number. It is easy to apply, as it only needs the modelling of a quantile of a return distribution.

Despite the strengths above, VaR has come under many criticisms because it is not a coherent risk measure because it does not always satisfy the subadditivity condition, and this has become a contentious issue. In some instances, it exhibits "superadditivity", a situation where the VaR of a portfolio is greater than the sum of the VaR of the constituent assets. Alexander (2008d) notes that being as VaR is measured in value terms it can be coherent under particular assumptions regarding the distribution of returns, i.e. if a normal distribution is assumed. However, she contends that the non-coherence nature of VaR stems from the fact that the quantiles, as opposed to the variance operator, do not obey subadditivity unless the returns have an elliptical distribution. Similarly, Acerbi et al. (2001) acknowledge that VaR is subadditive in the Gaussian world and some other special cases. They also point out that this subadditivity is as a result of everything in the Gaussian world being proportional to the standard deviation which in turn is subadditive. However, like Alexander (2008d) and Acerbi et al. (2001); Danielson et al. (2005) also recognise that VaR is subadditive in situations when asset returns are normally distributed in the area below the mean, or more generally for log-concave distributions; nevertheless they correspondingly point out that these are exceptional cases. Dhaene et al. (2003), quoted in Danielson et al. (2005) contend that VaR is subadditive most of the time and therefore there is no real need to adopt more complicated risk measure based exclusively on account of subadditivity.

Furthermore, they assert that "imposing subadditivity for all risks (including dependent risks) is not in line with what could be called best practice". Also, Danielson et al. (2013) observe that the axioms of coherence result in a very restrictive set of risk measures that cannot be used in practical situations. Correspondingly, Emmer et al. (2013) argue that the lack of subadditivity of VaR may not be a serious issue provided that the underlying risks have a finite variance, or in some cases, a finite mean. Despite subadditivity being perceived to be an important condition for a risk measure, Kou and Peng (2014) argue that this argument is inconclusive. They contend that the notion that subadditivity implies that "a merger does not create extra risk" may not be true as observed from the merger of Bank of America and Merrill Lynch in 2008 arguing that mergers can lead to the creation of institutions that may become "too big to fail". Secondly, they declare that the notion that diversification is beneficial maybe a fallacy as Fama and Miller (1972) demonstrate the ineffectiveness of diversification for assets returns which exhibit fat tails. It can be seen that the issue of subadditivity is controversial; however, what is clear is that VaR does often violate this axiom. The significance of the subadditive condition is highlighted by Szegö (2005) who argues that measuring risk without this axiom is akin to measuring the distance between two points using a rubber band instead of a ruler. For this reason and the preceding, the inadequacy of VaR due to the lack of subadditivity cannot be ignored.

Another criticism of VaR is that it ignores the severity, i.e., the magnitude of the losses beyond the confidence level (significant level). As outlined earlier, VaR is used to denote, either the maximum possible loss given a confidence level's best case or the minimum potential loss given a significant level's worst case, over a specific period. The implication is that portfolios or assets with more likelihood for significant loses may be deemed to be less risky than those with less likelihood for big losses. It is for this reason that Acerbi et al. (2001) refer to VaR as the *"best of the worst case scenarios"* and they note that this risk measure systematically underestimates the potential losses related with the specific level of probability. Although subadditivity was earlier said to be the major weakness of VaR, not taking into consideration losses beyond VaR is deemed to be a more severe deficiency particularly when one faces choices of various risks with different tails (Emmer et al., 2013). Because of the preceding reason, Bader (1995) contends that VaR can be dangerous despite its seductive simplicity. To overcome the above challenges of VaR, a popular alternative, expected shortfall (ES) can be employed. The study now examines the expected shortfall as a risk measure.

## 3.6. The Expected shortfall

In order to address the shortcoming of VaR, Artzner et al. (1997) and Acerbi et al (2001) advocate the expected shortfall (ES) which they define as a specified period's mean (or [conditional] expected value) of the losses of the portfolio related to a particular level of a probability's worst cases. In other words, this is the mean of the losses beyond VaR for a specific confidence level. Separately, Delbaen (1998) and Rockafellar and Uryasev (2002) propose a model akin to the expected shortfall referred to as the "Conditional VaR (CVAR) by the latter. Expected shortfall is also referred to as Conditional VaR (CVaR), Tail VaR (TVaR), Tail Conditional Expectation (TCE) and Conditional Tail Expectation (CTE). The purpose of the expected shortfall by definition is particularly to counter the concern of VaR's inability to consider losses beyond the VaR level, also referred to as "tail risk" (Yamai and Yoshiba, 2005). The expected shortfall is regarded as a special case of a broader class of statistics known as exceedance measures because it describes a statistical relationship conditional on one or more variables being in its tail. Because the expected shortfall is by characterisation an exceedance mean (Sheppard, 2013), it is considered to be harder to manipulate (Kondor, 2014). In contrast with VaR, it is a coherent risk measure being as it does not violate any of the axioms of coherent risk measures. This is particularly so for the subadditivity axiom which even though VaR does not breach it in some cases, expected shortfall does not violate it in all cases. Additionally, because expected shortfall is convex, it can be optimised unlike VaR (Rockafeellar and Uryasev (2000). As Acerbi et al. (2001) put it*," … in full generality that it's impossible to build examples for which the assessment of relative riskiness among portfolios is trivial and in which at the same time expected shortfall gives opposite results".* Therefore, ES can be applied without restrictions.

In the Gaussian world, i.e. under normal distributions, both VaR and expected shortfall provide virtually the same information since they are both scalar multiples of standard deviation (Yami and Yoshima, 2005). However, they argue that for non-normal profit and loss distributions, VaR may have tail risk due to non-linearity of the portfolio position or non-normality of the underlying asset prices. A massive difference between VaR and expected shortfall suggests that there is a fat tail loss distribution (Tasche, 2013). Moreover, Yamai and Yoshima (2002) (cited in Yamai and Yoshima, 2005) observe that VaR is consistent with expected utility maximisation when portfolios are ranked by first-order stochastic dominance, while expected shortfall is consistent with expected utility maximisation when portfolios are ranked by second-order stochastic dominance. This they argue makes VaR more likely to have an unanticipated effect on utility maximisation than the expected shortfall.

Given the definition of the expected shortfall above, it can be formally defined as (Yamai and Yoshima, 2005):

$$ES_{\alpha} = E[X \,|X \,\geq\, Var_{\alpha}(X)] \tag{3.14}$$

From the definition, Equation (5.11) shows that ES is the expected value of a loss given that VaR has been exceeded. Yamai and Yoshima (2005) and Hull (2015) show that when the loss distribution is normal, the expected shortfall is calculated as follows:

$$ES_{\alpha} = E[X \,|X \,\geq\, VaR_{\alpha}(X)] = \frac{1}{\alpha\sigma_x\sqrt{2\pi}}\int_{VaR_{\alpha}(X)}^{\infty} t\,.\,e^{-t^2/2\sigma_x^2}dt = \frac{e^{-Y_{\alpha}^2/2}}{\alpha\sqrt{2\pi}}\sigma_x \tag{3.15}$$

Where $Y_{\alpha}$ is the 100α percentile of standard normal distribution (Yami and Yoshima , 2005). This is the point on a normal distribution with a mean of 0 and a standard deviation of 1 that has a probability $\alpha$ of being exceeded (Hull, 2015). Therefore, when a 0 mean is assumed, expected shortfall, like VaR, is proportional to the standard deviation.

It was established earlier that a good risk measure has to satisfy the axioms of coherent risk measures. In addition, it can be ascertained whether a risk measure satisfies the conditions of spectral risk measures and these are explored below.

### 3.6.1. Spectral risk measures

A risk measure can be characterised by the weights it allocates to percentiles (or quantiles) of the loss distribution such risk measures are referred to as spectral risk measures. There are conditions that need to be satisfied to make these spectral risk measures coherent. Dowd (2005) citing (Acerbi, 2004) shows that the following conditions have to be satisfied by a weighting function $\emptyset(p)$ to make a risk measure coherent. The first condition is non-negativity which means that all weights should be positive. The second condition is normalisation which states that for each risk measure all the weights in a portfolio ought to add up to one. The last condition is increasingness (weakly increasing) which says that if some probability $p_2$ exceeds another probability $p_1$, then $p_2$ must have weight equal to or greater than that of $p_1$.

According to Hull (2015), a spectral risk measure is coherent if the weights allocated to the qth quantile of the loss distribution is a non-decreasing function of q. Therefore, the third condition is what ensures coherence in that a risk measure ought to give higher losses at least the same weight as lower losses (Dowd, 2005). This reflects a user's risk aversion and Dowd (2005) argues that for a 'well behaved' risk-aversion function, then the weights will rise smoothly, and the rate at which the weights rise is proportional to risk aversion. If VaR at the $\alpha$ confidence level is given by $VaR_\alpha = q_\alpha,$ VaR gives a 100% weighting to the $\alpha$ quantile and zero to other quantiles (Dowd, 2008 and Hull, 2015). This is because by definition, VaR is a single quantile and is considered to be degenerate since it assigns an infinite value of the probability density function (pdf) to the $\alpha$ quantile and a zero value elsewhere (Dowd, 2005). Correspondingly, VaR does not satisfy the third condition and therefore does not qualify as a spectral risk measure. This is because it suggests that the user is risk-loving since no weight is given to losses above VaR. In other words, the VaR risk measures suggests negative risk aversion in the tail loss region. On the other hand, the ES at the confidence level $\alpha$ is $ES_a = \frac{1}{1-\alpha}\int_\alpha^1 q_p dp$ (Dowd, 2008). This means that ES assigns an equal weight of $\frac{1}{1-\alpha}$ to all quantiles greater than the $\alpha$ quantile and zero weight to all quantiles below the $\alpha$ (non-tail) quantile (Hull, 2015). Put differently, ES assigns equal weight on tail losses thereby satisfying the condition of a coherent spectral risk measure. Then again, despite ES being a coherent and spectral risk measure, like VaR, it does not display risk aversion. Instead, it exhibits risk neutrality since it assigns equal weights to losses above the VaR threshold

(Dowd, 2005). Despite ES being a coherent spectral risk measure, there are problems associated with expected shortfall as examined below:

### 3.6.2 Problems with expected shortfall

The key advantage that VaR has over expected shortfall is the ease of backtesting. While expected shortfall deals with the incoherence of VaR as a risk measure; its major criticisms stem from the difficulty of backtesting. The challenge in the backtesting of ES arises because it lacks elicitability (Gneiting, 2011).

#### 3.6.2.1. Elicitability

Elicitability is a property that enables a [risk] measure to have a scoring function that forecasts future losses and also makes a comparison of different models possible (Sherif, 2015 and Chen, 2014). The principles behind elicitability are explained using Kou and Peng (2014) and the references therein. Elicitability is said to provide a decision-theoretic foundation for effective evaluation of point forecasting procedures. The starting point is describing a point forecasting problem which is viewed as the measurement of risk of $X$ using $\rho$. This is where a distribution $F_x$ is summarised by the risk measurement $\rho(X)$ (or $\rho(F_X)$) using a real number. Kou and Peng (2014) observe that there is a need to find an estimate $\widehat{F_x}$ to use for forecasting the unknown true value $\rho(X)$ because the true distribution of $F_X$ is unknown. Because there may be a number of methods or procedures that can be used to forecast $\rho(F_X)$, evaluating, which procedure provides a better forecast of $\rho(F_X)$ is of the essence.

Suppose one wants to forecast the realisation of a random variable $Y$ using a point $x$, without knowing the true distribution $F_Y$. The expected forecasting error is given by:

$$ES(x, Y) = \int S(x, y) d\, F_X(y) \tag{3.16}$$

Where $S(x, y)$: $\mathbb{R}^2 \rightarrow \mathbb{R}$ is a forecasting objective function. The optimal point forecast corresponding to $S$ is

$$\rho^*(F_Y) = \arg\min(x)\ E[S(x, Y)] \tag{3.17}$$

For example, where $S(x, y) = (x - y)^2$ and $S(x, y) = |x - y|$, the optimal forecast of the mean functional (square error) and the median functional (absolute error) are $\rho^*(F_Y) = E(Y)$ and $\rho^*(F_Y) = F_Y^{-1}\left(\frac{1}{2}\right)$ respectively.

A statistical functional $\rho$ is elicitable if there exists a forecasting objective function $S$, such that it is minimising the expected forecasting error yield $\rho$. In other words, the mean score of the forecasting error should be as low as possible (as shown in Equation 5.14). Elicitability of $\rho$ enables the evaluation of two point forecasting methods by comparing their respective expected forecasting errors, $ES(x, Y)$. As the distribution $F_Y$ is unknown, the expected forecasting error can be approximated by the mean, $\frac{1}{n}\sum_{i=1}^{n} S(x, Y)$, where $Y_1, \cdots, Y_n$ are samples that have the distribution $F_Y$ and $x_1, \cdots, x_n$ are the corresponding point forecasts.

If a statistical functional $\rho$ is not elicitable, then for any objective function $S$, the minimisation of the expected forecasting error does not yield the true value $\rho(F)$. This means that it is not possible to ascertain which of the competing point forecasts for $\rho(F)$ performs the best by comparing their forecasting errors, regardless of what objective function $S$ is used.

Kou and Peng (2014) trace the concept of elicitability as dating back to the pioneering work of Savage (1971), Thomson (1979), Osband (1985), Lambert, Pennock and Shoham (2008) and Gneiting (2011). Kou and Peng (2014) recognise the importance of the specification of an objective function (i.e. the function $S$) being consistent for the target functional. They quote Gneiting (2011) as pointing out that "*in issuing and evaluating point forecasts, it is essential that either the objective function be specified ex ante, or an elicitable target functional be named, such as an expectation or a quantile, and objective functions be used that are consistent for the target functional.*"

It is possible to backtest a VaR model using the most recent data. This can be achieved by simply observing whether the number of exceptions (exceedances) that would have been encountered if the model had been used in the past is significantly different from what is expected (Hull, 2014). This is, however, not that straight forward in the case of ES. By definition ES measures all risks in the tail of a distribution, i.e., it is concerned with the average losses beyond a certain VaR threshold. Included in this are theoretically impossible losses not observable during backtesting (Chen, 2014). This infers that in backtesting ES, the prediction is an entire distribution while the realisation is a single scenario (Acerbi and Szekely 2014). Put differently; what is being tested is an expectation rather than a single quantile (Danielsson 2011). Consequently, this poses a difficulty because it entails averaging of multiple scenarios in backtesting in order to estimate the expected shortfall (Emmer et al., 2013). For this reason, unlike VaR, ES has a deficiency in terms of forecasting.

The standard solution to the backtesting conundrum of ES is utilising ES to carry out calculations then employ VaR after that do the backtesting. In this regard, despite the proposal by the Basel Committee on Banking Supervision (2013) to replace VaR with ES, they opted to keep VaR as a backtesting measure. This approach still leaves the tails untested (Acerbi and Szekely, 2014) and makes for a somewhat awkward position in which the risk measure being backtested is quite different from the one used to calculate capital (Hull, 2014). Besides, backtesting ES by making use of the violation ratios for VaR makes the shortfall procedure less reliable than that of the VaR backtest (Daielsson, 2011 and Acerbi and Szekely, 2014). This is because the indicated test is a joint test of the accuracy of VaR and the expectation beyond this VaR. For this reason, the estimation errors in the ES backtest also have to be taken into consideration. This suggests that the ES backtest is more likely to be less accurate than that of the VaR backtest (Danielsson, 2011). Accordingly, the expected shortfall backtests needs a bigger sample size as compared to the VaR. Danielsson (2011) argues that in instances where VaR is subadditive, the ES is obtained directly from VaR and results in the same signal as VaR, and it is, therefore, better to utilise VaR.

Some other methods have been proposed in practice to mitigate the deficiency above of ES (Emmer, 2013; Acerbi and Szekely, 2014 and Fissler, Ziegel and Gneiting, 2016). For example, Emmer et al. (2013) propose the approximation of ES with quantiles or expectiles hence rendering it possible to make use of backtesting methods for VaR. However, Acerbi and Szekely (2014) contend that while expectiles are coherent and elicitable alternatives to ES, their underlying concept is less intuitive than the concept of VaR and ES. Moreover, another problem that quantiles present is that they may display diversification where there is none (Tasche, 2013).

Despite numerous criticisms levelled against ES, Kerkhof and Melenberg (2004) observe that it is not necessarily harder to backtest than VaR. More recently, Acerbi and Szekely (2014) and Fissler, Ziegel and Gneiting (2016) argue that while elicitability is relevant for model selection, it is not relevant for model testing and it, therefore, has nothing to do with backtesting. They contend that elicitability permits the comparison, naturally, of different models that forecast statistics in the precise same sequence of events while recording only point predictions. This, they assert is model selection, not model testing and that *"it is a relative ranking, not an absolute validation".* Acerbi and Szekely (2014) further declare that even a hypothesis test based on elicitability still needs either the collection of the predictive distribution or strong distributional assumptions, with no guarantee of better power *a priori.* Besides VaR backtests entail counting exceptions, despite being elicitable. Moreover, they highlight that the simplicity of these tests and the recording of a point estimates lies in the quantiles defining a Bernoulli random variable as opposed to the elicitability of VaR. In another study, Fisseler, Ziegel and Gneiting (2016) prove that actually, expected shortfall is jointly elicitable with VaR. They show that the two backtests, i.e. the traditional backtest which is aimed at model verification by testing whether the estimates by the risk measure are correct and the comparative backtest whose purpose is to test the estimates of a particular risk measure are as good as another (e.g. one provided by a regulator). They argue that unlike the comparative backtest, the traditional backtest does not require elicitability as it looks at model validation as pointed out earlier. However, elicitability is crucial for the comparative backtesting as the purpose is to compare risk measures and in this instance, the ES and VaR can be used together as they are jointly elicitable.

#### 3.6.2.2. Estimation errors

Yamai and Yoshima (2005) show that even though the ES has better properties than VaR where tail risk is concerned, it does not always generate better results than VaR when simulation methods are adopted. Both VaR and ES estimates are affected by estimation error; however, the sampling fluctuation resulting from sample size is not substantial enough in VaR (Kondor, 2014). As the underlying loss distribution becomes more fat-tailed, the ES estimates become more varied due to infrequent and significant losses. Because of this, the ES estimation error grows more substantial than that of VaR (Yamai and Yoshima, 2005). The remedy to the high estimation error in the expected shortfall is to increase the sample size used in the simulations (Emmer et al., 2013 and Yamai and Yoshima, 2005) to obtain the same level of accuracy as VaR. Failure to do this, the optimisation for portfolios will become unfeasible (Kondor, 2014). Likewise, Sheppard (2013) observes that ES can only be measured when there is a VaR exceedance and that about four years of data would only produce about 50 observations where this is true. Therefore, the ES evaluation is constrained by the lack of data in the tails and this could result in the failure to reject a hypothesis in many cases even when using badly misspecified ES models (ibid). Because of this estimation error, for Gaussian tails and where there are heavier tails, a 1%VaR is equivalent to 2.5% ES (Acerbi and Szekely, 2014).

## 3.7. Data, methodology and analysis

### 3.7.1. Data

In order to perform market risk analysis, frequent data, such as intra daily or daily returns are required. For this reason, this study has made use of REITs returns, to apply market risk modelling to real estate because they are more susceptible to analysis owing to the availability of frequent data. For this reason, direct real estate returns were not considered due to infrequency data availability. Five (5) international REIT Indices spanning three continents, namely, the US, UK, France, Australia and Canada were used. The criterion for picking these markets was purely based on data availability i.e., those that had daily data from 1996 and this spans from January 1st, 1996 to December 31st, 2016. This is equivalent to 5480 data points of returns giving a total of 21 years, which includes the global financial crisis period. Like Zhou and Anderson (2012), it is noteworthy that apart from the US, not all the other markets had REITs in the period of investigation, for example, REITs in the UK were only introduced in 2007 but their returns have been backdated. This was done by making use of returns of property operating companies before the introduction of REITs in these markets. All the data for the REIT markets under consideration were obtained from Thompson Reuters Datastream®.

#### Descriptive statistics

Table 3.1 shows the descriptive statistics of the five REIT markets along with an equally-weighted portfolio created from these REIT markets. As expected the mean return for all the markets are about zero. The statistics show that the returns from these markets are not normally distributed. The returns are skewed as they all have non-zero third moments, i.e. the skewness. Additionally, the fourth moment, i.e. kurtosis and the Jarque Bera probability reveal the existence of fat tails because they show values over 3 and are very large respectively. The reported probabilities for the Jarque-Bera statistics are significant therefore leading to the rejection of the null hypotheses of normal distribution. This is in line with one of the stylised facts of financial assets, which state that returns for financial assets are not normally distributed and exhibit fat tails.

| | Australia | Canada | France | UK | US | Portfolio |
|---|---|---|---|---|---|---|
| Mean | 0.000116 | 0.000257 | 0.000451 | 0.000136 | 0.000206 | 0.000292 |
| Median | 0.000000 | 0.000000 | 0.000000 | 0.000119 | 0.000104 | 0.000516 |
| Maximum | 0.079842 | 0.091943 | 0.144561 | 0.103992 | 0.171574 | 0.060999 |
| Minimum | -0.120779 | -0.090097 | -0.225182 | -0.164104 | -0.206765 | -0.074997 |
| Std Deviation | 0.011914 | 0.009858 | 0.015628 | 0.013737 | 0.016602 | 0.008417 |
| Skewness | -0.704946 | -0.405114 | -0.405114 | -0.499407 | -0.154563 | -0.669984 |
| Kurtosis | 14.350740 | 12.615460 | 18.835200 | 13.094350 | 27.145300 | 12.652250 |
| Jarque-Bera | 29872.17 | 21337.65 | 57405.3 | 23494.03 | 133139.2 | 21682.86 |
| Probability | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |

Table 3.1…Descriptive statistics for international REIT indices

Note: This table is for the log returns of international REITs for the entire sample period, i.e., January 1st, 1996 to December 31st, 2016. The portfolio shown is equally -weighted.

The other stylised fact as highlighted earlier is that concerned with volatility clustering. This can be seen visually in Exhibit 3.1 that plots the returns of the five REIT markets and shows that periods of high volatility bunch up together indicating that the returns of the REITs are also prone to volatility clustering in the same way as mainstream financial asset returns.

Another stylised fact about financial returns has to do with the predictability of returns. As earlier pointed out, the returns of financial assets show little autocorrelation or serial correlations; however, squared returns, on the other hand, have a more significant serial correlation. Exhibits 6.2 and 6.3 show correlograms that provide a graphical display of estimates of serial correlation through the Autocorrelation Functions (ACFs). This demonstrates that at least 5% of the estimated correlations for returns lie outside the blue lines, i.e. the 95% confidence intervals. The squared returns show even more evidence of autocorrelation thus confirming volatility clustering[25]. REIT returns are, therefore in line with this stylised fact suggesting that it is difficult to predict returns for REITs compared to volatility as the squared returns have higher autocorrelations. This can be shown by utilising formal statistical tests like the Durbin–Watson test and the Ljung–Box test. For the data in question, the Durbin-Watson statistic is 1.2393 meaning that there is a positive serial correlation as it is below 2 (Johnston and DiNardo, 1997). The presence of serial correlation is supported by the Ljung-Box test whose results are shown in the correlogram in Exhibit 3.4 which indicate substantial and persistent autocorrelation in the residuals. No serial correlations happen when AC and PAC at all lags are near zero and all Q-statistics should not be significant as shown by the probability.

[25] This is perhaps with the exception of France where not all the points lie outside the blue lines but the majority do.

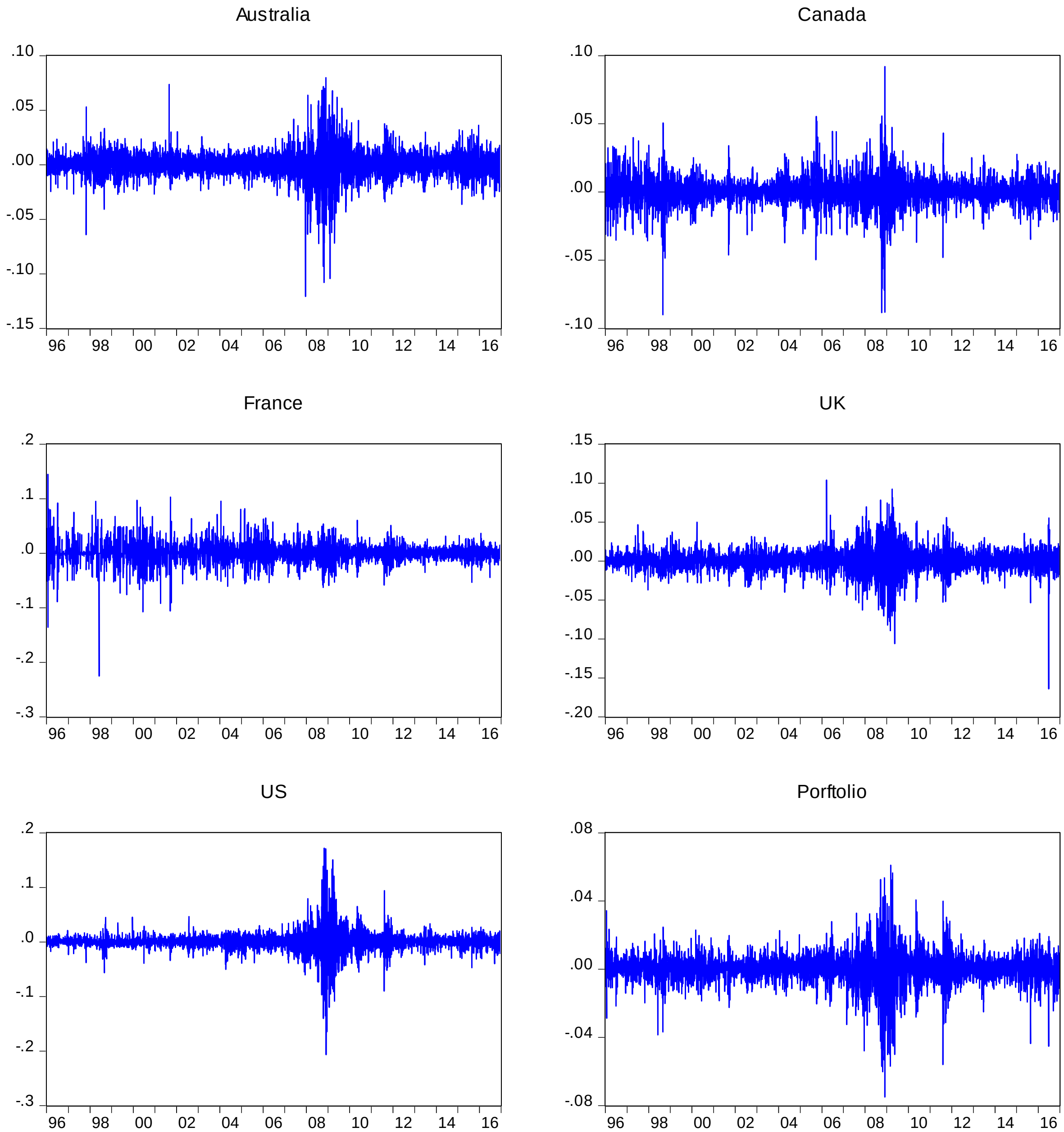



Exhibit 3.1: REITs log returns

Note: this exhibit shows plots of REIT log eturns for the different REIT markets for the period from 1996to 2016 .

ACF for returns

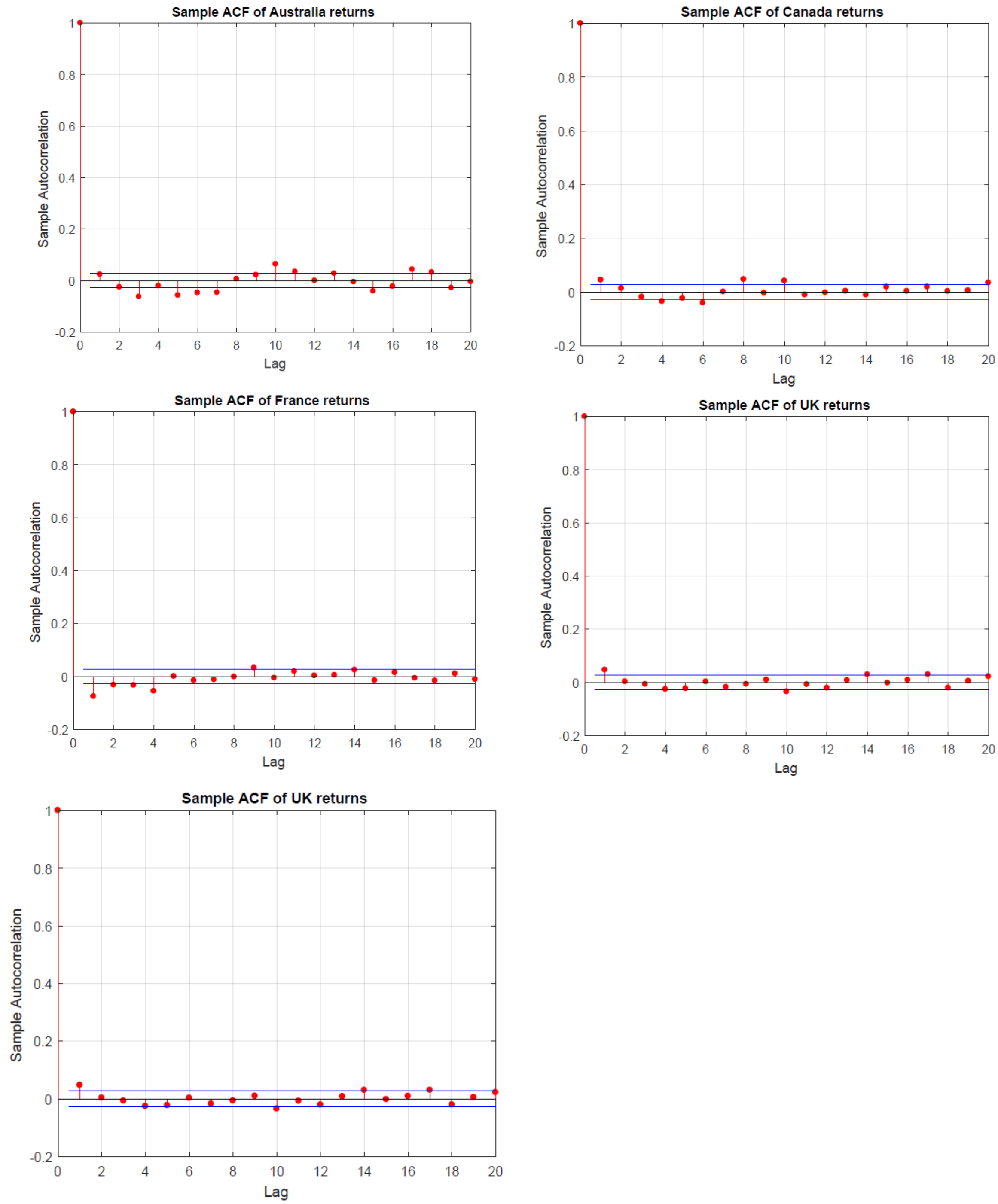



Exhibit 3.2.Autocorrelation Function (ACF) for REIT returns.
Note: The correlogram or autocorrelation plot shows the autocorrelation function of returns with 20 lags. The blue lines marking the standard 95% confidence intervals for the autocorrelations of a process of i.i.d. finite-variance random variables

ACF squared returns

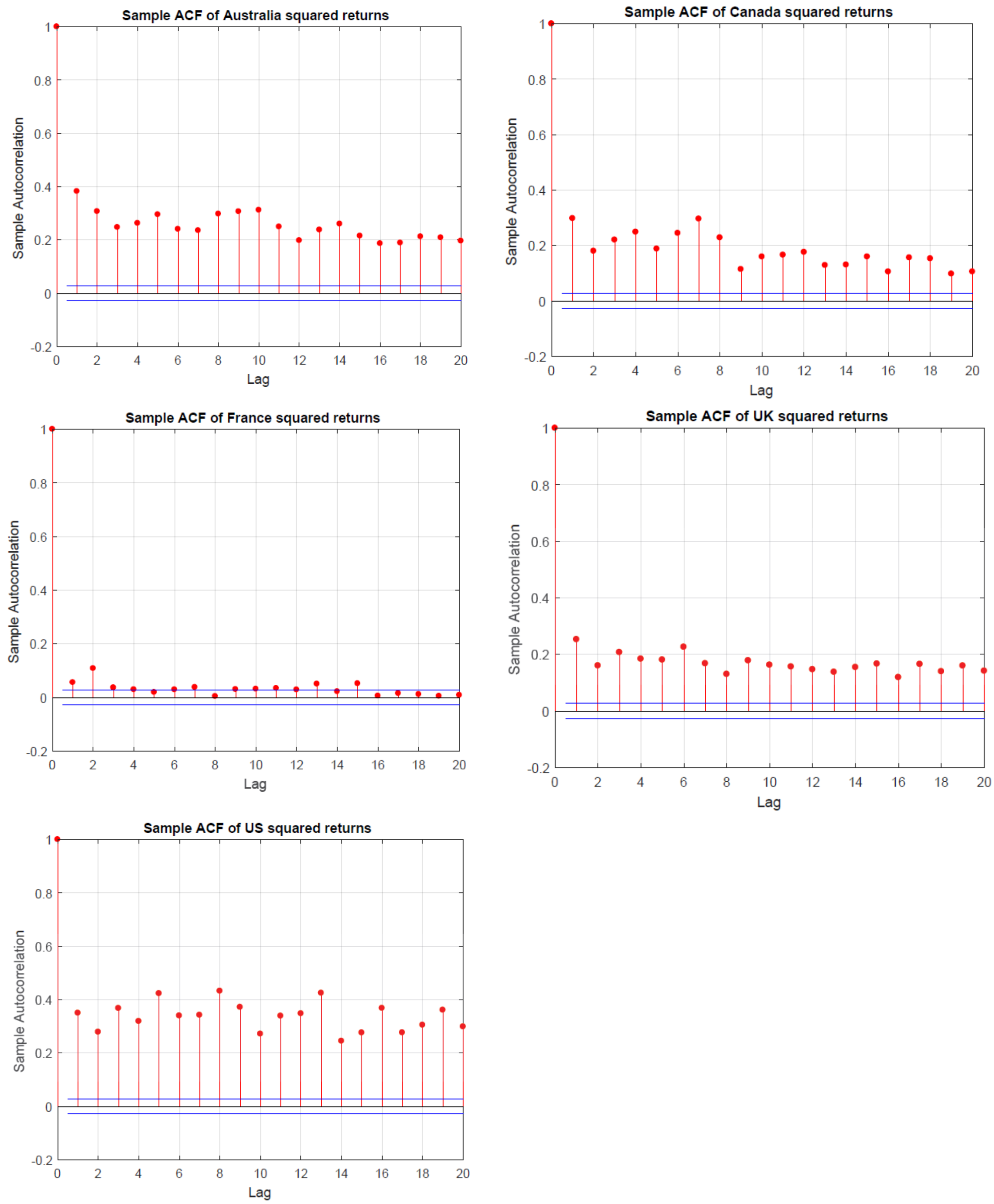



Exhibit 3.3 Autocorrelation function (ACF) for squared returns

Note: The correlogram or autocorrelation plot shows the autocorrelation function of squared returns with 20 lags. The blue lines marking the standard 95% confidence intervals for the autocorrelations

As earlier highlighted, fat tails characterise financial returns. This occurs because, compared to a normally distributed random variable, an asset or portfolio shows more extreme outcomes even when the mean and variance (of the random variable and the asset or portfolio) are the same. The Quantile-

Quantile (QQ) plots in Exhibit 3.5 demonstrate that the REIT returns are not normally distributed given that not all points lie alone the red line. These REIT returns also display leptokurtosis or fat tails since the points on each end of the QQ plots do not lie on the red line. In other words, these distributions are more narrow in the centre but have longer and heavier tails than the normal distribution. Fat tails imply that the extreme values will be expected compared to those from returns obtained from a normal distribution (McNeil et al., 2005). In this instance, the implication is that similar to the returns of financial assets; the normal distribution is not suitable to model these returns

Sample: 1/01/1996 30/12/2016
Included observations: 5480

| Autocorrelation | Partial Correlation | | AC | PAC | Q-Stat | Prob |
|---|---|---|---|---|---|---|
| | | 1 | 0.380 | 0.380 | 793.07 | 0.000 |
| | | 2 | 0.321 | 0.206 | 1357.1 | 0.000 |
| | | 3 | 0.339 | 0.200 | 1987.8 | 0.000 |
| | | 4 | 0.292 | 0.103 | 2456.1 | 0.000 |
| | | 5 | 0.302 | 0.120 | 2957.9 | 0.000 |
| | | 6 | 0.300 | 0.100 | 3452.3 | 0.000 |
| | | 7 | 0.277 | 0.065 | 3874.1 | 0.000 |
| | | 8 | 0.294 | 0.088 | 4348.7 | 0.000 |
| | | 9 | 0.301 | 0.087 | 4845.6 | 0.000 |
| | | 10 | 0.269 | 0.037 | 5242.9 | 0.000 |
| | | 11 | 0.281 | 0.060 | 5678.0 | 0.000 |
| | | 12 | 0.334 | 0.125 | 6289.3 | 0.000 |
| | | 13 | 0.331 | 0.098 | 6891.2 | 0.000 |
| | | 14 | 0.252 | -0.024 | 7239.5 | 0.000 |
| | | 15 | 0.263 | 0.018 | 7620.0 | 0.000 |
| | | 16 | 0.302 | 0.075 | 8120.1 | 0.000 |
| | | 17 | 0.245 | -0.013 | 8451.4 | 0.000 |
| | | 18 | 0.304 | 0.083 | 8960.7 | 0.000 |
| | | 19 | 0.273 | 0.019 | 9371.5 | 0.000 |
| | | 20 | 0.261 | 0.019 | 9746.0 | 0.000 |

Exhibit 3.4: Ljung-Box Correlogram for REITs – (in Eviews)

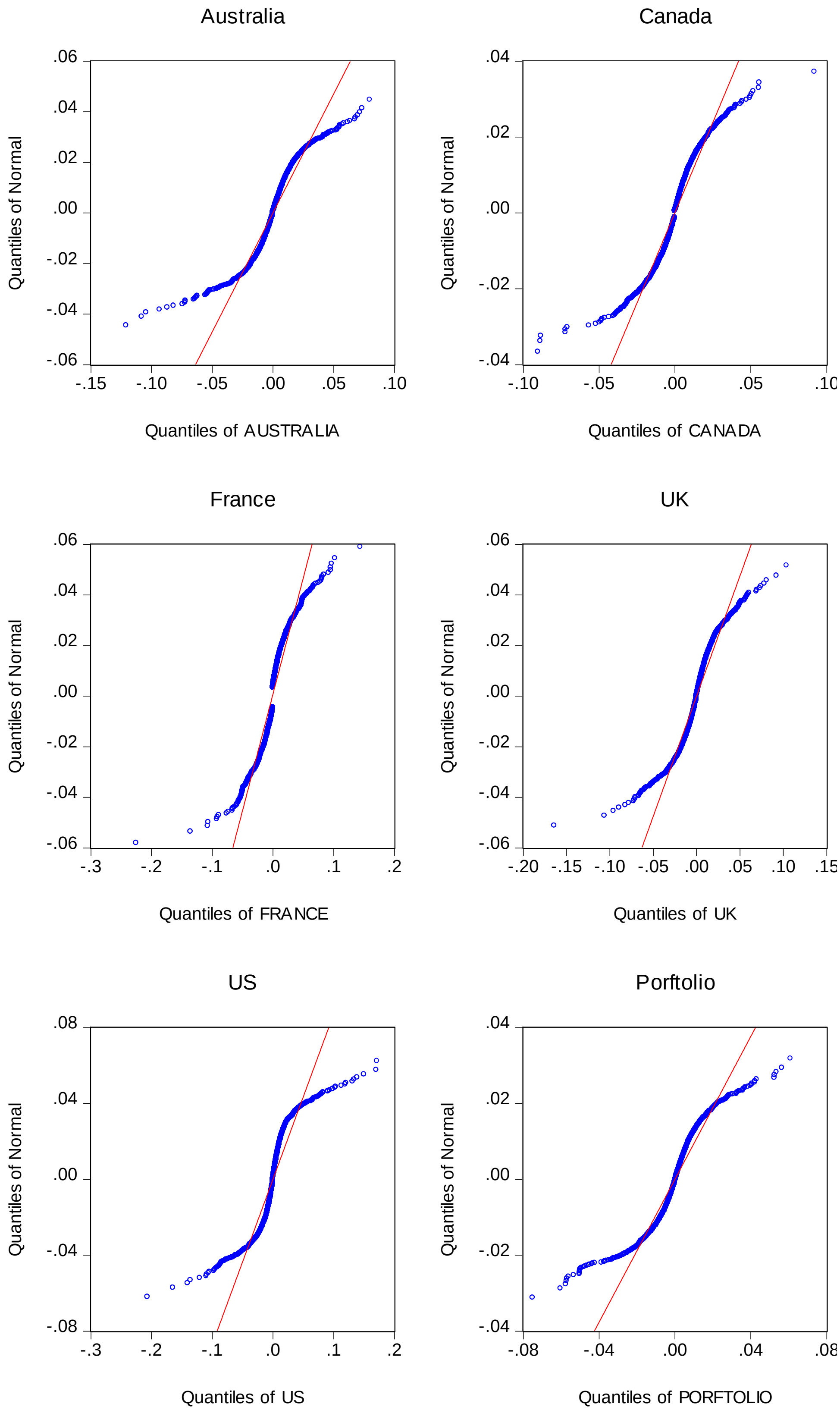



Exhibit 3.5: QQ plots for 5 International REIT indices and an equally weighted portfolio

Note: The Quantile-Quantile (QQ) plots show if the distribution of the data is normal in which case all the points lie on the redline. This is because the red line represents the theoretical normal distribution.

### 3.7.2. Methodology

The study estimates VaR and ES to ascertain its performance in all the five REIT markets at both univariate (asset) and multivariate (portfolio) levels. The univariate level estimates VaR and ES based on the equally-weighted, historical simulation, EWMA, and GARCH(1,1) models and backtesting undertaken by making use of the Risk Management toolbox in MATLAB®[26] and a method proposed by Danielsson (2011) for VaR and ES respectively. At the portfolio level, the Risk Management toolbox, as well as Danielsson's (2011) method, is utilised in order to assess the performance of the models. Further analysis is carried out taking the stylized facts of returns into consideration. Here, the GARCH-EVT-copula is be implemented to address the nonnormality, fat tails, non-linear dependences and also leverage effects. Backtesting is then implemented using methods proposed by Acerbi and Szesky (2014) to test the accuracy of the risk measures. In order to implement the above, MATLAB is utilised. The further analysis will use 99% and 97.5% losses at 1 day holding period estimation[27] for VaR and ES respectively. In order to reduce the estimation error, 1000 daily historical returns are used to forecast the VaR and ES. The student t distribution is employed to model the REIT data because of the unsuitability of the normal distribution in modelling returns of financial assets highlighted before (McNeil et al., 2005 and Koliai, 2016) which can lead to the underestimation of market risk.

### 3.7.3. Backtesting

The performance of the market risk models can be tested or validated by a process called backtesting. Christoffersen (2012) defines backtesting as a process that considers the ex-ante risk measure forecast from a model (in this case VaR or ES) and compares them with the ex-post realised returns. More formally, Dowd (2005) describes backtesting as the quantitative methods used in ascertaining whether a model's forecasts are consistent with the assumptions on which it is based, or in ranking against each other, a group of models. In other words, backtesting looks at assessing the accuracy of a model had it been applied in the past. This is achieved by comparing the realised or actual ex-post (i.e. historical) losses with the forecasted or potential (ex-ante) losses (for VaR for instance), for a particular data point. If the realised loss is more than the VaR, then there has been a violation of the forecast, and these are considered as exceptions or exceedances (Hull, 2015). For example, for a daily VaR with a 95% confident interval, the expected exceedances or exceptions should not be more than 5% of the days in

[26] MATLAB developed by MathWorks, is a numerical computing environment and proprietary programming language.
[27] A 5 and 10 day holding period was also going to be done but this was abandoned due to the protracted time that the simulations take. For example a 1 day holding period took about 8 days to run

the sample. Violations in excess of 5% of the days imply VaR underestimation of the losses while exceedances lower than 5% indicates an overestimation of losses. Both have implications particularly for capital allocations with the former likely to result in large losses while the latter may result in too much risk averseness and potentially missing out on opportunities. It is noteworthy, however; that while backtesting can identify the weakness of risk forecasting models, it does not provide information about the causes of the weaknesses (Danielson. 2011). This research is based on daily returns, and it is assumed that there are no intra-day changes in the value of the asset or portfolio. That said, the reality is that the value of an asset or portfolio changes during the day; however, for easy of calculations this will not be incorporated in this study.

In order to undertake the backtesting the following notation is used:

$W_T$ Testing window size

$W_E$ Estimation window

$T = W_T + W_E$ Number of observations in the sample

An estimation window, $W_E$ (i.e. the number of observations used to forecast the risk) of 1000 days is used similar to Kuester, et al (2006, 2005). A relatively high estimation window is chosen with EVT in mind, which according to Danielsson (2011) requires at least a sample size of 1000 as a rule of thumb. This is in order to capture the violations as they are normally observed infrequently. The data sample over which the forecasts have been made is called the testing window ($W_T$) and this corresponds to 17 years' worth of observations i.e. between January 2000 and December 2016. This means that the VaR and ES forecasts have been evaluated using a sample size of 5480 observations over a period of 21 years which is the sum of the estimation window and the test window. The backtesting procedure is summarised in Exhibit 3.6.

In essence, the VaR and ES are estimated over the testing period using the estimation window and the forecast will be compared to the return after a holding period (H) of one day. This results in the ($VaR_{W_E+1}$ for the first iteration. The process will be repeated for the number of 4435 iterations in the testing window. Even though both VaR and ES are both backtested, for simplicity the notations used to illustrate the backtesting process is based on VaR.

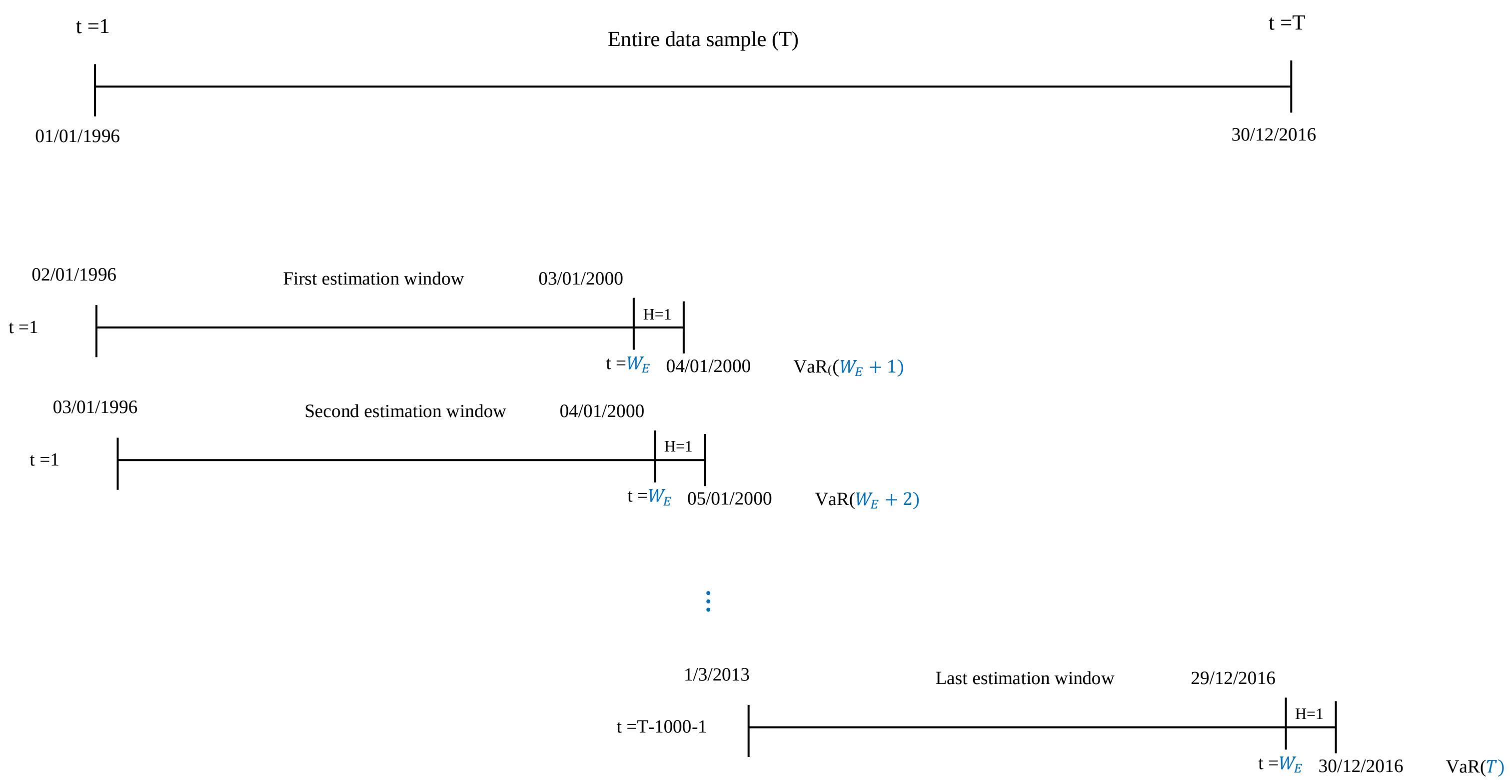



Exhibit 3.6: Backing procedure

Note: The entire data spans about 21 years and 1000 daily observations are used to forecast the risk measures and backtesting is done over 17 years.

### Definition of violations

As previously stated, a violation occurs when the actual observed return exceeds the forecast VaR. In other words, when the VaR limit has been violated (Danielson, 2011) and therefore there has been an exceedance. Each violation is given a value of 1 whereas a value of 0 is given when there is no violation. Formally, the VaR violation are denoted as presented by Danielson (2011) and Jäger (2015) as

$$\hat{I}_{t+H}^{H} = \begin{cases} 1 & if\ L_{t+H} \geq VaR_c^{t,H} \\ 0 & if\ L_{t+H} \leq VaR_c^{t,H} \end{cases} \tag{3.18}$$

Where $L_{t+H}$ represents the observed loss at point $t + H$ and $VaR_c^{t,H}$ is the forecasted VaR at time $H$ but for point $t + H$.

The number of violations for the test period is given by

$$\hat{I} = \sum_{t=1}^{T} I_{t+H}^{H} \tag{3.19}$$

Dowd (2005) highlights that formal statistical backtesting is based on standard hypothesis testing. This involves stating the null hypothesis together with an alternative hypothesis that is to be accepted in the case of the rejection of the null hypothesis. The hypothesis test is undertaken at a specific significant level which specifies the probability associated with the point at which the hypothesis is 'accepted' or is 'true' or formally where the decision is to 'fail to reject' the null hypothesis. The hypothesis is 'accepted' if the estimated value of this probability, exceeds the chosen significance level, and is rejected otherwise (Dowd, 2005). The higher the significance level, the more likely the null hypothesis will be accepted, and the less likely a Type I error will be made, i.e. the incorrect rejection of an accurate model. However, this increases the possibility of occurrence of a Type II error, i.e. the incorrect acceptance of a false model. Any test, therefore, involves a trade-off between these two types of possible errors. Preferably, a significance level that takes account of the likelihood of these errors (and, in theory, their costs as well) and strikes an appropriate balance between them, should be selected. Nevertheless, it is customary in practice for an arbitrary significance level, such as 5%, to be selected and applied in all the tests. A significant level of this magnitude gives the model a particular benefit of the doubt suggesting that the model would only be rejected in instances where the evidence against it is reasonably strong (Dowd, 2005).

As stated earlier, in order to assess the performance of VaR and ES, backtesting was undertaken using at univariate and multivariate level, i.e., asset and portfolio levels respectively. Before undertaking these backtests, the respective backtesting methods for both VaR and ES are reviewed

### 3.7.4. Backtesting VaR

#### 3.7.4.1. Binomial Test

The primary frequency test also referred to as the binomial test, introduced by Kupiec (1995), is the most widely used because it is the most straightforward. It tests whether the number of observed losses $x$, exceeding VaR (observed frequency of tail losses) is consistent with those predicted by the model (Dowd, 2005). In other words, the aim is to test whether the fraction of violations is significantly different from that promised fraction (Chrisoffersen, 2012). As indicated before, when performing a backtest, a violation; exception or exceedance takes a value of 1 and a value of 0 when there is no violation as shown by $\hat{I}_{t+H}^{H}$ (see Equation 6.1). The result is therefore, a sequence of ones and zeros. The hit sequence of violation should be completely unpredictable and therefore distributed independently over time as a variable that takes the value of 1 with probability $\alpha$ and the value 0 with probability$(1-\alpha)$. When done over the testing window, $(\widehat{I}_{t+H}^{H})_{t=1,\ldots,T}$ is a Bernoulli process of i.i.d. with a success probability $\alpha$. The Bernoulli process has two aspects. The first aspect tests that the number of violations is correct on average and is achieved by utilising a standard one-sided binomial test. The other aspect checks if the i.i.d. property is satisfied (Jäger, 2015). The former is what the Basel Committee demands. The observed significance level is defined as:

$$\alpha' = \frac{Observed\ number\ of\ violation}{Number\ of\ trading\ days\ in\ the\ backtest\ period} = \frac{I}{T} \tag{3.20}$$

The null hypothesis $H_0$ checks whether the percentage of the violation $\alpha'$ is greater than or equal to $\alpha$ i.e. $(1-CL)$. So formally this translates to :-

$$H_0: a \leq \alpha' \tag{3.21}$$

The test statistic is given by the number of violations, which in the testing window, is binomially distributed if $H_0$ is satisfied; therefore

$$Z_0\left(\vec{L}^H = \sum_{t=1}^{T} \hat{I}_{t+H}^H \sim B(T,\ \alpha')\right) \tag{3.22}$$

The probability of getting precisely $k$ violations in $T$ trials is given by the density of the binomial distribution:

$$f(k,T,\alpha) = \binom{T}{k}\alpha^k(1-\alpha)^{T-k}, \binom{T}{k} = \frac{T!}{k!(T-k)!} \tag{3.23}$$

Where $\alpha$ is equal to 1 minus the confidence level, $T$ is the sample size and $k$ the number of observations whose losses exceeding the VaR.

$$Z_{bin} = \frac{x - Np}{\sqrt{Np(1-p)}}$$

The rejection of the null hypothesis implies that the VaR forecast has either been underestimated or overestimated. However, $a \neq \alpha'$does not necessarily mean that it a model is bad because the statistical deviations of the estimation of $\alpha'$ may not be out of the ordinary (Jäger,2015). For this reason, the it is necessary to measure the significance of this departure and this can be achieved by the application of the Basel Committee's "traffic light" system which consists of red, yellow and green zones.

#### 3.7.4.2. The traffic light Test

Based on a testing window (T=250) and $\alpha = 1\%$ the Basel Committee defines a risk model in the red zone as having k violations producing a Type I error with a probability of, at most, 0.01%. This is equivalent to a probability of, at most, k violations amounting to at least 99.99% in which case the model must be rejected, as it is significant. A model is in the green zone when the probability of, at most, k violations amounts to less than 95%. This model can be used as it is not significant. The yellow zone lies between the green and red zones and calls for the calibration of with a factor for further calculation. Table 3.2 below summarises the definitions of the Basel Committee traffic zones based on 250 days test window.

| Basel Zone | Violations (k) | Probability zone | Decision |
|---|---|---|---|
| Red | 10 ≤ k | $P(X \leq k) = \sum_{i=0}^{k} \binom{n}{i} \alpha'(1-\alpha)^{T-i} \geq 99.99\%$ | Reject model |
| Yellow | 5 ≤ k ≤ 9 | $95\% \leq P(X \leq k) = \sum_{i=0}^{k} \binom{n}{i} \alpha'(1-\alpha)^{T-i} < 99.99\%$ | Calibrate model with factor for further calculation |
| Green | K ≤ 4 | $P(X \leq k) = \sum_{i=0}^{k} \binom{n}{i} \alpha'(1-\alpha)^{T-i} < 95\%$ | Use model |

Table 3.2: Traffic light test. Source: Danielsson (2011) and Jäger (2015)

Note: The definitions of the traffic light system in the above table is based on a 250 days (trading days) window. Green, Yellow and Red signify the number violations that are 4 or less, between 5 and 9, and at least 10 respectively

#### 3.7.4.3. Kupiec's proportion of failures

The proportion of failures (POF) test is a variation of the binomial test that was introduced by Kupiec (1995). This test through a likelihood ratio tests whether the probability of violations corresponds with the probability p suggested in the confidence level of the VaR. The VaR model is accepted or rejected if the probability of violation (exceptions) are in line or different with p respectively. In its basic form, similar to equation (6.2) the observed proportion of failures is defined as:

$$p' = \frac{Observed\ number\ of\ violation}{Number\ of\ trading\ days\ in\ the\ backtest\ period} = \frac{I}{T} \tag{3.24}$$

The null hypothesis $H_0$ checks whether the proportion of failures (percentage of the violation) $p'$ is greater than or equal to $p$ i.e. $(1 - CL)$. So formally this translates to :-

$$H_0: p \leq p' \tag{3.25}$$

#### 3.7.4.4. Time Until First Failure (TUFF)

The time Until First Failure is a simple test centred on the time when the first exceedance occurs. Its recommended use is when long runs of data are unavailable, for instance in the period shortly after the introduction of a new risk model or major changes to an existing one have been made (Dowd, 2005). The model is more likely to be rejected, the shorter the time to the first failure. The weakness of this test is that it ignores whatever happens after the first violation. This test, therefore, leaves out a lot of information. Give a probability of an exceedance $p$, the probability of observing the first exceedance in period $T$ is $p(1-p)^{T-1}$, and the probability of observing the first exceedance by period $T$ is $1-(1-p)^{T}$ thus obeying a geometric distribution (Dowd, 2005).

The previous two tests are basic, whose strengths lie in their simplicity, easy of application and parsimonious nature in that they do not require a lot of information, i.e. they can be implemented by knowing only n, p and x. Their foremost weakness, however, is that they are not able to identify bad models because they lack the power unless applied to large samples. This stems from the earlier mentioned problem of disregarding potentially useful information. Dowd (2005) identifies the two types of discarded information below:

- The tests discard information regarding the temporal pattern of exceedances, as the frequency of exceedances is the only focus. This is so because many risk models predict that exceedances are i.i.d. i.e. the probability of a tail loss is constant and independent of whether or not an exceedance occurred the previous period.
- Frequency tests discard information on the sizes of tail losses predicted by models. The implication is that a 'bad' risk model that generates an acceptably accurate frequency of exceedances is likely to pass despite having poor forecasts of losses larger than VaR.

In order to address the weaknesses above, the fundamental frequency tests are rewritten in the likelihood ratio (LR) form based on Christoffersen (1998). The test here is that the model predicts the 'correct' frequency of exceedances or in other words looks at the prediction of correct unconditional coverage (Dowd, 2005). If $x$ is the number of exceedances in a sample, and $n$ is the number of observations, then the observed frequency of exceedances is $x/n$ (similar to Equation (3.24)). Given that, the predicted probability of exceedances is $p$ (i.e. 1 minus the confidence level); the earlier tests can be expressed in terms of LR test. Under the hypothesis/prediction of correct unconditional coverage, the tests statistics of the POF and TUFF are given in Equations (3.26) and (3.27) respectively:

$$LR_{POF} = -2ln\left(\frac{(1-p)^{N-x}p^{x}}{\left(1-\frac{x}{N}\right)^{N-x}\left(\frac{x}{N}\right)^{x}}\right) = -2\left[(N-x)log\left(\frac{N(1-p)}{N-x}\right) + xlog\left(\frac{Np}{x}\right)\right] \quad (3.26)$$

Where $x$ and $N$ represent the number of failures and the number of observations respectively. $p = 1-CL$ (CL is the confidence level)

$$LR_{TUFF} = -2ln\left(\frac{p(1-p)^{n-1}}{\left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^{n-1}}\right)$$

$$= -2(\log(p) + (-1)\log(1-p) + nlog(n) - (n-1)(\log(n-1) \qquad (3.27)$$

In Equation (3.27), with $n$ represents the number of days until the first exceedance. This test statistic is asymptotically distributed as a chi-square variable with one degree of freedom (Kupiec 1995).

#### 3.7.4.5. Coverage tests

##### Unconditional coverage test

A test of the null hypothesis that is assumed to follow an i.i.d. Bernoulli process and has a constant 'success' probability equal to the significance level of the VaR, $\alpha$, is referred to as unconditional coverage test ( Alexander, 2008d). Here, the test statistic is a likelihood ratio statistic given by

$$LR_{uc} = \frac{\pi_{exp}^{n_1}\left(1-\pi_{exp}\right)^{n_0}}{\pi_{obs}^{n_1}(1-\pi_{obs})^{n_0}} \qquad (3.28)$$

Where $\pi_{exp}$ is the expected proportion of exceedances, $\pi_{obs}$ is the observed proportion of exceedances, $n_1$ is the observed number of exceedances and $n_0 = n - n_1$ where $n$ is the sample size of the backtest. Put differently, $n_0$ is the number of returns with indicator 0 (i.e. where there are no exceedances), $\pi_{exp}$ = $\alpha$ and $\pi_{obs}$ = $n_1/n$. The asymptotic distribution of $-2\ln LR_{uc}$ is chi-squared with one degree of freedom.

Alexander (2008b) observes that in order to reduce rounding errors, it is better to compute the log of likelihood ratio statics directly as shown below, then taking the log afterwards.

$$\ln(LR_{uc}) = n_1\ln\left(\pi_{exp}\right) + n_0\ln\left(1-\pi_{exp}\right) - n_1\ln(\pi_{obs}) + n_0\ln(1-\pi_{obs}) \qquad (3.29)$$

As observed earlier, one of the stylised facts is that financial returns usually present volatility clustering. This can result in the clustering of exceedances indicating that the VaR model is not sufficiently responsive to changing market conditions (Alexander 2008d). Consequently, a VaR model could still be rejected if the exceedances are not independent despite passing the unconditional coverage test. This is because violations that are clustered together increase the probability of bankruptcy compared to those that are spread randomly over time (Christoffersen, 2012). For this reason, Christoffersen (1998) developed the independence test.

### Independence test

Unlike the unconditional test, when exceedances are not independent the probability of an exceedance tomorrow, given there has been an exceedance today, is no longer equal to $\alpha$. Notations by Alexander (2008d) are used to describe the independence test. As before, let $n_1$ be the observed number of exceedances and $n_0 = n - n_1$ be the number of 'good' returns as in the conditional test. Let $n_{ij}$ be the number of returns with indicator value $i$ followed by indicator value $j$, i.e. $n_{00}$ is the number of times a good return is followed by another good return. Likewise, $n_{01}$ represents the number of times a good return precedes an exceedance, $n_{10}$ the number of times with an exceedance followed by a good return, and $n_{11}$ the number of times with an exceedance followed by another exceedance. So $n_1 = n_{11} + n_{01}$ and $n_0 = n_{10} + n_{00}$. Also let

$$\pi_{01} = \frac{n_{01}}{n_{00} + n_{01}} \; and \; \pi_{11} = \frac{n_{11}}{n_{10} + n_{11}}$$

i.e. $\pi_{01}$ is the proportion of exceedances, given that the last return was a 'good' return and $\pi_{11}$ is the proportion of exceedances, given that the last return was an exceedance. The independence test statistic as derived by Christoffersen (1998) is:

$$LR_{ind} = \frac{\pi_{obs}^{n_1}(1 - \pi_{obs})^{n_0}}{\pi_{01}^{n_{01}}(1 - \pi_{01})^{n_{00}}\,\pi_{11}^{n_{11}}(1 - \pi_{11})^{n_{10}}} \tag{3.30}$$

The asymptotic distribution of $-2\ln LR_{ind}$ is chi-squared with one degree of freedom.

The independence test only works if exceedances are consecutive because it is based on a first-order Markov chain only. In order to detect exceedances that are not consecutive, an extension is applied to a higher order Markov chain that allows more than first-order dependence. In such cases, the conditional coverage test, which is a combined test for both unconditional coverage and independence, can be employed. This is given by:-

$$LR_{cc} = \frac{\pi_{exp}^{n_1}\left(1 - \pi_{exp}\right)^{n_0}}{\pi_{01}^{n_{01}}(1 - \pi_{01})^{n_{00}}\,\pi_{11}^{n_{11}}(1 - \pi_{11})^{n_{10}}} \tag{3.31}$$

The asymptotic distribution of $-2\ln LR_{cc}$ is chi-squared with two degrees of freedom. It follows that $LR_{cc} = LR_{uc} + LR_{ind}$ and hence $-2\ln LR_{cc} = -2\ln LR_{uc} - 2\ln LR_{ind}$

## 3.7.5. Backtesting results

The study has split the backtesting results into two. The first set of results utilise the Risk Management Toolbox from MATLAB and incorporate the results for a simple ES test proposed by Danielsson (2011). This looks at results for both univariate and multivariate levels. The second set of backtesting is undertaken at the multivariate level only and involves backtest procedures proposed by Acerbi and Szekely (2014).

### 3.7.5.1.0 Univariate (Asset) level

The market risk modelling of the five International REIT indices was undertaken in MATLAB using the "Risk Management" toolbox. Both VaR and ES are estimated by making use of the Normal VaR, Historical Simulation (HS), EWMA and GARCH(1,1) at 95% and 99% confidence levels. Exhibits 3.7 and 3.8 show the performance of VaR and ES for these approaches at 95% confidence level when compared to the returns for each of the five REIT markets.

As expected, it can be seen from both Exhibits 3.7 and 3.8 that the Normal VaR and Historical Simulation for both 95% and 99% VaR estimates are generally stable over time because equally-weighted volatility is assumed. They are generally low but increase after the 2008 financial crisis and mostly remaining high until around 2013. The EWMA and GARCH volatility models result in more realistic VaR and ES estimate because they are time-varying, unlike the Normal and Historical Simulation. In terms of whether the models are good or bad, formal backtests whose results are summarised in Exhibits 6.8 to 6.16, were undertaken.

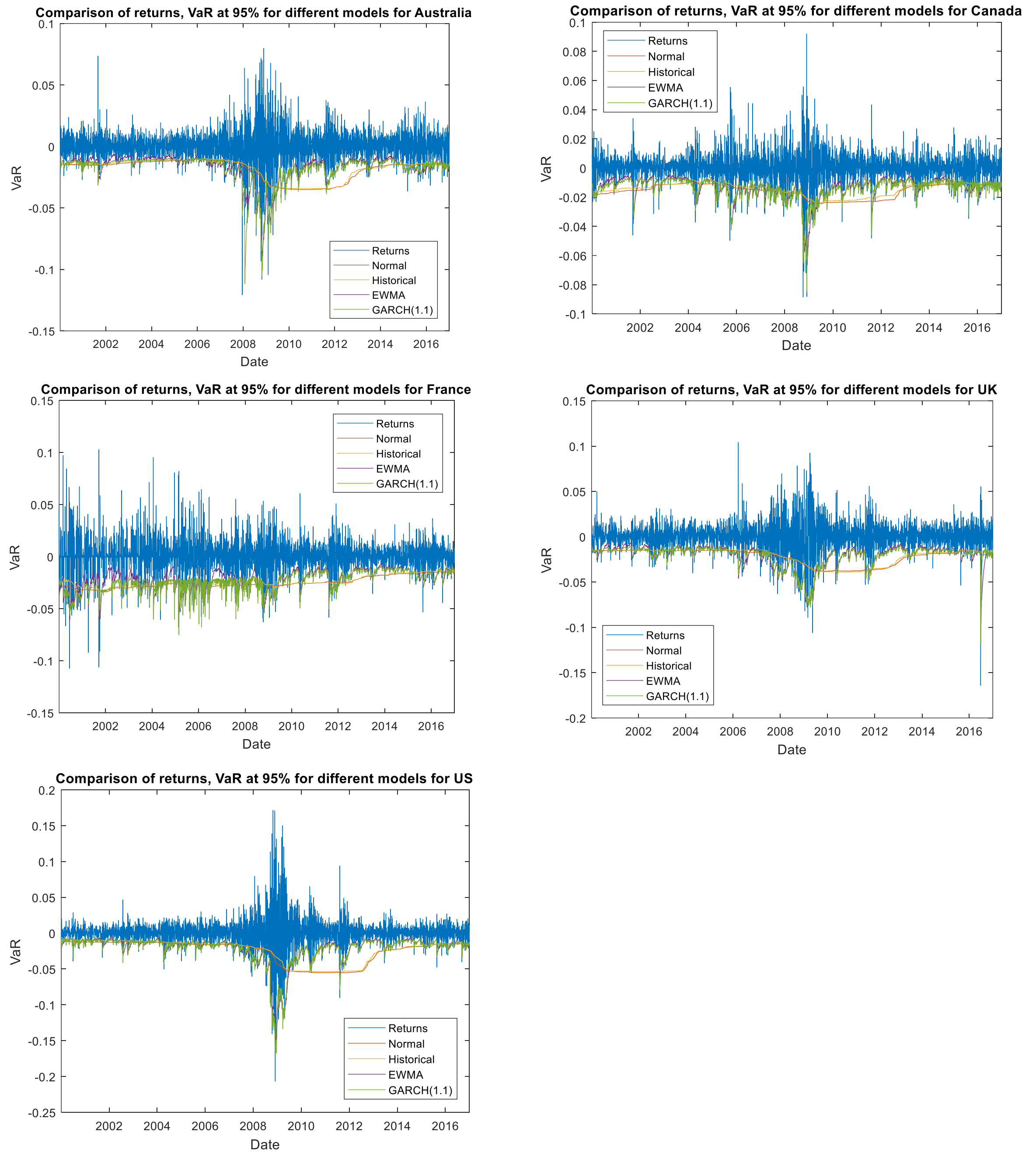



Exhibit 3.7… Comparison or returns and 95% VaR for international REIT indices

Note: The exhibit above shows returns of different REIT indices compared to the 95% VaR predictions estimated by the different VaR models, namely normal VaR, historical simulation, EWMA and GARCH(1,1)

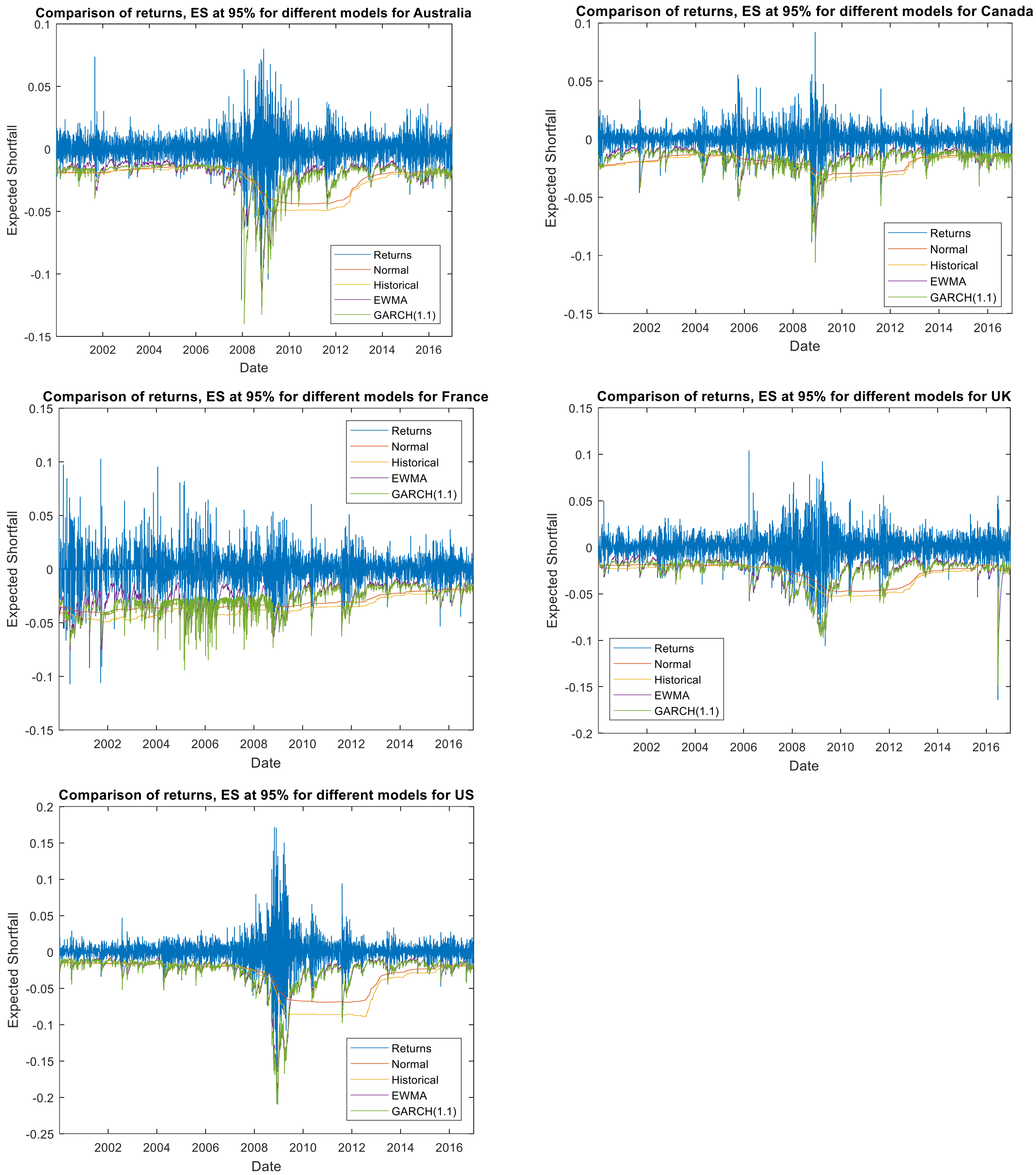



Exhibit 3.8: Comparison or returns and 95% ES for international REIT Indices

Note: The exhibit above shows returns of different REIT indices compared to the 95% VaR predictions estimated by the different VaR models, namely normal VaR, historical simulation, EWMA and GARCH(1,1)

#### 3.7.5.1.1. Formal VaR backtesting

Exhibits 3.9 to 3.11 show the formal backtesting results for 95% and 99% VaR for the Normal, Historical Simulation, EWMA and GARCH models for all Australia, Canada, France, UK and the US REITs markets. Exhibit 3.9 shows the detailed backtesting results for the Normal 95% VaR. The results for the other VaR models, i.e. historical simulation, EWMA and GARCH(1,1) and also the 99% VaR in Appendix D Panel (a) of Exhibit 3.9 comprises of the basic results for the backtests. The expected [failure] obtained by multiplying the number of observations in the backtest by the significant level, illustrates the number of failures anticipated if the model is accurate. The failures are the actual number of violations or exceedances produced by the model. The observed level is the ratio between the actual failures and the number of observations. If the tests are accurate, the observed level for 95% VaR and 99% VaR should be result in observed levels of 0.95 and 0.99 respectively. The ratio simply represents the failures as a proportion of the expected failures and should result in a value of 1 if the model is accurate.

Panels (b) and (c) shows the test statistics and the p-values for the VaR backtests methods described previously, namely; the binomial, traffic light system, proportion of failure, time until first failure, conditional coverage and conditional coverage independence tests. Lastly Panel (d) summarises the hypothesis tests using a test level of 95% for each of the volatility models.

Exhibits 3.10 and 3.11 summarises the backtest results for both 95% VaR and 99% VaR as stated above; the detailed results are in Appendix D.

#### 3.7.5.1.2. 95% VaR backtesting results

##### Normal VaR

The traffic light system mostly results in green and yellow regions resulting in the acceptance of the model. The hypothesis is rejected for all the REIT markets except Australia and Canada for the binomial and the proportion of failure test. The time until first failure tests are not significant for all the markets and therefore the hypothesis is accepted. The results for the conditional coverage and the conditional coverage independent test are significant for all the REITs markets considered.

### Historical simulation VaR

Exhibit 3.10 shows that the traffic light system resulted in green and yellow for Canada, France and Australia, the UK respectively resulting in a decision to fail to reject the null hypothesis; however, it is rejected for the US that is in the red zone. The binomial and proportion of failure produce the same results that show that the test is not significant for all of the REITs markets except UK and US whose p-values are statistically significant leading to rejecting the null hypothesis that the model accurately predicts VaR. For the time until to first failure, only France is significant meaning that the model accurately predicted the VaR for the other markets. Similar to the normal, the conditional coverage and conditional coverage independent are also significant leading to the reject of the hypotheses.

### EWMA

The traffic light system suggests that the model accurately predicts VaR with all the markets falling in the green zone except the UK that is in the yellow zone but is still acceptable as discussed earlier. The binomial, proportion of failure, time until first failure are not significant for all the REITs markets. All the REITs markets are significant under the conditional coverage tests while for the conditional coverage independent tests only the UK is not significant.

### GARCH

All the REITs markets are in the green zone for the traffic light system. Similarly, the binomial and proportion of failure tests are not significant suggesting that null hypotheses of the model are correctly predicting the VaR should not be rejected. The time until first failure test results in the falling to reject the hypotheses for all the REITs markets except for France. Concerning the conditional coverage test, the results were not significant for all the REITs markets except Australia and Canada. For the conditional coverage independent test, all the results for the backtest are significant except for the US.

In general, the less stringent tests of the traffic light, binomial, proportion of failures, time until first failure backtesting tests end in the hypotheses being accepted more times than the more stringent test of conditional coverage and conditional coverage independence tests. This is notably so for the more robust VaR estimated from EWMA and GARCH volatility models. The conditional coverage and the conditional coverage independence tests result in the rejection of hypotheses for all the REITs markets for the VaR estimated from a normal distribution and historical simulation. The time until first failure tests generally suggest that the VaR models are accurate for most markets except for the historical simulation and GARCH VaR estimate in which one market results in the rejection of the model.

| | Observations | Expected | Failures | Observed Level | Ratio |
|---|---|---|---|---|---|
| Australia | 4435 | 221.75 | 235 | 0.94701 | 1.0598 |
| Canada | 4435 | 221.75 | 204 | 0.95400 | 0.9200 |
| France | 4435 | 221.75 | 177 | 0.96009 | 0.7982 |
| UK | 4435 | 221.75 | 256 | 0.94228 | 1.1545 |
| US | 4435 | 221.75 | 261 | 0.94115 | 1.1770 |

(a)

| | Binomial | | Traffic Light | | Proportion of Failure | | Time Until First Failure | | |
|---|---|---|---|---|---|---|---|---|---|
| | Z-score | P Value | Probability | Type I Error | Likelihood Ratio | P-Value | Likelihood Ratio | P-Value | First Failure |
| Australia | 0.9129 | 0.1807 | 0.8286 | 0.1892 | 0.8181 | 0.3657 | 3.3215 | 0.0684 | 2 |
| Canada | -1.2229 | 0.1107 | 0.1164 | 0.8969 | 1.5350 | 0.2154 | 0.8654 | 0.3522 | 7 |
| France | -3.0832 | 0.0010 | 0.0008 | 0.9994 | 10.1820 | 0.0014 | 0.6813 | 0.4092 | 8 |
| UK | 2.3598 | 0.0091 | 0.9906 | 0.0112 | 5.3161 | 0.0211 | 0.3153 | 0.5744 | 11 |
| US | 2.7042 | 0.0034 | 0.9963 | 0.0045 | 6.9370 | 0.0084 | 3.3215 | 0.0684 | 2 |

(b)

| | Conditional Coverage | | Conditional Coverage independence | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Likelihood Ratio | P-Value | Likelihood Ratio | P-Value | N00 | N10 | N01 | N11 |
| Australia | 52.27 | 0.0000 | 51.45 | 0.0000 | 4006 | 193 | 193 | 42 |
| Canada | 66.69 | 0.0000 | 65.16 | 0.0000 | 4066 | 164 | 164 | 40 |
| France | 19.44 | 0.0000 | 9.26 | 0.0000 | 4096 | 161 | 161 | 16 |
| UK | 37.74 | 0.0000 | 32.43 | 0.0000 | 3961 | 217 | 217 | 39 |
| US | 60.38 | 0.0000 | 53.45 | 0.0000 | 3960 | 213 | 213 | 48 |

(c)

| | TL | Bin | POF | TUFF | CC | CCI |
|---|---|---|---|---|---|---|
| Australia | Green | accept | accept | accept | reject | reject |
| Canada | Green | accept | accept | accept | reject | reject |
| France | Green | reject | reject | accept | reject | reject |
| UK | Yellow | reject | reject | accept | reject | reject |
| US | Yellow | reject | reject | accept | reject | reject |

(d)

Exhibit 3.9: Backtesting results for Normal 95% VaR

Note: Exhibit 3.9 shows the outcomes of the binomial, traffic light system, proportion of failure, time until first failure, conditional coverage and conditional coverage independence backtesting methods of VaR. Panel (a) consists of the basic results for the backtests. Panels (b) and (c) shows the test statistics and the p-values while Panel (d) summarises the hypothesis tests using a test level of 99% for each of the volatility models.

| | TL | Bin | POF | TUFF | CC | CCI |
|---|---|---|---|---|---|---|
| Australia | green | 0.1807 | 0.3657 | 0.0684 | 0.0000 | 0.0000 |
| Canada | green | 0.1107 | 0.2154 | 0.3522 | 0.0000 | 0.0000 |
| France | green | 0.0010 | 0.0014 | 0.4092 | 0.0001 | 0.0023 |
| UK | yellow | 0.0091 | 0.0211 | 0.5744 | 0.0000 | 0.0000 |
| US | yellow | 0.0034 | 0.0084 | 0.0684 | 0.0000 | 0.0000 |

(a) Normal VaR

| | TL | Bin | POF | TUFF | CC | CCI |
|---|---|---|---|---|---|---|
| Australia | yellow | 0.0302 | 0.0654 | 0.0684 | 0.0000 | 0.0000 |
| Canada | green | 0.1173 | 0.2403 | 0.3522 | 0.0000 | 0.0000 |
| France | green | 0.0585 | 0.1109 | 0.0144 | 0.0021 | 0.0018 |
| UK | yellow | 0.0001 | 0.0004 | 0.5744 | 0.0000 | 0.0000 |
| US | red | 0.0000 | 0.0000 | 0.0684 | 0.0000 | 0.0000 |

(b) Historical Simulation

| | TL | Bin | POF | TUFF | CC | CCI |
|---|---|---|---|---|---|---|
| Australia | green | 0.3848 | 0.7703 | 0.0684 | 0.0060 | 0.0015 |
| Canada | green | 0.3848 | 0.7703 | 0.3522 | 0.0002 | 0.0000 |
| France | green | 0.0626 | 0.1311 | 0.4092 | 0.0067 | 0.0054 |
| UK | yellow | 0.0410 | 0.0873 | 0.2948 | 0.0357 | 0.0530 |
| US | green | 0.1807 | 0.3657 | 0.6565 | 0.0053 | 0.0019 |

(c) EWMA

| | TL | Bin | POF | TUFF | CC | CCI |
|---|---|---|---|---|---|---|
| Australia | green | 0.0670 | 0.1278 | 0.0684 | 0.0012 | 0.0008 |
| Canada | green | 0.0982 | 0.1903 | 0.3522 | 0.0001 | 0.0001 |
| France | green | 0.1107 | 0.2154 | 0.0144 | 0.0520 | 0.0364 |
| UK | green | 0.3334 | 0.6681 | 0.2948 | 0.0592 | 0.0193 |
| US | green | 0.3848 | 0.7703 | 0.0684 | 0.2663 | 0.1095 |

(d) GARCH(1,1)

Exhibit 3.10: 95% backtesting results for 95% VaR

Note: This exhibit presents the results for 95% VaR for the following tests, Traffic Light (TL), Binomial (Bin), Proportion of Failure (POF), Time Unitll First Failure (TUFF), Conditional Coverage (CC) and Conditional Coverage Independence (CCI). Only p-values are presented apart from the traffic light results.

#### 3.7.5.1.3. 99% VaR backtesting results

##### Normal VaR

The 99% VaR is more stringent than the 95% VaR. The backtest outcome for the traffic light, binomial, proportional of failures, conditional coverage and as well as the conditional coverage independence result in the rejection of the null hypotheses for all the REIT markets as they are all in the red traffic light zone and the p-values are significant. The time until first failure test, however, is only significant for Australia and Canada.

### Historical simulation

The backtesting results for 99% historical simulation VaR resulted in Australia and the US laying in the red zone, and therefore the hypothesis is rejected for these markets. For the binomial and proportion of failures, the results are significant for all the markets except for Canada and France. The times until first failure the hypotheses are accepted for all the markets except Canada. Both the conditional coverage and the conditional coverage independence tests are all significant for all the markets and therefore result in the rejection of the null hypotheses.

### EWMA

For the EWMA the hypotheses were rejected for all the markets for the traffic light, binomial, proportion of failure and conditional coverage. The time until first failure test, on the other hand, give rise to results that are not significant for all the markets apart from Australia. Lastly, the results for conditional coverage independence test are not significant except Canada and the US.

### GARCH

All the markets are in the red zone except for Australia that is yellow under the traffic light system. The results for binomial and the proportion of failure tests are significant for all the markets. Only Australia and the UK have significant results for the time until first failure and the conditional coverage independence test respectively. For the conditional coverage tests, the results for all the markets are significant except for Australia.

Unlike the 95% VaR the 99% VaR considers VaR at a very high confidence level. Consequently, overall, the basic backtesting tests like the traffic light system, binomial and proportion of failures tests lead to the rejection of the null hypotheses that the models used to estimate VaR, except the historical simulation, are accurate.

| | **TL** | **Bin** | **POF** | **TUFF** | **CC** | **CCI** |
|---|---|---|---|---|---|---|
| Australia | red | 0.0000 | 0.0000 | 0.0110 | 0.0000 | 0.0000 |
| Canada | red | 0.0000 | 0.0000 | 0.0484 | 0.0000 | 0.0000 |
| France | red | 0.0000 | 0.0000 | 0.5947 | 0.0000 | 0.0035 |
| UK | red | 0.0000 | 0.0000 | 0.6959 | 0.0000 | 0.0045 |
| US | red | 0.0000 | 0.0000 | 0.7828 | 0.0000 | 0.0000 |

(a) Normal VaR

| | TL | Bin | POF | TUFF | CC | CCI |
|---|---|---|---|---|---|---|
| Australia | red | 0.0000 | 0.0001 | 0.4778 | 0.0000 | 0.0013 |
| Canada | yellow | 0.0394 | 0.0912 | 0.0484 | 0.0001 | 0.0001 |
| France | green | 0.1038 | 0.1927 | 0.5947 | 0.0464 | 0.0350 |
| UK | yellow | 0.0001 | 0.0004 | 0.6959 | 0.0002 | 0.0293 |
| US | red | 0.0000 | 0.0001 | 0.7828 | 0.0000 | 0.0000 |

(b) Historical Simulation

| | TL | Bin | POF | TUFF | CC | CCI |
|---|---|---|---|---|---|---|
| Australia | red | 0.0000 | 0.0000 | 0.0110 | 0.0001 | 0.1701 |
| Canada | red | 0.0000 | 0.0000 | 0.0582 | 0.0000 | 0.0195 |
| France | red | 0.0000 | 0.0000 | 0.5947 | 0.0000 | 0.2247 |
| UK | red | 0.0000 | 0.0000 | 0.9376 | 0.0000 | 0.0567 |
| US | red | 0.0000 | 0.0000 | 0.7828 | 0.0000 | 0.0157 |

(c) EWMA

| | TL | Bin | POF | TUFF | CC | CCI |
|---|---|---|---|---|---|---|
| Australia | yellow | 0.0135 | 0.0353 | 0.0110 | 0.1062 | 0.8134 |
| Canada | red | 0.0000 | 0.0000 | 0.0582 | 0.0001 | 0.5636 |
| France | red | 0.0000 | 0.0000 | 0.5947 | 0.0000 | 0.6788 |
| UK | red | 0.0000 | 0.0002 | 0.9376 | 0.0001 | 0.0324 |
| US | red | 0.0000 | 0.0000 | 0.3114 | 0.0000 | 0.0929 |

(d) GARCH(1,1)

Exhibit 3.11: 99% backtesting results for 95% VaR

Note: This exhibit presents the results for 95% VaR for the following tests, Traffic Light (TL), Binomial (Bin), Proportion of Failure (POF), Time Unitll First Failure (TUFF), Conditional Coverage (CC) and Conditional Coverage Independence (CCI). Only p-values are presented apart from the traffic light results.

#### 3.7.5.2.0. Multivariate (Portfolio) level

At the portfolio, level the backtesting outcomes at the 95% VaR are in yellow and green zones for the normal, historical simulation and the EWMA, GARCH VaR models respectively. In contrast, at 99% VaR all the VaR models fall in the red zone, apart from the historical simulation that is in the yellow zone. For the binomial and percentage of failures tests for 95% VaR, the normal and historical simulation VaR model backtests are significant leading to the rejection of the null hypotheses of a good model while the EWMA, and GARCH backtesting results were not significant. However, all the models are significant for 99% VaR, and therefore the null hypotheses are rejected for all models for the two backtesting tests. The hypotheses for the time until first failure of 95% VaR are all rejected suggesting that all the models are good predictors of VaR. In contrast, the same test for 99% VaR give rise to significant results for all the VaR models. As expected, because the VaR used a very high confidence level (99%), the first failures are delayed significantly compared to those of the 95% VaR (see panel (b)) of Exhibits 3.12 and 3.13. For both 95% and 99% VaR, the backtest for the conditional coverage and the conditional coverage independence tests were all significant, leading to the rejection of the null hypotheses.

| | Observed Level | Observations | Failures | Expected | Ratio |
|---|---|---|---|---|---|
| Normal 95 | 0.9430 | 4435 | 253 | 221.75 | 1.1409 |
| Historical 95 | 0.9391 | 4435 | 270 | 221.75 | 1.2176 |
| EWMA 95 | 0.9459 | 4435 | 240 | 221.75 | 1.0823 |
| GARCH 95 | 0.9468 | 4435 | 236 | 221.75 | 1.0643 |

(a)

| | Binomial | | Traffic Light | | Proportion of Failure | | Time Until First Failure | | |
|---|---|---|---|---|---|---|---|---|---|
| | Z-score | P-Value | Probability | Type I Error | Likelihood Ratio | P-Value | Likelihood Ratio | P-Value | First Failure |
| Normal 95 | 2.1531 | 0.0157 | 0.9843 | 0.0185 | 4.4428 | 0.0350 | 0.8654 | 0.3522 | 7 |
| Historical 95 | 3.3243 | 0.0004 | 0.9995 | 0.0007 | 10.3650 | 0.0013 | 0.8654 | 0.3522 | 7 |
| EWMA 95 | 1.2574 | 0.1043 | 0.9007 | 0.1116 | 1.5416 | 0.2144 | 0.8654 | 0.3522 | 7 |
| GARCH 95 | 0.9818 | 0.1631 | 0.8452 | 0.1714 | 0.9450 | 0.3310 | 0.8654 | 0.3522 | 7 |

(b)

| | Conditional Coverage | | Conditional Coverage independence | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Likelihood Ratio | P-Value | Likelihood Ratio | P-Value | N00 | N10 | N01 | N11 |
| Normal 95 | 96.0960 | 0.0000 | 91.6540 | 0.0000 | 3986 | 195 | 195 | 58 |
| Historical 95 | 98.3750 | 0.0000 | 88.0090 | 0.0000 | 3955 | 209 | 209 | 61 |
| EWMA 95 | 16.6160 | 0.0002 | 15.0750 | 0.0001 | 3982 | 212 | 212 | 28 |
| GARCH 95 | 21.1750 | 0.0000 | 20.2300 | 0.0000 | 3992 | 206 | 206 | 30 |

(c)

| | TL | Bin | POF | TUFF | CC | CCI |
|---|---|---|---|---|---|---|
| Normal 95 | yellow | reject | reject | accept | reject | reject |
| Historical 95 | yellow | reject | reject | accept | reject | reject |
| EWMA 95 | green | accept | accept | accept | reject | reject |
| GARCH 95 | green | accept | accept | accept | reject | reject |

(d)

Exhibit 3.12: 95% Portfolio VaR

Note: Exhibit 3.12 shows the outcomes of the binomial, traffic light system, proportion of failure, time until first failure, conditional coverage and conditional coverage independence backtesting methods of VaR. Panel (a) consists of the basic results for the backtests. Panels (b) and (c) shows the test statistics and the p-values while Panel (d) summarises the hypotheses tests using a test level of 95% for each of the volatility models.

| | Observed Level | Observations | Failures | Expected | Ratio |
|---|---|---|---|---|---|
| Normal 99 | 0.9736 | 4435 | 117 | 44.35 | 2.6381 |
| Historical 99 | 0.9851 | 4435 | 66 | 44.35 | 1.4882 |
| EWMA 99 | 0.9820 | 4435 | 80 | 44.35 | 1.8038 |
| GARCH 99 | 0.9829 | 4435 | 76 | 44.35 | 1.7136 |

(a)

| | Binomial | | Traffic Light | | Proportion of Failure | | Time Until First Failure | | |
|---|---|---|---|---|---|---|---|---|---|
| | Z-score | P-Value | Probability | Type I Error | Likelihood Ratio | P-Value | Likelihood Ratio | P-Value | First Failure |
| Normal 99 | 10.9640 | 0.0000 | 1.0000 | 0.0000 | 82.9030 | 0.0000 | 0.0762 | 0.7825 | 75 |
| Historical 99 | 3.2673 | 0.0005 | 0.9991 | 0.0013 | 9.2825 | 0.0023 | 0.0262 | 0.8713 | 117 |
| EWMA 99 | 5.3802 | 0.0000 | 1.0000 | 0.0000 | 23.3760 | 0.0000 | 0.0762 | 0.7825 | 75 |
| GARCH 99 | 4.7765 | 0.0000 | 1.0000 | 0.0000 | 18.7990 | 0.0000 | 0.0762 | 0.7825 | 75 |

(b)

| | Conditional Coverage | | Conditional coverage independence | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **Likelihood Ratio** | **P-Value** | **Likelihood Ratio** | **P-Value** | **N00** | **N10** | **N01** | **N11** |
| Normal 99 | 112.8100 | 0.0000 | 29.9090 | 0.0000 | 4216 | 101 | 101 | 16 |
| Historical 99 | 18.0300 | 0.0001 | 8.7475 | 0.0031 | 4307 | 61 | 61 | 5 |
| EWMA 99 | 29.0180 | 0.0000 | 5.6412 | 0.0175 | 4279 | 75 | 75 | 5 |
| GARCH 99 | 25.2300 | 0.0000 | 6.4310 | 0.0112 | 4287 | 71 | 71 | 5 |

(c)

| | TL | Bin | POF | TUFF | CC | CCI |
|---|---|---|---|---|---|---|
| Normal 99 | red | reject | reject | accept | reject | reject |
| Historical 99 | yellow | reject | reject | accept | reject | reject |
| EWMA 99 | red | reject | reject | accept | reject | reject |
| GARCH 99 | red | reject | reject | accept | reject | reject |

(d)

Exhibit 3.13: 99% Portfolio VaR

Note: Exhibit 3.13 shows the outcomes of the binomial, traffic light system, proportion of failure, time until first failure, conditional coverage and conditional coverage independence backtesting methods of VaR. Panel (a) consists of the basic results for the backtests. Panels (b) and (c) shows the test statistics and the p-values while Panel (d) summarises the hypotheses tests using a test level of 99% for each of the volatility models.

### 3.7.5.3. Backtesting Expected shortfall

As discussed before the main challenge for the expected shortfall lies in backtesting. Backtesting ES is not as straight forward as that of VaR. In order to backtest ES for univariate variables, this study utilises a methodology proposed by Danielsson (2011) which is similar to violation ratios for VaR. The methodology is described below:

Backtesting ES using Danielsson (2011) approach involves calculating the normalised shortfall (NS). If $ES_t$ denotes the observed ES on day $t$, for days when VaR is violated, normalised this is given as

$$NS_t = \frac{y_t}{ES_t} \tag{3.32}$$

From the definition of ES, given that VaR is violated, the expected $y_t$ is:

$$\frac{E[Y_t|Y_t \ < -VaR_t}{ES_t} = 1 \tag{3.33}$$

The average$NS$, denoted by $\overline{NS}$ should be 1. The null hypothesis is therefore:

$$H_0{:}\ \overline{NS} = 1 \tag{3.34}$$

Exhibits 3.14 and 3.15 present the results of the ES backtesting for Danielsson's simple test which show that the null hypotheses should be rejected for all the ES approaches. This is because all the values are not equal to 1 although generally those for the historical simulation followed by GARCH at both 95% and 99% ES were the closest to 1. For both 95% and 99% ES, the Normal ES resulted in figures furthermost from 1. As stated before the backtesting of ES is not an easy undertaking because what is being tested is an expectation rather than a single figure. This said, Acerbi and Szerkely (2015) have proposed more robust ways of backtesting ES, and these were applied in the study when looking at multivariate volatility modelling below.

| 95%ES | Australia | Canada | France | UK | US |
|---|---|---|---|---|---|
| Normal | 1.25 | 1.26 | 1.19 | 1.22 | 1.30 |
| Historical | 1.13 | 1.07 | 0.99 | 1.08 | 1.12 |
| EWMA | 1.09 | 1.16 | 1.16 | 1.11 | 1.15 |
| GARCH(1,1) | 1.08 | 1.15 | 1.12 | 1.10 | 1.14 |

Exhibit 3.14: Backtesting results for 95% ES

Note: This shows the backtest for 95% ES of the 4 different models using the method proposed by Danielsson (2011. According to this method, a good ES estimate should have a value of 1.

| 99% ES | Australia | Canada | France | UK | US |
|---|---|---|---|---|---|
| Normal | 1.26 | 1.27 | 1.12 | 1.26 | 1.33 |
| Historical | 1.06 | 1.06 | 0.96 | 1.03 | 1.06 |
| EWMA | 1.09 | 1.21 | 1.17 | 1.18 | 1.20 |
| GARCH(1,1) | 1.12 | 1.19 | 1.10 | 1.17 | 1.16 |

Exhibit 3.15: Backtesting results for 95% ES

Note: This shows the backtest for 99% ES of the 4 different models using the method proposed by Danielsson (2011. According to this method, a good ES estimate should have a value of 1.

Before implementing the portfolio level risk measures of VaR and ES, the EVT, copula and non-symmetric GARCH are first explored.

#### 3.7.5.3.1. Extreme Value Theory

Danielsson (2011) observes that most statistical models are based on modelling the entire distribution of a quantity of interest. In these instances, the estimation process is dominated by observations in the centre of the distribution, since there are ordinarily insufficient observations that are extreme (or uncommon). Central to parametric VaR is the normality assumption that is based on the central limit theorem[28]. This, however, is only applicable to quantiles and probabilities in the central mass of the density function or distribution, as opposed to the tails (Dowd, 2005). While this may result in a good estimation of the distribution of data for common events, the estimation of the distribution of the tails is likely to be inaccurate. In order to model extremes (or tails), a semi-parametric approach called Extreme Value Theory (EVT) should be applied as it extends the central limit theorem but does not assume normality since the assumption of the distribution of returns is not required.

Generally, the normal distribution underestimates potential losses at high confidence levels. It is therefore difficult to estimate VaR reliably because empirical distributions typically have insufficient data in the tails (Jorion 2007); however, EVT can be used in these instances because it explicitly focuses

[28] The key assumption of the central limit theorem is normal distribution. It postulates the property that normalised sum of independent random variables tends towards a normal distribution. This is despite the fact that these original variables themselves are not normally distributed.

on the tails. Hull (2015) defines EVT as the term used to describe the science of estimating the tails of a distribution that can be used to improve VaR estimates at very high confidence levels. Because EVT concentrates on modelling the tails of the distribution of returns, it takes care of the possible occurrence of extreme values, particularly huge negative returns suddenly arising. This is done by smoothing and extrapolating the tails of an empirical distribution (Hull, 2015). This said, because EVT only applies to the tails, it is not accurate for the centre of distributions (Jorion, 2007). According to Christoffersen (2012), the fundamental result in EVT states that the extreme tail of a wide range of distributions share common properties and thus can be approximately described by a relatively simple distribution called the Generalised Pareto Distribution (GPD)[29]. GDP incorporates other known distributions, including the Pareto and normal as exceptional cases (Jorion, 2007). While the concept of EVT can be quite complicated, Hull (2015) simplifies it, and therefore this paper shows it as presented in there.

Supposed the function $F(v)$ is the cumulative distribution function (cdf) for a variable $v$ for example the loss on a portfolio over a certain period and $u$ is a value of $v$ in the right-hand tail of the distribution. The probability that $v$ lies between $u$ and $u+y$, $(y > 0)$ is $F(u+y) - F(u)$. The probability that $v$ is greater than $u$ is $1 - F(u)$. $F_u(y)$ is defined as the probability that $v$ lies between $u$ and $u+y$ conditional on $v > u$. This is

$$F_u(y) = \frac{F(u+y) - F(u)}{1 - F(u)} \tag{3.35}$$

The variable $F_u(y)$ defines the right tail of the probability distribution. It is the cumulative probability distribution for the amount by which $v$ exceeds $u$ given that it does exceed $u$.

The results state that for a wide class of distribution $F_u(y)$ converges to a generalised Pareto distribution as the threshold $u$ is increased. The generalised Pareto (cumulative) distribution is

$$G_{\xi,\beta}(y) = \begin{cases} \left[1 + \xi \dfrac{y}{\beta}\right]^{-1/\xi} & if\ \xi > 0 \\ \left[1 + \exp\left(\dfrac{y}{\beta}\right)\right] & if\ \xi = 0 \end{cases} \tag{3.36}$$

29 Generalised Pareto Distribution is a distribution capable of modelling tails of a wide variety of distributions based on theoretical arguments (MathWorks, 2016)

The distribution has two parameters that have to be estimated from the data. These are ξ and β. The parameter ξ is the shape parameter and determines the heaviness of the tail of the distribution. The parameter β is a scale parameter.

Where the underlying variable $v$ has a normal distribution, $\xi = 0$ and the tails disappear at an exponential speed (Jorion, 2007). As the tails of the distribution become heavier, the value of ξ increases since the tail disappears more slowly than the normal. For most financial data, $\xi > 0$ and in the range of 0.1 to 0.4 (Dowd, 2005 and Jorion 2007).

### Estimating ξ and β

The parameters ξ and β can be estimated using maximum likelihood methods. The probability density function, $g_{\xi,\beta}(y)$, of the cumulative distribution in Equation (3.36) is calculated by differentiating $G_{\xi,\beta}(y)$ with respect to y. It is

$$g_{\xi,\beta}(y) = \frac{1}{\beta}\left(1 + \frac{\xi y}{\beta}\right)^{-1/\xi-1} \tag{3.37}$$

A value for $u$ has to be chosen first (usually a value close to the 95th percentile point of the empirical distribution usually works well). The observations are then ranked on $v$ from the highest to the lowest and the attention is focused on those observations for which $v > u$. Suppose there are $n_u$ such observations and they are $v_i$ $(1 \leq i \leq n_u)$. The likelihood function (assuming that $\xi \neq 0$) is

$$\prod_{i=1}^{n_u} \frac{1}{\beta}\left(1 + \frac{\xi(v_i - u)}{\beta}\right)^{-1/\xi-1} \tag{3.38}$$

Maximising this function is the same as maximising its logarithm:

$$\sum_{i=1}^{n_u} ln\left[\frac{1}{\beta}\left(1 + \frac{\xi(v_i - u)}{\beta}\right)^{-1/\xi-1}\right] \tag{3.39}$$

Standard numerical procedures can be used to find the value of ξ and β that maximise this expression

### Estimating the tail of the distribution

The probability that $v > u + y$ conditional that $v > u$ is 1-$G_{\xi,\beta}(y)$. The probability that $v > u$ is $1 - F(u)$. The unconditional probability that $v > u$ (when $v > u$) is therefore

$$[1 - F(u)][1 - G_{\xi,\beta}(x - u)] \tag{3.40}$$

If $n$ is the total number of observations, an estimate of $1 - F(u)$, calculated from the empirical data, is $\frac{n_u}{n}$. The unconditional probability that $v > x$ is therefore

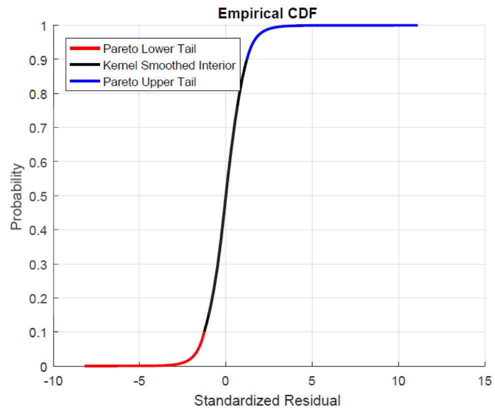
$$Prob(v > x) = \frac{n_u}{n}[1 - G_{\xi,\beta}(x - u)] = \frac{n_u}{n}\left[1 + \xi\frac{x - u}{\beta}\right]^{-1/\xi} \tag{3.41}$$

### Estimating semi-parametric CDFs with EVT

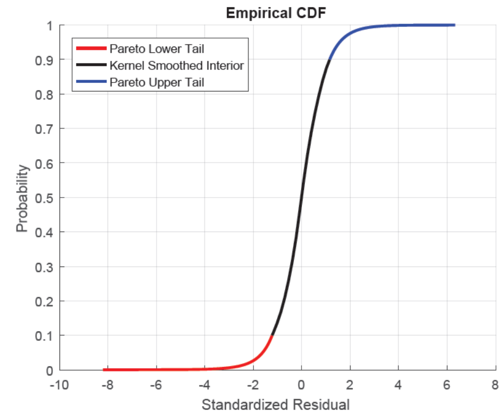
Given the fat tails revealed earlier by the REIT data (see Exhibit 3.5) EVT was used in order to model the data to get GPD fit resulting in the semi-parametric CDF for each market as shown in Exhibit 3.16. This was done by replicating Mathworks (2016) and Jäger (2015) where the negative log-likelihood function is optimised to estimate parameters of the GPD by utilising MATLAB[30]. Despite the normal distribution not being adequate for modelling market risk, the kernel of the CDF is approximately normally distributed necessitating the application of the normal distribution for estimating the interior of the distribution and EVT for the fat tails (Jäger, 2015). Exhibit 3.17 displays the empirical CDF (for the 5 REIT markets) of the upper 10% tail exceedances of the residuals together with the CDF fitted by the GPD in order to show the importance of the GPD fit. The GPD model seems to be a good choice as the fitted distribution of each REIT market follows the exceedance data.

Semi parametric CDF

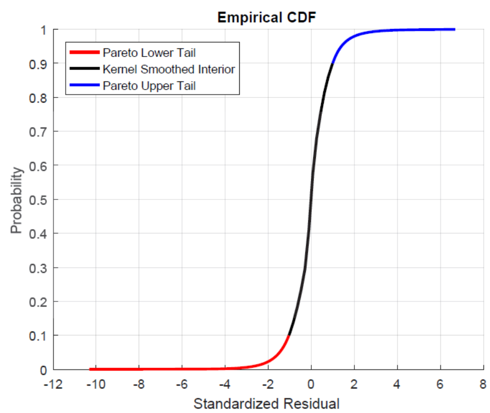
[30] MATLAB code used form Mathworks using the POT method with threshold $u = 10\%$ by making use of the paretotails function of the Statistics toolbox in MATLAB



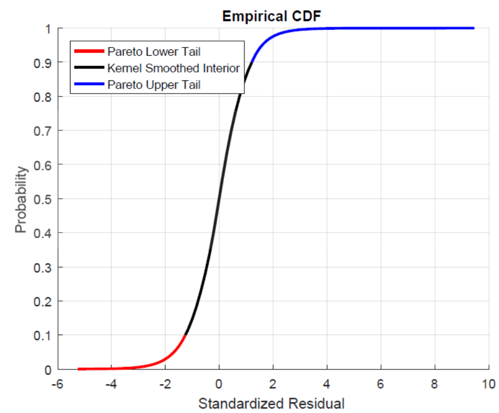

Australia



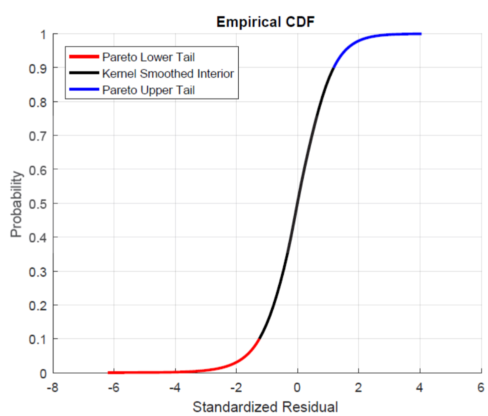

Canada




France




UK




US

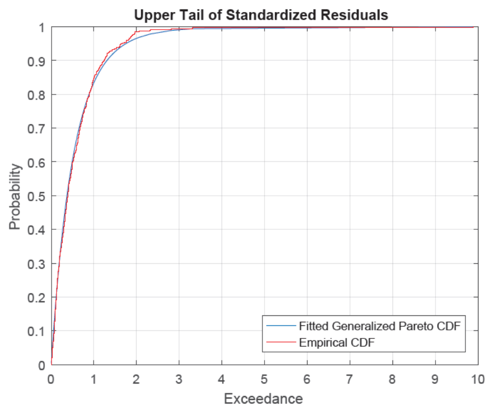
Exhibit 3.16: REIT CDF of standardised residuals

Note: Due to the presence of fat tails, the REIT data for each index used EVT to obtain the GPD fit therefore leading to a semi-parametric CDF.

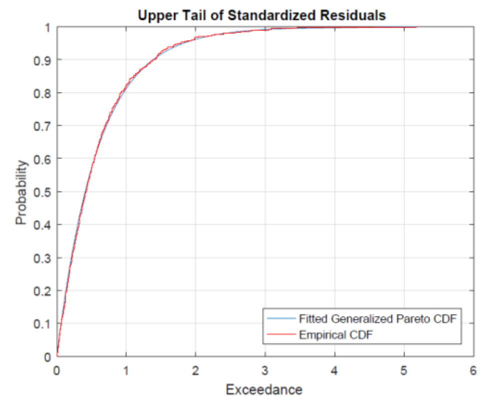
The model described above is the Peak Over Threshold (POT). POT is based on models for all large observations that exceed a high threshold and therefore makes better use of data on extreme values (Danielsson, 2011). The other method used to model extreme values is the block maximum models. Similar to Danielsson (2011) and Jäger (2015) this paper utilises POT because the block maxima model is said to be rather wasteful of data and requires the largest observation collected from large samples of identically distributed observations. Additionally, the interior of the financial returns can be modelled by a normal distribution (Jäger, 2025).

#### 3.7.5.3.2. Nonlinear dependence (Copulas)

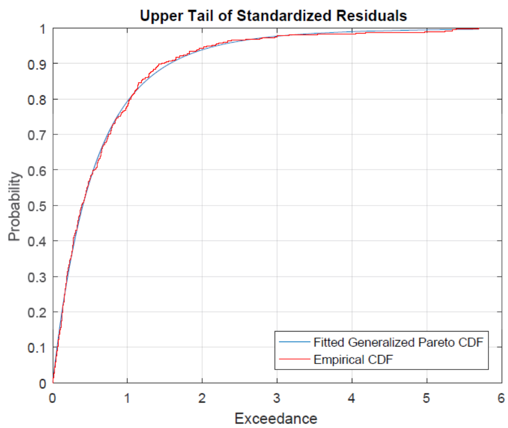
As the study is looking at creating a portfolio of different REIT markets, it is important to consider the co-dependencies of these REIT markets. This is because every joint distribution contains both a description of the marginal behaviour of the individual risk factors and a description of their dependence structure. A way of isolating the description of the dependence structure is by using statistical functions, developed by Sklar (1959), referred to as copulas (McNeil et al., 2005). Because normally distributed portfolio returns are assumed, it is in turn usually assumed that the returns for constituent assets in a portfolio have a multivariate normal distribution. Furthermore, there is an assumption that each asset return follows an i.d.d. process and that the joint distribution of the variables is elliptical (Alexander, 2008b). In this case, a variance-covariance framework can easily be applied by making use of linear correlations. However, this will tend to underestimate the joint probability of simultaneous negative extreme values across constituent assets in a portfolio. This can, in turn, exaggerate the benefits of portfolio diversification (Christoffersen 2012), and therefore lead to misleading results. In reality, most assets or portfolios do not satisfy the assumptions above as their returns are rarely normal nor do they have an elliptical distribution but possess either asymmetric marginal distributions, non-linear dependence or both (Alexander 2008b). On that account, correlation cannot be used as a measure of association or dependence. Dowd (2005) observes that correlation is usually misused and applied to situations for which it is unsuitable and therefore warns against the tendency of using it in risk measurement as though it were an all-purpose dependence measure.



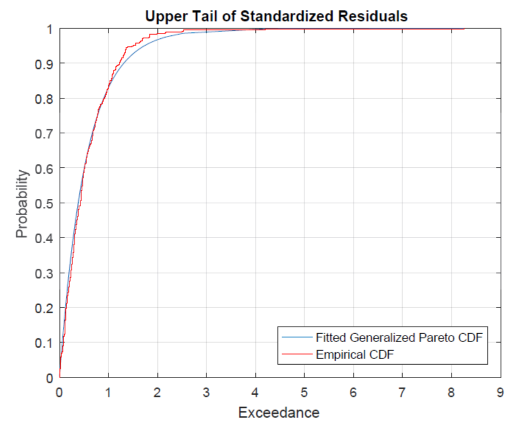

Australia



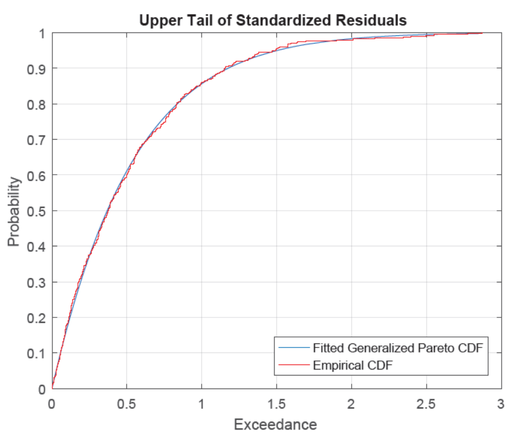

Canada




France




UK




US

Exhibit 3.17: Upper tail of standardised residuals

Note: This shows the Upper tail of standardised residuals along with the CDF fitted by the GPD in order to show the importance of the GPD fit.

Furthermore, constituent assets in a portfolio could be from distributions that may be distinct from a normal distribution and that these assets may even exhibit fat tails or excess kurtosis and volatility clustering. This implies that such assets are likely to have joint distributions that are non-linear in nature. For this reason, linear dependence measures such as correlation will not reflect accurate associations particularly in the tails. The most common way of modelling nonlinear dependence is with copulas. A Copula is a function that joins a multivariate distribution function to a collection of univariate marginal distribution functions (Dowd, 2005). The joint distribution of two i.i.d. random variables $X$ and $Y$ is the bivariate distribution function that gives the probabilities of both $X$ and $Y$ taking certain values at the same time. However, the joint distribution of two or more variables can be built by first specifying the stand-alone distributions (i.e. marginal distributions) and then using copulas to represent the dependencies between these variables (Alexander 2008b). Expressing these dependences using quantiles rather than correlations that other multivariate volatility forecasting models use can achieve this. Consequently, this gives universal validity to copula approaches meaning that marginal distributions are taken (each of which describes the way in which a random variable moves 'on its own') and the copula function is what expresses how they 'come together' to determine the multivariate distribution (Dowd, 2005).

Explicitly, a copula enables one to construct a multivariate distribution function from the marginal distribution function in a way that allows for very general dependence structure. Here, the dependence structure is isolated from the structure of the marginal distributions making it possible to apply copulas with any marginal distributions even when they are different for each return. For example, this can be a combination of assets with a range of different distributions such as; t-distribution with 10 degrees of freedom, chi-squared distribution with 15 degrees of freedom, a gamma distribution. (Alexander, 2008b). Furthermore, Dowd (2005) observes that while variance-covariance approaches are only valid in instances where the dependence measure used is the correlation, copula approaches are universally valid and can be utilised in instances where correlation-based approaches cannot and are therefore statistically universally correct way to estimate risk measures from the position level. Because copulas expresses dependence in a quantile scale, it makes it useful for describing extreme outcomes dependence and thus applies to VaR in a natural way (McNeil et al., 2005). This is because VaR also looks at risk in quartile scale of loss distributions hence making copulas useful in risk management. Another profound advantage for copulas arises when estimating an integrated risk measure across several diverse types of risk measures which have two different distributions. The result is different univariate density functions while most multivariate methods would assume a single multivariate density function. This presents a challenge particularly for risk integration and enterprise-wide risk management. Copulas, on the other hand, is able to tackle this challenge. An additional attractive property of copula that linear correlation does not possess is "scale invariance" to reasonable transformations of the random variables and /or their distribution function. This means that converting Profit and Loss to returns, for example, does not affect the copula, because it does not respond to changes in the units of measurements (Dowd, 2005).

This paper examines copulas based on Dowd (2005) and the references therein.

Consider two random variables, $X$ and $Y$. If $F(x,y)$ is a joint distribution function with continuous marginals $F_x(x) = u$ and $F_y(y) = v$, then $F(x,y)$ can be written in terms of a unique function $C(u,v)$:

$$F(x,y) = C(u,v) \qquad (3.42)$$

Where $C(u,v)$ is known as the copula of $F(x,y)$. The copula function describes how the multivariate function $F(x,y)$ is derived from or coupled with the marginal distribution functions $F_x(x)$ and $F_y(y)$, and the copula can be interpreted as giving the dependence structure of $F(x,y)$.

This result for the copula is fundamental as it enables the construction of joint distribution functions from marginal distribution functions in a way that takes account of the dependence structure of random variables. To model the joint distribution function, all that is needed is to specify the marginal distributions, choose a copula to represent the dependence structure, estimate the parameters involved, and then apply the copula function to the marginals. Upon modelling the joint distribution function, any risk measures in principle can be estimated from it.

### Types of copulas

There are various kinds of copulas, the simplest ones are

$C_{ind}(u,v) = uv$ - The independence (or product) copula used where X and Y are independent

$C_{min}(u,v) = \min[u,v]$ - The minimum (or comonotonicity) copula that is used where X and Y are positively dependent or comonotonic (meaning they rise or fall together).

$C_{max}(u,v) = \max[u+v-1,0]$ - The maximum (or countermonotonicity) copula used where X and Y are negatively dependent or countermonotonic – meaning as one rises the other falls and vice versa.

Other important copulas are:

### Gaussian (or normal) copula

$$C_p^{Ga}(x,y) = \int_{-\infty}^{\Phi^{-1}(x)} \int_{-\infty}^{\Phi^{-1}(y)} \frac{1}{2\pi(1-\rho^2)^{0.5}} exp\left\{\frac{-(s^2-2\rho st+t^2)}{2(1-\rho^2)}\right\} dsdt \qquad (3.43)$$

Where $-1 \leq \rho \leq 1$ and $\Phi$ is the univariate standard normal function. Note that this copula depends only on the correlation coefficient, $\rho$, which confirms that $\rho$ is sufficient to determine the whole dependence structure. For variables that have standard normal marginal distributions, this dependence structure will then be multivariate normally distributed. Regrettably, the Gaussian copula does not have a closed – form solution, so the copula has to be estimated by numerical methods.

### t-copulas

The t-copula for $v$ degrees of freedom is a straightforward generalisation of the normal one:

$$C_{v,p}^{t}(x,y) = \int_{-\infty}^{t_v^{-1}(x)} \int_{-\infty}^{t_v^{-1}(y)} \frac{1}{2\pi(1-\rho^2)^{0.5}} exp\left\{1+\frac{(s^2-2\rho st+t^2)}{2(1-\rho^2)}\right\}^{-\frac{v+2}{v}} dsdt \qquad (3.44)$$

Where $t_v^{-1}(x)$ is the universe of the distribution function of the standard univariate t-distribution for $v$ degrees of freedom.

### Gumbel (logistic) copula

$$C_\beta^{Gu}(x,y) = exp\left[-\left\{(-\log x)^{\frac{1}{\beta}} + (-\log y)^{\frac{1}{\beta}}\right\}^{\beta}\right] \qquad (3.45)$$

Where $\beta$ satisfies $0 < \beta \leq 1$ and determines the amount of dependence between our variables: $\beta = 1$ indicates that the variables are independent, $\beta > 0$ indicates limited dependence, and the limiting value of 0 indicates perfect dependence. Multivariate extremes can reasonably be modelled by this copula because in contrast to the Gaussian copula, it is consistent with EVT.

### Tail dependence

Copulas can also be used to investigate tail dependence, which is an asymptotic measure of the dependence of extreme values of the joint distribution. Unlike the ordinary density function that is always positive and presents greater values in the centre, copula densities used in finance often have higher values in the corners (Alexander 2008b). This highlights the importance of dependence in the tails because extreme events are often related (i.e., disasters often come in pairs or more), (Dowd, 2005) and models that fail to accommodate their dependence can lead a firm to disaster.

If marginal distributions are continuous, a coefficient of (upper) tail dependence of $X$ and $Y$ can be defined as the limit, as $\alpha \to 1$ from below, of

$$\Pr\left[ Y > F_y^{-1}(\alpha) \mid X > F_x^{-1}(\alpha) \right] = \lambda \tag{3.46}$$

provided such limit exists. If $0 < \lambda \leq 1$, $X$ and $Y$ are asymptotically dependent in the upper tail. In other words, it represents the conditional probability that one variable takes a value in the upper tail, given that also the other variable takes a value in its upper tail. In this instance, the copula is said to have upper tail dependence for a, $X$ and $Y$, when $\lambda > 0$, and the higher the value of the dependence coefficient, the stronger the upper tail dependence (Alexander 2008b). If $\lambda = 1$, $X$ and $Y$ are asymptotically independent i.e., the upper and lower tail dependence coefficients are different, therefore presenting asymmetric tail dependence.

### Estimating copulas

Estimating copulas entails choosing the copulas functional form and then estimating the parameters concerned by using either parametric or non-parametric methods to find estimators that best fit the data by some criterion. This is done by use of maximum likelihood estimators for the parametric method. Copulas can be challenging to implement (please see Alexander 2008b) and it is for this reason that this study makes use of readily available functions in MATLAB®.

### Calibrate the t-Copula

The following two sections replicate and use the approach and code provided by Jäger (2015) and MathWorks (2016). Using the standardised residual and the fitted tails with the GPD, the standardised residuals are transformed to uniform variates using the semi-parametric CDF, and the t-copula is fitted to the transformed data[31.]

### Simulate portfolio returns with a t-Copula

Having obtained the parameters of a t-copula, the dependent financial returns for each REIT market matching to the dependence of the standardised residual is then simulated[32]. This was done by simulating 10,000 independent random trails of dependent standardised REIT residuals. Each column of the simulated standardized residuals array represents an i.i.d. stochastic process when viewed in isolation, whereas each row shares the rank correlation induced by the copula (Mathwork and Jäger (2015). Following this, the random sample was transformed into the original scale. The autocorrelation and heteroscedasticity that were eliminated by the ARCH(1) and GJR-GARCH(1,1) model are then reintroduced to satisfy the EVT assumption (Jäger, 2015)[33.]

Finally, the continuous portfolio returns for each time point is calculated and the distribution for the resulting portfolio returns is transformed into a loss distribution from which the VaR and ES are calculated. Exhibit 3.18 displays the results of the CDF of the portfolio returns.

[31] This uses the copulafit function of the statistic toolbox which uses the maximum likelihood approach for a t-copula to estimate the parameter for the copula

[32]This is done by making use of the MATLAB function ‘copularnd’ of the Statistics and Machine Learning toolbox.

[33] This is done using the Econometrics Toolbox™ filter function in MATLAB.

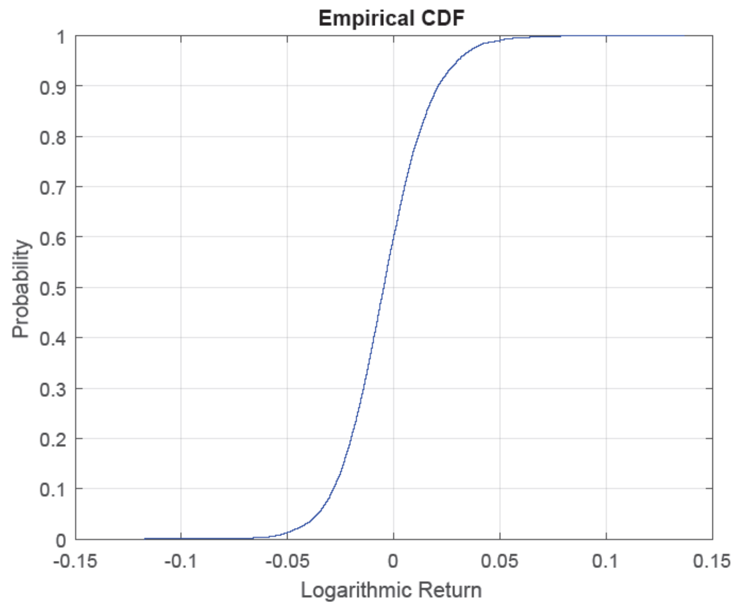



Exhibit 3.18: CDF plot of portfolio returns

Alexander (2008b) argues that an entire joint distribution of returns should be utilised. One way of accommodating such conditions is to utilise parametric methods, i.e. where risk is estimated by fitting probability curves to the data and then inferring a risk measure (Dowd, 2005). While parametric methods result in straightforward VaR and ES measures; they are prone to errors if the assumed density function does not adequately fit the data. As previously discussed, a way of taking account of volatility clustering is by fitting a normal distribution to the data conditional on a GARCH volatility process.

#### 3.7.5.3.3. Backtesting Expected shortfall (Acerbi and Szekely, 2014)

As indicated previous, the backtesting of ES is contentious because there is no agreed way to undertake it. The VaR backtesting methods cannot be applied because ES is a conditional mean and therefore, it is not possible to create a Bernoulli trial as a test statistic. Furthermore, elicitability has been the main issue surrounding the challenges of backtesting ES. Acerbi and Szekely (2014) nevertheless, assert that elicitability is not a necessary condition and have proposed three methods of backtesting ES. Similar to Jäger (2015), this study uses the two methodologies that are based on and are akin to the Basel backtesting framework for VaR. Acerbi and Szekely (2014) implement a standard hypothesis-testing framework for unconditional coverage of ES similar to the standard Basel VaR setting. This section is based on Acerbi and Szekely (2014) and the annotation used are those in Jäger (2015).

It is assumed that distributions are continuous and strictly increasing. In line with issues to do with estimation error in expected shortfall highlighted earlier, $ES_{2.5\%}$ has been chosen similar to the Basel Committee to equal to $VaR_{1\%}$ for Gaussian tails and to penalise heavier tails. Consistent with the Basel VaR test, meant to detect only excesses of VaR exceptions, Acerbi and Szekely's null hypothesis generally assumes that the prediction is correct whilst the alternative hypotheses chosen are only in the direction of risk underestimation.

### Test 1: Testing ES after VaR

The inspiration of this test emanates from the conditional definition of Expected Shortfall.

$$ES_c^{t,H} = -\mathbb{E}\left(L^H \middle| L^H + VaR_c^{t,H}\right) \tag{3.47}$$

Jäger (2015) as has rewritten this as:

$$ES_c^{t,H} = \left(L^H \middle| L^H \geq VaR_c^{t,H}\right)$$

$$\Leftrightarrow 1 = \mathbb{E}(\frac{L^H}{ES_c^{t,H}} | L^H \geq VaR_c^{t,H})$$

$$\Leftrightarrow \mathbb{E}\left(\frac{L^H}{ES_c^{t,H}} - 1 \middle| L^H \geq VaR_c^{t,H}\right) = 0 \tag{3.48}$$

Where $VaR_c^{t,H}$ backtest of the Basel Committee has been satisfied already, the magnitude of the realised exceptions against the model predictions can be tested separately. Defining $\hat{I}_{t+H}^H = \left(X_t + VaR_c^{t,H}\right)$ as the indicator function, the test statistic is defined as:

$$Z_1\left(\vec{L}^H\right) = \frac{\sum_{t=1}^{T} \frac{L_t^H \cdot \hat{I}_{t+H}^H}{ES_c^{t,H}}}{\hat{I}^{W_{BT}}} - 1 \tag{3.49}$$

If $\hat{I}^{W_{BT}} = \sum_{t=1}^{HT} \hat{I}_{t+H}^H > 0$.

For this test, Acerbi and Szekely choose the null hypothesis as;

$$H_0 : P_t^{[\alpha]} = F_t^{[\alpha]}, \forall t \tag{3.50}$$

Where $P_t^{[\alpha]}(x) = \min(1, P_t(x)/\alpha)$ is the distribution tail for $x > VaR_c^{t,H}$. The alternatives are

$$H_1 : ES_{c,F}^{t,H} \geq ES_c^{t,H}, \text{ for all } t \text{ and } ES_{c,F}^{t,H} > ES_c^{t,H} \text{ for some } t$$

$$VaR_{c,F}^{t,H} = VaR_c^{t,H}, \text{ for all } t$$

Acerbi and Szekely show that VaR is still correct under $H_1$ given that VaR has been tested. It is expected that the realised value $Z_1(\vec{L}^H)$ is zero and it rejects the null hypothesis when it is negative. Therefore, it is $\mathbb{E}_{H_0}(Z_1|\hat{I} > 0) = 0$ and $\mathbb{E}_{H_0}(Z_1|\hat{I} > 0) < 0$) (Jäger, 2015). This is as proved in proposition A.2 by Acerbi and Szekely (2014).

### Test 2: Testing ES directly

The second test emanates from the unconditional expectation. This is taken from Acerbi and Szekely (2014) but also uses the notations from Jäger (2015).

$$ES_c^{t,H} = \mathbb{E}\left(\frac{L_t^H \cdot \hat{I}_{t+H}^H}{\alpha}\right)$$

$$\Leftrightarrow 1 = \mathbb{E}\left(\frac{L_t^H \cdot \hat{I}_{t+H}^H}{\alpha \cdot ES_c^{t,H}}\right)$$

$$\mathbb{E}\left(\frac{L_t^H \cdot \hat{I}_{t+H}^H}{\alpha \cdot ES_c^{t,H}}\right) - 1 = 0 \tag{3.51}$$

Using the indictor function $\hat{I}_{t+H}^H$, following the Equation 57, the test statistics has been defined as:

$$Z_2(\vec{L}^H) = \sum_{t=1}^{T}\left(\frac{L_t^H \cdot \hat{I}_{t+H}^H}{T \cdot \alpha \cdot ES_c^{t,H}}\right) - 1 \tag{3.52}$$

Similar to Test 1, Acerbi and Szekely present the hypothesis as:

$$H_0 : P_t^{[\alpha]} = F_t^{[\alpha]}, \forall t$$

$$H_1 : ES_{c,F}^{t,H} \geq ES_c^{t,H}, \text{ for all } t \text{ and } ES_{c,F}^{t,H} > ES_c^{t,H} \text{ for some } t$$

$$VaR_{c,F}^{t,H} = VaR_c^{t,H}, \text{ for all } t$$

This test can directly be used for ES without testing VaR first (Jäger, 2015). As proved in Acerby and Szekely (2014) proposition A.3, $\mathbb{E}_{H_0}(Z_2) = 0$ and $\mathbb{E}_{H_1}(Z_2) < 0$. The implication is that the expected value of $Z_2(\vec{L}^H)$ is zero and the test rejects the null hypothesis when it is negative (Jäger, 2015).

### Test 3: Estimating ES from realised rank

Unlike the two tests above, test 3 utilises of quantiles. Acerbi and Szekely (2014) argue that it is possible to backtest the tails of a model by checking if the observed ranks $U_t = P_t(X_t)$ are i.i.d. as suggested by Berkowitz (2011). The rank values are uniformily distributed in the interval (0,1) if the distribution assumptions are correct. In order to convert this into a specific test for ES, each quantile must be assigned its dollar importance that depends on the shape of its tail. Acerbi and Szekely provide a sample estimator of the ES for a sample $Y_1, \cdots, Y_N$ as:

$$ES_c^T = -\frac{1}{[Tc]}\sum_{i}^{Nc} Y_{i:N} \tag{3.53}$$

They define the quantile test statistic as:

$$Z_2(\vec{L}^H) = \frac{1}{T}\sum_{t=1}^{T}\frac{ES_c^T(P_t^{-1}(U))}{E_V[ES_c^T(P_t^{-1}(V))} - 1 \tag{3.54}$$

Where $V$ is i.i.d. $U(0,1)$.

The denominator can be computed analytically as

$$E_V[ES_c^T(P_t^{-1}(V)) = \frac{T}{[Tc]}\int_0^1 I_{1-p}(T-[Tc],[Tc]P_t^{-1}(p)dp \tag{3.55}$$

Where $I_c(a,b)$ is a regularised incomplete beta function.

The goal here is that ES is recalculated as a mean above the quantile $P_t^{-1}(U)$, for each day $t, \cdots, T$. After that, the average of the result is then taken. Therefore, the hypothesis for test 3 involes the entire distributions;

$$H_0: P_t = F_t, \forall t$$

$$H_1: P_t \succeq F_t, \text{ for all } t \text{ and } \succ \text{ for some t}$$

Where ($\succeq$) $\succ$ denotes (weak) first order stochastic dominance

While test 3 is extremely general, it is less intuitive compared to tests 1 and 2. Please see Acerbi and Szekely (2014) for more details on this test.

#### Significance

The 'goodness' of a model is typically tested by checking whether its deviations from the estimated $\mathbb{E}_{H_0}(Z_1|\hat{I} > 0)$ and $\mathbb{E}_{H_0}(Z_2) = 0$. However, this deviation, akin to the Basel Committee VaR backtest, does not always indicate that a model is bad (Jäger, 2015). Statistical deviations are normal and for the Basel backtest, the distribution of the test statistic was known under $H_0$, the binomial distribution. However, for the significance of the ES backtest described above must be simulated because the distribution of the tests is unknown (Jäger, 2015). Acerbi and Szekely (2014) show that for all tests $Z = Z_i$, the distribution $P_z$ is simulated under $H_0$, to compute the p-value $p = P(Z(\vec{x}))$ of the realisation $Z(\vec{x})$ (notations from Jäger (2015) have been used here for consistence:

simulate independent $\quad L_t^i \sim P_t, \forall_t, \forall_i = 1, \ldots, M$

compute $\quad Z^i = Z(\vec{L}^i)$

estimate $\quad p = \sum_{i=1}^{M} \frac{(Z^i < Z(\vec{l}))}{M}$

where M is a suitably large number of scenarios, given a preassigned significant level $\alpha$, the test is finally accepted if $p > \alpha$ and rejected if $p \leq \alpha$.

From the previous procedure, Acerbi and Szekely (2014) observe that to backtest ES exceptions it may be essential to keep the memory of the entire distributions $P_t$, unlike the VaR backtest where it is sufficient to record a single number $I_t$per day. For this reason, the key differences between VaR and ES backtesting is the storage of more information (a cumulative distribution function per day) (Acerbi and Szekely, 2014) and the simulation of a large number of scenarios (Jäger, 2015).

The power of a test is the probability of rejecting the null hypothesis when the alternative is valid and this defined as 1 minus the probability of Type II error (Sheppard, 2013). This is computed similar to the significance with the difference that the distribution $P_z$ is simulated under $H_1$ (Jäger, 2015).

#### 3.7.5.3.4. ES backtesting results

Similar to Jäger (2015), this section displays the results of backtesting VaR and two tests for ES based on the 21 years of REIT data. Though the backtesting period for the Basel framework is usually one year, this study carries out the test on a more extended period as well, i.e. 17 years. Both use a holding period of one trading day. Exhibits 3.19 and 3.20 present the graphic illustration of the 1 year (2016) and 17 years backtests respectively for the estimation of $VaR_{2.5\%}^{t,H}$, $VaR_{1\%}^{t,H}$ and $ES_{2.5\%}^{t,H}$. In both backtests, a violation takes place were the ex post or observed loss (in blue) touches the ex ante VaR or ES estimates. It is clearly discernible that the $VaR_{2.5\%}^{t,H}$ has more violation while the $VaR_{1\%}^{t,H}$ and $ES_{2.5\%}^{t,H}$ estimates are very close over both the backtesting windows. This is in conformity with the Basel Committee choosing the 2.5% significant level for ES as an equivalent for the 1% VaR significant level. The 2.5% is only used here as an indication of what the losses would have been had the same level been used as the ES. The test statistics $Z_1$ and $Z_2$ were simulated in MATLAB 10000 times and Table 3.3 displays these formal test results. Exhibits 3.21 and 3.22 display the probability density functions (PDFs) whilst Exhibits 3.23 and 3.24 present the cumulative distribution functions (CDFs) of both backtests.

Concerning the Basel traffic light system, the 1-year backtest resulted in 8 violation shown in Table 3.3. This test falls in the yellow traffic zone. Despite being successful and acceptable there may be, need to calibrate the risk model as this zone indicates some concerns. The 17-year backtest fell in the green traffic zone with 55 violations observed in the period. Therefore, for both backtesting periods (i.e. 1-year and 17-years), the VaR backtest was successful.

The ES backtest resulted in 12 violations and 127 violations for the 1-year and 17-years backtesting windows respectively. While the increase in the number of violations in the ES compared to the VaR test was 50% in the 1-year backtest, there was a 131% increase in the 17-year backtest. Test 1 for the ES revealed that both the test statistic $Z_1$ and the expectation of $Z_1$ are not significant at 5% for the 1-year backtest window. Therefore, this result is in favour of the null hypothesis so the decision is to "fail to reject" $H_0$. In other words, the backtest was successful indicating that the model correctly predicts the risk. The 17-year backtest window for the Test 1 for ES is however, significant at 5%. This means that the ES backtest is unsuccessful leading to the rejection of the null hypothesis. This could be due to the longer time over which the backtest covers. It is interesting that the VaR test was successful for both the 1-year and 17-years backtests but the Test 1 for the ES was only successful in the 1-year backtest window and unsuccessful in the 17-year one. The mixed results seems to suggest that Test 1 for ES is not accurate over the long period given that there is an inconsistency between the losses forecasted by ES and the observed. However, since the ES test is more robust than the VaR test it should still be used as it brings up more exceedances compared to the VaR.

The test statistics of Test 2 for ES in both the backtesting windows (i.e. 1-year and 17-years) are not significant, leading to "Fail to Reject" the null hypothesis. This means there is consistency between the losses forecasted by this ES model and the observed losses, consequently the test is successful.

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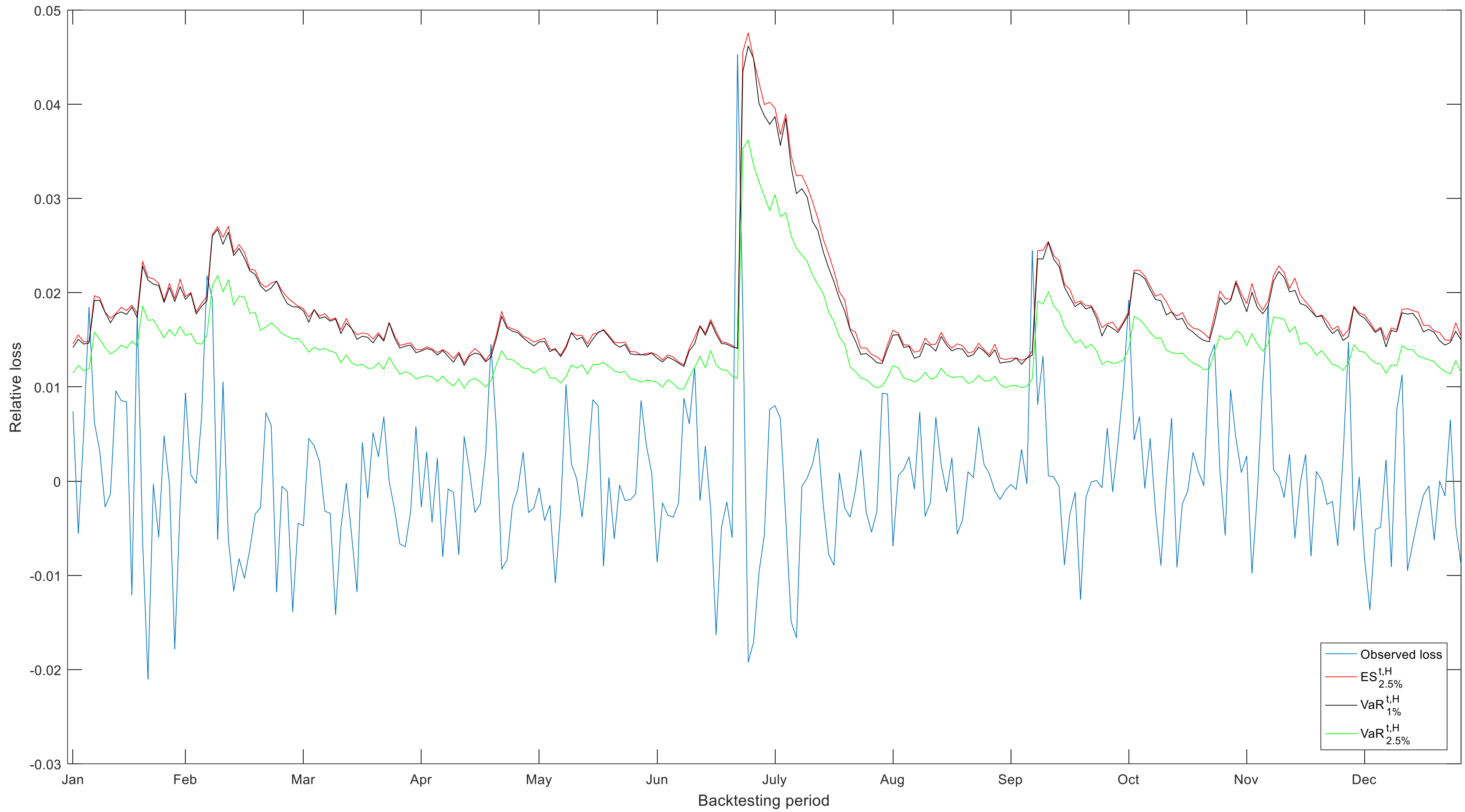



Exhibit 3.19:

Backtest results for 1 year, i.e. Jan 1st 2016 to Dec 31st, 2016

Note: The graph above shows the backtesting results and plots the observed losses against the losses estimated by ES, 99%VaR and 97.5% VaR for one year ranging from January 1, 2016 to December 31, 2016

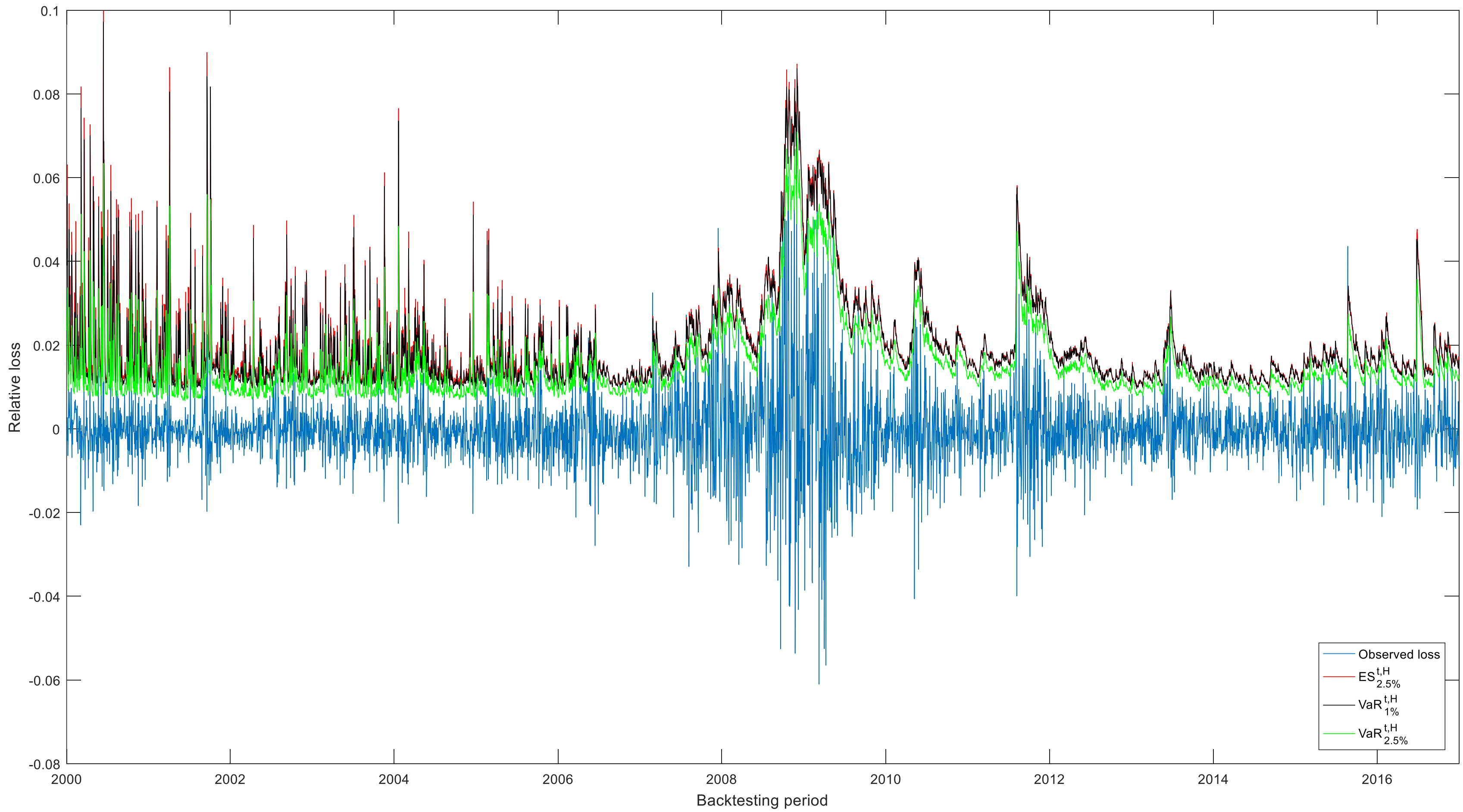
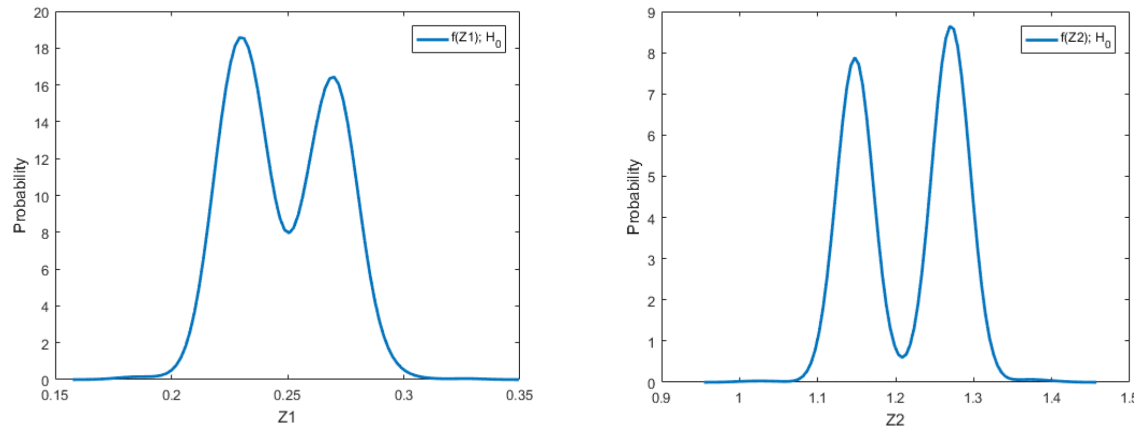



Exhibit 3.20: Backtest results for 17 years, i.e. Jan 1st 2000 to Dec 31st, 2016

Note: The graph above shows the backtesting results and plots the observed losses against the losses estimated by ES, 99%VaR and 97.5% VaR for 17 years ranging from January 1, 2000 to December 31, 2016

| Parameter | Testing Window 1 year | Test Window – 17 Yr |
|---|---|---|
| Testing Window | 260 | 4435 |
| Calculation time | 17 ours | 8 days |
| Value-at-Risk Basel Traffic Light | | |
| $Z_0(\vec{L}^H) = \hat{I}_{1\%}$ | 8 | 55 |
| $\alpha'_{1\%}$ | 3.0769 | 1.2404% |
| Probability for up to k violations | 99.8615% | 94.9983 |
| Cumulative Type1 Error for up to k Violation | 0.50757% | 6.6256% |
| Basel zone | Yellow | Green |
| Expected Shortfall Test 1 | | |
| $Z_1(\vec{L}^H)$ | 0.23798 | 0.015883 |
| $\mathbb{E}(Z_1(\vec{L}^H))$ | 0.24842 | 0.01717276 |
| $\hat{I}_{2.5\%}$ | 12 | 127 |
| Simulated p-value of realization $z(\vec{x})$ | 43.7437% | 38.5385% |
| Test results with M simulations and $\alpha_{z_1} = 5\%$ | Accept $H_0$ | Reject $H_0$ |
| | | |
| Expected Shortfall Test 2 Results | | |
| $Z_2(\vec{L}^H)$ | 1.2855 | 0.16389 |
| $\mathbb{E}(Z_2(\vec{L}^H))$ | 1.2133 | 0.17136 |
| $\hat{I}_{2.5\%}$ | 12 | 127 |
| Simulated p-value of realization $z(\vec{x})$ | 43.74% | 27.7277% |
| Test results with M simulations and $\alpha_{z_1} = 5\%$ | Accept $H_0$ | Accept $H_0$ |

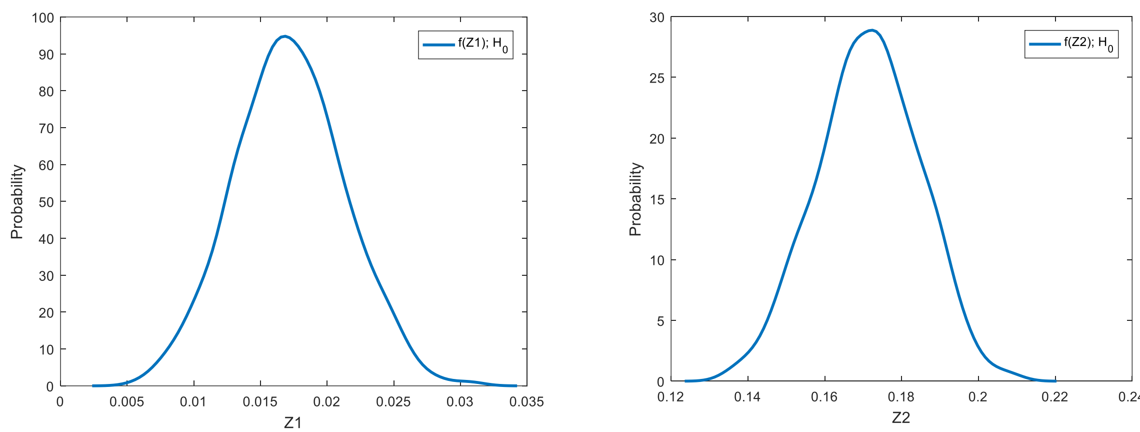
Table 3.3: Results from backtesting VaR and ES




**(a) Pdf for $Z_1(\vec{L}^H)$** **(b) Pdf for $Z_2(\vec{L}^H)$**

Exhibit 3.21: Probability Density Function for $Z_i(\vec{L}^H)$ for Jan 1st, 2016 to Dec 31st, 2016

Note: This probability density function (PDF) plot shows the distribution of Test 1 and Test 2 of Acerbi and Szekely's, (2014) backtests for the 1-year testing window

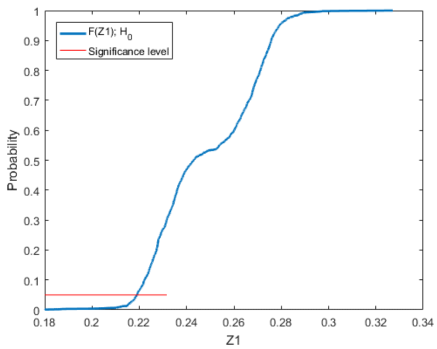



**(a) Pdf for $Z_1(\vec{L}^H)$** **(b) Pdf for $Z_2(\vec{L}^H)$**

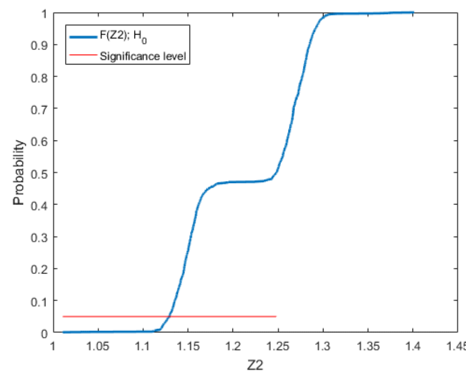
Exhibit 3.22: Probability Density Function for $Z_i(\vec{L}^H)$ for Jan 1st, 2000 to Dec 31st, 2016

Note: This probability density function (PDF) plot shows the distribution of Test 1 and Test 2 of Acerbi and Szekely's, (2014) backtests for the 17-year testing window

The pdf and the CDF can perhaps provide more explanation of the results presented by the test. The simulated pdf for the 1-year backtesting period show PDF that is non-Gaussian like while that for the 17-year backtesting window resembles that of a Gaussian PDF as shown in Exhibits 6.21and 6.22 respectively. This is probably due to the long time frame of the 17-year window as compared to the 1-year window. Exhibits 23 and 24 display the simulated CDFs for both backtesting windows for the two tests for ES. Consistent with the pdf the 1-year backtesting windows for both ES tests do not show a normal distribution while those for the 17-year backtesting window show normal distribution. The red line on each of the CDFs represents the 5% significant level that represents the lower limit of the green zone when it touches the CDF. For both the tests, $Z_1$ and $Z_2$ are higher than where the 5% significant line touches the CDF curve. This shows that the tests are not significant



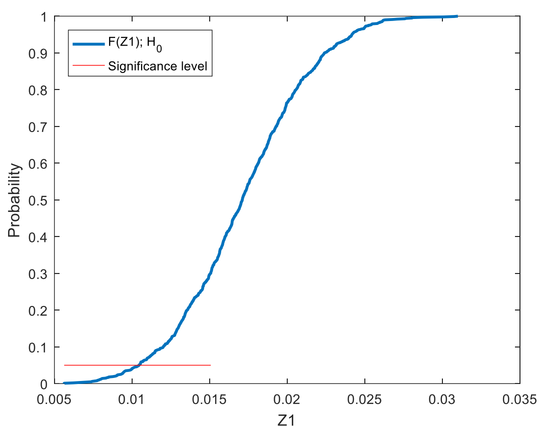

**(a) CDF for $Z_1(\vec{L}^H)$**



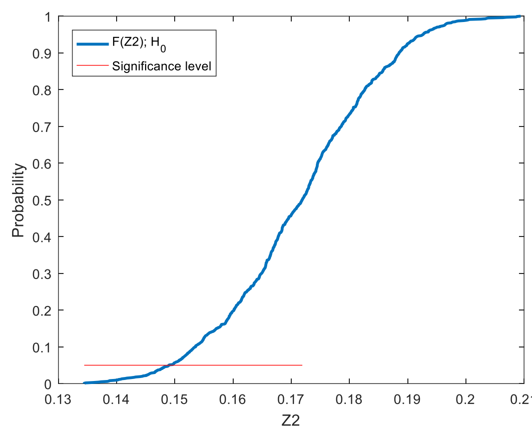

**(b) CDF for $Z_2(\vec{L}^H)$**

Exhibit 3.23: Cumulative Density Function for $Z_i(\vec{L}^H)$ for Jan 1st, 2016 to Dec 31st, 2016

Note: This cumulative density function (CDF) plot shows the cumulative distribution of Test 1 and Test 2 of Acerbi and Szekely's, (2014) backtests for the 1-year testing window




**(a) CDF for $Z_1(\vec{L}^H)$**




**(b) CDF for $Z_2(\vec{L}^H)$**

Exhibit 3.24: Cumulative Density Function for $Z_i(\vec{L}^H)$ for Jan 1st, 2000 to Dec 31st, 2016

Note: This cumulative density function (CDF) plot shows the cumulative distribution of Test 1 and Test 2 of Acerbi and Szekely's, (2014) backtests for the 17-year testing window

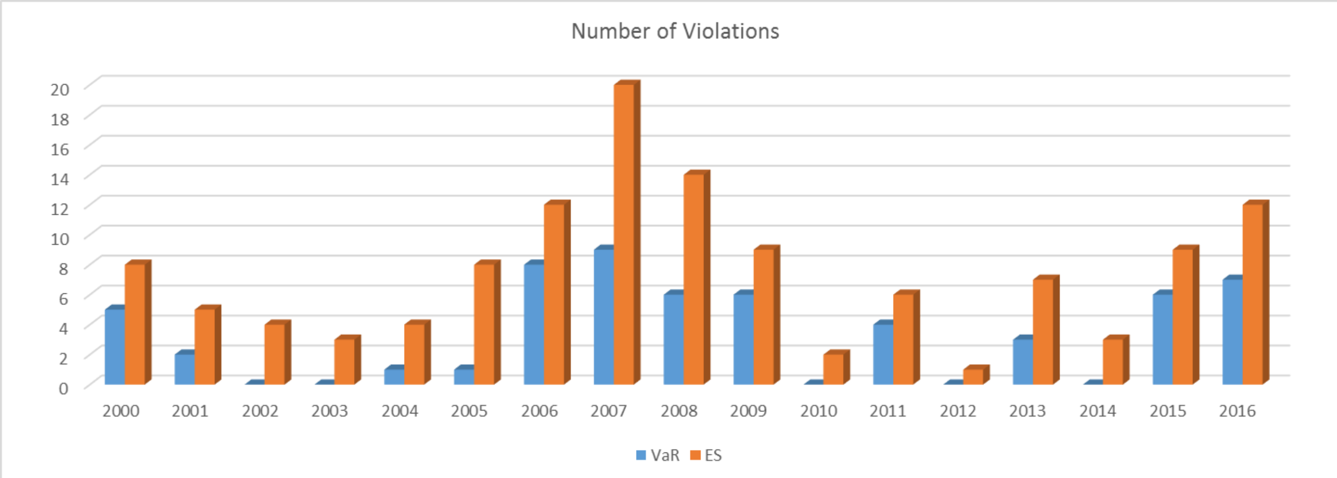



Exhibit 3.25: Number of violations for VaR and ES from 2000 to 2016

| Year | VaR Zone | $\mathbb{E}(Z1)$ Result | $\mathbb{E}(Z2)$ Result |
|---|---|---|---|
| 2000 | Green | Accept Ho | Accept Ho |
| 2001 | Green | Accept Ho | Accept Ho |
| 2002 | Green | Accept Ho | Accept Ho |
| 2003 | Green | Accept Ho | Accept Ho |
| 2004 | Green | Accept Ho | Accept Ho |
| 2005 | Green | Accept Ho | Accept Ho |
| 2006 | Yellow | Accept Ho | Accept Ho |
| 2007 | Yellow | Accept Ho | Accept Ho |
| 2008 | Yellow | Reject Ho | Accept Ho |
| 2009 | Yellow | Reject Ho | Accept Ho |
| 2010 | Green | Accept Ho | Accept Ho |
| 2011 | Green | Accept Ho | Reject Ho |
| 2012 | Green | Reject Ho | Accept Ho |
| 2013 | Green | Reject Ho | Accept Ho |
| 2014 | Green | Accept Ho | Accept Ho |
| 2015 | Yellow | Accept Ho | Accept Ho |
| 2016 | Yellow | Accept Ho | Accept Ho |

Table 3.4: Results for the yearly backtests for VaR and ES

The backtesting procedures were also applied to all the individual years of the backtesting period. Exhibit 3.25 shows the yearly breakdown of the violations for both VaR and ES for the backtesting period under investigation. As expected ES captured more violation compared to VaR with the big disparity being during the global financial crisis period this is consistent with the finding of Almudha (2018). The Acerbi and Szekely (2014) backtesting methods were also applied to these individual years with the summary shown in Table 6.4. The results mostly fell in the green zone, and a few in the yellow zone thus suggests that the model accurately predicts the VaR. Concerning ES, most of the results revealed that the test should fail to reject the null hypothesis for Test 1 meaning that there is consistency between the predicted ES and the observed values, therefore, suggesting that the model correctly predicts the ES. However, this is with the exception of four years that include the global financial crisis 2008/9 period and for 2012 and 2013 were the null hypothesis is rejected. Test 2 is only significant for 2011 leading to the decision to reject the hypothesis that the model correctly predicts the ES in this year only. The other years suggest that the null hypothesis should be "accepted".

## 3.8. Conclusion

The paper set out to explore the performance of market risk models namely, i.e. the value at risk and expected shortfall. Both the measures are used to quantify the risk of an investment and can be estimated using different parametric, non-parametric and semi-parametric models. The models range from the simplistic normal VaR and historical simulation to the complicated models that involve the combination of conditional volatility models like GARCH and with some that incorporate EVT to account for extreme returns while others utilise Monte Carlo Simulation. Various research has been undertaken on the performance of VaR in both non-REITs and REITs, and generally, the findings are that VaR performs differently for different models. In some instances, the historical simulation works just fine while in others the semi-parametric models, e.g. Filtered Historical Simulation and GARCH EVT perform better.

Despite the wide application of VaR for measuring market risk, owing to its simplicity, it has come under increasing pressure due to the major shortcoming that stems from the lack of subadditivity leading it not to qualify as a coherent risk measure. This is in addition to its failure to considers losses beyond VaR, being its other major weakness, which has the potential of underestimating the market risk of an asset or a portfolio. This has led to a call for its replacement with another market risk model called expected shortfall (ES) which, in contrast to VaR, is a coherent risk measure and also considers losses beyond VaR. Notwithstanding the elegance of ES, its main weakness hinges on its problematical implement of backtesting stemming from backtesting an expectation rather than a specific VaR value. Distinct from VaR which has several fully developed methods of backtesting, there is no agreed method for backtesting ES mainly due to its lack of elicitability. This difficulty has however been challenged with Acerbi and Szekely (2014) arguing that elicitability is not necessarily needed to backtest ES and with that, they have developed some approaches of backtesting ES, some of which this paper has applied to the REIT market. This study models market risk and investigate whether ES is a better estimator of market risk compared to VaR.

The study is based on REIT data from 5 international indices namely, Australia, Canada, France, the UK and the US for the period spanning from January 1996 to Dec 2016. The examination of the models is done at both univariate and multivariate levels. The univariate risk measures are estimated at 95% and 99% confidence levels using the normal, historical simulation, EWMA and GARCH(1,1) and backtested using a look back period of 1000 days. Various backtesting tests are employed for VaR, name; traffic light system, binomial, the proportion of failure, time until first failure, conditional coverage and conditional coverage independence test while a test by proposed by Danielson (2011) is utilised for ES.

The findings for the study are mixed, but overall the simple tests showed that the models are accurate at 95% VaR in more times than the stringent test while at 99% most of the basic tests show that the models are inaccurate estimates of market risk for all the models except the non-parametric historical simulation which is accurate. This is consistent with the literature from different studies. At multivariate level, the results are mixed as well, but overall, the EWMA and GARCH come out to be accurate estimators of market risk at 95% VaR level for the simple tests, but the stringent tests show that the models are inaccurate.

The univariate ES, on the other hand, shows that results in most of the estimates are not accurate for all the tests except for the time until first failure which is accurate for both the 95% and 99% levels.

The multivariate backtesting is extended by apply GARCH-GJR, EVT and copulas in order to incorporate challenges of data in terms of autocorrelation, fat tails and also non-linear dependence estimating VaR and ES. To this effect, the backtesting is performed using Acerbi and Szekely's (2014) Test1 and Test 2 using a 1-year and 17-year window. The finding is that the VaR appeared to be an accurate estimator of market risk in both periods while test 1 accepted ES and rejected it for the 1-year and 17 years backtesting windows respectively, the estimate for test 2 provided evidence that ES is good at forecasting market risk. The test as expected shows that ES can capture more violations than VaR so even though both tests seem to be accurate ES should be used because it captures more violation than VaR. This, therefore, suggests that ES is not as prone to the underestimation of market risk compared to VaR as evident from the yearly backtesting results.

The implication of the study is that ES should be used to measure market risk for REITs especially at portfolio level because its performance is more consistent compared to VaR. Furthermore, it results in less underestimation of market risk, unlike VaR due to its ability to capture losses in excess of VaR.

While previous studies have been undertaken in the estimation of market risk in REIT, this study's contribution is performing the analysis by considering stylised facts and using GARCH-EVT-and copulas at the multivariate (portfolio level) level and also for the first time applying the backtesting of ES to real estate securities using Acerbi and Szekely (2014) testing methods.

The limitation of the study is that only two tests of Acerbi and Szekely (2014) were undertaken due to implementation challenges of Test 3. Also due to time constraints, switching methods were not employed to take into consideration performances in different market conditions.

The market risk modelling exposed the researcher to other aspects of volatility that will be built on in the next chapter. The time-varying volatility and correlation are of particular interest, specifically as a means of further analysing, future volatility as implied by the options market. This has some bearing on the extent that returns and indeed volatility of an asset or market will move. The next chapter explores implied volatility of in public real estate.

# CHAPTER 4

# THE VOLATILITY TRANSMISSION OF THE UK IMPLIED VOLATILITY INDEX AND UK REITS WITH TRADED OPTIONS

## 4.1. Introduction

The complexity of risk management for REITs has increased due to the lack of traded hedging instruments (Cotter and Stevenson, 2006 and Diavatopoulos, Fodor et al. 2010). This is more so in the UK where only three companies in the REIT sector trade in options. Research conducted (Diavatopoulos, Fodor et al. 2010) has revealed that implied volatility in REITs has predictive power. This research was however undertaken in the US which has a more vibrant REIT market. Despite the UK market having more than 50 REITs, there are only three companies which trade in options. Research in the UK on implied volatility has mainly focused on the market index mainly using the VIX. Relatively little research has been undertaken with regards to the UK individual REIT companies.

A relationship between investor sentiment and their investment decision making has been established through the study of behavioural finance. Investors' decisions are, however, not rational because sentiments have a considerable bearing on them. It follows, therefore, that share price volatility is a function of investor sentiment. Implied volatility measures share price volatility, implied by the market and is used as an indication of investor sentiment. In general, high implied volatility is directly related with risk in the market where large market movements (either positive or negative) are expected and vice versa. In the US the Chicago Board of Options Exchange (CBOE) VIX, commonly referred to as the fear index, measures implied volatility relating to the market.

Generally, as the market becomes more unpredictable due to increased uncertainty, it becomes more important to focus on volatility rather than returns. Given the varied performance of UK REITs since their inception, volatility is of crucial importance as it is at the core in asset allocation, diversification and hedging instruments. There seems to be some consensus regarding REIT volatility which shows that it is time-varying and predictable (Cotter and Stevenson, 2006, 2007). Furthermore, research has also shown that REITs have characteristics that are unique in contrast to non-REITs (Diavantopoulos et al., 2010). Also, compared to non-REITs, REITs have higher gearing ratios because their share price falls, make them more susceptible to financial risk because of the increased level of debt relative to the value (Chung et al., 2016 and Kawaguchi, Sa Aadu and Shilling, 2016). Moreover, it is more challenging for REITs to reduce these gearing levels given their requirement to distribute a large proportion of their earnings to shareholders as dividends. While the high dividend payout is a key factor for investing in REITs since it is likely to translate in high yields, Kawaguchi, Sa Aadu, and Shilling (2016) argue that the increase in financial risk has negative consequences for risk-averse investors who may be better off investing in lower yielding, fixed income securities.

The performance of UK REITs since their introduction, just before the 2008 -2009 financial crisis, has been varied. The market generally witnessed an overall drop in the price of the REITs in the first two years, and after that, the trend has been upwards as shown in Exhibit 4.1. In return terms, there is volatility clustering of returns with some rather extreme rises and drops in these returns, implying that there is a high level of volatility as shown in Exhibit 4.2. It is evident that the performance of the UK REIT market exhibits high levels of unpredictability of both returns and volatility. The issue is how investors can predict the direction or magnitude of future prices, returns, and volatility of this market? Previous research (Diavatopoulos, Fodor et al. 2010, Chung, Fung et al. 2016, Cotter and Stevenson, 2007; Doran and Krieger, 2010; Busch, Christensen, and Nielsen, 2011 Anoruo and Murthy, 2016; 2017; Cotter and Stevenson 2006) shows that implied volatility contains some information that can be used in the prediction of future returns and volatility. That said, most of the previous research has been conducted on US REITs typically at the REIT market index level. This research, however, investigates the links or transmission by considering UK REIT companies to ascertain where there are spillover effects for the VFTSE and REIT companies iv and between the REIT companies. In order to undertake this task, only UK REIT companies that trade in options are considered.

The research proceeds as follows. The next section will explore implied volatility smiles and surfaces which will then be covered. The data and methodology will then follow before proceeding to the empirical analysis and the conclusion.


Exhibit 4.1: UK REIT Index Prices. Source: Bloomberg

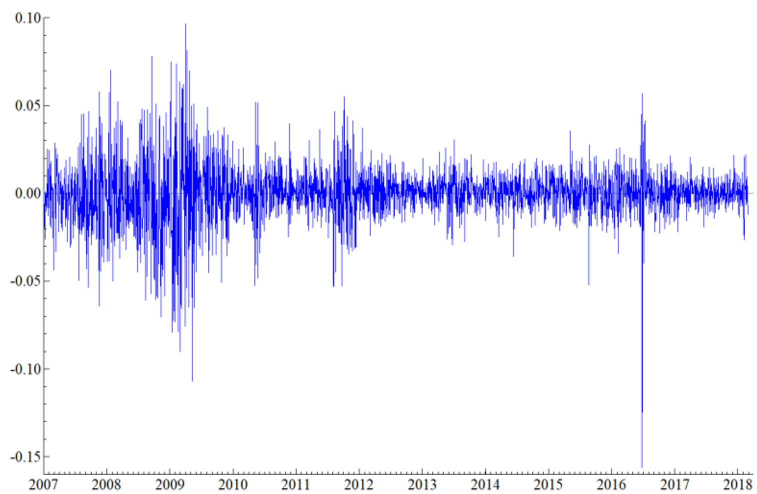



Exhibit 4.2: UK REIT index daily Returns

## 4.2. Implied volatility

The theoretical price of a standard European option is commonly calculated by the Black-Scholes-Merton (BSM) formula at any time prior to the option's expiry date. However, market prices for options are not determined by BSM formula, but rather the dynamics of supply and demand. The main uses of the BSM formula are twofold. The first is that it can be used for pricing standard European options that have no market prices available (or illiquid options). The second use of the BSM formula is in the calculation of implied volatility, gleaned from standard European call and put market prices.

The BSM model lays the foundation of the pricing of options by making some assumptions. Some of the assumptions are that the distribution for underlying is normal; the price follows a geometric Brownian motion and operates in markets that are complete and frictionless. However, this is not the case in reality and therefore market participants, i.e., Investors and options traders, do not believe in these assumptions The BSM model has several parameters namely, the price of the underlying, strike price, risk-free rate, dividend yield, time to expiry and volatility. While all these parameters are observable, the volatility is the only parameter that is not directly observable in the market. Also, as the price of the underlying, the risk-free rate and the dividend yield are the same for all options with the same expiry date, likewise it is assumed that the volatility should also be the same since it is based on the same underlying. Because volatility is not an observable parameter, traders reverse engineer the BSM model in order to calculate the volatility implied by market option prices. This is referred to as the "implied volatility". The implied volatility is calculated by using observed parameters in the BSM model and working out, the volatility that will result in an option price equal to the observed market option price. In other words, if the implied volatility is used as the volatility in the BSM model, it will result in the market option price. Alexander (2008c) observes that the implied volatilities for all options on the same underlying should be identical if market participants accept as true, the assumptions fundamental to the BSM model. This resulting unique implied volatility would possess the volatility of the geometric Brownian motion process[34]. Nevertheless, this is not so because different options on the same underlying result in implied volatilities that are not identical and this can be depicted graphically by plotting a volatility surface.

This research will discuss the implied volatility based on the standard European call option. As the right-hand side of the put-call parity[35] relationship is not dependent on volatility, a call and put of the same strike and maturity have the same implied volatility if the put-call parity holds on market prices (Alexander, 2008c). This is keeping in mind that the fundamental attribute of the put-call parity relationship is that it is based on a fairly simple no-arbitrage condition. Additionally, no probability distribution assumption of the underlying future price is required, making it true whether the underlying price distribution is lognormal or not lognormal (Hull, 2009). Hull (2009) illustrates this as follows:

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Suppose that, for a particular value of the volatility, $p_{BSM}$ and $c_{BSM}$ are he values of european put and call options calculated using the BSM model. Suppose further that $p_{mkt}$ and $c_{mkt}$ are the market values of these options. Because put-call parity holds for the BSM model, we should have

$$p_{BSM} + S_0 e^{-qT} = c_{BSM} + Ke^{-rT} \tag{4.1}$$

In the absence of arbitrage opportunities, put-call parity also holds for the market prices, so that

$$p_{mkt} + S_0 e^{-qT} = c_{mkt} + Ke^{-rT}$$

Subtracting these two equations, we get

$$p_{BSM} - p_{mkt} = c_{BSM} - c_{mkt} \tag{4.2}$$

This shows that the dollar pricing error when the BSM model is used to price a European put option should be exactly the same as the dollar pricing error when it is used to price a European call option with the same strike price and time to maturity.

Suppose that the implied volatility of the put option is 22%. This means that $p_{BSM} = p_{mkt}$, it follows that $c_{BSM} = c_{mkt}$ when the volatility is used. The implied volatility of the call is, therefore, also 22%. This argument shows that the implied volatility of a European call option is always the same as the implied volatility of a European put option when the two have the same strike price and maturity date. To put this another way, for a given strike price and maturity, the correct volatility to use in conjunction with the BSM model to price a European call should always be the same as that used to price a European put. This means that the volatility smile (i.e. the relationship between implied volatility and strike price for a particular maturity) is the same for calls and puts. It also means that the volatility term structure (i.e., the relationship between implied volatility and maturity for a particular strike) is the same for calls and puts.

From the BSM formula below,

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{4.3}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

the price and risk-free interest rate $r$ for a standard European call (or put) option with strike $K$ and time to maturity $T$ can be obtained from the market. To work out the implied volatility, $\sigma$ that agrees with

Equation (4.1) has to be calculated. This can be done using spreadsheet software like Microsoft excel or functions from more sophisticated programmes like MATLAB or R.

In contrast to historical volatilities that are backwards-looking, implied volatilities (also referred to as implicit volatilities) are forward-looking and provide an indication of the market's sentiment regarding the volatility of a particular underlying. Also, because options prices tend to be more variable than implied volatilities, traders frequently quote the implied volatility of options instead of their price (Hull, 2009). The importance of implied volatility has led to the creation of indices based on implied volatility, and the most popular is the Chicago Board of Options Exchange's (CBOE) volatility index (VIX) which is based on the implied volatility of 30-day options of the Standard and Poor (S&P) 500. Another volatility index is the VXN, but it is based on the NASDAQ 100 index. In the UK there is also a volatility index called VFTSE and is based on the (Financial Times Stock Exchange) FTSE 100 index (Whaley, 2009, Emna and Myriam, 2017). Implied volatility is essential because if the market implied volatility can be forecast successfully then the market price of options can also be forecast, and the option can, in turn, be hedged successfully. For this reason, Chiang (2012) argues that implied volatility indices provide superior means to investigate the relationship between market risk and return.

As stated earlier, a range of standard European options on the same underlying does not result in the same implied volatility. The relationship between the strike (moneyness[36]), maturity and implied volatility is depicted through volatility surfaces referred to as implied volatility and local volatility surfaces. These surfaces can be derived either from market prices or from prices based on a stochastic volatility model (Alexander 2008c). As pointed out previously, the BSM model is based on a number of assumptions, and if these assumptions were valid, then all options on the same underlying should end in the same market implied volatility resulting in a flat surface of market implied volatilities. This is not the case as market participants do not believe in the assumptions of the BSM model as suggested by the surface of the market implied volatilities which is not flat. A plot between the market implied volatility of all options on the same underlying with the same maturity and different strikes (or moneyness) results in a skewed smile shape and is called the volatility smile or volatility skew or smirk as shown in Exhibit 4.3.

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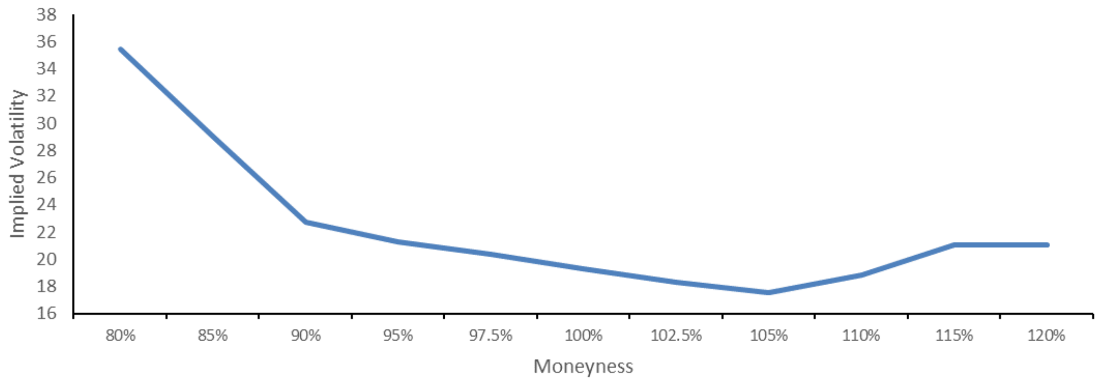


Exhibit 4.3: Volatility smile

Note: This shows is an example of a volatility smile for Land Securities option. The vertical and the horizontal axes represent the Implied volatility and the moneyness respectively. It can be seen that the implied volatility is not constant but changes with moneyness. (Data Source: Bloomberg, 2018)

Generally, implied volatility for equity indices is considerably higher for low strike (OTM puts and ITM calls) options than high strike (OTM calls and ITM puts). There is an inverse relationship between the strike and the implied volatility. Therefore, deep OTM put options are generally inexpensive because it is akin to buying insurance for an underlying one already holds. Furthermore, because the log price density is negatively skewed, the implied volatility for equity indices results in a negative skew. This negative skew emanates from the significantly high premium market participants are willing to pay for OTM puts as insurance to limit losses in the event of a market crash. This phenomenon, however, has only been in existence post the market crash in October 1987 with market participants worried about another crash of a similar magnitude hence the term “crashophobia” (coined by Mark Rubinstein (Hull, 2009). Hull (2009) also attributes the smile of equity options to leverage. There is an inverse relationship between price and volatility. A drop in an underlying's share price results in the decline of the value (market capitalisation) which in turn increases the leverage or gearing of a company. As a consequence, the volatility and hence the implied volatility will increase because the underlying is considered riskier. The opposite scenario (an increase in the share price) has the effect of lower leverage and ultimately the volatility of the underlying. Log price densities with heavier lower tails and lighter upper tails than the normal density tend to be pronounced in equity options corresponding to premiums traders charge for equity markets (Alexander, 2008c). In the BSM world, the risk-neutral log price density is normal since the BSM model assumes the geometric Brownian motion price process.

The underlying’s current price influences the foregoing relationship between the strike price and the implied volatility. For equities, the volatility skew is likely to move to the right or left when there is a price increase or decrease respectively. In order to stabilise the volatility smile, it is sometimes calculated as the relationship between the implied volatility and $K/S_0$ as opposed to the relationship between implied volatility and strike (Hull, 2009). $K/S_0$ is what was referred to as moneyness where ATM options have a value of 1.

A plot between the market implied volatility of all options on the same underlying with the same strike (or moneyness), but different maturities converges to long-term implied volatility and is called the term structure of implied volatility (see Exhibit 4.4). Typically, ATM options are used in conjunction with the maturity of the option to work out the term structure. Volatility should be constant according to the BSM model, suggests that implied volatility should also be constant over time. However, this is not the case as volatility is not constant and implied volatility changes over time. This term structure can be used when pricing options. In periods with historically low, short-dated volatilities, there is an expectation that volatilities will increase; hence the resulting volatilities tend to increase with maturity. Correspondingly, in periods with historically high, short-dated volatilities, the expectation is that volatilities will decrease, therefore resulting in volatilities that tend to decrease with maturity.

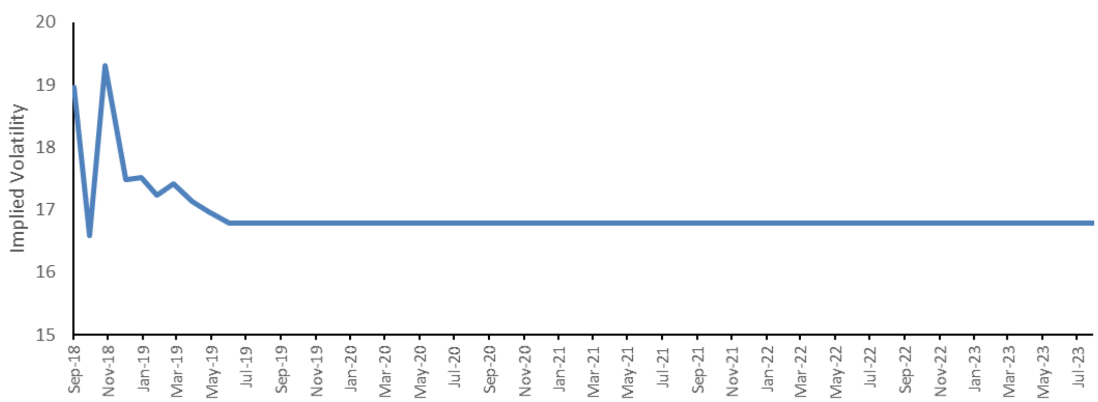


Exhibit 4.4: Implied volatility term structure

Note: This shows the term structure for Land securities which plots the relationship between the implied volatility and the time to maturity (term) on an option with the same strike price. The vertical and horizontal axes represent the implied volatility and time to maturity respectively. . (Data Source: Bloomberg, 2018)

The volatility smile and the volatility term structure when combined form the volatility surface (See Exhibit 4.5). Here, the implied volatility is a function of both the strike and the maturity. Again, if the assumption regarding volatility in the BSM model were correct, a flat volatility surface would result. However, the volatility surface is not flat because of the resulting implied volatility is not the same as shown in the volatility smile and volatility term structure plots. The importance of the volatility surface is that it can be used, through interpolation of the BSM model, to value options that have no market prices but have known strike and maturity.

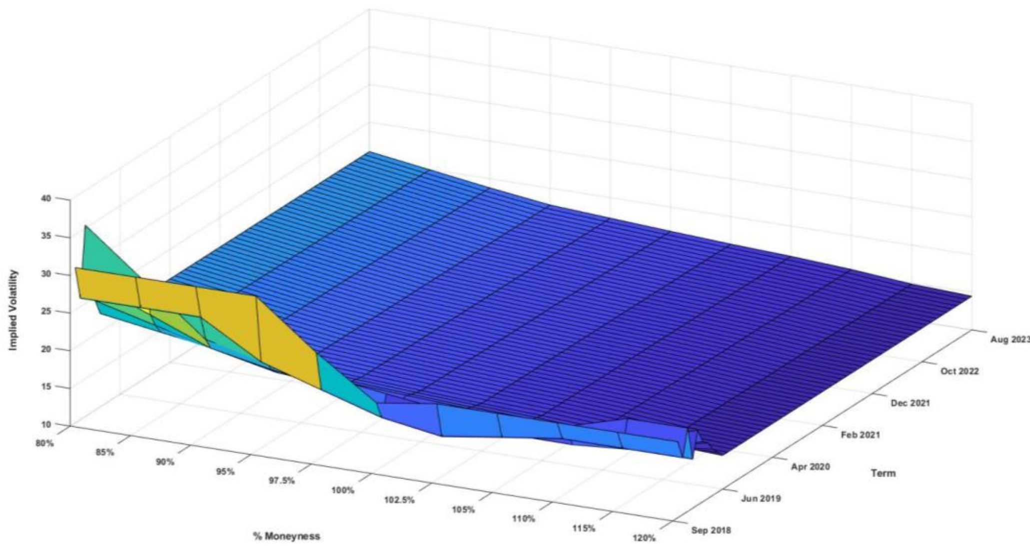


Exhibit 4.5: Volatility surface

Note: This is an example of a volatility smile of Land securities. This plots the implied volatility plotted against the moneyness and the term (time to maturity). (Data Source: Bloomberg, 2018)

The purpose of this research is to investigate the volatility spillover of the UK REITs with traded options by using implied volatility.

### 4.2.1. Implied volatility in real estate

As earlier indicated, implied volatility is the volatility implied from the price of options by reserve engineering the option formula to get the volatility that produces the market price for the option. Implied volatility is an indication of the market's view of the volatility. The focus of many studies hinges on the relationship between implied volatility and future stock returns and realised volatility. Instead of only observing the implied volatility of individual stocks, it is also possible to observe the implied volatility of a market through the volatility indices like the VIX.

There is evidence that these volatility indices provide information on the performance of the market as well as individual companies (Giot, 2005; Whaley, 2009;.Chiang, 2012; Emna, 2012; Gaert and Hoerova, 2014). While this is the case, there is also research that looks at the transmission or spillover between the volatility of a specific market to another market. The literature examined below is two fold, firstly it explores the relationship between implied volatility and real estate performance and then also explores volatility spillover both in the different markets as well as in real estate.

Numerous research has been undertaken that investigates implied volatility in real estate. One of the earliest studies in this respect is by Patel and Sing (2000) who estimate of implied volatility based on actual transaction data of property. They do this in order to address the failure of indices based on appraisals to adequately reflect specific property risk-return characteristics. They observe property is usually riskier than what the indices purport; therefore leading investors to carry further specific risk because the residual risk captured in some appraisals is nothing more than a simple adjustment of a local market trend. Patel and Sing (2000) use a one -factor option model to estimate the implied volatility and compare it to the actual volatility of the rental transactions of the UK commercial property sector. The estimation is based on the one factor real option proposed by Pindyck (1991) and Dixit and Pidyck (1994) and is analysed in the conditional variance in the mean framework proposed b Engle, Lilien, and Robins (1987) and Bollerslev, Engle, and Wooldrige(1988) and the exponential version of the model by Nelson (1991). They find that there is strong orthogonality in the information impounded in implied volatility estimates compared to that contained in historical standard deviations. However, this relationship is not clear-cut as other research present contrasting findings (Griffin and Lemmon, 2002; Clayton and MacKinnon, 2003), Guirguis, and Shilling, 2009).

While the work of Patel and Sing (2000) is based UK direct property, Diavatopoulos et al. (2010) examine equity REIT options and the predictive power of ex-ante risk measures taken from option prices. While implied volatility has been extensively investigated outside the real estate sector, Diavatopoulos et al. (2010) were the first to apply it to REITs. The study employed some simple regression as well as the reverse Fam and Macbath (1973) regressions. The findings of the study show that REIT implied volatility and implied idiosyncratic volatility distributions are like those of other listed equities and that the future realised volatility for REITs is related to both future and implied volatility.

Anoruo and Murthy (2016) examine the relationship between implied volatility from the US implied volatility index (VIX) and REIT returns by employing frequency a dependant regression and a frequency domain causality test allowing shocks to vary across frequency bands. The particular aim of the study is to investigate whether movements in US REIT returns at different frequencies, i.e. low-, medium -, and long-term, can be predicted by implied volatility. The results show that there is a negative relationship between the implied volatility and REIT returns at all the frequency levels. Extended analysis using causality tests reveal that causality runs from implied volatility to all and equity REIT returns in the short – and medium -term, frequently but not the other way round. In contrast, the results also indicate that in the medium term, causality runs from mortgage REITs to implied volatility.

Overall, conclude that knowledge of implied volatility can assist investors in predicting movements in the capital market.

Chung et al. (2016) focus on the global financial crisis (GFC) period in looking at the relationship between US REIT stock volatility and future returns by using an approach based on some regression models based on several regression models. The findings suggest on the one hand there is a negative relationship between the REIT implied volatility and contemporaneous stock returns. On the other hand, there is a significant positive relationship between REIT implied volatility and future stock volatility while the relationship between REIT implied volatility and future stock returns is significantly negative.

In another study, Akinsomi, Coskun, and Gupta (2018) investigate the impact of volatility and equity market uncertainty on herding behaviour in the UK REIT market through different market regimes by employing a Markov regime switching model. Here, the volatility of the equity market is obtained from the implied volatility index the VFTSE which is the UK version of the VIX. The findings are that the static model rejects the existence of herding in the UK REIT market whiles the regime-switching model estimates show significant evidence of herding and anti herding behaviour the low and high volatility regimes respectively. This suggests that the level of the equity market's volatility may provide a signal of herding related risk in UK REITs although this also depends on the market condition or regime in which the analysis is undertaken.

As previously state implied volatility is also used in the examining transmission or spillover effects between different markets within or outside real estate. This study now examines spillover effects focusing on volatility spillovers

## 4.3. Spillover effects

In general financial economics, volatility spillover research dates back to the 1990s and has exploded particularly after the introduction of Autoregressive Conditional Heteroscedasticity (ARCH) which make it possible to the first and second moments to be modelled concurrently (Stevenson, 2002). Most of the early research focuses on the stock market dynamics by examining the inter-linkages of return and volatility of international markets in studies such as Hamao et al. (1990), King, Sentana and Wadhani (1990), Bae and Karolyi (1994). The results are varied, for example, Bae and Karolyi (1994) reveal that the volatility spillovers between major stock markets are asymmetric whereas there is volatility spillover evidence among the US, the UK, Canada, Japan and Germany (Thedossiou and Lee, 1993). Research is not only limited to countries with developed stock markets, but other studies investigate the integration between in emerging countries, i.e. Turkey, Russia, Brazil, Korea, South Africa, and Poland (Mandaci and Torun, 2007). The results indicate the existence of significant co-integration between stock markets in Russian and Korea while some tests show a long- and short-term relationship between the Brazilian and Polish stock markets.

Motivated by the global financial crisis (GFC) Diebold and Yilmaz (2012)using US stock, bond, foreign exchange, and commodities markets propose spillover measures that build on from the previous study's (Diebold and Yilmaz, 2009 (DY09). They developed measures address the methodological and substantive limitations of their 2009 study. On the methodology side, they argue that their earlier study (DY09) could result in variance decompositions dependent on variable ordering. From the substantive side, only spillovers across identical assets can be measured by DY09. Diabold and Yilmz (2012) therefore propose a spillover measure, using a generalised vector autoregressive framework, whose forecast-error variance decompositions are variance ordering invariant and also propose measures of both the total and directional volatility spillovers. Their study covers a period from January 1999 to January 2010 and results reveal that cross-market volatility spillovers were quite limited up until the GFC, notwithstanding the significant volatility fluctuations during the sample period. They show that the spillovers increased as the crisis intensified.

Other research has looked at the financial integration between emerging markets i.e. Brazil, China, Russia, and Turkey; and developed markets of the the US, UK and Germany (Nasser and Hajilee, 2016) using the bounds testing approach to cointegration coupled with error-correction modelling in order to ascertain the short- and long-run relationship between these two markets. The results provide evidence of the existence of short-run integration between the two markets whereas on Germany displays a significant relationship with the long-run coefficients of all the stock markets in the emerging countries.

Spillover research is not only limited to stock markets but also between commodities like oil and stock markets (Malik and Hammoudeh, 2007) showing that there are spillovers from the oil markets to the Gulf equity markets except for Saudi Arabia that displays significant volatility spillover from the equity market to the oil market. In another research Bein (2017), using the DCC GARCH model investigates the time-varying co-movements and volatility transmission between stock market of three Baltic states (Estonia, Latvia and Lithuania) and two international crude oil indices (Brent and West Taxes

Intermediate (WTI)). Furthermore, the study also investigates the relationship between the two oil indices with the EU and the UK – major oil importing, and Norway and Russia – two exporting European countries. The results reveal that a positive albeit lower level of time-varying co-movement between the Baltics and the international markets. They also find contrary to the literature that impacts of oil shocks in the Baltics are also lower than in the European counties. Furthermore, the time-varying and volatility transmission with the oil importing and oil exporting countries shows the existence of high-level time-varying-movements. Bein (2017) also shows that the oil-exporting countries have high volatility transmission and magnitude of shocks from the oil markets. Volatility spillovers are also employed in research for commodity markets like steam coal (Li, Joyeux and Ripple, 2010; Papiez and Smiech, 2005; and Batten et al. 2019) which generally show the integration between different coal markets. Volatility spillover is also applied to studies on precious metals (Batten, Ciner and Lucy, 2014) and non-ferrous metals( Ciner, Lucey and Yarovaya, 2018).

Research on spillover effects also extends to real estate in general and REITs in particular. Stevenson (2002) examines the linkages between US REIT and other US capital markets asset classes, i.e. US based equities and fixed income sectors, by employing the second moment of the return distribution or volatility estimated using GARCH and EGARCH specifications on monthly data. The US REIT data in this study include equity, hybrid and mortgage REITs. Results show a causal relationship emanating from equity REITs to other REIT sectors with small-cap stocks and value stocks being the main influencing asset classes. Mortgage REITs, on the other hand, are generally not influenced by volatility in the fixed income sector. Cotter and Stevenson (2006) also examine the daily volatility dynamics between REITs sectors by employing a time-varying approach where they divide the sample into three periods. The first consists of a period with a lot of IPOs, the second covers the technology boom, and the third extends to the periods after the technology boom when there is a correction to the market. Like Stevenson (2002) they use GARCH based approaches and also investigate the influence of other US equities. Cotter and Stevenson (2006) provide an indication based on the results that the general market sentiment plays a more critical role than a more intuitive relationship within the capital market. This is because unlike, the monthly based results from previous studies, the findings indicate the linkages with the REIT sector and with related sectors like the value stocks, diminished. However, there is an enhancement of the influence of market sentiment from the large-cap indices. Whereas, Cotter and Stevenson (2008) and Cotter and Stevenson (2006) investigate the linkages between the volatility of different REIT subsectors and other equities sectors, the influence of leveraged or indexed traded funds (ETFs) on interday returns of REITs is examined by Bond and Hatch (2011) and Ertugrul, Sezer, and Sirmans (2008).

Anderson, Bhargava and Dania (2018) examine the interrelationships between the performances of several global REIT markets through determining the return linkages, volatility spillover effects and covariance. They divide the global REIT market into major world REIT indices, i.e. developed markets, the EU, and the far east; and secondly, into major REIT countries (Australia, Belgium, Canada, France, Germany Japan, Netherland and UK). They undertake this task by utilising unrestricted vector autoregressive analysis (VAR), GARCH, Threshold GARCH, and MGARCH and find that there is evidence to support high market integration between the major developed countries and the US REIT market except for Japan and Australia. Therefore there would be no benefit for a US investor to diversify through REITs of most of the major developed markets.

Using the US, UK and Australian markets, Hoesli and Reka (2013) utilise an asymmetric t-BEKK (Baba-Engle-Kraft-Kroner) covariance matrix specification when examining volatility transmission across markets. They analyse the relationship between local and global securitised real estate markets. Motivated by the fact that despite the underlying asset of real estate stocks being direct real estate, real estate stocks are stocks by definition. With this in mind, they also analyse the relationship between securitised real estate and common stock markets. To assess whether there is a different dynamic underlying the co-movements in the whole distributions including the extreme events in the series, correlations from the t-BKK model and tail dependence estimated using a time-varying copula framework are analysed. Unlike most research on volatility spillovers in real estate, this study is extended by also investigating market contagion by testing for tail dependence structural changes. The results show that the US has the most considerable spillover effects, both domestically and internationally. They also find that co-movements in tail distributions between markets seem to be important. Lastly, Hoesli and Reka (2013) show evidence of market contagion between the US and the UK markets after the subprime crisis.

More recently, Milcheva and Zhu (2018) distinguish between co-movement due to market risk exposure and co-movement due to linkages between markets, i.e. spillover risk by estimating a spatial multi-factor model (SMFM). This model which is estimated based on 14 developed countries’ real estate stock indices combines asset pricing techniques with spatial econometrics to assess systematic implications for REIT index returns. Milcheva and Zhu (2018) find that during the global financial crisis, spillover risk increases dramatically and explains up to 60% of total asset variation whereas unsystematic risks have been the predominant type of risk in real estate in the rest of the time. Furthermore, the results reveal that in comparison to traditional linkages such as geographical distance, economic integration performs a more pronounced role in the interconnectedness among markets. Another study for Liow and Huang (2018) investigate 10 established REIT markets and reveal that contrary to other findings in global REIT markets; there is less similar integration process occurring. They find that a significant source of REIT volatility integration shocks in 80% of the cases is the local stock market. In line with other literature, this is more pronounced in crisis periods compared to periods of relative calm.

## 4.4. Data, methodology and empirical analysis

### 4.4.1 Data

This research looks at the volatility spillover of implied volatility in the of the UK volatility index (VFTSE), UK REIT index and implied volatility of UK REITs. of returns and risk for UK REITs. Most research on implied volatility is around the non-REIT markets while that conducted around REITs focuses more on the US and other markets (Diavatopoulos et al., 2010; Anoruo and Murthy, 2016; Chung et al., 2016, Akinsomi, Coskun, and Gupta, 2018). Furthermore, the research above has mainly been conducted at the REIT index level for different countries. In contrast to the preceding, this research cascades to UK REITs companies that have traded options. While the UK market is growing steadily since its introduction in 2007, it lacks a vibrant options market. There are about 50 REITs in the UK (REITa, 2018) out of which only three have traded options, namely, British Land, Hammerson and Land Securities. These companies constitute about 30% of the market and are among the top five biggest UK REITs by market capitalisation. Except for Segro, the three REITs companies mentioned above have consistently been in the FTSE 100 since the introduction of REITs in the UK. Their long stock mark history and size make them stand far ahead of every other REIT. Consequently, the three companies have been chosen because they are the only UK REIT companies that have option related data like implied volatility. The three REITs have been trading in options for different lengths of time, i.e. May 2005, May 2008 and October 2013 for Land Securities, British Land and Hammerson respectively. Therefore, the data points are based on a common span and will range from October 2013 to February 2018. During this period all the three companies were listed on the Financial Times Share Exchange (FTSE) 100, however, from March 2018, Hammerson's was relegated from the FTSE 100 to the FTSE 250 due to a drop in its market capitalisation. US REITs were also considered however, Chung et al (2016) already undertook a study using examining the relationship between REIT implied volatility and stock market volatility. The researcher could not find REITs with traded options in other markets.

All data was sourced from Bloomberg, and this consists of daily data for the UK FTSE volatility index (VFTSE); UK REIT index, 30 days implied volatility for British Land, Hammerson and Land Securities the three REIT companies with traded options. The Implied volatility data was obtained specifically from the Bloomberg Option Monitor (OMON).

While the time series of assets can provide us with information for modelling the market, it is possible to use other liquid instruments with quoted prices. Such instruments and their parameters can be calibrated using particular models by making use of a parameter which most closely matches the observed prices. Calibrating models needs great care and all aspects, such as parameter stability or robustness, have to be taken into account as the parameters in question may change over time.

Traded options are an example of the instruments used in modelling assets prices. The price for traded options is observable in the market, and the Black-Scholes-Merton model (mentioned earlier) can be used to calculate or back-out the volatility resulting from this observed market option price. As stated previously, the backed out volatility is the volatility that is implied by the market; hence the eponymous 'implied volatility'. Implied volatility is said to contain information about the price movement, return and future realised volatility of an underlying and could thus be used in the calibration of assets performance. One of the assumptions of the Black-Scholes-Merton model is that volatility is constant, but this is not the case because implied volatility is not the same for all options on an underlying with different strike prices but the same maturity.

The distribution and descriptive statistics of the VFTSE and implied volatility of the are shown in Exhibit 4.6 and Table 4.1 respectively. The data is characterised by positive skew, the high kurtosis and high Jarque-Bera statistics suggesting that the data is not normally distributed and has fat tails. As a departure from most of the research associated with implied volatility in REITs, this research takes the absolute log of the daily changes in the implied volatility levels for the VFTSE, UK REITs and the 3 UK REITs, similar to Siriopoulos and Fasses (2013). Exhibit 4.7 and Table 4.2 show the plot for implied volatility changes and the descriptive statistics for the VFTSE and the 3 UK REITs. The plot for the changes of the implied volatility and UK REIT prices shows some volatility clustering. The mean for the actual log difference the mean is close to zero, and the median is virtually zero for all but Land Securities and the UK REIT index, though they are also almost zero.  The changes of the implied volatilities, and the UK REIT index prices are stationary, and this is supported by significant Augmented Dickey-Fuller (ADF) tests for all the variable which indicates that the data is stationary. The arch test is also significant indicating that there are Arch effects in data and therefore Arch family models like GARCH can be used with the data, this is despite the Jarque-Bera test not suggesting that the data is non-normal.

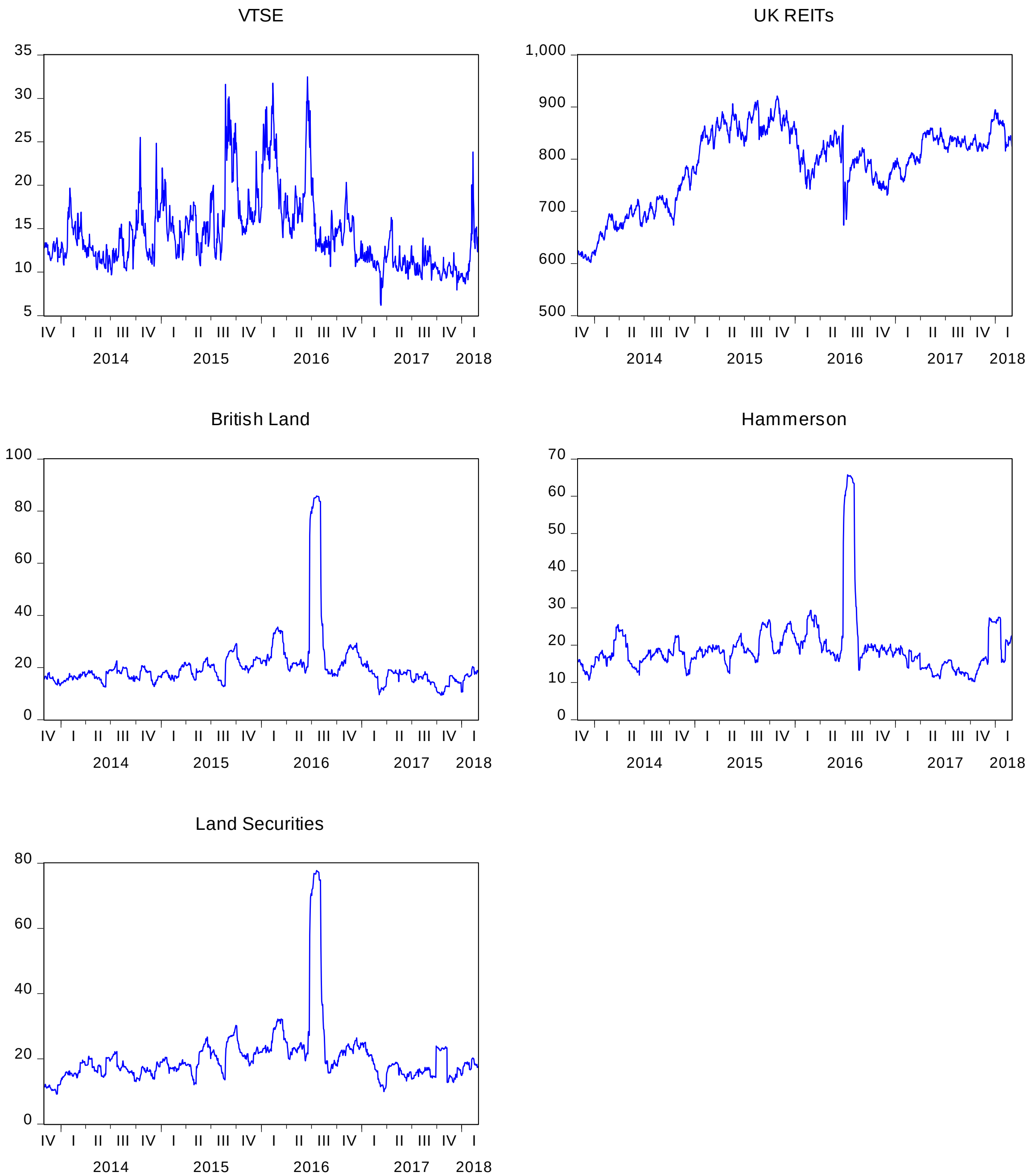


Exhibit 4.6: Price level for the VFTSE; Implied Volatility REIT companies and UK REITs

Note: Price levels of for VTSE; implied volatility for British Land, Hammerson and Land Securities; and UK REIT Index for the period 31 October 2013 to 28th February 2018

| | VTSE | UK_REITS | BRITISH LAND | HAMMERSON | LAND SECURITIES |
|---|---|---|---|---|---|
| Mean | 14.5070 | 789.9283 | 20.4900 | 19.3554 | 20.4483 |
| Median | 13.4115 | 808.7550 | 18.2165 | 17.9850 | 18.3590 |
| Maximum | 32.4780 | 921.4100 | 85.8170 | 65.7320 | 77.7220 |
| Minimum | 6.1940 | 602.0200 | 9.3730 | 10.2080 | 9.1280 |
| Std. Dev. | 4.2958 | 74.5550 | 11.1791 | 8.1973 | 9.8966 |

| | | | | | |
|---|---|---|---|---|---|
| Skewness | 1.4510 | -0.7022 | 4.5416 | 3.9104 | 4.1712 |
| Kurtosis | 5.2962 | 2.5885 | 25.6343 | 21.3874 | 23.1947 |
| Jarque-Bera | 645 | 101 | 28006 | 18799 | 22479 |
| Probability | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| Observations | 1130 | 1130 | 1130 | 1130 | 1130 |

Table 4.1: Descriptive statistics of the VFTSE and Implied Volatility levels British Land, Hammerson and Land Securities

| | VFTSE | UK_REITS | BRITISH_LAND | HAMMERSON | LAND_SECURITIES |
|---|---|---|---|---|---|
| Mean | 0.00342 | 0.00031 | 0.00132 | 0.00256 | 0.00137 |
| Median | 0.00000 | 0.00011 | 0.00000 | 0.00000 | -0.00108 |
| Maximum | 0.53753 | 0.05848 | 0.55573 | 0.40442 | 0.36124 |
| Minimum | -0.36276 | -0.14475 | -0.30014 | -0.30619 | -0.22502 |
| Std. Dev. | 0.08278 | 0.01100 | 0.05159 | 0.06603 | 0.04933 |
| Skewness | 0.81381 | -2.35678 | 1.50285 | 0.73237 | 1.07981 |
| Kurtosis | 8.01128 | 34.35104 | 23.22272 | 8.47509 | 10.73298 |
| Jarque-Bera | 1306 | 47282 | 19663 | 1511 | 3032 |
| Probability | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| Observations | 1129 | 1129 | 1129 | 1129 | 1129 |
| ADF | -27.13487*** | -20.54427*** | -20.54427*** | -25.70869*** | -27.48655*** |
| Arch Test | 25.96143*** | 159.6116*** | 30.58385*** | 52.13789*** | 70.28634*** |

Table 4.2: Descriptive statistics of implied volatility changes

Note: ADF is the Augmented Dickey-Fuller Test that test for stationarity and the Arch test checks if the data has arch effects to determine if the Arch family models can be used.

*** Represents statistical significance at the 1% level

** Represents statistical significance at the 5% level

* Represents statistical significance at the 10% level

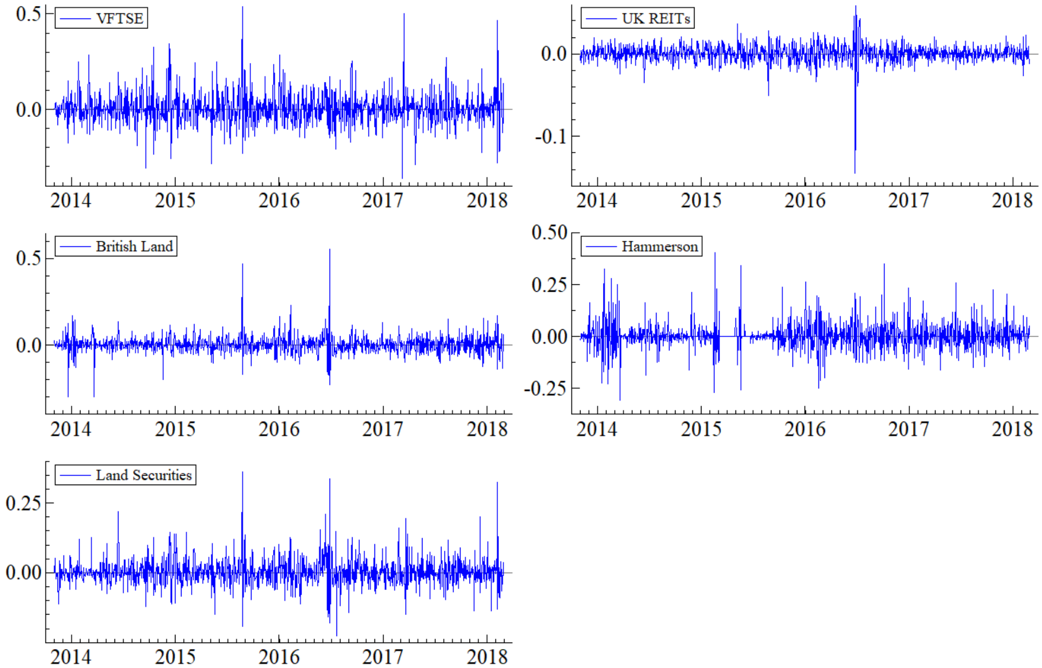


Exhibit 4.7: Implied volatility changes

Note: These are log changes for the VFTSE; UK REITs; and Implied volatility of British Land, Hammerson and Land Securities between 1 November 2013 to 28 February 2018

As stated before, this research investigates the volatility spillover in UK REITs through he use of implied volatility. Previous research has shown that implied volatility can be used to predict returns and investors can use this in their trading strategies. Hight levels of implied volatility indices are seen as an indicator of an instantaneous increase of the return for the share market (Giot, 2005). This is important for REITs owing to their characteristics. Chung et al. (2016) argue that unlike other non-REIT companies, REITs have a higher exposure to default risk especially in periods of financial turmoil due to their relatively high gearing ratios. The high gearing is more prolonged because of the challenges in paying down this debt brought about by one of the critical requirements of paying out most of their earnings as dividends. This, they argue, makes REITs riskier during such periods, therefore, increasing the likelihood of a negative correlation between the volatility and future return. This can be supported by Exhibit 4.8 that shows that the gearing was highest around the 2008 global financial crisis due to the general fall on share prices and hence market values. In such instances, REITs are likely to sell at a discount. The general trend for this gearing has been downward for the three REITs though Hammerson and British Land have shown some upward trend in some periods.

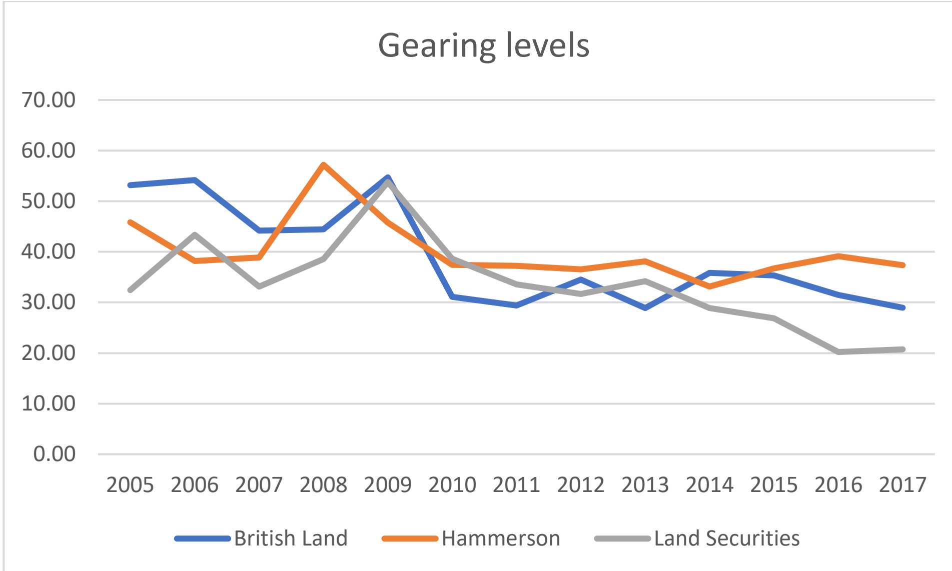


Exhibit 4.8: – Gearing levels for British Land, Hammerson, and Land Securities. Source: FAME (2018)

The notion of implied volatility has risen in importance so much so that it has led to the development of dedicated indices. The most popular is the CBOE volatility index (VIX), normally referred to as the fear index, which provides a gauge of the market anticipation of volatility implied by the S&P 500 index options. The VIX, introduced in 1993, was the first volatility index whose two primary purposes was first to provide a benchmark of expected short-term market volatility and secondly to provide an index upon which futures and options prices could be written (Whaley, 2000). As the VIX is based on implied volatility, its level was implied by the S&P 100 Index option prices. The VIX level, therefore, denotes expected future market volatility over the next 30 calendar days. Therefore it is forward-looking as it measures the volatility that investors expect to see (in 30 days) Emna and Myriam, 2017). This is unlike historical volatility, which looks at realised volatility and is, hence, backwards-looking. Generally, the VIX is high in crisis periods indicating the anxiety of investors with regards to likely drop in the market, and it is for this reason why it is referred to as the “investor fear gauge.” Whaley (2000) argues that the increase (decrease) in the expected market volatility leads to investors demanding higher (lower) rates of return which in turn results in a fall (rise) in the share prices. Therefore, this suggests that the association between the rate of change in the VIX and the S&P 500 index’s return is positive. However, the relationship is not as straightforward. In the same way that puts are bought as insurance to price drops; there is the same effect to the VIX level as implied volatility, as a change in the VIX should rise at a higher absolute rate in a share market fall than in a share market rise (Whaley, 2000). Beyond market prices, the implied volatility index provides a superior way of investigating the relationship between the risk and return of the market. There is a strong negative relationship between changes in the implied volatility index and returns (Chiang, 2012).

The current VIX is based on the S&P 500, but before 2003 the original VIX was based on the S&P 100 with the implied volatility calculated using the Black-Scholes-Merton model. The current VIX however, is model-free but based on on the notional of future variance (variance swaps) and arrived at directly from market observations. Other major indices like the NASDAQ, DAX and FTSE 100 also have equivalent volatility indices referred to as the VXN, VDAX (Germany) and VFTSE (UK) respectively. The other established exchanges with volatility indices are, Switzerland, Belgium, Paris, Canada, South Africa, India, Japan, Greece, Australia, and Korea[37].

Research on the European market by Emna and Myriam (2017) confirms that indeed the implied volatility indices contain relevant information regarding future share market volatility. However, they argue that this information is still insufficient in predicting the volatility as they do not display all the information about the markets. While implied volatility indices are regarded as forward-looking gauges of the volatility of the corresponding markets as well as indicators of investor sentiment, there is no consensus on their predictive power of return and volatility by implied volatility. It is argued that the latter, will differ from market to market and also the period of investigation. Siriopoulos and Fassas (2008) however, reveal that though the VFTSE was biased, it is an efficient predictor of realised volatility and unlike historical volatility, implied volatility contained all the information concerning the future volatility of the market. Siriopoulos and Fassas (2013) examined the spillover effects in international markets concerning implied volatility indices. They test expectations of market participants through implied volatility indices in order to undertake integration analysis of all available volatility indices. Their findings suggest that there is significant integration of investor's expectations regarding future uncertainty. Furthermore, their findings reveal a slight increase in conditional correlations for all the volatility indices under review over the years supporting the view that conditional correlations across implied volatility indices increase in periods financial markets turbulence.

37 See Siriopoulos & Fassas. (2013) for an exhaustive list of volatility indices

### 4.4.2. Methodology

Two methods can examine a variable's behaviour. The first utilises univariate time series where a variable’s behaviour is examined by only using the time value and nothing else. In other words, the dynamics of the subject variable examined is only based on how it has performed in the past. The other method uses structural models where the investigation is how the variable (dependent) behaves with other variable(s) (independent). In this instance, the time component is not taken into consideration and results in a simple regression where one can examine, for example, how the rate of interest rates influence an asset’s returns. The asset’s return is a dependent variable and rate of interest rates is the independent variable. Where there is more than one independent variable, this becomes a multiple regression. The combination of these two methods results in a multivariate time series, and this includes the lag of the time series and the lags of the independent variables or factors. Most research on REIT volatility has utilised either univariate models (Davaney, 2001; Stevenson, 2002, Cotter and Stevenson, 2004, Cotter and Stevenson, 2007) and structural models (Diavatopoulous et al., 2010 and Diavatopoulous et al., 2011). Others (Cotter and Stevenson, 2006; Case, Yang and Yildirim, 2010; Anderson, Bhargava and Dania, 2018) have, however, utilized multivariant GARCH related models on both volatility and returns of REITs. Cotter and Stevenson (2006) consider the return and volatility linkages between REIT sub-sectors and further examine the influence of other US equity series. Case, Yang and Yildirim (2010) on the other hand investigate the dynamics in the correlation of returns between publicly traded REITs and non-REIT shares. However, Anderson, Bhargava and Dania (2018) take a global view by examining the dynamic relations across REIT markets.

The methodology employed in this study is twofold. Given that the sample data for this study is stationary and also has Arch effects (see Table 4.2), multivariate GARCH modelling is employed. Secondly, variance decomposition using generalised vector decomposition as proposed by Diebold and Yilmaz (2012) is employed to examine the level of spillover effects. This research employs two multivariate GARCH models, namely; Constant Conditional Correlation (CCC) GARCH by Bollerslev (1990) and Engle’s (2002) Dynamic Conditional Correlation (DCC) specification, to examine the expectation interdependence of the VFTSE, UK REITs index and implied volatilities of three UK REITs with traded options. This is similar to Siriopoulos and Fassas (2013) who examines the volatility of different share indices unlike Case, Yang and Yildirim (2010) who mainly concentrate on returns. Both the CCC and DCC GARCH models are popular specification methods for volatility that is time-varying and dynamic asset correlations respectively. The dynamic structure of interrelated time series is depicted by Sims' (1980) vector autoregression (VAR). Share and Watson (2001) define a univariate autoregression as a single-equation, single –variable linear model in which its own lagged values explain the current value of the variable. The mathematical form of the VAR system as taken from Siriopoulos and Fasses (2013) is:

$$\Delta IV_t = \alpha + \sum_{i=0}^{p} b_i \Delta IV_{t-1} + u_t \tag{4.4}$$

Where $\Delta IV$ represents the endogenous variables in this case a vector of the daily changes of the implied volatility of each REIT, $p$ is the lag length, $\alpha$ and $b_i$ are matrices of coefficients to be estimated and $u_t$ is a vector of innovations that are not serially correlated. Yet these could be contemporaneously correlated with each other as well as correlated with the past prices of the endogenous variables (Siriopoulos and Fasses, 2013). The number of lags employed in the module can be deteremined by either the Akaike Information Criterion (AIC) or the Bayesian Information Criterion (BIC) also called Schwarz Information Criterion (SIC) (Sheppard, 2013).

The variance and hence, the volatility of returns are not constant but usually dependent on past volatility as can be seen from the clustering of data (Exhibit (9.2). In order to determine the dependence of the variables, it is essential that correlations are estimated similar to the volatilities. If volatility is not constant over time, the correlation as well should not be constant but somewhat dependent on past correlation. In other words, it should be conditional. Similar to the estimation of volatility, various models can be used to estimate this conditional correlation. One is the simple moving average[38]i. This is quite simple as all the past observations in the selected sample (for example a 30-day moving average) are given equal weighting, and those beyond are assigned no weight. The sample will fall off once the 30 days have passed and the resulting correlation will change not because of a particular phenomenon happening but simply because some observations have fallen off the sample period (30 days) while new ones have been added.

38 This is similar to the moving average that covered in chapter 5

The exponential weighted moving average (EWMA)[39] is another method for estimating the conditional correlation. Unlike the moving average, the method employees a parameter $\lambda$, which assigns more weight to the correlation of the recent observation while the other more past observations decline exponentially. $\lambda$ is normally choosen abitrarliy however, RiskMetrics have recommended that 0.94 is used as $\lambda$ for all assets but this has raised some controversy. In addition, the same $\lambda$ must be employed for all assets in a multivariate context in order to have a have a correlation matrix that is positive definite.

Engle (2002) points out that Orthogonal GARCH[40] method or principle component GARCH method (proposed by Alexander; 1998, 2001) can be used as an alternative approach in estimating multivariate models. Here, unconditionally uncorrelated linear combinations of a series $r$ is contructed. Thereafter, univariate GARCH models are estimated for all of these and the full covariance matrix is constructed by assuming the conditional correlations are all zero (Eagle, 2002). Furthermore, Engle (2002) suggests that a slight better method is to run this regression as a GARCH regression, thus getting residuals that are orthogonal in a generalised least square (GLS) metric.

### Multivariate GARCH

Multivariate GARCH (MGARCH) models like Vector GARCH by Bollerslev, Engle & Wooldridge (1988), BKK GARCH by Babam, Engle, Kraft and Kroner (1995), Constant Conditional Correlation (CCC) GARCH by Bollerslev (1990), later extended by Jeantheau (1998) and the Dynamic Conditional Correlation (DCC) GARCH model by Engle (2002) are natural generalisations that used to study relationships between variables. MGARCH models are mostly applied when studying relationships between volatilities and co-volatilities between markets. Laurent (2018) and Bauwens, Laurent and Rombouts (2006) identify the following specific issues relating to markets that MGARCH models are used to investigate:

- whether the volatility of one market leads to the volatility of other markets,
- the transmission of the volatility of an asset to another asset - directly or indirectly through the conditional variance or conditional covariances respectively.
- whether the impact is the same for negative and positive shocks of the same amplitude,

[39] Please see chapter 3 for more on EWMA.

[40] GARCH stands for generalised autoregressive conditional heteroscedasticity and this is introduced and discussed in chapter 3

- whether correlations between asset returns change over time, are they high during periods of high volatility.

Furthermore, they also highlight the application of MGARCH models in the assessment of the impact of financial markets' volatility on real variables like exports and output drought rates and their respective volatility. Another use of MGARCH is in the computation of time-varying hedge ratios.

Most of these models have been formulated in a way that the covariances and variances are linear functions of the squares and cross products of the data. In so doing, the aim is to specify the conditional variance matrix. However, the parameters for these matrices increase at a rapid rate as the dimension increases (Silvennoinen and Teräsvirta, 2007). The covariance matrix has to be invertible, but this becomes challenging in terms of computation, when the number of assets, $n$, exceeds the number of the time series, $t$[41].. It is therefore important that a multivariate GARCH model is parsimonious enough but still maintaining flexibility at the same time and the conditional covariance matrix must be positive definite.

Silvennoinen and Teräsvirta (2007) divide the different specifications of MGARCH into four groups:

1. Models of the conditional covariance matrix: The VEC-GARCH and the BEKK parametric models fall in this group. The VEC-GARCH model is a straightforward generalisation of the univariate GARCH model. Here, every conditional variance and covariance is a function of all lagged conditional variance and covariance, including the squared returns and cross – products of returns. Even if the model's generality is an advantage, it presents some disadvantages - one of which is that there exists only sufficient, somewhat restrictive conditions for the conditional covariance matrix to be positive definite (Silvennoinen and Teräsvirta, 2007). Although Bollerslev, Engle, and Wooldridge (1988) offer a simplified version of the model which makes the estimation less complicated than in the original VEC model, the parameters seem too restrictive because no interaction is allowed between the different conditional variances and covariances. The main challenge to the VEC model is

[41] For more explanation of this please see chapter 3

that the estimation of parameters is computationally demanding. The BEKK model is an alternative to the VEC model though it is viewed as a restrictive form of the VEC model. Its restriction ensures that the conditional covariance matrices are positive definite by construction. Despite this advantage, like the VEC model, the BEKK model still involves somewhat heavy computations because of several matrix inversions (ibid).

2. Factor models: these models emanate from economic theory. The name factor comes from the idea that the return process is assumed to be generated from several unobserved heteroscedastic factors. The number of these factors is small in relation to the return vector and therefore leads to reduced dimensionality. Parsimony is at the centre of conditional covariance matrices in the factor models. Uncorrelated factors are desirable in order to represent genuinely different common components driving returns. So the original observed series present in the returns are assumed to be linked to unobserved, uncorrelated variables or factors through a linear, invertible transformation (Silvennoinen and Teräsvirta, 2007). An example of a factor model is the Generalised Orthogonal (GO-) GARCH by van der Weide (2002).

3. Conditional variances and correlations: These models hinge on the decomposition of the conditional covariance matrix into conditional standard deviations and correlations. Examples of MGARCH models in this group are the Constant Conditional Correlation (CCC) GARCH and the Dynamic Conditional Correlation (DCC) GARCH models. These will be examined in more detail below.

4. Nonparametric and semiparametric approaches: These provide an alternative to the parametric estimation of the conditional covariance structure. In contrast to the parametric models, the nonparametric and semiparametric models do not impose a particular structure on the data. Silvennoinen and Teräsvirta (2007) note that nonparametric models, however, suffer from the 'curse of dimensionality' due to the lack of data in all directions of the multidimensional space thus the performance of the local

smoothing estimator deteriorates quickly as the dimension of the conditioning variable increases.

#### 4.4.2.1. Conditional variances and correlations

This research employs the conditional variance and correlation models i.e. CCC and DCC multivariate GARCH models. CCC GARCH is covered first as DCC GARCH builds on it. (The following is mainly taken from taken from Sheppard (2013) and the references therein). The conditional variance and correlation permit one to specify separately, the individual conditional variances on the one hand, and the conditional correlation matrix on the other. Put differently; the conditional covariance is decomposed into k conditional variances and conditional correlations. Even though theoretical results on stationarity, ergodicity and moments may not be that straightforward to obtain compared to the models in the other groups, they are parsimonious and therefore much easier to estimate (Laurent, 2018).

#### The Constant Conditional Correlation (CCC) GARCH model

The conditional variance matrix for this class of models (CCC and DCC GARCH) is specified hierarchically. First, one chooses a GARCH-type model for each conditional variance. Second, the conditional correlation matrix is then modelled based on conditional variances. The CCC GARCH assumes that these correlations are constant and hence the conditional covariances are proportional to the product of the corresponding standard deviations. The effect of this restriction highly reduces the number of unknown parameters. Therefore

$$H_t = D_t R D_t \tag{4.5}$$

where $D_t$ is the diagonal matrix of the conditional standard deviation of the $i^{th}$ asset in its $i^{th}$ diagonal position.

$$D_t = \begin{bmatrix} \sigma_{1,t} & 0 & 0 & \cdots & 0 \\ 0 & \sigma_{2,t} & 0 & \cdots & 0 \\ 0 & 0 & \sigma_{3,t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{k,t} \end{bmatrix} \tag{4.6}$$

Where $\sigma_{i,t} = \sqrt{\sigma_{ii,t}}$. The conditional variances are typically modelled using standard GARCH(1,1).

$$\sigma_{ii,t} = \omega_i + \alpha_i \mu_{i,t-1}^2 + \beta_i \sigma_{ii,t-1} \tag{4.7}$$

Other specifications such as TARCH or GJR GARCH or EGARCH can also be used to take into the asymmetric GARCH. It is also possible to model the conditional variances with different models for each asset, a distinct advantage over the VEC and related models (Sheppard, 2013).

The matrix containing the constant conditional correlations is represented by $R$.

$$R = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1k} \\ \rho_{12} & 1 & \rho_{23} & \cdots & \rho_{2k} \\ \rho_{13} & \rho_{23} & 1 & \cdots & \rho_{3k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{1k} & \rho_{2k} & \rho_{3k} & \cdots & 1 \end{bmatrix} \tag{4.8}$$

The conditional covariance matrix can be computed from the conditional standard deviations and the conditional correlations, and so all of the dynamics in the conditional covariance are attributable to changes in the conditional variances. The CCC GARCH model can be estimated in two steps. The first is the k conditional variance models (for example, GARCH) which produces the vector of standardised residuals $\mu_{i,t} = \epsilon_{i,t}/\sqrt{\hat{\sigma}_{ii,t}}$. The second step estimates the constant conditional correlation using the standard correlation estimator on the standardised residauls. Therefore the covariance in the constant conditional correlation GARCH model evolves according to:

$$H_t = \begin{bmatrix} \sigma_{11,t} & \rho_{12}\sigma_{1,t}\sigma_{2,t} & \rho_{13}\sigma_{1,t}\sigma_{3,t} & \cdots & \rho_{1k}\sigma_{1,t}\sigma_{k,t} \\ \rho_{12}\sigma_{1,t}\sigma_{2,t} & \sigma_{22,t} & \rho_{23}\sigma_{2,t}\sigma_{3,t} & \cdots & \rho_{2k}\sigma_{2,t}\sigma_{k,t} \\ \rho_{13}\sigma_{1,t}\sigma_{3,t} & \rho_{23}\sigma_{2,t}\sigma_{3,t} & \sigma_{33,t} & \cdots & \rho_{3k}\sigma_{3,t}\sigma_{k,t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{1k}\sigma_{1,t}\sigma_{k,t} & \rho_{2k}\sigma_{2,t}\sigma_{k,t} & \rho_{3k}\sigma_{3,t}\sigma_{k,t} & \cdots & \sigma_{kk,t} \end{bmatrix} \tag{4.9}$$

where $\sigma_{11,t}, i = 1, 2, \ldots, k$ evolves according to some univariate GARCH process on asset $i$, usually a GARCH(1,1). The CCC model contains $N(N+5)/2$ parameters. $H_t$ is positive definite if and only if all the $N$ conditional variances are positive and $R$ is positive definite (Bauwen, Laurent and Rombouts, 2006). The unconditional variances are easily obtained, as in the univariate case, however, due to nonlinearity of $H_t$, the covariances are difficult to calculate. The assumption that the conditional correlations are constant seems unrealistic in many empirical application and for this reason, Engle (2002) and Tse and Tsui (2002) developed a generalisation of the CCC model by making the conditional correlation matrix time dependent and this resulted in the dynamic conditional correlation (DCC) model.

### The Dynamic Conditional Correlation (DCC) GARCH model

The DCC GARCH model, introduced by Engle (2002) and Tse and Tsui (2002), extends the CCC GARCH by introducing simple, scalar BEKK-like dynamics to the conditional correlations. In the DCC model, the conditional correlation matrix is time-dependent. In the DCC, the correlation matrix,$R$, in the CCC is replaced with $R_t$. Like the CCC GARCH, the covariance matrix, $H_t$ (Equation (4.6)), can be decomposed into the conditional standard deviations, $D_t$, and a correlation matrix, $R_t$ (see Equation (4.10)). However, $D_t$ and $R_t$ are time varying in contrast with CCC GARCH where $R$, is constant. Therefore, the covariance in a DCC GARCH model (Engle, 2002) evolves according to

$$H_t = D_t R_t D_t \tag{4.10}$$

where

$$R_t = Q_t^* Q_t Q_t^* \tag{4.11}$$

$$Q_t = (1-a-b)\bar{R} + a u_{t-1} u_{t-1}' + b Q_{t-1,} \tag{4.12}$$

$$= (1-a-b)\bar{R} + a\left(R_{t-1}^{\frac{1}{2}} e_{t-1}\right)\left(R_{t-1}^{\frac{1}{2}} e_{t-1}\right)' + b Q_{t-1,} \tag{4.13}$$

$$Q_t^* = (Q_t \odot I_k)^{-\frac{1}{2}} \tag{4.14}$$

$u_t$is the $k\ by\ 1$ vector of standardised returns ($\mu_{i,t} = \epsilon_{i,t}/\sqrt{\hat{\sigma}_{ii,t}}$ and $\odot$denotes Hadamard multiplication[42] (element by element). $\{e_t\}$ are a sequence of i.i.d. innovations with mean 0 and covariance $I_k$, such as a standard multivariate normal or possibly a heavy tailed distribution. $D_t$is a diagonal matrix with the conditional standard deviation of asset $i$ on the $i^{th}$diagonal position. The conditional variance $\sigma_{11,t}, i = 1,2,\ldots,k$, evolve according to some univariate GARCH process for asset $i$, usually a GARCH(1,1) and are identical to Equation (4.7). The specification of the univariate GARCH models is not limited to the standard GARCH(p,q), but any GARCH process with normally distributed errors that satisfies appropriate stationarity conditions and non-negativity constraints can be included (Engle and Sheppard, 2001). For example to capture asymmetric effects in volatility, TARCH or GJR GARCH could be used.

42 The Hadamard product is an operation where two matrices of the dimensions are multiplied together to produce a new matrix (i.e. each element i,j in the new matrix is the product elements i,j of the original two matrices.

Equation (4.7) and (4.14) are needed to ensure that $R_t$is a correlation matrix with diagonal elements equal to 1. $\bar{R}$ is the $\overline{N \times N}$ unconditional variance matrix of $u_t$, and a and b are nonnegative scalar parameters satisfying $a + b < 1$. The $Q_t$ process is parameterised in a similar manner to the variance targeting BEKK which allows for three step estimation. The first two steps are identical to those of the CCC GARCH model. The third plugs in the estimate of the correlation into Equation (9.8) to estimate the parameters which govern the conditional correlation dynamics, a and b.

Unlike Tse and Tsui (2002), Engle(2002) formulates the conditional correlations as a weighted sum of past correlations. Indeed, the matrix $Q_t$ is written like a GARCH equation, and then transformed to a correlation matrix (Laurent, 2018).

A drawback of the DCC model is that a and b are scalars so that all the conditional correlations obey the same dynamics. This is necessary to ensure that $R_t$ is positive definite ∀t through sufficient conditions on the parameters. If the conditional variance are specfied as GARCH(1,1) models then the DCC models contain $(\mathrm{N} + 1)(\mathrm{N} + 4)/2$ parameters (Laurent, 2018),.

Both the CCC and DCC models can be estimated consistently in two steps which makes the approach feasible when N is high. Of course, when N is large, the restriction of common dynamics gets tighter, but for large N the problem of maintaining tractability also gets harder.

The DCC model permits flexible GARCH specifications in the variance part. Furthermore, because the conditional variances, jointly with the conditional correlations can be estimated using $\mathrm{N}$ univariate models, it makes it possible for the DCC GARCH model to be extended to more complex GARCH type models.

#### 4.4.2.2. Vector autoregressive and variance decomposition

While the multivariate GARCH (MGARCH) will be used to model the relationship and provide information on the risk measures and spillovers among the VFTSE, UK REITs, and the three UK REIT companies, the purpose of the of the generalised vector autoregressive (VAR) and the variance decomposition as proposed by Diebold and Yilmaz (2012), is to measure the total and directional spillovers. In contrast to their earlier model (Diebold and Yilmaz (2009) that relies on Cholesky factor decomposition but is order dependent Diebold and Yilmaz (2012) build propose a new approach that eliminates the possible dependence of the results on ordering. Based on the generalised VAR framework, the new approach computes the forecast error variance decomposition without the orthogonalisation of shocks (Katusiime, 2018). ). This is achieved by exploiting the generalised VAR framework of Koop, Pesaran, and Potter (1996) and Pesaran and Shin (1998) (in Diebold and Yilmaz, 2012) which they refer to as KPPS.The method by Diebold & Yilmaz (2012) results in the measurement of the total spillovers which culminates into a spillover index that can be presented as spillover tables and plots and therefore can provide information as to the net contributor and net recipient of the spillovers (Batten et al., 2019; Mensi at al.,2018; Diebold and Yilmaz, 2012).

### Variance decomposition

The own variance shares (or variance decompositions) are defined as the fractions of the $H$-step-ahead error varcs in forecasting $x_i$ that come about due to shocks to $x_i$ for $i = 1, 2, \cdots, N$, and spillovers (or cross variance shares) are the fractions of the $H$-step-ahead error varcs in forecasting $x_i$ due to shocks to $x_i$, for $i, j = 1, 2, \cdots, N$, such that $i \neq j$. Utilising the generalised VAR frame, the $H$-step-ahead generalised forecasts-error variance decomposition is expressed by Diebold and Yilmax (2012) as:

$$\theta_{ij}^{g}(H) = \frac{\sigma_{jj}^{-1} \sum_{h=o}^{H-1} \left(e_i' A_h \sum e_j\right)^2}{\sum_{h=o}^{H-1} \left(e_i' A_h \sum A_h' e_j\right)} \tag{4.15}$$

Where $\Sigma$ denotes the covariance matrix for the error vector $\varepsilon$; $\sigma_{jj}$ is the standard deviation of the error term for the $jth$ equation; $e_j$ is the selection vector, with one of the $jth$ element and zero otherwise. The sum of the elements in each row of the variance decomposition table is not equal to 1 i.e., $\sum_{j=1}^{N} \theta_{ij}^{g}(H) \neq 1$. Each entry of the variance decomposition matrix by row sum is normalised as:

$$\tilde{\theta}_{ij}^{g}(H) = \frac{\theta_{ij}^{g}(H)}{\sum_{j=1}^{N} \theta_{ij}^{g}(H)} \tag{4.16}$$

Where $\sum_{j=1}^{N} \theta_{ij}^{g}(H) = 1$ and $\sum_{ij=1}^{N} \theta_{ij}^{g}(H) = N$ by construction.

Diebold and Yilmaz (2012) construct the total volatility spillover index below, by utilising the volatility contributions from the KPPS variance decomposition. The total spillover index measures the contribution of spillovers of volatility shocks across all the markets to the total forecast error variance.

$$S^{P}(H) = \frac{\sum_{\substack{i,j=1 \\ i \neq j}}^{N} \tilde{\theta}_{ij}^{g}(H)}{\sum_{i,j=1}^{N} \tilde{\theta}_{ij}^{g}(H)} \; x\ 100 = \frac{\sum_{\substack{i,j=1 \\ i \neq j}}^{N} \tilde{\theta}_{ij}^{g}(H)}{N} \; x\ 100 \qquad (4.17)$$

It is also possible to identify which mark plays the dominant role in volatility spillovers between markets by considering directional spillovers (Mensi et al., 2018). This is done by examining spillovers from one market to another, e.g. market $i$ to market $j$ and vice versa. The two categories of direction volatility spillovers are "from" and "to" and are calculated using equation (4.18) and (4.19) respectively.

$$S_{i}^{g}(H) = \frac{\sum_{\substack{i,j=1 \\ i \neq j}}^{N} \tilde{\theta}_{ij}^{g}(H)}{\sum_{i,j=1}^{N} \tilde{\theta}_{ij}^{g}(H)} \; x\ 100 = \frac{\sum_{\substack{i,j=1 \\ i \neq j}}^{N} \tilde{\theta}_{ij}^{g}(H)}{N} \; x\ 100 \qquad (4.18)$$

$$S_{i}^{g}(H) = \frac{\sum_{\substack{i,j=1 \\ i \neq j}}^{N} \tilde{\theta}_{ji}^{g}(H)}{\sum_{i,j=1}^{N} \tilde{\theta}_{ji}^{g}(H)} \; x\ 100 = \frac{\sum_{\substack{i,j=1 \\ i \neq j}}^{N} \tilde{\theta}_{ji}^{g}(H)}{N} \; x\ 100 \qquad (4.19)$$

The net spillovers can also be calculated by getting the difference between the gross volatility shocks transmitted "to" and those received "from" all the markets, i.e.:

$$S_{i}^{g}(H) = \left( = \frac{\sum_{\substack{i,j=1 \\ i \neq j}}^{N} \tilde{\theta}_{ji}^{g}(H)}{\sum_{i,j=1}^{N} \tilde{\theta}_{ji}^{g}(H)} \; x\ 100 \right) - \left( \frac{\sum_{\substack{i,j=1 \\ i \neq j}}^{N} \tilde{\theta}_{ij}^{g}(H)}{\sum_{i,j=1}^{N} \tilde{\theta}_{ij}^{g}(H)} \; x\ 100 \right) \qquad (4.20)$$

The net spillovers can be extended to enable one to calculate the net pairwise volatility spillovers. For example, the net pairwise volatility spillover between market $i$ and market $j$ is the difference between the gross volatility shock transmitted from market $i$ to market $j$ and those from market $j$ to market $i$. This is illustrated in equation (4.21):

$$S_{i}^{g}(H) = \left( \frac{\tilde{\theta}_{ji}^{g}(H)}{\sum_{i,j=1}^{N} \tilde{\theta}_{ik}^{g}(H)} - \frac{\tilde{\theta}_{ij}^{g}(H)}{\sum_{i,j=1}^{N} \tilde{\theta}_{jk}^{g}(H)} \right) x\ 100 = -\left( \frac{\tilde{\theta}_{ji}^{g}(H) - \tilde{\theta}_{ij}^{g}(H)}{N} \right) x\ 100 \qquad (4.21)$$

### 4.4.3. Empirical analysis

The analysis is divided into two parts. The first part examines the relationship between the VFTSE, UK REITs index and the implied volatility of the REIT companies with traded options using the multivariate

GARCH. The second section of the analysis examines volatility spillover effects by utilising variance decomposition as proposed by Diebold and Yilmaz (2012).

#### 4.4.3.1. Multivariate GARCH

The first part examines the result and analysis of the constant conditional correlation (CCC) and dynamic conditional correlation (DCC) generalised autoregressive conditional heteroscedasticity (GARCH) models. This empirical analysis was undertaken using Oxmetrics, a specialised statistical and econometric software programme. Table 4.3 shows a summary of the estimated coefficient and p-values of the two multivariate GARCH models. Of the two models, the CCC is considered to generate the data best as it has low values for both the AIC and BIC compared to the DCC. While both models display conditional correlations that are significantly different from zero, the DCC model consistently has lover correlations. Both models show that the highest correlation is between Land Securities and British Land. This implies that there is more interconnection between the implied volatilities of these REIT companies. The lowest correlation for both models is between British Land and the UK REITs index suggesting that there is very little spillover between the price of the REIT market and that of the implied volatility of British Land meaning that the change in the implied volatility of British Land is affected by other factors other than the UK REIT market prices. While the implied volatility index for the FTSE (VFTSE) shows, statistically significant correlation with the three REIT companies, there are little spillover effects with the UK REIT market. Interestingly there is negative correlations between the REIT market and the implied volatility of the three REIT companies indicating low volatility spillovers. Both models show evidence of interconnectedness between the implied volatility changes of the three REIT companies with Land Securities and British Land having the highest correlation while Land Securities and Hammerson showed the least connectedness among the three.

Exhibit 4.9 displays the conditional variance plots estimated by both the CCC and DCC GARCH and shows that Hammerson experienced marked volatility during the sample period (perhaps that is why it was relegated from the FTSE 100 at the beginning of the second quarter of 2018). The VFTSE has the second highest conditional volatility; and, the UK REIT index has the least volatility while the pattern for the conditional volatility of British Land and Land Securities is similar.

| | | CCC | | DCC | |
|---|---|---|---|---|---|
| | | Coefficient | p-Value | Coefficient | p-Value |
| | | Panel A - GARCH Results | | | |
| ω | VFTSE | 0.0015*** | 0.0001 | 0.0015*** | 0.00010 |
| α | VFTSE | 0.1250*** | 0.0063 | 0.1250*** | 0.00630 |
| β | VFTSE | 0.6531*** | 0.0000 | 0.6531*** | 0.00000 |
| ω | UK-REITs | 0.0000** | 0.02780 | 0.0000** | 0.02780 |
| α | UK-REITs | 0.1724*** | 0.00120 | 0.1724*** | 0.00120 |
| β | UK-REITs | 0.7632*** | 0.00000 | 0.7632*** | 0.00000 |
| ω | British Land | 0.0004** | 0.02050 | 0.0004* | 0.06840 |
| α | British Land | 0.3193*** | 0.00830 | 0.2696*** | 0.00010 |
| β | British Land | 0.5470*** | 0.00010 | 0.7042*** | 0.00000 |
| ω | Hammerson | 0.0003* | 0.06840 | 0.0003* | 0.06330 |
| α | Hammerson | 0.2696*** | 0.00010 | 0.2441*** | 0.00580 |
| β | Hammerson | 0.7042*** | 0.00000 | 0.6397*** | 0.00000 |
| ω | Land Securities | 0.0001*** | 0.00010 | 0.0001*** | 0.00010 |
| α | Land Securities | 0.3087** | 0.02110 | 0.3087** | 0.02110 |
| β | Land Securities | 0.2158 | 0.16970 | 0.2158 | 0.16970 |
| | | Panel B - Conditional Correlation Results | | | |
| | ρUR_VF | -0.4269*** | 0.00000 | -0.3839*** | 0.00000 |
| | ρBL_VF | 0.4314*** | 0.00000 | 0.3884*** | 0.00000 |
| | ρHS_VF | 0.2170*** | 0.00000 | 0.1892*** | 0.00020 |
| | ρLS_VF | 0.4900*** | 0.00000 | 0.4508*** | 0.00000 |
| | ρBL_UR | -0.4876*** | 0.00000 | -0.4500*** | 0.00000 |
| | ρHS_UR | -0.3406*** | 0.00000 | -0.3185*** | 0.00000 |
| | ρLS_UR | -0.4718*** | 0.00000 | -0.4499*** | 0.00000 |
| | ρHS_BL | 0.2717*** | 0.00000 | 0.2275*** | 0.00000 |
| | ρLS_BL | 0.5363*** | 0.00000 | 0.4943*** | 0.00000 |
| | ρLS_HS | 0.2666*** | 0.00000 | 0.2649*** | 0.00000 |
| | | Panel C - Diagnostics | | | |
| | df | 4.5062 | 0.0000 | 4.4801 | 0.0000 |
| | AIC | -20.2299 | | -20.2921 | |
| | BIC | -20.0250 | | -20.0783 | |

Table 4.3: estimates for the CCC(1,1) and DCC(1,1) GARCH models

This table provides the summary for the multivariate models, i.e. the estimated coefficients and p-values for the CCC and DCC GARCH models. The GARCH univariate parameters (ω, α and β) are estimated for the VFTSE, UK REIT index, British Land, Hammerson and Land Securities. The conditional correlations are also provided, and the abbreviations are V F= VFTSE, UR = UK REITs, BL = British Land, HS = Hammerson and LS = Land Securities. Df is the degree of freedom, and the AIC and BIC are the Akaike Information Criterion and Schwartz Criterion respectively.

*** Represents statistical significance at the 1% level

** Represents statistical significance at the 5% level

* Represents statistical significance at the 10% level

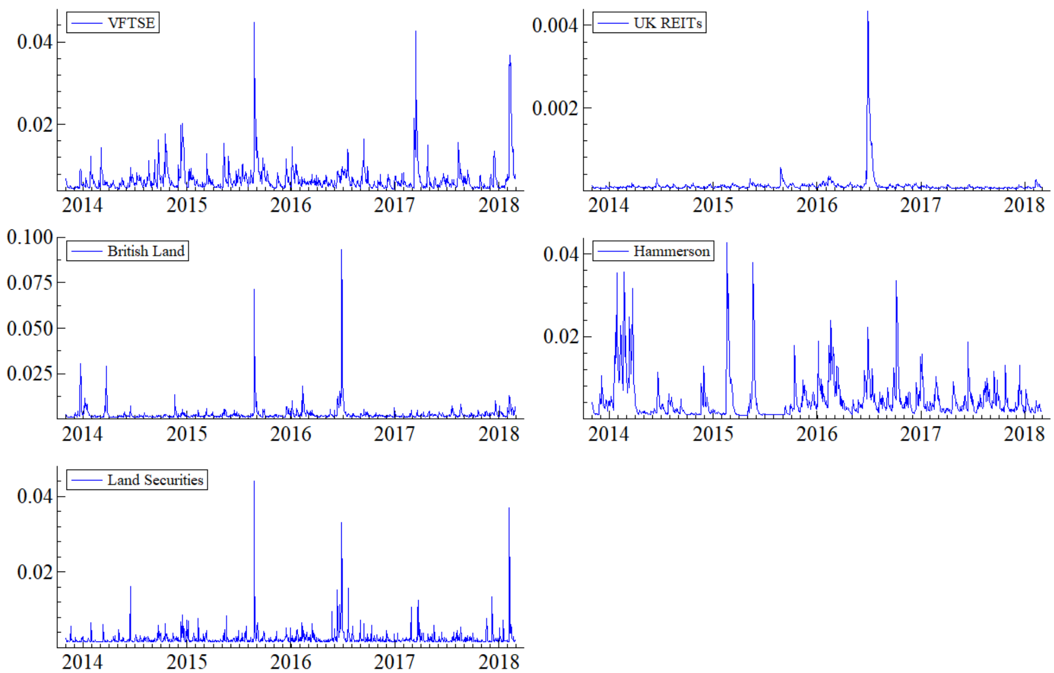


Exhibit 4.9: Conditional Variances

Note: This shows the conditional variances estimated by the DCC GARCH. Both the CCC and DCC show the same estimates for the conditional variances.

Table 4.4 presents the unconditional correlation matrix for the implied volatility changes in VFTSE, UK REIT Index British Land, Hammerson, and Land Securities; measured using the Pearson's correlation coefficient. The null hypothesis of no relationship between the variable is rejected, as the correlation coefficients are all not equal to zero. There appears to be a moderate[43] positive correlation between the UK REIT index with British Land and Land Securities. Hammerson, on the other hand, has a very weak positive correlation with VFTSE; and a weak positive correlation with British Land and Land Securities. Land Securities and British Land have the highest unconditional correlation (though moderate) as they deal in the same property sectors namely Retail, offices and mix use while Hammerson mainly focuses on the retail sector. Unlike the conditional correlations which have several negative correlations, the unconditional correlations are all positive. Exhibit 4.10 shows the plots in the conditional correlation for the DCC GARCH model. By definition, the CCC results in constant conditional correlations which as shown in Table 4.3. The Dynamic conditional correlations show that the correlations vary over time with the biggest range being that between VFTSE and British Land and the smallest range is the that of VFTSE and the UK REIT index. These conditional correlations have

[43] Evans (1996) provides a guide that the strength of the correlation is: "very weak" between 0.0 and 0.2; "weak" between 0.2 and 0.4; "moderate" between 0.4 and 0.6; "strong" between 0.6 and 0.8; and "very strong" between 0.8 and 1.0"

varied between -0.26 to 0.71 over the sample period, and this is significantly different from the average conditional correlations shown in Table 4.4. The average conditional correlations are all not equal to zero, and hence the null hypothesis that there is no association amongst the changes in implied volatilities and price of the UK REIT index is rejected. This suggests that there is integration or transmission or spillovers amongst the implied volatilities of the VFTSE, UK REITs and those of the three REITs albeit very little in some instances. Though there is some integration, this is moderate as the highest average conditional correlation is 0.53, i.e., that between Land Securities and British Land as shown by the CCC model. The least integration is between British Land and the UK REIT index. As shown in Table 4.5, in some periods this conditional correlation is quite high meaning they move together in general, there is a positive relationship among the VFTSE, British Land, Hammerson and Land securities as suggested by their implied volatilities changes.

A comparison between the conditional correlation and the unconditional correlation reveals discrepancies that are almost uniform as all but one conditional correlation i.e. that between Hammerson and VFTSE; are marginally higher than the unconditional correlations

| | VFTSE | UK REITs | British Land | Hammerson | Land Securities |
|---|---|---|---|---|---|
| VFTSE | 1 | | | | |
| UK REITs | 0.2982 | 1 | | | |
| British Land | 0.3473 | 0.4427 | 1 | | |
| Hammerson | 0.0373 | 0.2161 | 0.1742 | 1 | |
| Land Securities | 0.3942 | 0.4447 | 0.4917 | 0.1059 | 1 |

Table 4.4: Unconditional Correlation matrix VFTSE, British Land, Hammerson and Securities implied volatility changes

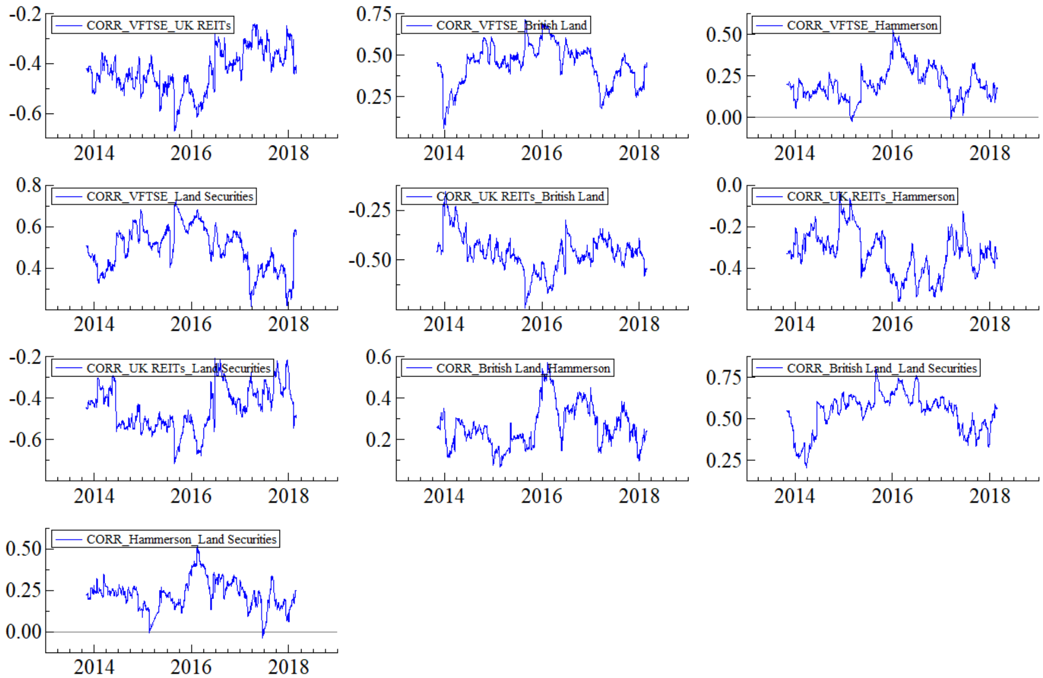


Exhibit 4.10 Conditional Correlation plots for the changes implied volatility

| | ρUR_VF | ρBL_VF | ρHS_VF | ρLS_VF | ρBL_UR | ρHS_UR | ρLS_UR | ρHS_BL | ρLS_BL | ρLS_HS |
|---|---|---|---|---|---|---|---|---|---|---|
| max | -0.2560 | 0.6253 | 0.4333 | 0.6399 | -0.2585 | -0.1015 | -0.2630 | 0.4826 | 0.7148 | 0.4521 |
| min | -0.5880 | 0.1625 | 0.0252 | 0.2522 | -0.6493 | -0.4977 | -0.6470 | 0.0885 | 0.2325 | 0.0747 |
| range | 0.3319 | 0.4627 | 0.4081 | 0.3876 | 0.3908 | 0.3961 | 0.3839 | 0.3940 | 0.4823 | 0.3774 |

Table 4.5: Conditional correlation ranges

While the multivariant GARCH models provide information about the relationship regarding the correlation of volatilities across the assets and markets under investigation, further analysis can be undertaken by considering the generalised vector autoregressive model and volatility decomposition to assess the total and directional volatility spillovers. The next section, therefore, extends the study by considering the above-mentioned models.

#### 4.4.3.2. Volatility decomposition

This section builds on the volatility spillover analysis by make use of a framework proposed by Diebold and Yilmax (2012)[44]. The study examines both static spillovers and rolling spillovers by analysing the

[44] This was undertaken my utilising R code from https://github.com/ which is free and under a General Public Licence

spillover index, directional spillovers, net and pairwise spillovers. The analysis uses the same data used in the preceding MGARCH analysis.

| | VFTSE | UK REITs | British Land | Hammerson | Land Securities | From others | Net | Conclusion |
|---|---|---|---|---|---|---|---|---|
| VFTSE | 58.41 | 10.61 | 11.83 | 2.66 | 16.49 | 41.59 | -2.56 | Net recipient |
| UK.REITs | 10.15 | 57.45 | 13.34 | 7.47 | 11.59 | 42.55 | 3.21 | Net contributor |
| British.Land | 10.63 | 13.91 | 53.52 | 4.12 | 17.82 | 46.48 | 2.19 | Net contributor |
| Hammerson | 3.41 | 10.21 | 5.69 | 77.45 | 3.23 | 22.54 | -5.95 | Net recipient |
| Land.Securities | 14.84 | 11.03 | 17.81 | 2.34 | 53.99 | 46.02 | 3.11 | Net contributor |
| To others | 39.03 | 45.76 | 48.67 | 16.59 | 49.13 | 199.18 | | |
| Including own | 97.44 | 103.21 | 102.19 | 94.04 | 103.12 | **39.8%** | **Total spillover index** | |

Table 4.6: Volatility spillovers across VFTSE, UK REITs, British Land, Hammerson and Land Securities

Note: From others – directional measure of spillovers from all $markets_j$ to $market_i$

To others – directional measure of spillovers from all $markets_j$ to $market_i$

Including own – directional measure of spillovers from $market_i$ to all $markets_j$ including from own market

Table 4.6 shows that the total volatility spillover calculated using Equation (4.17) is about 39.8%. Land Securities contributes the most to others and therefore has the highest influence on the volatility contributing about 49%. This is followed very closely by British Land whose contribution to the others is roughly the same as Land Securities at 48.7% contribution. This suggests that the transmission of risk to the other markets and companies under investigation is high for Land Securities and British Land and the UK REITs is at 45.8%. The volatility spillover between Land Securities and British Land is the highest for all off-diagonal values in the table. This is consistent with the findings in the MGARCH analysis which showed a generally high conditional correlation between these two companies. The contribution to others by the VFTSE and Hammerson is relatively low with Hammerson showing the least volatility spillovers to others. The three highest contributors to others all have positive net values and are therefore net contributors as more volatility spillover is going “to others” than they are receiving “from others”.

Further analysis shows that the total volatility spillover is not constant over time as shown in Exhibit 4.11. This is done by taking a 100-day rolling data to analyse the volatility spillovers. Total volatility spillover started at a value of about 25% and peaked close to 70% at the beginning of 2016, before starting its descent and reaching its lowest at about 10% toward second quarter of 2017.


Exhibit 4.11: Total volatility spillovers Index

While the total spillover provides a pattern of the level of volatility spillovers, it does not show the direction of the spillovers. Exhibits 4.12 and 4.13 display the rolling data for directional “from others” and “to others” respectively. Like the total volatility spillover, the directional “from others” varies over though in a similar pattern except Hammerson whose distribution is different. The trend is the same also for the directional “to others” with Hammerson again being quite different compared to the VFTSE, UK REIT index, British Land and Land Securities. It is interesting to see both directional rolling volatility spillover that all have the highest value around 2016 and after, this could be attributed to the collapse in the European stock market (Mensi, 2018) and looming Brexit vote and the after effects of the Brexit result.

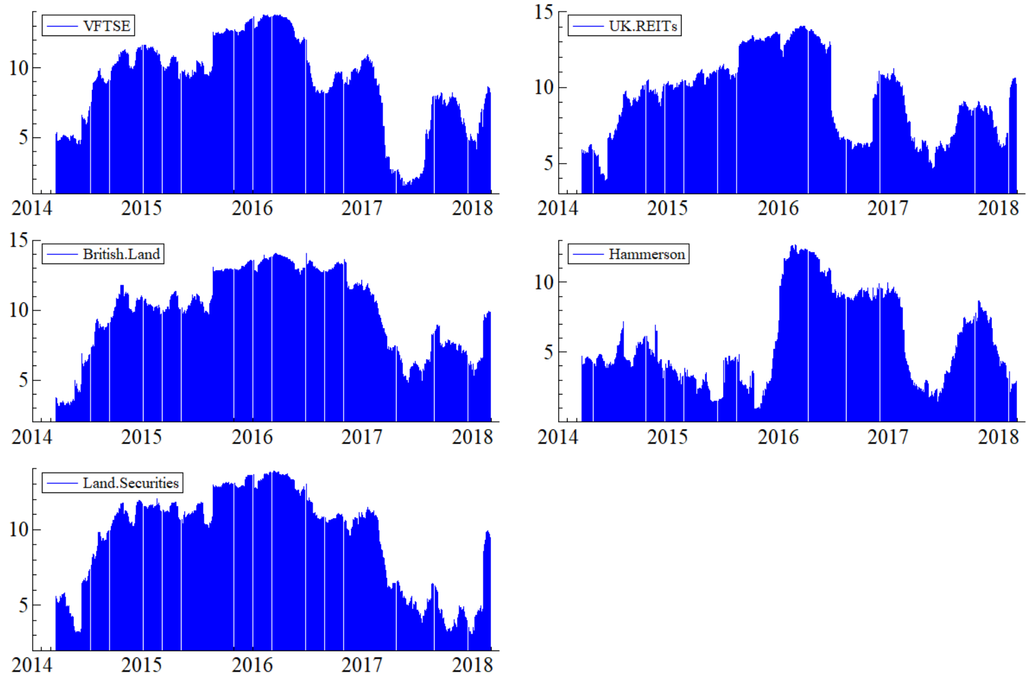


Exhibit 4.12: Directional volatility spillovers – From others

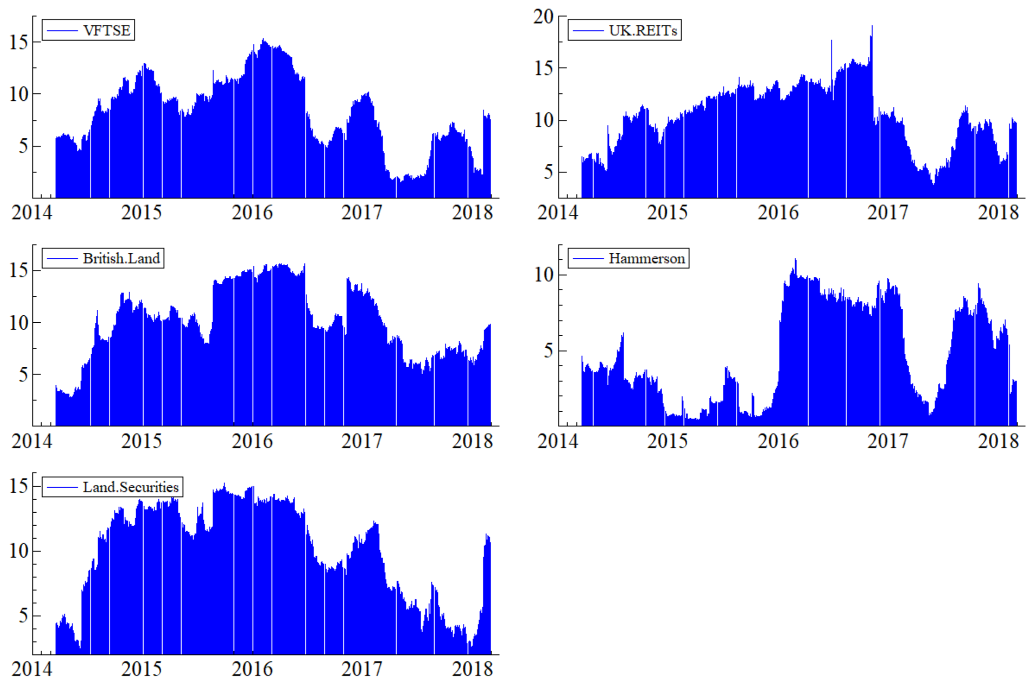


Exhibit 4.13: Directional volatility spillovers – To others

Exhibit 4.14 presents the changes in the net rolling volatility spillover and shows that the UK REIT index, Land Securities and British Land are net transmitters or contributors of volatility spillover. The VFTSE and Hammerson are net recipients. This more so before mid-2017 when they display persistent negative net volatility spillovers. This evolution of the volatility spillovers over time is consistent with the results in Table 4.6. Lastly, the volatility spillover dynamics are examined by making use of the net pairwise volatility spillovers displayed in 9.10. Over time, the net pairwise volatility spillover varies greatly. As in the net volatility spillover, Hammerson and UK REIT index show that they are mainly recipients of the volatility spillovers when matched with others. The VFTSE has shown that it is mainly a contributor to the volatility spillovers; this is expected as it represents a much wider market. However, what is unexpected is the UK REIT to contribute more to the volatility of the REIT companies.

Overall, the MGARCH and the variance decomposition approaches show similar results in terms of the spillover. The CCC and DCC GARCH models revealed that there is volatility transmission markets and REIT companies under investigation as they are correlated whichever of the model are used. This conditional correlation suggests that there is volatility spillover though quite weak for Hammerson and UK REITs. Diebold and Yilmax (2012) framework also show volatility spillover with Land Securities, British Land and UK REITs being the net contributors while VFTSE and Hammerson are net recipients. The pattern is the same even for rolling volatility spillovers.

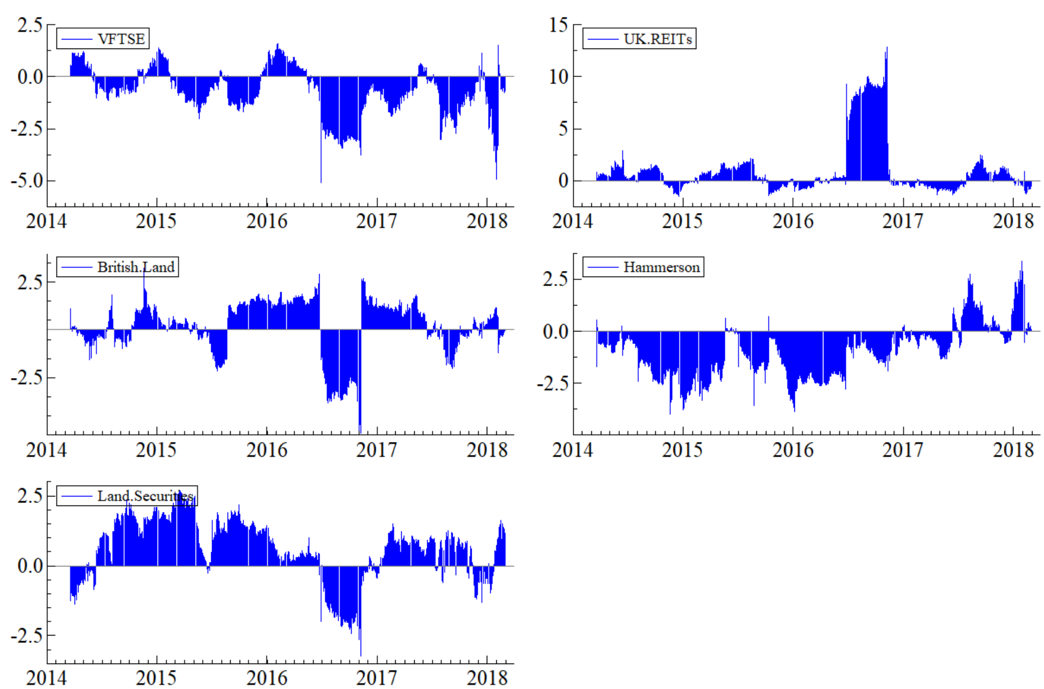


Exhibit 4.14: Net volatility spillovers

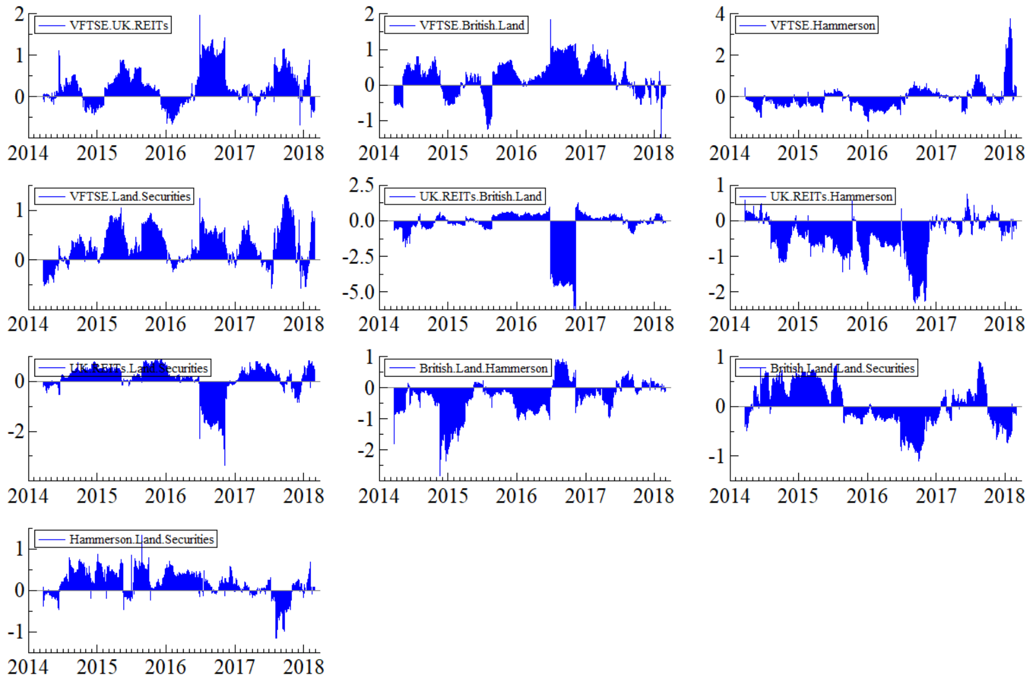


Exhibit 4.15: Pairwise volatility spillovers

## 4.5. Conclusion

The UK REIT market has been growing over the years from a hand full of property companies converting to REITs after their introduction in 2007 to about 50 in 2018. Despite this growth, there only exists three companies that implied volatilities could be obtained as they are the only one that trades in options. Implied volatility has been examined in the real estate context by largely investigating its predictive power to both return and historical volatility. This study set to examine whether there is transmission or spillover effects between the UK stock market implied volatility index VFTSE and the implied volatilities of British Land, Hammerson and Land Securities as well as the UK REIT index. Previous studies have been undertaken to examine spillover effects in real estate, and some have utilised implied volatility of the UK or US stock exchanges and the influence on the REIT market. This study examines spillover effects in REITs by utilising implied volatility and using a multivariate GARCH approach in order to model conditional volatility and also generalised VAR and volatility decomposition as proposed by Diebold and Yilmax (2002).

While other researchers have employed the multivariate generalised autoregressive conditional heteroscedasticity (MGARCH) approach to index or sector level, this research contributes to the volatility analysis in REITs by utilising the Constant Conditional Correlation (CCC) GARCH by Bollerslev (1990) and Engle's (2002) Dynamic Conditional Correlation (DCC) specification. This is applied to the UK volatility index, VFTSE, UK REIT index and three UK REIT companies with traded options in order to determine the volatility transmission among them directly through conditional variances and correlations. Volatility decomposition as proposed by Diebold and Yilmax (2012)is also employed to examine the total, directional, net and pairwise volatility spillovers.

The findings are that there is by both MGARCH models reveals that there is transmission as revealed by significant conditional correlations. Despite there being volatility spillovers it weak for Hammerson and VFTSE. While the MGARCH provide information regarding the presence of volatility transmission, the models do not show the extent and direction of these spillovers. The analysis is extended by employing volatility decomposition which showed evidence of volatility spillovers with British Land, Land Securities and UK REIT index being net contributors to others while Hammerson and VFTSE are the net recipients. This is undertaken by rolling the data, and the pattern is similar. The weak volatility spillovers by Hammerson and VFTSE are consistent with the findings of the two MGARCH models.

The implication of these findings is that general low transmission or spillovers particularly with the VFTSE could be good investors trying to diversify between REITs and the other non-REIT companies in the UK stock exchange. As British Land and Land Securities have the highest volatility spillovers, one can avoid allocating their money in the other if one already has it as an asset or is intending to invest. So one should just invest in either but not both at the same time.

The contribution to the research is the application of both the multivariate GARCH and variance decomposition at the same on REITs in order to examine volatility spillovers.

# CHAPTER 5

# CONCLUSION

The purpose of the research was to investigate ways that risk can be used in the investment and management of real estate. An examination of the asset allocation methods shows that most decisions are undertaken based on the returns rather than the volatility. Risk parity is an allocation method that focuses on volatility as opposed to returns. This allocation method is not free from controversy, however, the results of the research revealed that it was a good method in some instance as it solves the shortcoming presented by the mean-variance methods. The conventional allocation methods have challenges of leading to allocations that have corner solutions thereby going against the spirit of diversification. Additionally, return based allocation are prone to estimation errors when forecasting returns. Despite not giving the best performance all the time, risk parity can benefit the real estate market due to its diversification benefits, and it can easily be implemented in the public real estate perhaps in a multi-asset portfolio of which real estate is part. This said the application of risk parity to direct real estate is questionable due to the lumpiness inherent in direct real estate.

The risk modelling for real estate revealed that REITs had similar characteristics to that of equities. This is good because both value at risk (VaR) and Expected Shortfall (ES) can be used to measure risk in addition to using the conventional standard deviation. While the expected shortfall has advantages over VaR, its backtesting is still a challenge although one can use specialist software such as MATLAB, R, and OxMetric in order to undertake this backtesting. VaR has more tool to help in the backtesting because unlike ES were the expectation is not being calculated but rather a value – which is easier to compare. The results in the modelling using ES and VaR were mixed; however, ES was more superior in that it was able to account for the performance in the tails as opposed to VaR. With regards to the modelling of dynamic variance and correlations that were covered at the end, the findings for the transmission of volatility is consistent with that of Cotter and Stevenson ( 2006,2007), Diavatopolous et al. (2013). The results reject the null hypothesis of independence in the volatility changes of the VFTSE, UK REIT index, British Land, Hammerson and Land Securities. This said a higher average conditional correlation was expected between the changes in the implied volatilities of the three UK REITs with traded options, particularly Land Securities and British Land. The variance decomposition also showed volatility spillovers with Land Securities, British Land and UK REITs being the net volatility contributors or transmitters while Hammerson and VFTSE are net recipients.

## Areas of further research

1. Extending the analysis of risk parity by using VaR and ES as risk measures and also testing it on different markets and perhaps also combining this analysis with implied volatility.
2. Extend the implied volatility analysis by also using stochastic volatility models and also variance swaps

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Ziegel, J.F. (2013) Coherence and elicitability (available at: http://arxiv.org/abs/1303.1690) accessed 20/12/2015

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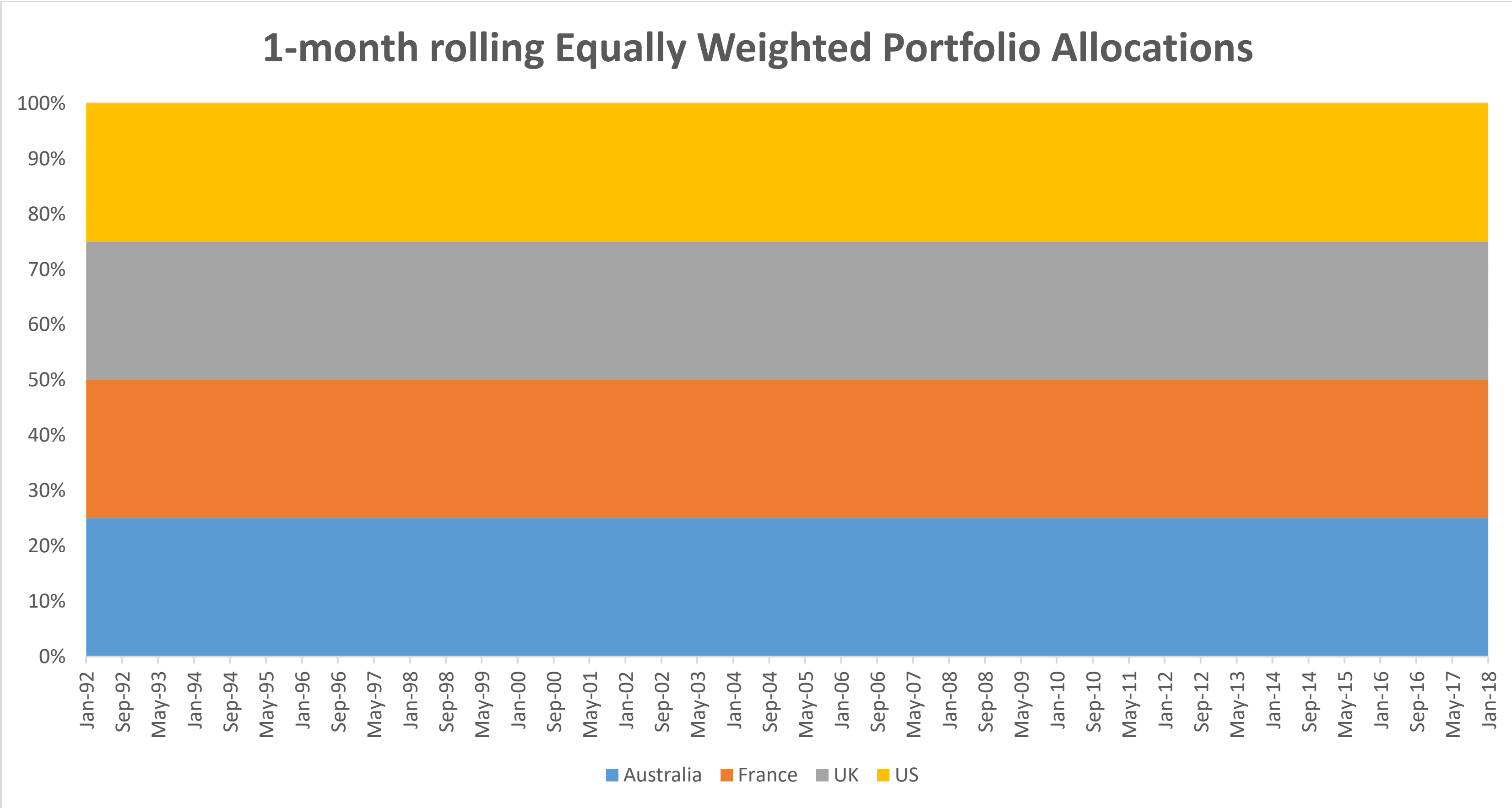
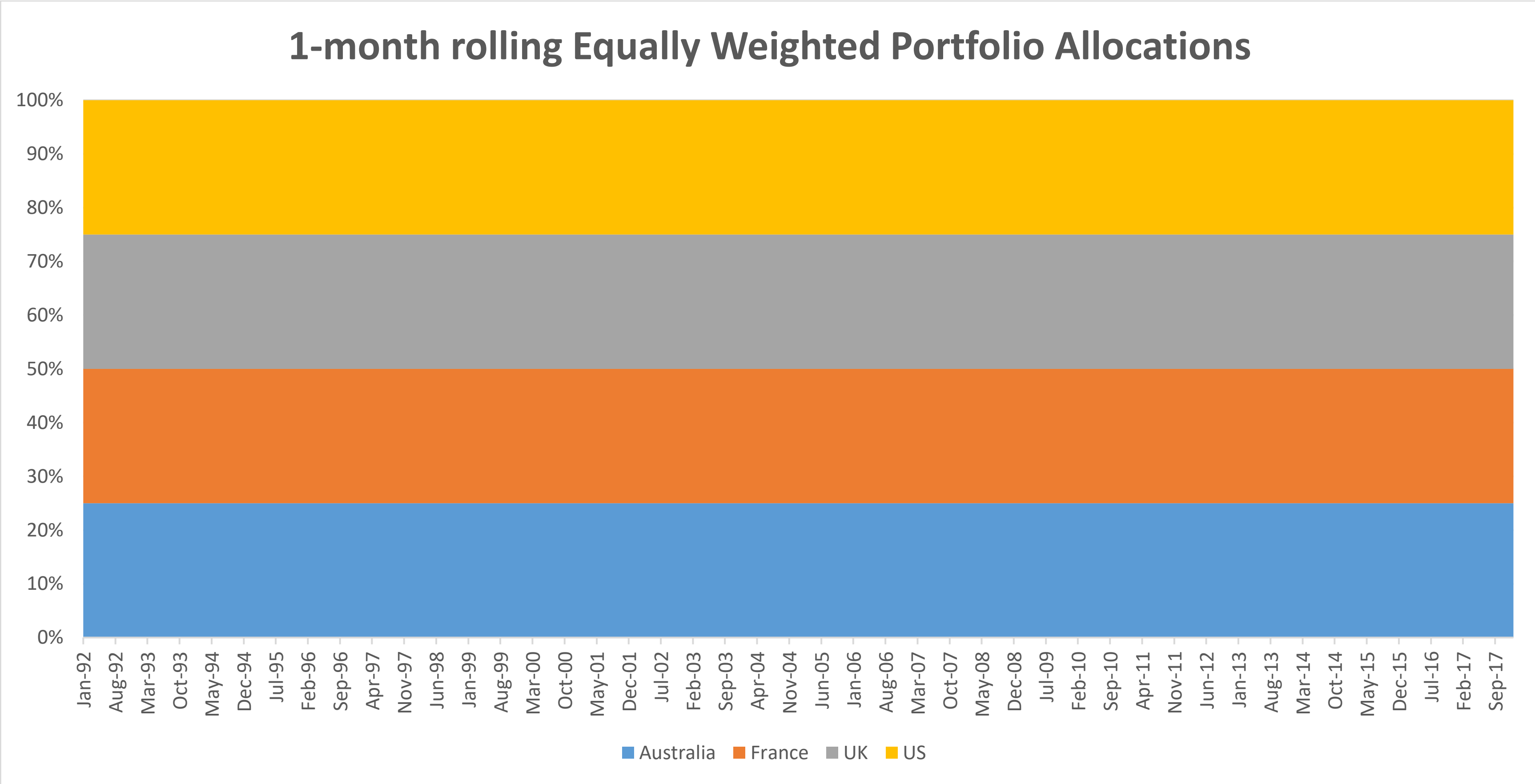
APPENDIX A: PORTFOLIO ALLOCATIONS




(a)



1-month rolling Equally Weighted Portfolio Allocations
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Jan-92
Aug-92
Mar-93
Oct-93
May-94
Dec-94
Jul-95
Feb-96
Sep-96
Apr-97
Nov-97
Jun-98
Jan-99
Aug-99
Mar-00
Oct-00
May-01
Dec-01
Jul-02
Feb-03
Sep-03
Apr-04
Nov-04
Jun-05
Jan-06
Aug-06
Mar-07
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May-08
Dec-08
Jul-09
Feb-10
Sep-10
Apr-11
Nov-11
Jun-12
Jan-13
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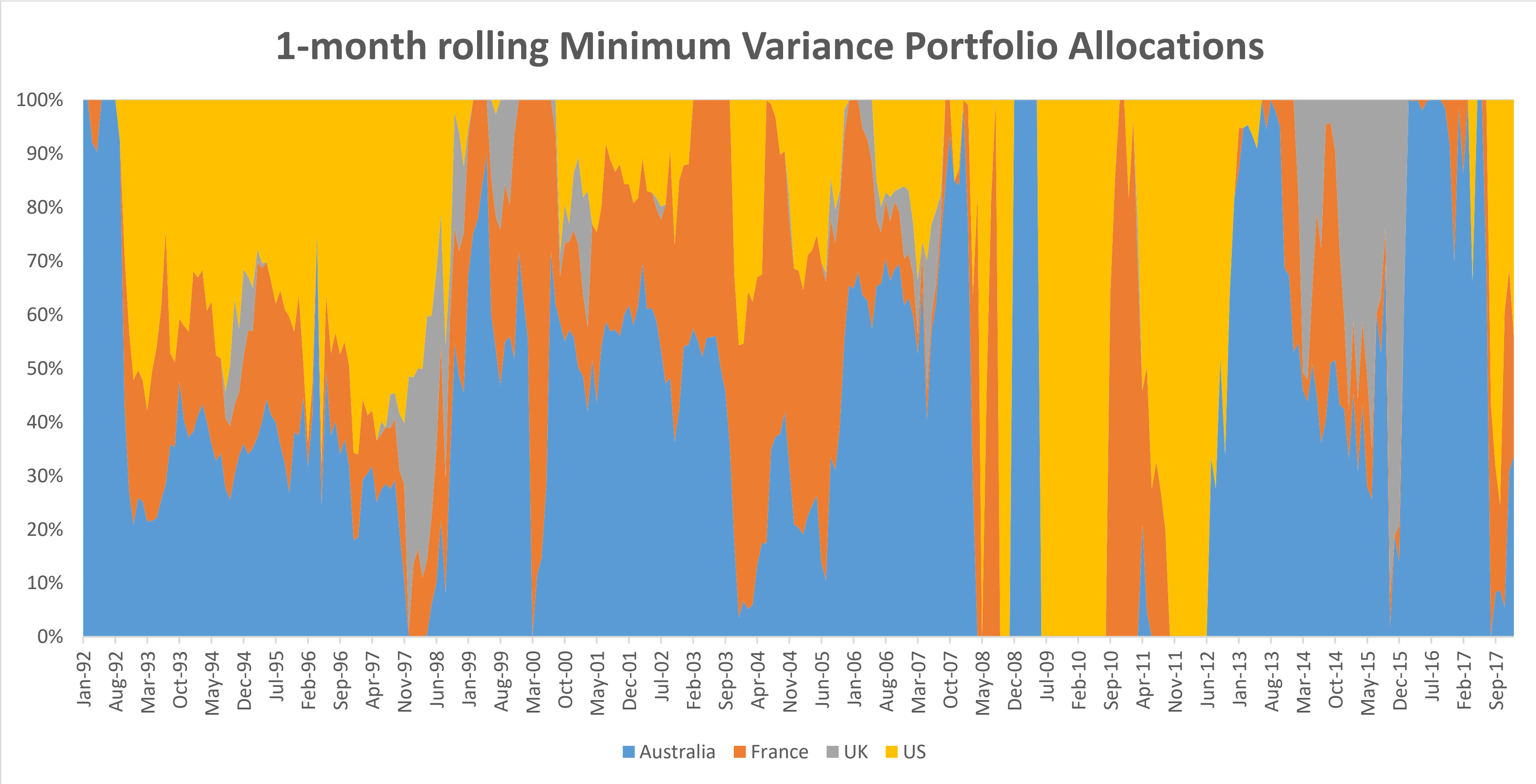
May-15
Dec-15
Jul-16
Feb-17
Sep-17
Australia
France
UK
US


(b)


1-month rolling Minimum Variance Portfolio Allocations
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Jan-92
Aug-92
Mar-93
Oct-93
May-94
Dec-94
Jul-95
Feb-96
Sep-96
Apr-97
Nov-97
Jun-98
Jan-99
Aug-99
Mar-00
Oct-00
May-01
Dec-01
Jul-02
Feb-03
Sep-03
Apr-04
Nov-04
Jun-05
Jan-06
Aug-06
Mar-07
Oct-07
May-08
Dec-08
Jul-09
Feb-10
Sep-10
Apr-11
Nov-11
Jun-12
Jan-13
Aug-13
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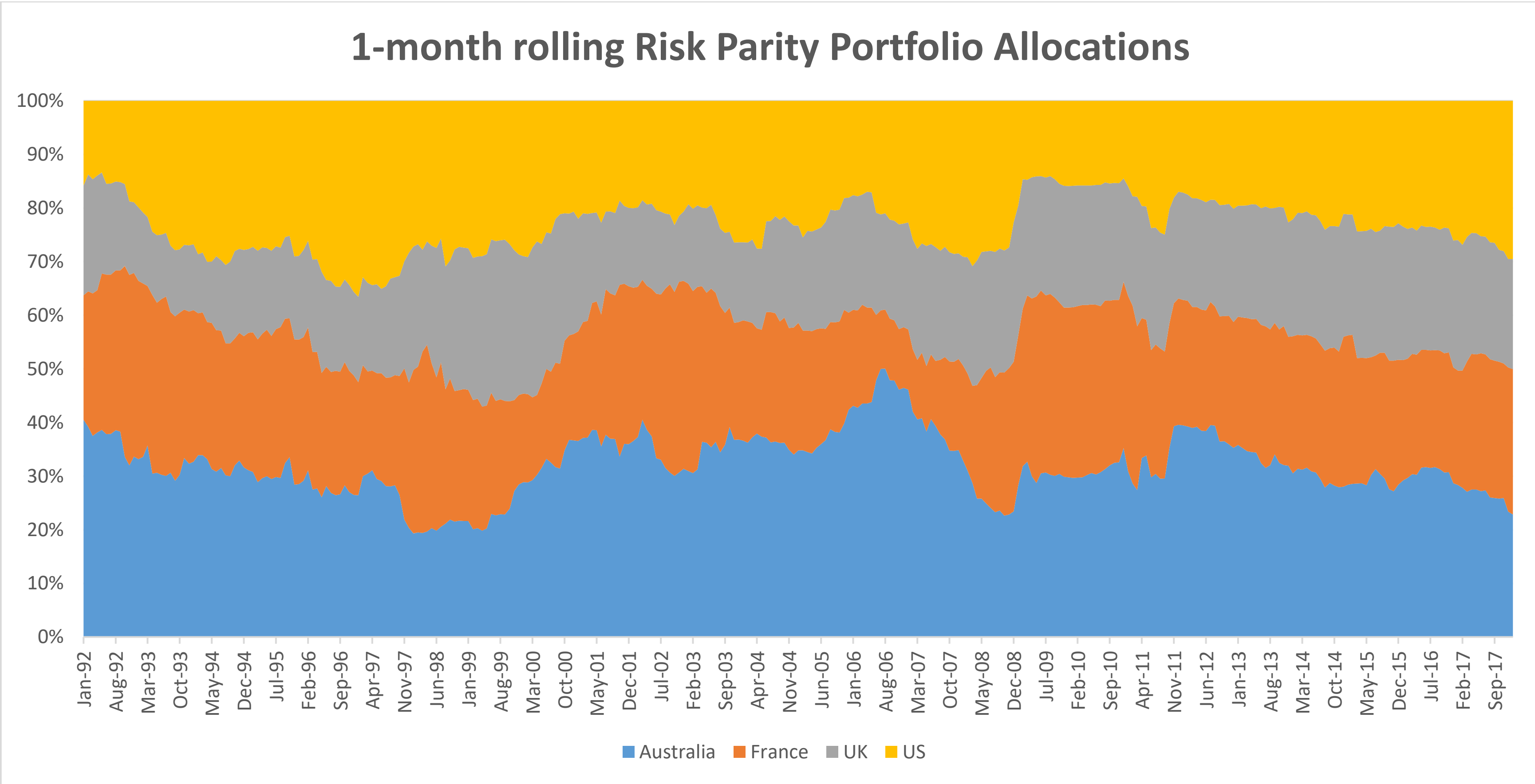
Oct-14
May-15
Dec-15
Jul-16
Feb-17
Sep-17
Australia
France
UK
US

(c)



1-month rolling Risk Parity Portfolio Allocations
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Jan-92
Aug-92
Mar-93
Oct-93
May-94
Dec-94
Jul-95
Feb-96
Sep-96
Apr-97
Nov-97
Jun-98
Jan-99
Aug-99
Mar-00
Oct-00
May-01
Dec-01
Jul-02
Feb-03
Sep-03
Apr-04
Nov-04
Jun-05
Jan-06
Aug-06
Mar-07
Oct-07
May-08
Dec-08
Jul-09
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Sep-10
Apr-11
Nov-11
Jun-12
Jan-13
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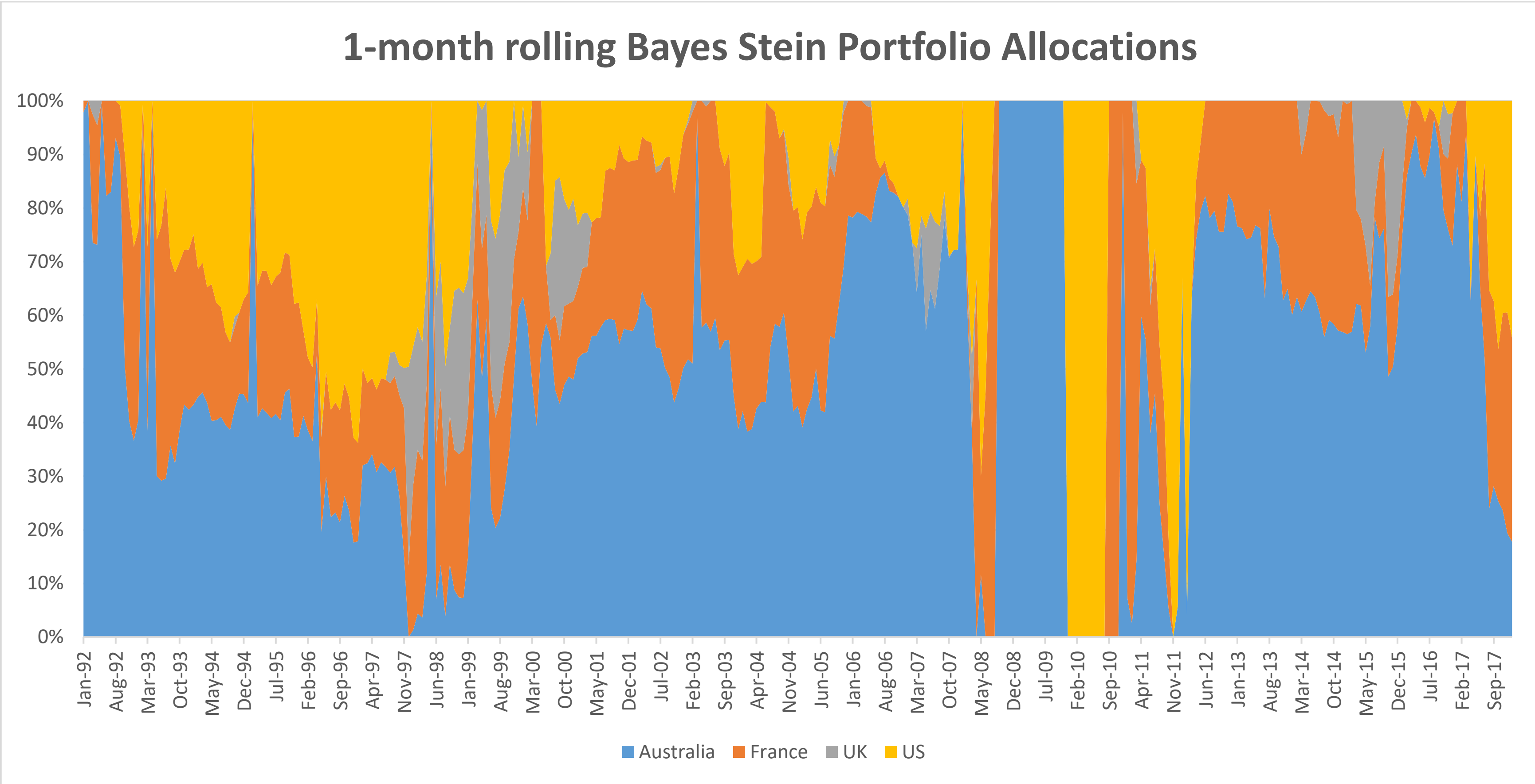
May-15
Dec-15
Jul-16
Feb-17
Sep-17
Australia
France
UK
US


(d)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100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Jan-92
Aug-92
Mar-93
Oct-93
May-94
Dec-94
Jul-95
Feb-96
Sep-96
Apr-97
Nov-97
Jun-98
Jan-99
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Apr-11
Nov-11
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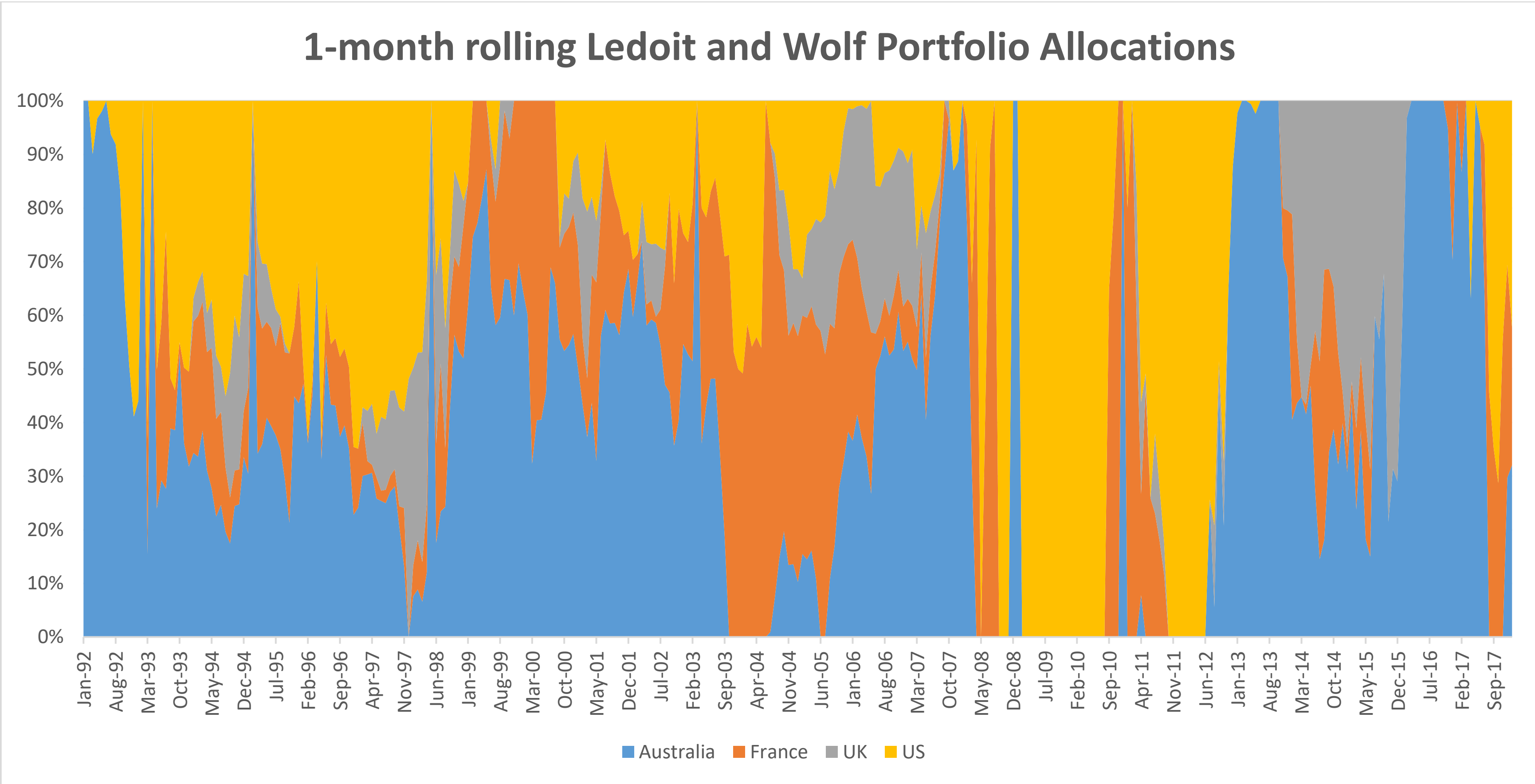
Sep-17
Australia
France
UK
US

(e)


1-month rolling Ledoit and Wolf Portfolio Allocations
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90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Jan-92
Aug-92
Mar-93
Oct-93
May-94
Dec-94
Jul-95
Feb-96
Sep-96
Apr-97
Nov-97
Jun-98
Jan-99
Aug-99
Mar-00
Oct-00
May-01
Dec-01
Jul-02
Feb-03
Sep-03
Apr-04
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Jun-05
Jan-06
Aug-06
Mar-07
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Apr-11
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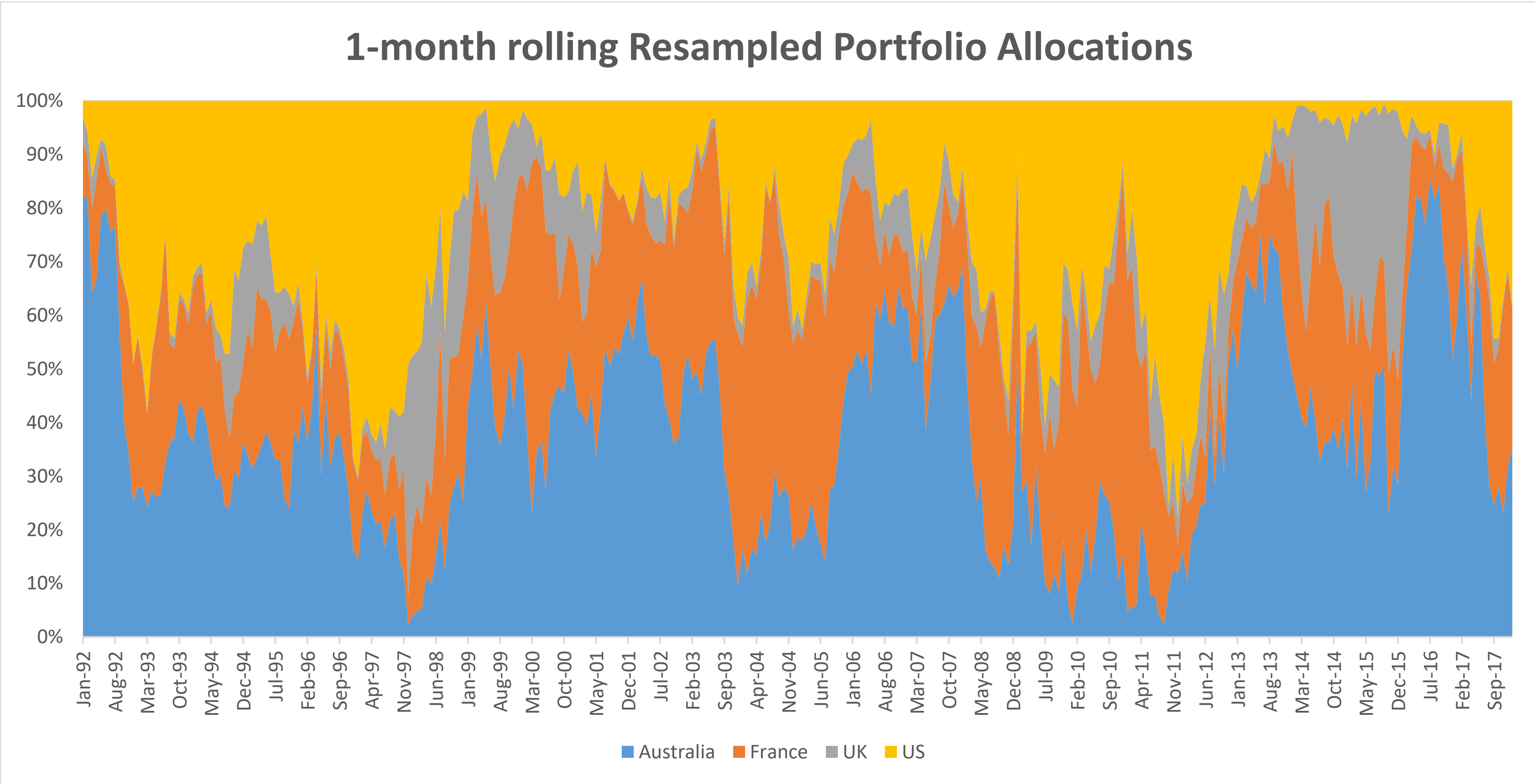
Dec-15
Jul-16
Feb-17
Sep-17
Australia
France
UK
US

(f)


1-month rolling Resampled Portfolio Allocations
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Jan-92
Aug-92
Mar-93
Oct-93
May-94
Dec-94
Jul-95
Feb-96
Sep-96
Apr-97
Nov-97
Jun-98
Jan-99
Aug-99
Mar-00
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Jul-02
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Sep-03
Apr-04
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Mar-07
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May-08
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Apr-11
Nov-11
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Oct-14
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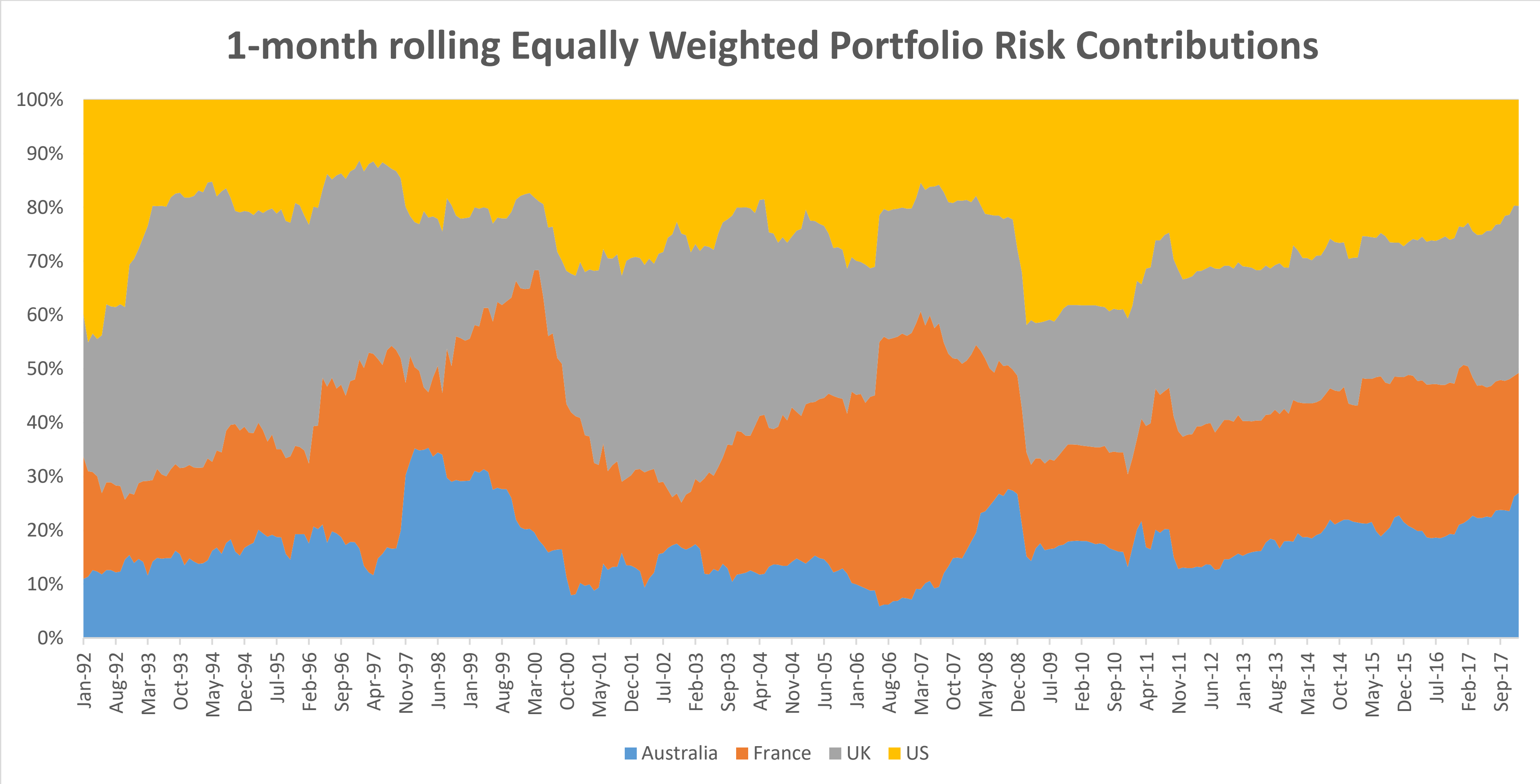
Jul-16
Feb-17
Sep-17
Australia
France
UK
US

(g)

# APPENDIX B: PORTFOLIO RISK CONTRIBUTIONS

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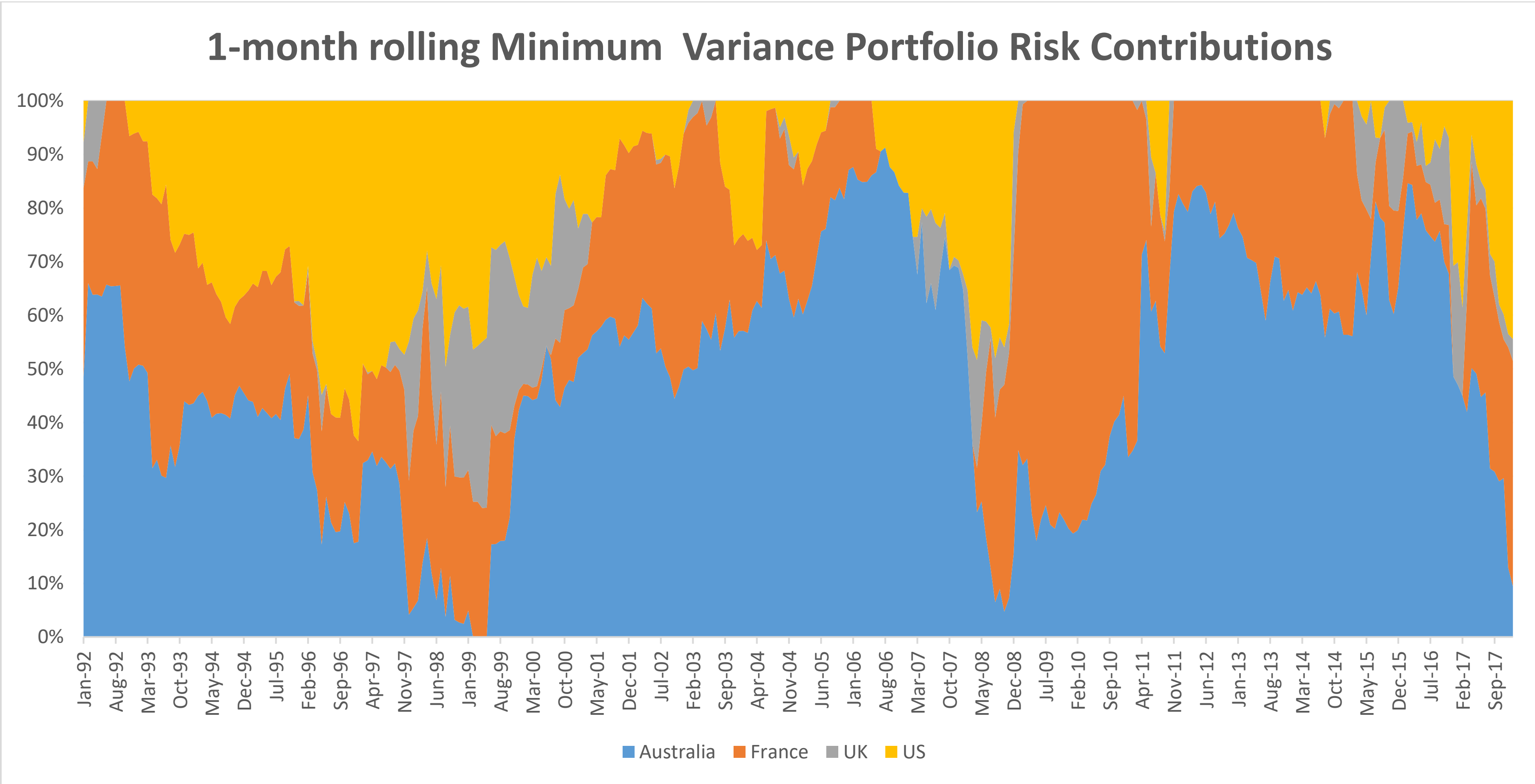

1-month rolling Minimum Variance Portfolio Risk Contributions
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Jan-92
Aug-92
Mar-93
Oct-93
May-94
Dec-94
Jul-95
Feb-96
Sep-96
Apr-97
Nov-97
Jun-98
Jan-99
Aug-99
Mar-00
Oct-00
May-01
Dec-01
Jul-02
Feb-03
Sep-03
Apr-04
Nov-04
Jun-05
Jan-06
Aug-06
Mar-07
Oct-07
May-08
Dec-08
Jul-09
Feb-10
Sep-10
Apr-11
Nov-11
Jun-12
Jan-13
Aug-13
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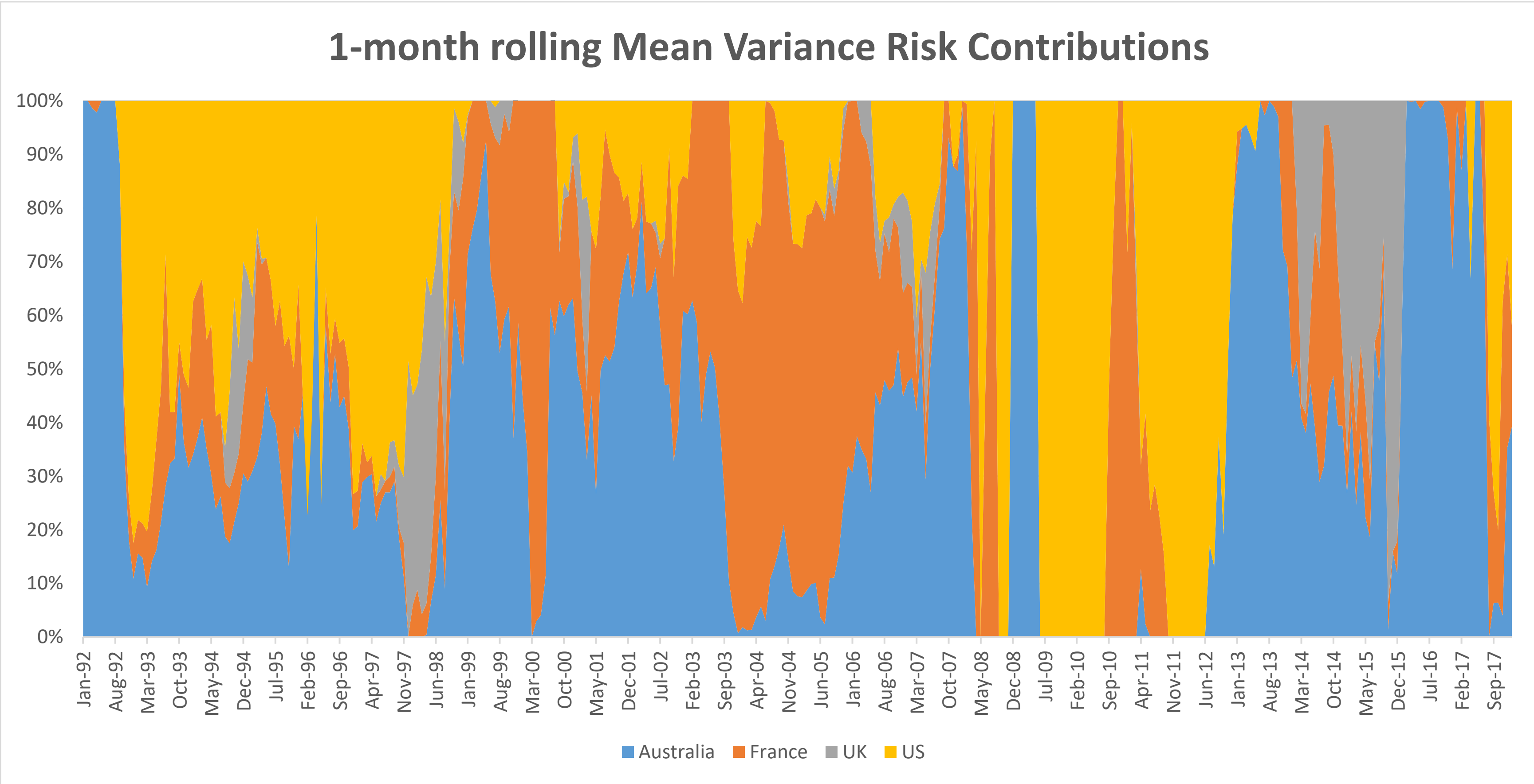
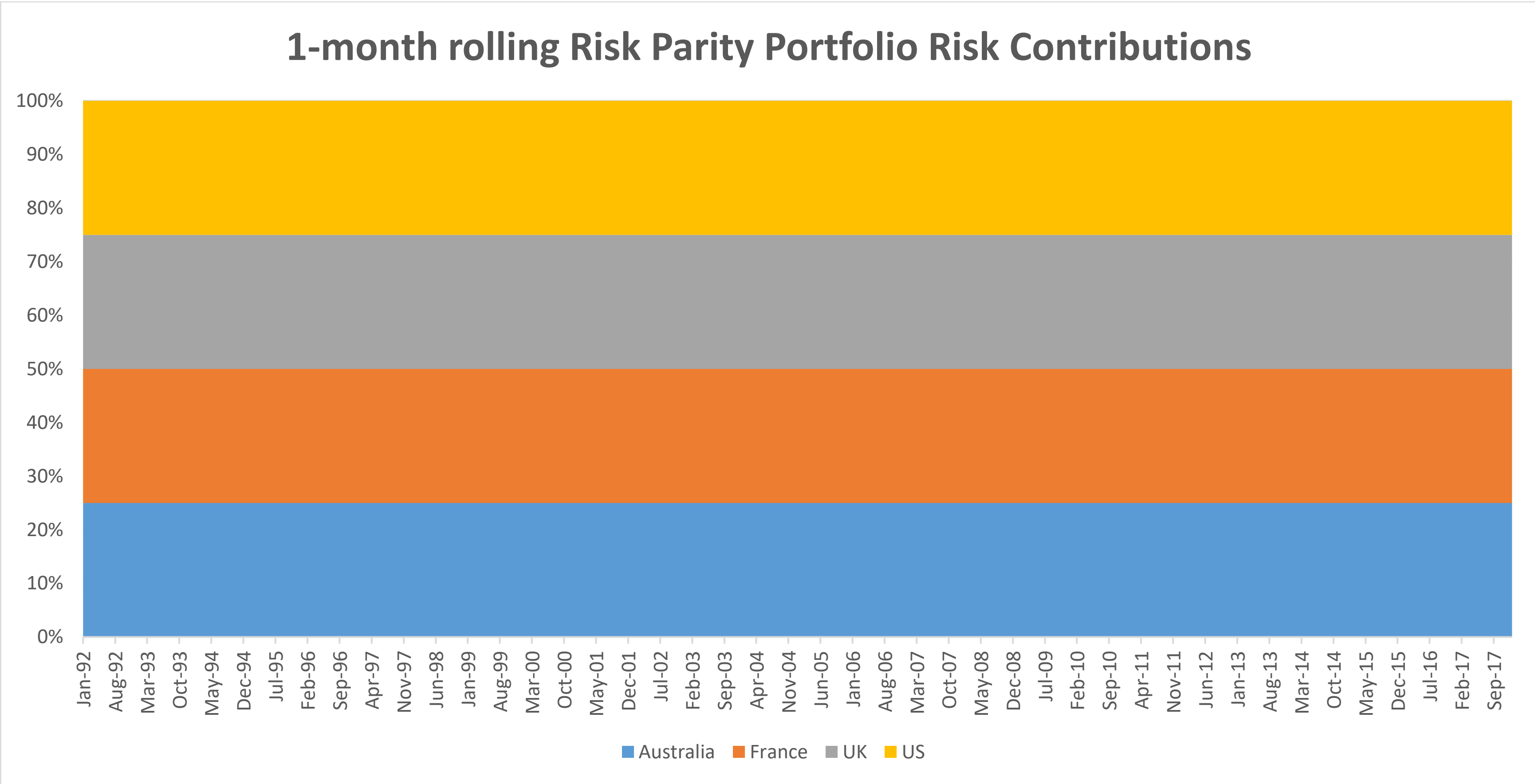
Oct-14
May-15
Dec-15
Jul-16
Feb-17
Sep-17
Australia
France
UK
US


(ii)



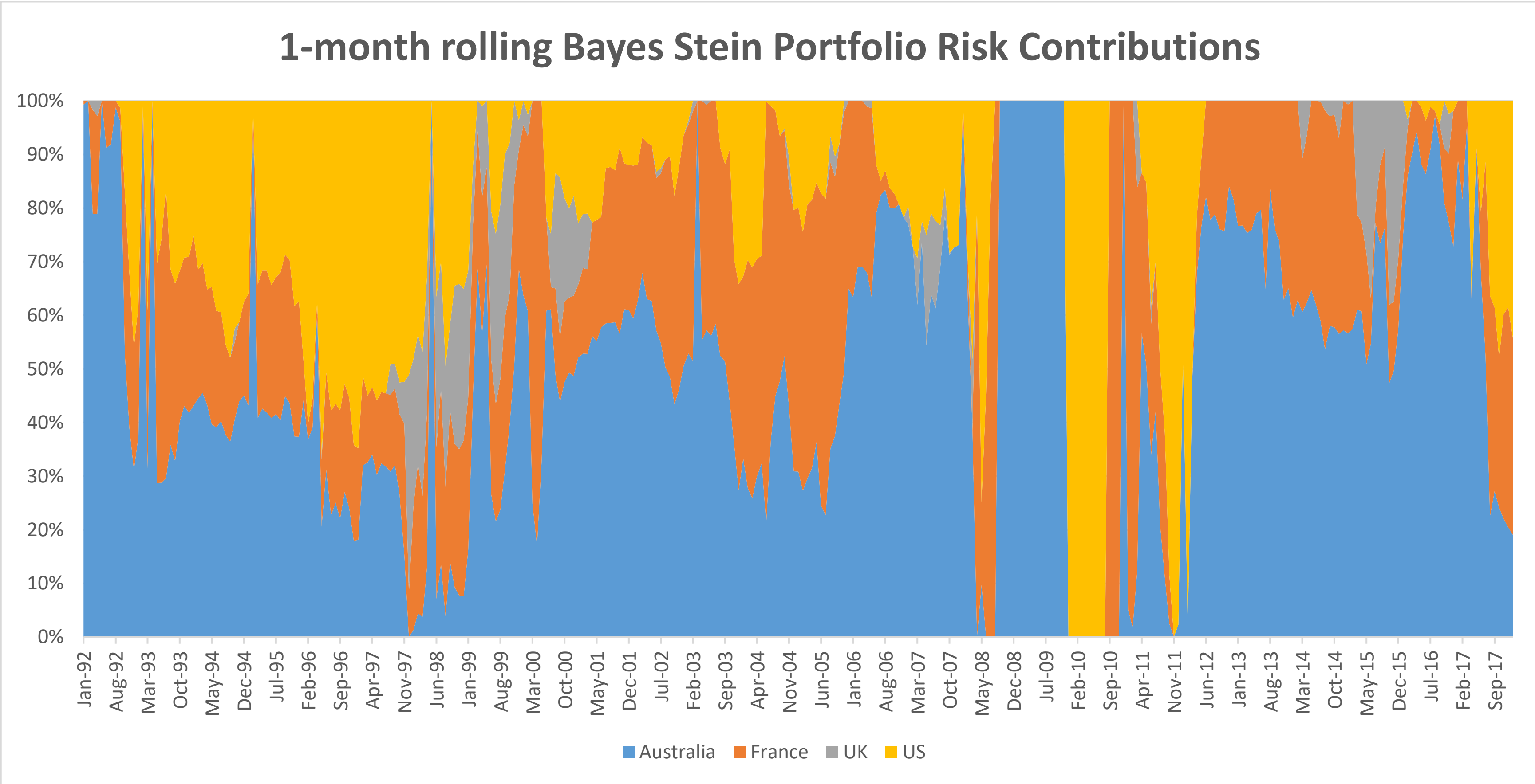
1-month rolling Mean Variance Risk Contributions
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Jan-92
Aug-92
Mar-93
Oct-93
May-94
Dec-94
Jul-95
Feb-96
Sep-96
Apr-97
Nov-97
Jun-98
Jan-99
Aug-99
Mar-00
Oct-00
May-01
Dec-01
Jul-02
Feb-03
Sep-03
Apr-04
Nov-04
Jun-05
Jan-06
Aug-06
Mar-07
Oct-07
May-08
Dec-08
Jul-09
Feb-10
Sep-10
Apr-11
Nov-11
Jun-12
Jan-13
Aug-13
Mar-14
Oct-14
May-15
Dec-15
Jul-16
Feb-17
Sep-17
Australia
France
UK
US

(iii)



1-month rolling Risk Parity Portfolio Risk Contributions
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Jan-92 Aug-92 Mar-93 Oct-93 May-94 Dec-94 Jul-95 Feb-96 Sep-96 Apr-97 Nov-97 Jun-98 Jan-99 Aug-99 Mar-00 Oct-00 May-01 Dec-01 Jul-02 Feb-03 Sep-03 Apr-04 Nov-04 Jun-05 Jan-06 Aug-06 Mar-07 Oct-07 May-08 Dec-08 Jul-09 Feb-10 Sep-10 Apr-11 Nov-11 Jun-12 Jan-13 Aug-13 Mar-14 Oct-14 May-15 Dec-15 Jul-16 Feb-17 Sep-17
Australia
France
UK
US


(iv)



1-month rolling Bayes Stein Portfolio Risk Contributions
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Jan-92
Aug-92
Mar-93
Oct-93
May-94
Dec-94
Jul-95
Feb-96
Sep-96
Apr-97
Nov-97
Jun-98
Jan-99
Aug-99
Mar-00
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May-01
Dec-01
Jul-02
Feb-03
Sep-03
Apr-04
Nov-04
Jun-05
Jan-06
Aug-06
Mar-07
Oct-07
May-08
Dec-08
Jul-09
Feb-10
Sep-10
Apr-11
Nov-11
Jun-12
Jan-13
Aug-13
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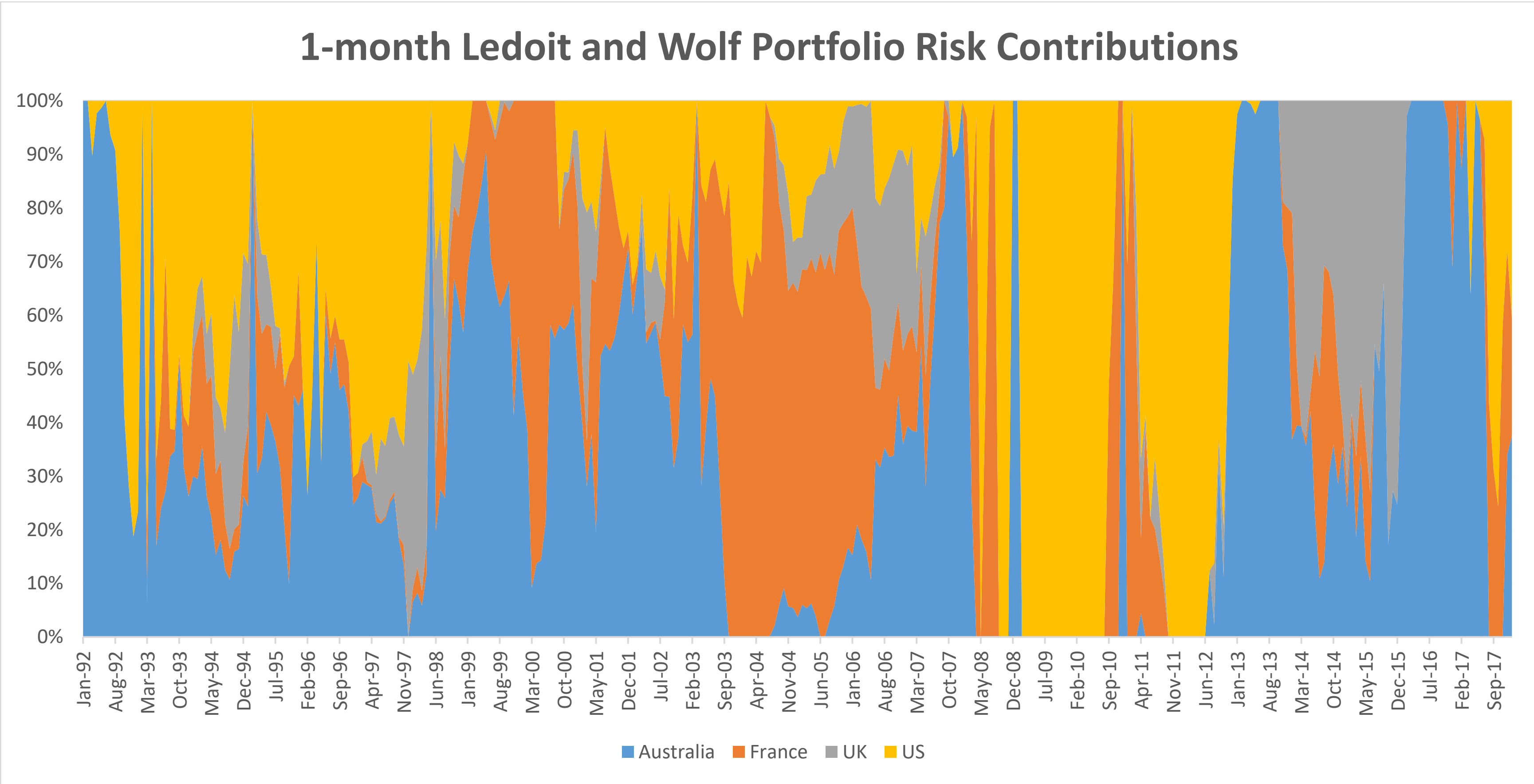
May-15
Dec-15
Jul-16
Feb-17
Sep-17
Australia
France
UK
US


(v)


1-month Ledoit and Wolf Portfolio Risk Contributions
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Jan-92
Aug-92
Mar-93
Oct-93
May-94
Dec-94
Jul-95
Feb-96
Sep-96
Apr-97
Nov-97
Jun-98
Jan-99
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Mar-00
Oct-00
May-01
Dec-01
Jul-02
Feb-03
Sep-03
Apr-04
Nov-04
Jun-05
Jan-06
Aug-06
Mar-07
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May-08
Dec-08
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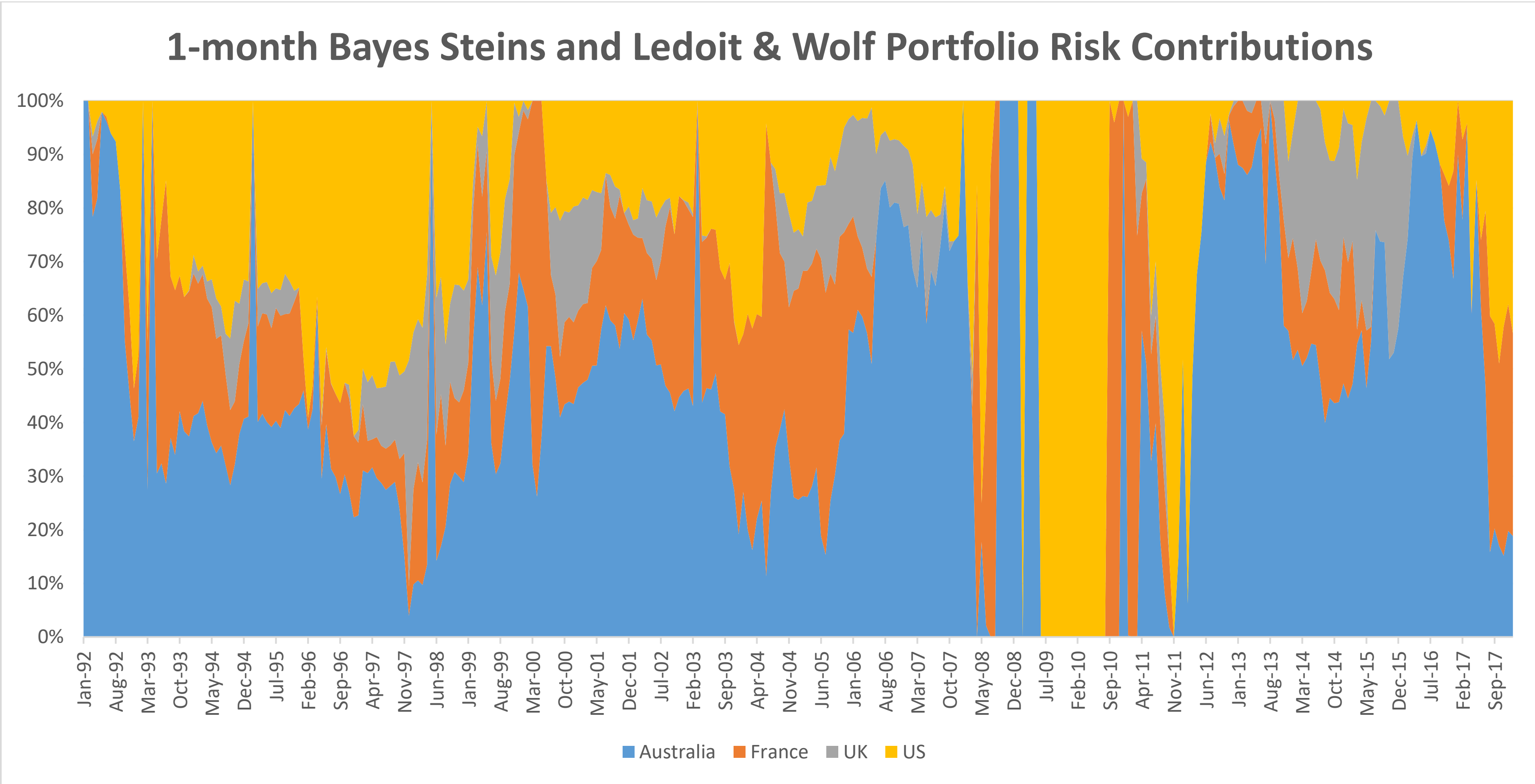
Sep-10
Apr-11
Nov-11
Jun-12
Jan-13
Aug-13
Mar-14
Oct-14
May-15
Dec-15
Jul-16
Feb-17
Sep-17
Australia
France
UK
US
(vi)



1-month Bayes Steins and Ledoit & Wolf Portfolio Risk Contributions
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Jan-92
Aug-92
Mar-93
Oct-93
May-94
Dec-94
Jul-95
Feb-96
Sep-96
Apr-97
Nov-97
Jun-98
Jan-99
Aug-99
Mar-00
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Jul-02
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Apr-04
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Jul-09
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Sep-10
Apr-11
Nov-11
Jun-12
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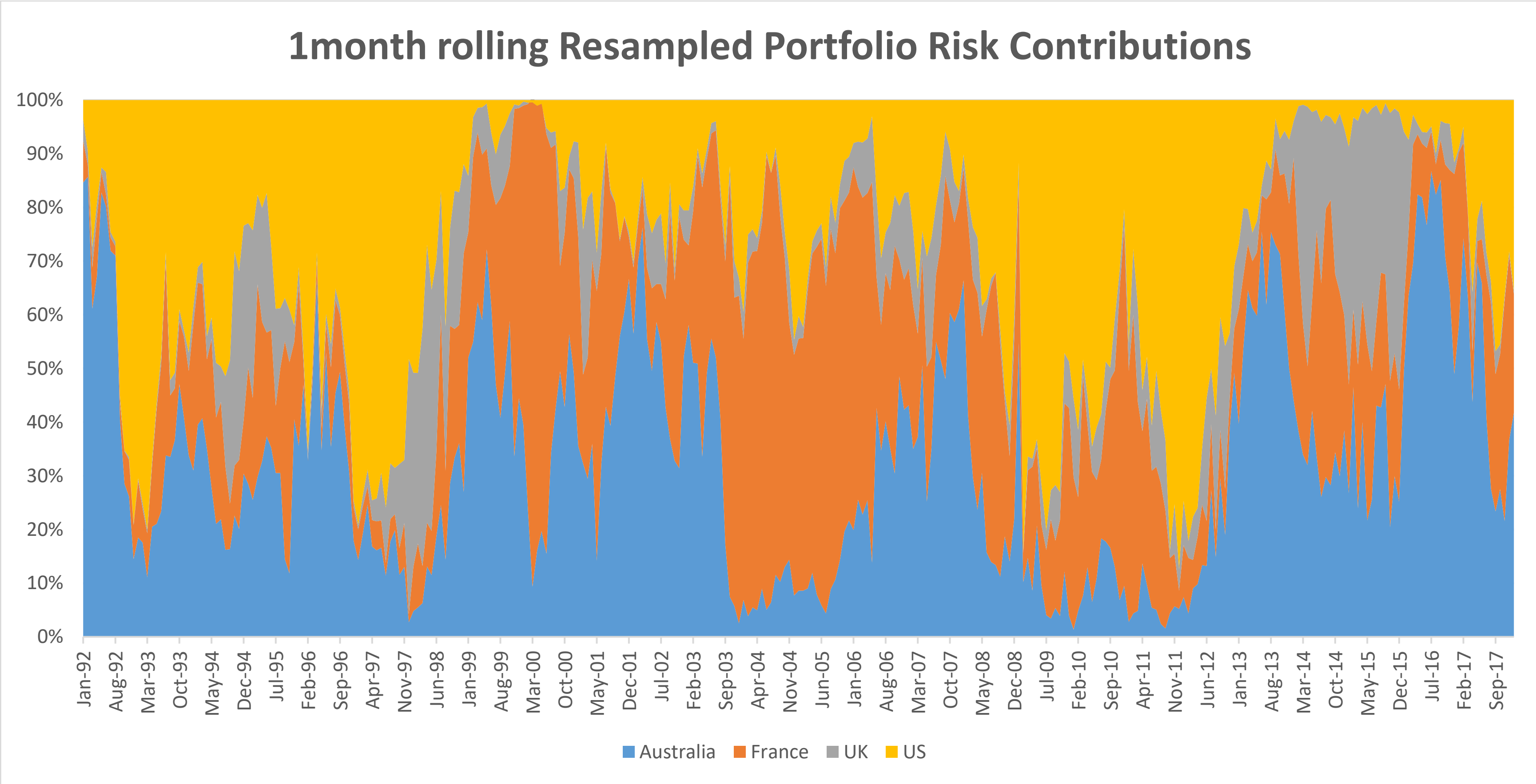
Oct-14
May-15
Dec-15
Jul-16
Feb-17
Sep-17
Australia
France
UK
US


(vii)



1month rolling Resampled Portfolio Risk Contributions
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Jan-92
Aug-92
Mar-93
Oct-93
May-94
Dec-94
Jul-95
Feb-96
Sep-96
Apr-97
Nov-97
Jun-98
Jan-99
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Sep-10
Apr-11
Nov-11
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Mar-14
Oct-14
May-15
Dec-15
Jul-16
Feb-17
Sep-17
Australia
France
UK
US


(viii)

# APPENDIX C: HYPOTHESES TESTING

## 1-Month rolling period

**Jobson -Korkie Test for Comparative Performance In-sample**

| | Equally Weighted | Minimum Variance | Mean Variance | Bayes Stein (BS) | Ledoit & Wolf (LW ) | BS + LW | Resampling | Risk Parity |
|---|---|---|---|---|---|---|---|---|
| Equally Weighted | | | | | | | | |
| Minimum Variance | -1.131 | | | | | | | |
| Mean Variance | -3.723*** | -2.556** | | | | | | |
| Bayes Stein (BS) | -1.754* | 0.508 | 3.836*** | | | | | |
| Ledoit & Wolf (LW ) | -4.017*** | -2.743*** | -0.869 | -4.091*** | | | | |
| BS + LW | -2.643*** | 0.088 | 3.667*** | -2.485** | 3.981*** | | | |
| Resampling | -3.92*** | -2.223** | 2.223** | -4.121*** | 2.759*** | -3.893*** | | |
| Risk Parity | 0.183 | 1.711* | 3.707*** | 1.898* | 3.938*** | 2.66*** | 3.953*** | |

**Ledoit and Wolf (2008) Hypothesis testing In-sample**

| | Equally Weighted | Minimum Variance | Mean Variance | Bayes Stein (BS) | Ledoit & Wolf (LW ) | BS + LW | Resampling | Risk Parity |
|---|---|---|---|---|---|---|---|---|
| Equally Weighted | | | | | | | | |
| Minimum Variance | 0.234 | | | | | | | |
| Mean Variance | 4.287*** | 4.931*** | | | | | | |
| Bayes Stein (BS) | 1.148 | 0.474 | 5.516*** | | | | | |
| Ledoit & Wolf (LW ) | 1.229 | 3.135*** | 0.181 | 5.511*** | | | | |
| BS + LW | 2.841*** | 0.167 | 4.994*** | 1.607 | 0.850 | | | |
| Resampling | 3.778*** | 5.739*** | 4.560*** | 5.639*** | 4.407*** | 0.732 | | |
| Risk Parity | 0.261 | 0.521 | 5.294*** | 3.716*** | 4.987*** | 5.795*** | 5.365*** | |

**Jobson -Korkie Test for Comparative Performance out-of-sample**

| | Equally Weighted | Minimum Variance | Mean Variance | Bayes Stein (BS) | Ledoit & Wolf (LW ) | BS + LW | Resampling | Risk Parity |
|---|---|---|---|---|---|---|---|---|
| Equally Weighted | | | | | | | | |
| Minimum Variance | -.1044 | | | | | | | |
| Mean Variance | .2739 | .3674 | | | | | | |
| Bayes Stein (BS) | .1111 | .2277 | -.2943 | | | | | |
| Ledoit & Wolf (LW ) | .4043 | .4451 | .2099 | .392 | | | | |
| BS + LW | .2132 | .318 | -.1174 | .2698 | -.2858 | | | |
| Resampling | .4495 | .4855 | -.0428 | .1809 | -.2199 | .0492 | | |
| Risk Parity | -.0513 | .1174 | -.297 | -.1328 | -.4189 | -.2401 | -.5151 | |

**Ledoit and Wolf (2008) Hypothesis testing out-of-sample**

| | Equally Weighted | Minimum Variance | Mean Variance | Bayes Stein (BS) | Ledoit & Wolf (LW ) | BS + LW | Resampling | Risk Parity |
|---|---|---|---|---|---|---|---|---|
| Equally Weighted | | | | | | | | |
| Minimum Variance | 0.366 | | | | | | | |
| Mean Variance | 1.181 | 1.302 | | | | | | |
| Bayes Stein (BS) | 0.421 | 0.653 | 1.019 | | | | | |
| Ledoit & Wolf (LW ) | 1.590 | 1.365 | 0.842 | 1.724* | | | | |
| BS + LW | 0.822 | 0.776 | 0.335 | 0.968 | 1.328 | | | |
| Resampling | 1.666* | 1.650* | 0.225 | 0.607 | 0.741 | 0.155 | | |
| Risk Parity | 0.148 | 0.432 | 1.216 | 0.468 | 1.562 | 0.839 | 1.991** | |

**Jobson -Korkie Test for Comparative Performance** **out-of-sample with transaction cost**

| | Equally Weighted | Minimum Variance | Mean Variance | Bayes Stein (BS) | Ledoit & Wolf (LW ) | BS + LW | Resampling | Risk Parity |
|---|---|---|---|---|---|---|---|---|
| Equally Weighted | | | | | | | | |
| Minimum Variance | .7014 | | | | | | | |
| Mean Variance | 1.4892 | 1.0405 | | | | | | |
| Bayes Stein (BS) | .9377 | .5067 | -.5203 | | | | | |
| Ledoit & Wolf (LW ) | 1.6051 | 1.0155 | .0719 | .5381 | | | | |
| BS + LW | 1.1188 | .6479 | -.3223 | .433 | -.3613 | | | |
| Resampling | 6.2936 | 2.3886 | .8977 | 1.2807 | .9128 | 1.1089 | | |
| Risk Parity | .6262 | -.6825 | -1.4105 | -.8637 | -1.4754 | -1.0337 | -5.3858 | |

**Ledoit and Wolf (2008) Hypothesis testing** **out-of-sample with transaction cost**

Test stat

| | Equally Weighted | Minimum Variance | Mean Variance | Bayes Stein (BS) | Ledoit & Wolf (LW ) | BS + LW | Resampling | Risk Parity |
|---|---|---|---|---|---|---|---|---|
| Equally Weighted | | | | | | | | |
| Minimum Variance | 1.976** | | | | | | | |
| Mean Variance | 5.355*** | 4.078*** | | | | | | |
| Bayes Stein (BS) | 3.021*** | 1.417 | 1.556 | | | | | |
| Ledoit & Wolf (LW ) | 5.322*** | 3.937*** | 0.236 | 1.792* | | | | |
| BS + LW | 3.566*** | 1.706* | 0.789 | 1.076 | 1.008 | | | |
| Resampling | 7.948*** | 4.724*** | 2.101** | 5.121*** | 2.160** | 3.691*** | | |
| Risk Parity | 2.122** | 1.877* | 5.332*** | 2.629*** | 5.257*** | 3.183*** | 7.489*** | |

## Appendix D - Backtesting VaR

| | Observations | Expected | Failures | Observed Level | Ratio |
|---|---|---|---|---|---|
| Australia | 4435 | 221.75 | 235 | 0.94701 | 1.0598 |
| Canada | 4435 | 221.75 | 204 | 0.95400 | 0.9200 |
| France | 4435 | 221.75 | 177 | 0.96009 | 0.7982 |
| UK | 4435 | 221.75 | 256 | 0.94228 | 1.1545 |
| US | 4435 | 221.75 | 261 | 0.94115 | 1.1770 |

(a)

| | Binomial | | Traffic Light | | Proportion of Failure | | Time Until First Failure | | |
|---|---|---|---|---|---|---|---|---|---|
| | Z-score | P Value | Probability | Type I Error | Likelihood Ratio | P-Value | Likelihood Ratio | P-Value | First Failure |
| Australia | 0.9129 | 0.1807 | 0.8286 | 0.1892 | 0.8181 | 0.3657 | 3.3215 | 0.0684 | 2 |
| Canada | -1.2229 | 0.1107 | 0.1164 | 0.8969 | 1.5350 | 0.2154 | 0.8654 | 0.3522 | 7 |
| France | -3.0832 | 0.0010 | 0.0008 | 0.9994 | 10.1820 | 0.0014 | 0.6813 | 0.4092 | 8 |
| UK | 2.3598 | 0.0091 | 0.9906 | 0.0112 | 5.3161 | 0.0211 | 0.3153 | 0.5744 | 11 |
| US | 2.7042 | 0.0034 | 0.9963 | 0.0045 | 6.9370 | 0.0084 | 3.3215 | 0.0684 | 2 |

(b)

| | Conditional Coverage | | Conditional Coverage independence | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Likelihood Ratio | P-Value | Likelihood Ratio | P-Value | N00 | N10 | N01 | N11 |
| Australia | 52.27 | 0.0000 | 51.45 | 0.0000 | 4006 | 193 | 193 | 42 |
| Canada | 66.69 | 0.0000 | 65.16 | 0.0000 | 4066 | 164 | 164 | 40 |
| France | 19.44 | 0.0000 | 9.26 | 0.0000 | 4096 | 161 | 161 | 16 |
| UK | 37.74 | 0.0000 | 32.43 | 0.0000 | 3961 | 217 | 217 | 39 |
| US | 60.38 | 0.0000 | 53.45 | 0.0000 | 3960 | 213 | 213 | 48 |

(c)

| | TL | Bin | POF | TUFF | CC | CCI |
|---|---|---|---|---|---|---|
| Australia | Green | accept | accept | accept | reject | reject |
| Canada | Green | accept | accept | accept | reject | reject |
| France | Green | reject | reject | accept | reject | reject |
| UK | Yellow | reject | reject | accept | reject | reject |
| US | Yellow | reject | reject | accept | reject | reject |

(d)

Backtesting results for Normal 95% VaR

| | Observations | Expected | Failures | Observed Level | Ratio |
|---|---|---|---|---|---|
| Australia | 4435 | 221.75 | 249 | 0.9439 | 1.1229 |
| Canada | 4435 | 221.75 | 239 | 0.9461 | 1.0778 |
| France | 4435 | 221.75 | 199 | 0.9551 | 0.8974 |
| UK | 4435 | 221.75 | 275 | 0.9380 | 1.2401 |
| US | 4435 | 221.75 | 304 | 0.9315 | 1.3709 |

(a)

| | Binomial | | Traffic Light | | Proportion of Failure | | Time Until First Failure | | |
|---|---|---|---|---|---|---|---|---|---|
| | Z-score | P Value | Probability | Type I Error | Likelihood Ratio | P-Value | Likelihood Ratio | P-Value | First Failure |
| Australia | 1.8775 | 0.0302 | 0.9704 | 0.0344 | 3.3960 | 0.0654 | 3.3215 | 0.0684 | 2 |
| Canada | 1.1885 | 0.1173 | 0.8884 | 0.1249 | 1.3791 | 0.2403 | 0.8654 | 0.3522 | 7 |
| France | -1.5674 | 0.0585 | 0.0608 | 0.9473 | 2.5408 | 0.1109 | 5.9915 | 0.0144 | 1 |
| UK | 3.6688 | 0.0001 | 0.9998 | 0.0002 | 12.5470 | 0.0004 | 0.3153 | 0.5744 | 11 |
| US | 5.6669 | 0.0000 | 1.0000 | 0.0000 | 28.9260 | 0.0000 | 3.3215 | 0.0684 | 2 |

(b)

| | Conditional Coverage | | Conditional Coverage independence | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **Likelihood Ratio** | **P-Value** | **Likelihood Ratio** | **P-Value** | **N00** | **N10** | **N01** | **N11** |
| Australia | 58.32 | 2.164E-13 | 54.93 | 1.251E-13 | 3982 | 203 | 203 | 46 |
| Canada | 83.04 | 9.303E-19 | 81.66 | 1.618E-19 | 4008 | 187 | 187 | 52 |
| France | 12.31 | 2.118E-03 | 9.77 | 1.770E-03 | 4056 | 180 | 179 | 19 |
| UK | 41.32 | 1.067E-09 | 28.77 | 8.155E-08 | 3925 | 234 | 234 | 41 |
| US | 78.74 | 7.963E-18 | 49.82 | 1.688E-12 | 3882 | 248 | 248 | 56 |

(c)

| | **TL** | **Bin** | **POF** | **TUFF** | **CC** | **CCI** |
|---|---|---|---|---|---|---|
| Australia | yellow | accept | accept | accept | reject | reject |
| Canada | green | accept | accept | accept | reject | reject |
| France | green | accept | accept | reject | reject | reject |
| UK | yellow | reject | reject | accept | reject | reject |
| US | red | reject | reject | accept | reject | reject |

(d)

Backtesting results for Historical Simulation 95% VaR

| | **Observations** | **Expected** | **Failures** | **Observed Level** | **Ratio** |
|---|---|---|---|---|---|
| Australia | 4435 | 221.75 | 226 | 0.9490 | 1.0192 |
| Canada | 4435 | 221.75 | 226 | 0.9490 | 1.0192 |
| France | 4435 | 221.75 | 244 | 0.9450 | 1.1003 |
| UK | 4435 | 221.75 | 247 | 0.9443 | 1.1139 |
| US | 4435 | 221.75 | 235 | 0.9470 | 1.0598 |

(a)

| | Binomial | | Traffic Light | | Proportion of Failure | | Time Until First Failure | | |
|---|---|---|---|---|---|---|---|---|---|
| | Zscore | P Value | Probability | Type I Error | Likelihood Ratio | P-Value | Likelihood Ratio | P-Value | First Failure |
| Australia | 0.2928 | 0.3848 | 0.6317 | 0.3944 | 0.0852 | 0.7703 | 3.3215 | 0.0684 | 2 |
| Canada | 0.2928 | 0.3848 | 0.6317 | 0.3944 | 0.0852 | 0.7703 | 0.8654 | 0.3522 | 7 |
| France | 1.5330 | 0.0626 | 0.9398 | 0.0686 | 2.2791 | 0.1311 | 0.6813 | 0.4092 | 8 |
| UK | 1.7397 | 0.0410 | 0.9602 | 0.0458 | 2.9235 | 0.0873 | 1.0977 | 0.2948 | 6 |
| US | 0.9129 | 0.1807 | 0.8286 | 0.1892 | 0.8181 | 0.3657 | 0.1978 | 0.6565 | 30 |

(b)

| | Conditional Coverage | | Conditional Coverage independence | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Likelihood Ratio | P-Value | Likelihood Ratio | P-Value | N00 | N10 | N01 | N11 |
| Australia | 10.216 | 0.0060 | 10.13 | 0.0015 | 4005 | 203 | 203 | 23 |
| Canada | 17.473 | 0.0002 | 17.39 | 0.0000 | 4009 | 199 | 199 | 27 |
| France | 10.024 | 0.0067 | 7.74 | 0.0054 | 3970 | 220 | 220 | 24 |
| UK | 6.6674 | 0.0357 | 3.74 | 0.0530 | 3961 | 226 | 226 | 21 |
| US | 10.466 | 0.0053 | 9.65 | 0.0019 | 3988 | 211 | 211 | 24 |

(c)

| | TL | Bin | POF | TUFF | CC | CCI |
|---|---|---|---|---|---|---|
| Australia | green | accept | accept | accept | reject | reject |
| Canada | green | accept | accept | accept | reject | reject |
| France | green | accept | accept | accept | reject | reject |
| UK | yellow | accept | accept | accept | reject | accept |
| US | green | accept | accept | accept | reject | reject |

(d)

Backtesting results for EWMA 95% VaR

| | Observations | Expected | Failures | Observed Level | Ratio |
|---|---|---|---|---|---|
| Australia | 4435 | 221.75 | 200 | 0.9549 | 0.9019 |
| Canada | 4435 | 221.75 | 203 | 0.9542 | 0.9155 |
| France | 4435 | 221.75 | 204 | 0.9540 | 0.9200 |
| UK | 4435 | 221.75 | 228 | 0.9486 | 1.0282 |
| US | 4435 | 221.75 | 226 | 0.9490 | 1.0192 |

(a)

| | Binomial | | Traffic Light | | Proportion of Failure | | Time Until First Failure | | |
|---|---|---|---|---|---|---|---|---|---|
| | **Z-score** | **P Value** | **Probability** | **Type I Error** | **Likelihood Ratio** | **P-Value** | **Likelihood Ratio** | **P-Value** | **First Failure** |
| Australia | -1.4985 | 0.0670 | 0.0699 | 0.9392 | 2.3188 | 0.1278 | 3.3215 | 0.0684 | 2 |
| Canada | -1.2918 | 0.0982 | 0.1032 | 0.9090 | 1.7154 | 0.1903 | 0.8654 | 0.3522 | 7 |
| France | -1.2229 | 0.1107 | 0.1164 | 0.8969 | 1.5350 | 0.2154 | 5.9915 | 0.0144 | 1 |
| UK | 0.4306 | 0.3334 | 0.6819 | 0.3428 | 0.1838 | 0.6681 | 1.0977 | 0.2948 | 6 |
| US | 0.2928 | 0.3848 | 0.6317 | 0.3944 | 0.0852 | 0.7703 | 3.3215 | 0.0684 | 2 |

(b)

| | Conditional Coverage | | Conditional Coverage independence | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **Likelihood Ratio** | **P-Value** | **Likelihood Ratio** | **P-Value3** | **N00** | **N10** | **N01** | **N11** |
| Australia | 13.52 | 0.0012 | 11.21 | 0.0008 | 4054 | 180 | 180 | 20 |
| Canada | 18.02 | 0.0001 | 16.30 | 0.0001 | 4051 | 180 | 180 | 23 |
| France | 5.91 | 0.0520 | 4.38 | 0.0364 | 4043 | 188 | 187 | 16 |
| UK | 5.65 | 0.0592 | 5.47 | 0.0193 | 3998 | 208 | 208 | 20 |
| US | 2.65 | 0.2663 | 2.56 | 0.1095 | 3999 | 209 | 209 | 17 |

(c)

| | TL | Bin | POF | TUFF | CC | CCI |
|---|---|---|---|---|---|---|
| Australia | green | accept | accept | accept | reject | reject |
| Canada | green | accept | accept | accept | reject | reject |
| France | green | accept | accept | reject | accept | reject |
| UK | green | accept | accept | accept | accept | reject |
| US | green | accept | accept | accept | accept | accept |

(d)

Backtesting results for GARCH 95% VaR

| | Observations | Expected | Failures | Observed Level | Ratio |
|---|---|---|---|---|---|
| Australia | 4435 | 44.35 | 110 | 0.9752 | 2.4803 |
| Canada | 4435 | 44.35 | 93 | 0.9790 | 2.0970 |
| France | 4435 | 44.35 | 80 | 0.9820 | 1.8038 |
| UK | 4435 | 44.35 | 106 | 0.9761 | 2.3901 |
| US | 4435 | 44.35 | 121 | 0.9727 | 2.7283 |

(a)

| | Binomial | | Traffic Light | | Proportion of Failure | | Time Until First Failure | | |
|---|---|---|---|---|---|---|---|---|---|
| | Z-score | P Value | Probability | Type I Error | Likelihood Ratio | P-Value | Likelihood Ratio | P-Value | First Failure |
| Australia | 9.9076 | 0.0000 | 1.0000 | 0.0000 | 69.5270 | 0.0000 | 6.4579 | 0.0110 | 2 |
| Canada | 7.3421 | 0.0000 | 1.0000 | 0.0000 | 40.9720 | 0.0000 | 3.8944 | 0.0484 | 442 |
| France | 5.3802 | 0.0000 | 1.0000 | 0.0000 | 23.3760 | 0.0000 | 0.2831 | 0.5947 | 56 |
| UK | 9.3040 | 0.0000 | 1.0000 | 0.0000 | 62.2910 | 0.0000 | 0.1528 | 0.6959 | 66 |
| US | 11.5680 | 0.0000 | 1.0000 | 0.0000 | 90.9360 | 0.0000 | 0.0760 | 0.7828 | 130 |

(b)

| | Conditional Coverage | | Conditional Coverage independence | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Likelihood Ratio | P-Value | Likelihood Ratio | P-Value | N00 | N10 | N01 | N11 |
| Australia | 99.07 | 0.0000 | 29.54 | 0.0000 | 4229 | 95 | 95 | 15 |
| Canada | 75.43 | 0.0000 | 34.46 | 0.0000 | 4262 | 79 | 79 | 14 |
| France | 31.90 | 0.0000 | 8.53 | 0.0035 | 4280 | 74 | 74 | 6 |
| UK | 70.35 | 0.0000 | 8.06 | 0.0045 | 4230 | 98 | 98 | 8 |
| US | 112.02 | 0.0000 | 21.09 | 0.0000 | 4206 | 107 | 107 | 14 |

(c)

| | TL | Bin | POF | TUFF | CC | CCI |
|---|---|---|---|---|---|---|
| Australia | red | Reject | reject | reject | reject | reject |
| Canada | red | Reject | reject | reject | reject | reject |
| France | red | Reject | reject | accept | reject | reject |
| UK | red | Reject | reject | accept | reject | reject |
| US | red | Reject | reject | accept | reject | reject |

(d)

Backtesting results for Normal 99% VaR

| | Observations | Expected | Failures | Observed Level | Ratio |
|---|---|---|---|---|---|
| Australia | 4435 | 44.35 | 73 | 0.9835 | 1.6460 |
| Canada | 4435 | 44.35 | 56 | 0.9874 | 1.2627 |
| France | 4435 | 44.35 | 36 | 0.9919 | 0.8117 |
| UK | 4435 | 44.35 | 70 | 0.9842 | 1.5784 |
| US | 4435 | 44.35 | 72 | 0.9838 | 1.6234 |

(a)

| | Binomial | | Traffic Light | | Proportion of Failure | | Time Until First Failure | | |
|---|---|---|---|---|---|---|---|---|---|
| | **Z-score** | **P Value** | **Probability** | **Type I Error** | **Likelihood Ratio** | **P-Value** | **Likelihood Ratio** | **P-Value** | **First Failure** |
| Australia | 4.3237 | 0.0000 | 1.0000 | 0.0000 | 15.6460 | 0.0001 | 0.5038 | 0.4778 | 45 |
| Canada | 1.7582 | 0.0394 | 0.9627 | 0.0502 | 2.8537 | 0.0912 | 3.8944 | 0.0484 | 442 |
| France | -1.2601 | 0.1038 | 0.1157 | 0.9128 | 1.6971 | 0.1927 | 0.2831 | 0.5947 | 56 |
| UK | 3.8710 | 0.0001 | 0.9999 | 0.0002 | 12.7440 | 0.0004 | 0.1528 | 0.6959 | 66 |
| US | 4.1728 | 0.0000 | 1.0000 | 0.0001 | 14.6500 | 0.0001 | 0.0760 | 0.7828 | 130 |

(b)

| | Conditional Coverage | | Conditional Coverage independence | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **Likelihood Ratio** | **P-Value** | **Likelihood Ratio** | **P-Value** | **N00** | **N10** | **N01** | **N11** |
| Australia | 26.01 | 0.0000 | 10.36 | 0.0013 | 4294 | 67 | 67 | 6 |
| Canada | 18.98 | 0.0001 | 16.13 | 0.0001 | 4328 | 50 | 50 | 6 |
| France | 6.14 | 0.0464 | 4.44 | 0.0350 | 4364 | 34 | 34 | 2 |
| UK | 17.49 | 0.0002 | 4.75 | 0.0293 | 4298 | 66 | 66 | 4 |
| US | 33.13 | 0.0000 | 18.48 | 0.0000 | 4298 | 64 | 64 | 8 |

(c)

| | **TL** | **Bin** | **POF** | **TUFF** | **CC** | **CCI** |
|---|---|---|---|---|---|---|
| Australia | red | Reject | reject | accept | reject | reject |
| Canada | yellow | Accept | accept | reject | reject | reject |
| France | green | Accept | accept | accept | reject | reject |
| UK | yellow | Reject | reject | accept | reject | reject |
| US | red | Reject | reject | accept | reject | reject |

(d)

Backtesting results for Historical Simulation 99% VaR

| | Observations | Expected | Failures | Observed Level | Ratio |
|---|---|---|---|---|---|
| Australia | 4435 | 44.35 | 74 | 0.98331 | 1.66850 |
| Canada | 4435 | 44.35 | 81 | 0.98174 | 1.82640 |
| France | 4435 | 44.35 | 96 | 0.97835 | 2.16460 |
| UK | 4435 | 44.35 | 77 | 0.98264 | 1.73620 |
| US | 4435 | 44.35 | 79 | 0.98219 | 1.78130 |

(a)

| | Binomial | | Traffic Light | | Proportion of Failure | | Time Until First Failure | | |
|---|---|---|---|---|---|---|---|---|---|
| | Z-score | P Value | Probability | Type I Error | Likelihood Ratio | P-Value | Likelihood Ratio | P-Value | First Failure |
| Australia | 4.4747 | 0.0000 | 1.0000 | 0.0000 | 16.6700 | 0.0000 | 6.4579 | 0.0110 | 2 |
| Canada | 5.5311 | 0.0000 | 1.0000 | 0.0000 | 24.5850 | 0.0000 | 3.5893 | 0.0582 | 7 |
| France | 7.7948 | 0.0000 | 1.0000 | 0.0000 | 45.5790 | 0.0000 | 0.2831 | 0.5947 | 56 |
| UK | 4.9274 | 0.0000 | 1.0000 | 0.0000 | 19.9040 | 0.0000 | 0.0061 | 0.9376 | 108 |
| US | 5.2292 | 0.0000 | 1.0000 | 0.0000 | 22.1930 | 0.0000 | 0.0760 | 0.7828 | 130 |

(b)

| | Conditional Coverage | | Conditional Coverage independence | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Likelihood Ratio | P-Value | Likelihood Ratio | P-Value | N00 | N10 | N01 | N11 |
| Australia | 18.55 | 0.0001 | 1.8823 | 0.1701 | 4289 | 71 | 71 | 3 |
| Canada | 30.04 | 0.0000 | 5.4547 | 0.0195 | 4277 | 76 | 76 | 5 |
| France | 47.05 | 0.0000 | 1.4742 | 0.2247 | 4246 | 92 | 92 | 4 |
| UK | 23.54 | 0.0000 | 3.6316 | 0.0567 | 4284 | 73 | 73 | 4 |
| US | 28.03 | 0.0000 | 5.8320 | 0.0157 | 4281 | 74 | 74 | 5 |

(c)

| | TL | Bin | POF | TUFF | CC | CCI |
|---|---|---|---|---|---|---|
| Australia | red | reject | reject | reject | reject | accept |
| Canada | red | reject | reject | accept | reject | reject |
| France | red | reject | reject | accept | reject | accept |
| UK | red | reject | reject | accept | reject | accept |
| US | red | reject | reject | accept | reject | reject |

(d)

Backtesting results for EWMA 99% VaR

| | Observations | Expected | Failures | Observed Level | Ratio | First Failure |
|---|---|---|---|---|---|---|
| Australia | 4435 | 44.35 | 59 | 0.9867 | 1.3303 | 2 |
| Canada | 4435 | 44.35 | 76 | 0.9829 | 1.7136 | 7 |
| France | 4435 | 44.35 | 81 | 0.9817 | 1.8264 | 56 |
| UK | 4435 | 44.35 | 71 | 0.9840 | 1.6009 | 108 |
| US | 4435 | 44.35 | 83 | 0.9813 | 1.8715 | 30 |

(a)

| | Binomial | | Traffic Light | | Proportion of Failure | | Time Until First Failure | | |
|---|---|---|---|---|---|---|---|---|---|
| | Z-score | P Value | Probability | Type I Error | Likelihood Ratio | P-Value | Likelihood Ratio | P-Value | First Failure |
| Australia | 2.2109 | 0.0135 | 0.9860 | 0.0197 | 4.4291 | 0.0353 | 6.4579 | 0.0110 | 2 |
| Canada | 4.7765 | 0.0000 | 1.0000 | 0.0000 | 18.7990 | 0.0000 | 3.5893 | 0.0582 | 7 |
| France | 5.5311 | 0.0000 | 1.0000 | 0.0000 | 24.5850 | 0.0000 | 0.2831 | 0.5947 | 56 |
| UK | 4.0219 | 0.0000 | 1.0000 | 0.0000 | 13.6830 | 0.0002 | 0.0061 | 0.9376 | 108 |
| US | 5.8329 | 0.0000 | 1.0000 | 0.0000 | 27.0780 | 0.0000 | 1.0246 | 0.3114 | 30 |

(b)

| | Conditional Coverage | | Conditional Coverage independence | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **Likelihood Ratio** | **P-Value** | **Likelihood Ratio** | **P-Value** | **N00** | **N10** | **N01** | **N11** |
| Australia | 4.48 | 0.1062 | 0.0557 | 0.8134 | 4317 | 58 | 58 | 1 |
| Canada | 19.13 | 0.0001 | 0.3335 | 0.5636 | 4284 | 74 | 74 | 2 |
| France | 24.76 | 0.0000 | 0.1715 | 0.6788 | 4274 | 79 | 79 | 2 |
| UK | 18.26 | 0.0001 | 4.5776 | 0.0324 | 4296 | 67 | 67 | 4 |
| US | 29.90 | 0.0000 | 2.8226 | 0.0929 | 4272 | 79 | 79 | 4 |

(c)

| | **TL** | **Bin** | **POF** | **TUFF** | **CC** | **CCI** |
|---|---|---|---|---|---|---|
| Australia | yellow | reject | reject | reject | accept | accept |
| Canada | red | reject | reject | accept | reject | accept |
| France | red | reject | reject | accept | reject | accept |
| UK | red | reject | reject | accept | reject | reject |
| US | red | Reject | reject | accept | reject | accept |

(d)

Backtesting results for GARCH 99% VaR