Model risk in the over-the-counter market

Model Risk in the Over-the-Counter Market*

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Abstract

We propose a methodology to measure the parameter estimation risk and model specification risk of pricing models, as well as model selection risk of model classes, based on realized payoffs, for products in the over-the-counter market. Lévy jump models and affine jump-diffusion models are applied in estimating the fair variance strikes of variance swaps and forward starting option prices. Our results show that both parameter estimation risk and model specification risk are significant for variance swaps, while model specification risk is dominant when pricing forward starting options. We also find that the size of the model selection risk is substantial for both products.

Keywords: risk management; robustness and sensitivity analysis; forward starting option; variance swap; model risk

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1 Introduction

The use of models in financial markets is accompanied by their own risk, which leads to various attempts to identify and measure model risk. The academic research on model risk in continuous-time finance dates back to Derman (1996). Measuring model risk is an essential concern in the industry as well (Basel Committee on Banking Supervision, 2009; Federal Reserve Board of Governors, 2011; European Banking Authority, 2012).

The focus of this paper is the measurement of model risk for pricing models in the over-the-counter (OTC) market, where market prices are not available. Specifically, we consider exotic option pricing models, under both the physical and risk-neutral probability measures. Based on Bayesian methods, we investigate the parameter estimation risk (PER) and model specification risk (MSpR) of models used to price exotic options; and, by taking PER into consideration, we study the model selection risk (MSeR) of a model class for both long and short positions in the OTC market; in an empirical analysis, we apply our proposed methodology for variance swap (VS) rates and forward starting (FS) option prices to estimate the model risk of affine jump-diffusion (AJD) models and Lévy jump models.

Many previous studies differentiate between PER and MSpR as essential components of model risk. For example, Green and Figlewski (1999) find that option writers are exposed to considerable model risk due to imperfect model specification and inaccurate parameter estimation. However, only a handful of studies attempt to measure PER and MSpR. Kerkhof et al. (2010) propose a worst-case type risk measure over a set of models. They compute PER based on the confidence interval of parameters and quantify MSpR as compared to a reference model in the model set. In addition, Lazar et al. (2020) evaluate PER and MSpR of continuous-time finance models with expected shortfall (ES) type risk measures based on Bayesian estimators.

Moreover, the majority of studies use point-wise estimation methods. However, such an approach would ignore PER and underestimate model risk. Compared with the asymptotic distribution based measure of PER in Kerkhof et al. (2010), a Bayesian estimation approach is a more reliable way to measure PER. Jacquier and Jarrow (2000) study PER based on Bayesian estimation methods for the Black-Scholes (BS) model. Jacquier et al. (2002) further apply Bayesian estimators for stochastic volatility models and find that the Bayesian approach outperforms the moments and likelihood estimators for pricing. Johannes and Polson (2010) point out that the marginal posterior distribution obtained via the Bayesian approach can quantify PER, also see Chung et al. (2013) and Tunaru (2015). Tunaru and

Zheng (2017) apply Bayesian methods to study the PER of the BS and Merton jump-diffusion (MJD) models. Most recently, Leung et al. (2021) find that it is essential to consider parameter uncertainty for risk management; and they suggest using a Bayesian estimation method for this. We follow Lazar et al. (2020) and compute PER in a Bayesian framework, using an ES-type measure based on the estimated price distribution; such a measure would highlight the risk that market players are exposed to due to the risk of inaccurate parameter estimation.

"All models are wrong, but some are useful" (Box et al., 1987), and no models will fit market prices perfectly. Chen and Hong (2011) emphasize the importance of model specification. MSpR is regarded as the main source of model risk (Derman, 1996). Lazar et al. (2020) propose a measure of MSpR for liquid products; however, this approach requires market prices, which are generally not available in the OTC market. As such, the framework put forward in this paper relaxes the need for option prices because it only requires payoffs which are easily calculated, so it works for thinly traded exotic options in the OTC market. OTC options are the result of a private transaction between the buyer and the seller and deals are not always done at the best price. Moreoever, the OTC options have no secondary market where investors buy and sell products they already own. Typically option traders in the OTC market hold options until maturity, and holders have to enter into separate transactions to offset losses or leverage gains. Compared with exchange-traded options traders, OTC options investors are typically "buy and hold investing" (Deuskar et al., 2011), since holders realize their profits or losses at expiry. Fabozzi (1997) also state that OTC options investors intend to hold options to expiration because of the illiquidity of the OTC market. Given the reasons above, it is valid to use realized payoffs to compute MSpR in the OTC market.

Our first contribution is that we propose an approach to assess the PER and MSpR of models used to price exotic options in the OTC market in a Bayesian framework. To the best of our knowledge, this is the first study to study MSpR in the OTC market, and we approximate MSpR of models using realized payoffs of derivatives. The measurement also enables us to investigate the PER for long and short positions. Additionally, the time evolution of PER and MSpR of models can be obtained; which can reflect the models' ability to capture the market dynamics.

Our second contribution concerns MSeR of a model class. The MSeR has been often defined in terms of price differences within a set of models and measured using the worst-case approach (Cont, 2006; Barrieu and Scandolo, 2015; and Coqueret and Tavin, 2016).

However, Turner (2015) states that model uncertainty should also be considered when selecting models. In line with this, we take PER into consideration when assessing MSeR. As such, we propose a framework that computes time-varying MSeR using information on PER, which allows for differentiation between MSeR for long and short positions. A detailed discussion of MSeR can be found in Section 2.

Our third contribution is that we investigate the model risk in estimating fair variance strikes of variance swaps and forward starting option prices. These two types of exotic options have the advantage that closed-form pricing formulae are available for the continuous-time finance models we investigate in this paper. We find that modeling jumps, especially for variance jumps, reduces the MSpR in estimating fair variance strikes of variance swaps significantly. On the contrary, the distribution of jumps affects the performance of models in estimating forward starting option prices differently; and the log-stable jumps are preferred. Moreover, we show that the size of the MSeR is considerable.

The paper is organized as follows: Section 2 proposes the model risk measurement framework; Section 3 introduces the models and the related derivative pricing methods; the empirical study is presented in Section 4; and the last section concludes.

2 Model Risk Measures

This section introduces the model risk measurement framework for products in the OTC market. We consider model risk from two aspects: (1) individual model risk of a single pricing model, which can be decomposed into PER and MSpR, and here our methodology extends the framework of Lazar et al. (2020) using Bayesian inference; (2) the model risk associated with a class of models, named MSeR, building on the works of Cont (2006), Barrieu and Scandolo (2015) and Coqueret and Tavin (2016).

Before introducing the model risk measures, we summarise the main properties that model risk measures of products in the OTC market should satisfy:

1. *Time Variability*: model risk is time-varying. As market conditions evolve, the suitability of a model to fit prices also changes (for example, if a model provides a good fit in a stable market, it might provide a poor fit in volatile market conditions).

2. *Symmetry of MSpR*: the MSpR of a model is the same for both long and short positions. This concerns the fit of a model to prices of a financial product, so this is not dependent on the position taken in the product.

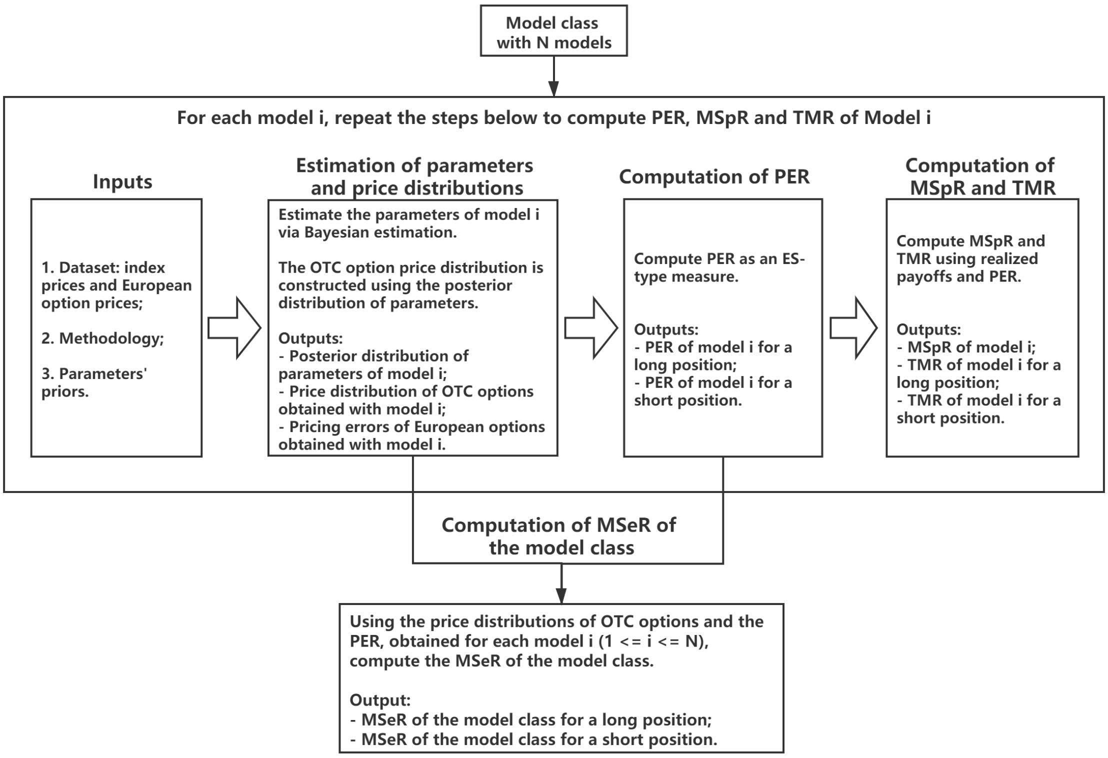

Figure 1: Computation of Model Risk in the OTC Market.

3. *Asymmetry of PER and MSeR*: the PER of a model and the MSeR of a model class can be different for long and short positions. Under normal circumstances, prices must be positive, which makes the distribution of the returns asymmetric (for example, the loss of a call option seller is unlimited but the the loss of a buyer is bounded).

Figure 1 presents the diagram of the computation of the different components of model risk. In the following sections, we discuss the computation in more detail. We introduce the measurement of PER and MSpR in Section 2.1 and MSeR in Section 2.2.

2.1 PER and MSpR of Models

The sources of model risk of models have been discussed in several studies, from an industrial (Federal Reserve Board of Governors, 2011) or academic (Green and Figlewski, 1999; and Morini, 2011) perspective; but there is no unified framework on the sources of model risk. However, most studies differentiate between PER and MSpR as different sources of model risk. As such, it is essential to measure these two components as the decomposition of risks can enhance risk management (Schilling et al., 2020). Additionally, there is no "true" model; a "super" model (which considers all possible factors) might have a lower MSpR, but it might be affected by a large PER if the parameters are difficult to be estimated accurately.

For this reason, the nested models of the "super" model would be better choices. One can only achieve the above inference by measuring PER and MSpR of the models. In our setup, the total model risk (TMR) is the sum of PER and MSpR.

The PER is measured with the posterior distribution of estimated prices,[1] which is denoted as $\widetilde{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K})$, and can be obtained through a Bayesian estimation approach (see the Supplementary Appendix for details). Here $\mathcal{H}$ represents the OTC product (such as an exotic option) conditional on model $\mathcal{M}_\vartheta$ with parameter vector Θ_ϑ at time t, given an observed dataset $\mathcal{D}$ covering the historical series of the underlying asset observations and the prices of the benchmark instruments (the vanilla option prices in this paper.[2]). For clarity, we also insist on the notational $\mathcal{K}$ for different computational methodologies (including estimation, calibration and pricing). Let $\widehat{F}$ represent an estimated price of the target option. The model price adjusted posterior distribution $\widetilde{\Lambda}$ for long (L) and short (S) positions is defined as:

$$\begin{aligned}\widetilde{\Lambda_t}(\mathcal{H}, L; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) &= \widetilde{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) - \widehat{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}), \\ \widetilde{\Lambda_t}(\mathcal{H}, S; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) &= \widehat{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) - \widetilde{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}).\end{aligned} \tag{1}$$

The estimated price of $\mathcal{H}$ can be computed as the expected value of the posterior price distribution with respect to Θ as below:

$$\widehat{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) = \mathrm{E}_\Theta \left[\widetilde{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K} \right]. \tag{2}$$

Let $VaR^{PER}_{\eta,t}(\mathcal{H}, \hbar; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K})$ denote the value-at-risk (VaR) at a critical level $\eta \in (0, 1)$, computed as the absolute value of the η quantile of the adjusted posterior distribution $\widetilde{\Lambda_t}(\mathcal{H}, \hbar; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K})$ defined in (2), where $\hbar = L$ for a long position and $\hbar = S$ for a short position. The ES-type PER model risk measure, at level η, for option $\mathcal{H}$, based on model $\mathcal{M}_\vartheta$ with parameter vector Θ_ϑ, dataset $\mathcal{D}$, methodology $\mathcal{K}$, and a long(short) position $\hbar$ is:

$$\rho^{PER}_{\eta,t}(\mathcal{H}, \hbar; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) = \frac{1}{\eta} \int_0^\eta VaR^{PER}_{x,t}(\mathcal{H}, \hbar; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) dx. \tag{3}$$

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[1] Several studies (e.g., Jacquier and Jarrow, 2000; Jacquier et al., 2002; Chung et al., 2013 and Leung et al., 2021) state that the posterior distributions of parameters retain information about PER.

[2] Our setup relies on data on the underlying as well as vanilla option prices to jointly estimate the model parameters with a Bayesian approach, similar to Yu et al. (2011), and compute VS rates as well as FS option prices. More details can be found in the Supplementary Appendix A, B and C. The vanilla options are commonly used to calibrate models when pricing exotic options, see Hull and Suo (2002), Coqueret and Tavin (2016) and Detering and Packham (2016).

The PER of option $\mathcal{H}$ for a given model $\mathcal{M}_\vartheta$ is defined as the average PER of long and short positions:

$$\rho_{\eta,t}^{PER}(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K}) = \frac{\rho_{\eta,t}^{PER}(\mathcal{H},L;\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K}) + \rho_{\eta,t}^{PER}(\mathcal{H},S;\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})}{2}. \tag{4}$$

Algorithm 1: PER and MSpR for the model $\mathcal{M}_\vartheta(\Theta_\vartheta)$ with parameter vector Θ_ϑ

Inputs: dataset $\mathcal{D}$; model $\mathcal{M}_\vartheta(\Theta_\vartheta)$; methodology $\mathcal{K}$; $n^* > 1$ (the burn-in); $N^* > n^*$ (the total iteration number); and parameters' priors (introduced in the Supplementary Appendix).

Outputs: $\widehat{F}_t(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$; $\rho_{\eta,t}^{PER}(\mathcal{H},\hbar;\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$; $\rho_{\eta,t}^{PER}(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$; $\rho_{\eta,t}^{TMR}(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$; and $\rho_{\eta,t}^{MSpR}(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$.

Initialization: $\iota = 1$;

Estimation: with Bayesian Approach (methodology $\mathcal{K}$)

while $\iota \leq N^*$ **do**

  Update parameters with the methods described in the Supplementary Appendix and obtain $\Theta_\vartheta^{(\iota)}$;

  if $\iota > n^*$ **then**

    Use updated parameter vector $\Theta_\vartheta^{(\iota)}$ and compute the model price for option $\mathcal{H}$ at time t, denoted as $F_t(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta^{(\iota)}),\mathcal{D},\mathcal{K})$.

Construct the posterior distribution of parameters (denoted as $\widetilde{\Theta_\vartheta}$) with $\left\{\Theta_\vartheta^{(\iota)}\right\}\Big|_{\iota=n^*+1}^{N^*}$.
Compute $\widehat{F}_t(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$ with expected values of $\widetilde{\Theta_\vartheta}$.
Construct $\widetilde{F}_t(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$ with $\left\{F_t(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta^{(\iota)}),\mathcal{D},\mathcal{K})\right\}\Big|_{\iota=n^*+1}^{N^*}$.
Compute $\rho_{\eta,t}^{PER}(\mathcal{H},\hbar;\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$, $\rho_{\eta,t}^{PER}(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$, $\rho_{\eta,t}^{TMR}(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$, and $\rho_{\eta,t}^{MSpR}(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$ with Equation (3), (4), (6) and (7), respectively.
return: $\widehat{F}_t(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$; $\rho_{\eta,t}^{PER}(\mathcal{H},\hbar;\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$; $\rho_{\eta,t}^{PER}(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$; $\rho_{\eta,t}^{TMR}(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$; and $\rho_{\eta,t}^{MSpR}(\mathcal{H};\mathcal{M}_\vartheta(\Theta_\vartheta),\mathcal{D},\mathcal{K})$.

The method in Lazar et al. (2020) measures the model risk of models used to price vanilla options, based on market prices. However, in the OTC market, only bid and ask prices are available; even those with limited access only. Also, these prices are often affected by demand and supply, so considering them fair is unrealistic. As such, it is difficult to obtain a realistic evaluation of MSpR for exotic options because the fair market prices of products in the OTC market are not readily available and the methodology of Lazar et al. (2020) cannot be used. Moreover, in the illiquid OTC market, option traders usually follow the "buy and hold" strategy; therefore, as opposed to option prices (which are typically not available), the realized payoffs of options at expiration are informative about the profits and losses, and thus reveal the risks related to holding such products.

Here, we propose a method to evaluate the MSpR for products in the OTC market, especially for exotic options with complicated structures based on the realized payoff. In order to approximate the MSpR at time t, we need to determine an observation period, $\mathcal{T}^*$, before time t with a fixed number of trading days $\mathcal{U}$, thus, $\mathcal{T}^* = \{t-q\}_{q=1}^{\mathcal{U}}$. Although the fair market price of the option, denoted by $\mathcal{H}$, is not accessible, the realized payoffs for exotic option $\mathcal{H}^q$ during $\mathcal{T}^*$ can be calculated, and these are denoted as $\{P_{t-q}(\mathcal{H}^q)\}_{q=1}^{\mathcal{U}}$. Here $\mathcal{H}^q$ has the same settings as $\mathcal{H}$ but expires at $t-q$. For all q with $1 \leqslant q \leqslant \mathcal{U}$ we define the distribution $\Lambda_{t,q}$ adjusted with $P_{t-q}(\mathcal{H}^q)$ for long and short positions by using the linear functions below:

$$\begin{aligned}\Lambda_{t,q}(\mathcal{H}, L; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) &= \widetilde{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) - P_{t-q}(\mathcal{H}^q),\\ \Lambda_{t,q}(\mathcal{H}, S; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) &= P_{t-q}(\mathcal{H}^q) - \widetilde{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}).\end{aligned} \tag{5}$$

The VaR at level η of the profit and loss distribution of a long(short) position, denoted by $VaR_{\eta,t,q}(\mathcal{H}, \hbar; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K})$, is computed as the absolute value of the η quantile of $\Lambda_{t,q}(\mathcal{H}, \hbar; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K})$ with $\hbar$ being L (denoting long positions) or S (short positions). The ES-type model risk measure of the TMR, $\rho^{TMR}_{\eta,t}(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K})$, for exotic option $\mathcal{H}$, given a model $\mathcal{M}_\vartheta$ with parameter vector Θ_ϑ, dataset $\mathcal{D}$, and methodology $\mathcal{K}$ is defined as the average of the ES measures of the adjusted distributions for long and short positions in (5), averaged over all values of q, given by:

$$\begin{aligned}&\rho^{TMR}_{\eta,t}(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) =\\ &\frac{\sum_{q=1}^{\mathcal{U}} \left[\int_0^\eta VaR_{x,t,q}(\mathcal{H}, L; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K})dx + \int_0^\eta VaR_{x,t,q}(\mathcal{H}, S; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K})dx\right]}{2\eta\mathcal{U}}.\end{aligned} \tag{6}$$

Then, the MSpR of pricing the exotic option $\mathcal{H}$, using model $\mathcal{M}_\vartheta$ with parameter vector Θ_ϑ, dataset $\mathcal{D}$, and methodology $\mathcal{K}$ is measured as the difference between TMR and PER:

$$\rho^{MSpR}_{\eta,t}(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) = \rho^{TMR}_{\eta,t}(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) - \rho^{PER}_{\eta,t}(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}). \tag{7}$$

As such, the MSpR depends on the estimates of PER. For example, if a model has low PER but high TMR, then the model must be poorly specified with large MSpR. Whilst a model's PER only depends on the model's estimated price distribution, the MSpR (and TMR) also relies on the realized payoffs of the instruments in the OTC market. Due to the symmetry of MSpR and asymmetry of PER, the TMR for $\hbar = L(S)$ is the sum of PER for $\hbar = L(S)$ and MSpR:

$$\rho^{TMR}_{\eta,t}(\mathcal{H}, \hbar; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) = \rho^{PER}_{\eta,t}(\mathcal{H}, \hbar; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) + \rho^{MSpR}_{\eta,t}(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}). \tag{8}$$

Algorithm 1 provides the detailed explanation of computing the PER and MSpR of models. First, PER is computed based on (4). Next, TMR is obtained based on (6). Then, MSpR is calculated using (7). In our setup, the estimates for MSpR depend on PER.

2.2 MSeR of Model Classes

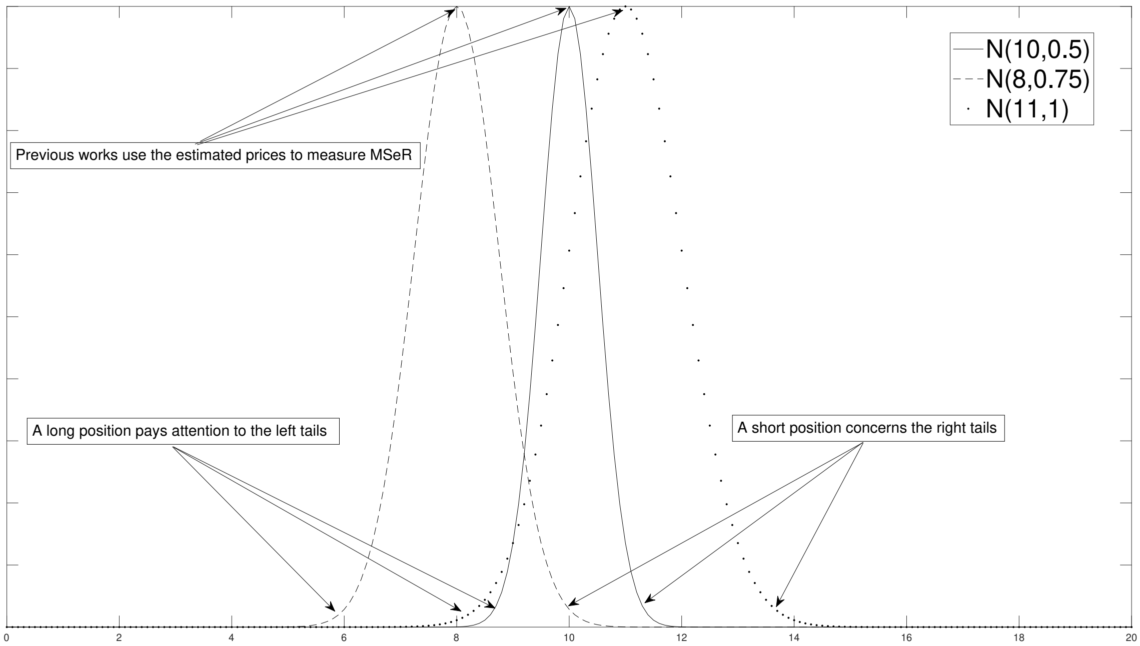

Figure 2: Example of Model Selection Risk.

This figure provides an example of MSeR. We assume that there are three models used to estimate the price of one product; and these three models obtain three estimated posterior distributions following $\mathbb{N}(10, 0.5)$, $\mathbb{N}(8, 0.75)$ and $\mathbb{N}(11, 1)$, respectively.

Several studies consider the price range of model prices as a measure of the MSeR for exotic options. However, these are based on point-wise estimation and provide the same MSeR for long and short positions. Here we propose a method to measure MSeR of long and short positions, by considering an adjustment to the range, based on the PER of models. As such, if the PER is different for long and short positions (due to the asymmetry of the price distribution), then the MSeR could also be different for long and short positions. Another reason for the asymmetry of MSeR is that the adjustment for PER is positive for short positions (market makers quote ask prices based on the right tail of the estimated price distribution when selling) and negative for long positions (market makers quote bid prices based on the left tail of the estimated price distribution when buying).

MSeR is measured using the worst-case approach in Cont (2006), Barrieu and Scandolo (2015) and Coqueret and Tavin (2016). Given a universe of models $\Upsilon = \{\mathcal{M}_1(\Theta_1), \mathcal{M}_2(\Theta_2), \ldots, \mathcal{M}_m(\Theta_m)\}$ of size m, the MSeR measure in the above studies are based on the estimated model prices $\widehat{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K})$. For example, Cont (2006) and Coqueret and Tavin (2016) measure the MSeR as:

$$\max_{\mathcal{M}_\vartheta(\Theta_\vartheta)\in\Upsilon}\left[\widehat{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K})\right] - \min_{\mathcal{M}_\vartheta(\Theta_\vartheta)\in\Upsilon}\left[\widehat{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K})\right]. \tag{9}$$

However, the above studies do not measure MSeR for long and short positions separately. The differentiation between the MSeR for long and short positions is especially important in the OTC market because of the wider role of the market makers. Prices of the OTC products are not determined by the market, but provided by the market makers. Market makers are allowed to take either long (as a buyer) or short (as a seller) positions in the securities. They will not use the point-wise estimated prices directly, because that can put them at a disadvantage. As Copeland and Galai (1983) point out market makers optimize their positions by setting favorable bid and ask prices for themselves; and the bid-ask spread in the OTC market is generally high as discussed by Deuskar et al. (2011). In order to protect themselves, buyers (sellers) provide quotes based on the lower (upper) tail of the estimated price distribution. In the measurement we propose, given a trust to a set of models, participants also need to make favorable prices, which can help them to mitigate the PER.[3] Thus, sellers pay close attention to the right tail of the estimated posterior distribution. At the same time, buyers are more concerned by the left tail of estimated prices, as demonstrated in Figure 2.

The MSeR for market makers, of option $\mathcal{H}$, given a model class Υ, dataset $\mathcal{D}$, methodology $\mathcal{K}$, and a long(short) position $\hbar$ can be calculated as:

$$\begin{aligned}\rho_{\eta,t}^{MSeR}(\mathcal{H},\hbar;\Upsilon,\mathcal{D},\mathcal{K}) = &\max_{\mathcal{M}_\vartheta(\Theta_\vartheta)\in\Upsilon}\left[\widehat{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) + \mathcal{I}\times\rho_{\eta,t}^{PER}(\mathcal{H},\hbar;\mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K})\right] \\ &- \min_{\mathcal{M}_\vartheta(\Theta_\vartheta)\in\Upsilon}\left[\widehat{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K}) + \mathcal{I}\times\rho_{\eta,t}^{PER}(\mathcal{H},\hbar;\mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K})\right], \\ &\text{where } \mathcal{I}=1 \text{ if } \hbar=S,\ \mathcal{I}=-1 \text{ if } \hbar=L.\end{aligned} \tag{10}$$

We are still in a worst-case framework, but for a long (short) position, the MSeR is computed based on the left (right) tail. This a more realistic approach, because it takes the asymmetry of the estimate price distribution into account. Algorithm 2 gives the procedure to compute MSeR for a given pricing model class.

[3]If buyers and sellers make prices according to the point estimated model prices $\widehat{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K})$, using the same model and the same estimation methods, then there should be no ask-bid spread unless they use different datasets to estimate parameters.

Algorithm 2: MSeR for the model class Υ

Inputs: dataset $\mathcal{D}$; model class Υ of size m; methodology $\mathcal{K}$; $n^* > 1$ (the burn-in); $N^* > n^*$ (the total iteration number); and parameters' priors (introduced in the Supplementary Appendix).

Outputs: $\rho_{\eta,\,t}^{MSeR}(\mathcal{H}, \hbar; \Upsilon, \mathcal{D}, \mathcal{K})$.

for $\vartheta = 1$ *to* m **do**

  Obtion $\mathcal{M}_\vartheta(\Theta_\vartheta) \in \Upsilon$;

  Algorithm 1 ($\mathcal{D}$, $\mathcal{M}_\vartheta(\Theta_\vartheta)$, $\mathcal{K}$, n^*, N^*, priors).

Compute $\rho_{\eta,\,t}^{MSeR}(\mathcal{H}, \hbar; \Upsilon, \mathcal{D}, \mathcal{K})$ with Equation (10).

return: $\rho_{\eta,\,t}^{MSeR}(\mathcal{H}, \hbar; \Upsilon, \mathcal{D}, \mathcal{K})$.

3 Models and Derivatives Pricing

This section describes the set of models used in this paper under both probability measures, the physical measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$ with the detailed estimation methods of the models considered described in the Supplementary Appendix. The change of measure is also discussed in this section. Moreover, the closed-form pricing formulae for variance swaps and forward starting options are presented in 3.3 and 3.4, respectively.

3.1 Stochastic Volatility Models

The models we compare are the stochastic volatility (SV) model, the stochastic volatility model with Poisson price jumps (SVJ), the stochastic volatility model with contemporaneous jumps in price and volatility (SVCJ), the stochastic volatility model with variance-gamma jumps (SVVG) and the stochastic volatility model with log-stable jumps (SVLS).

Taking $Y_t = \ln(S_t)$ as the logarithm of the asset price, (11) gives a general expression of the dynamics of Y_t under $\mathbb{P}$:

$$\begin{aligned} dY_t &= \mu dt + \sqrt{V_t} dW_t^Y(\mathbb{P}) + dJ_t^Y(\mathbb{P}), \\ dV_t &= \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_t^V(\mathbb{P}) + dJ_t^V(\mathbb{P}), \end{aligned} \tag{11}$$

where $W_t^Y(\mathbb{P})$ and $W_t^V(\mathbb{P})$ are correlated standard Brownian motions with $dW_t^Y(\mathbb{P})dW_t^V(\mathbb{P}) = \rho dt$; μ is the mean return and V_t denotes the instantaneous variance at t. The variance is a mean-reversion process, where κ represents the reversion speed; θ is the long-run mean of the variance and σ_V denotes the volatility of volatility. $J_t^Y(\mathbb{P})$ and $J_t^V(\mathbb{P})$ represent jump components in price and variance processes, respectively; the jump components are independent of the diffusion and variance processes.

SVJ and SVCJ are AJD models (Duffie et al., 2000) with Poisson processes to describe large discontinuous jumps. For jump sizes, $\xi^V \sim \mathbb{EXP}(\mu_V)$ and $\xi^Y|\xi^V \sim \mathbb{N}(\mu_J, \sigma_J^2)$.[4] For SVJ, $J_t^V(\mathbb{P}) = 0$ and the process of the jump $J_t^Y(\mathbb{P})$ has the same specification as SVCJ. For SVCJ, $dJ_t^Y(\mathbb{P}) = \xi^Y dN_t^Y$ and $dJ_t^V(\mathbb{P}) = \xi^V dN_t^V$, where $\{N_t^Y\}_{t\geq 0}$ and $\{N_t^V\}_{t\geq 0}$ are Poisson processes. Here, we assume that $N_t^Y = N_t^V = N_t$ with a constant intensity λ.[5] The SV model is obtained by setting $\lambda = 0$.

AJD models only allow finite-activity jump processes, while the Lévy processes can achieve infinite jump arrival rates. We take the SVVG model of Madan et al. (1998) and the SVLS model of Carr and Wu (2003) as proxies of Lévy jump models.

The variance-gamma process is given by:

$$X_t^{VG}(\sigma, \gamma, \nu) = \gamma G_t^\nu + \sigma W_{G_t^\nu}, \tag{12}$$

where $\{X^{VG}\}$ is an arithmetic Brownian motion with drift γ and volatility σ; $\{G_t^\nu\}_{t\geq 0}$ is a gamma process with unit mean rate and variance rate ν; $\{W_t\}_{t\geq 0}$ is a standard Brownian motion and independent of G_t^ν. Setting $J_t^Y(\mathbb{P}) = X_t^{VG}(\sigma, \gamma, \nu)$ and $J_t^V(\mathbb{P}) = 0$ reduces (11) to SVVG.

The log-stable process follows an α-stable distribution (S_α):

$$X_t^{LS}(\alpha, \sigma) - X_s^{LS}(\alpha, \sigma) \sim S_\alpha(\beta, \sigma(t-s)^{\frac{1}{\alpha}}, \gamma),\ t > s, \tag{13}$$

where $\alpha \in (0, 2]$ is the stability parameter; $\beta \in [-1, 1]$ is the skewness parameter; $\sigma \geq 0$ is the scale parameter; and $\gamma \in \mathbb{R}$ is the location parameter. Following Carr and Wu (2003), we set $\beta = -1$ and $\gamma = 0$. Setting $J_t^Y(\mathbb{P}) = X_t^{LS}(\alpha, \sigma)$ and $J_t^V(\mathbb{P}) = 0$ reduces (11) to SVLS.

A more detailed description of these models can be found in Li et al. (2008). In addition to the stochastic volatility models, we also use the MJD model, which is a constant volatility model with Poisson jumps, as a comparison. Equation (11) reduces to MJD by setting $V_{t+1} = V_t = \sigma_{MJD}^2$.[6]

---

[4] $\mathbb{EXP}$ denotes the exponential distribution. Moreover, we assume the volatility and asset return jump sizes are uncorrelated for brevity.

[5] This is a standard setting that has been applied in many studies (e.g., Duffie et al., 2000; Pan, 2002; Eraker, 2004; Yu et al., 2011; and Pun et al., 2015), Bates (2000) finds that state-dependent intensities models are significantly misspecified whilst Andersen et al. (2002) find no evidence to support time-varying intensity. Besides, we assume that only one jump could occur per trading day as we are working with daily data, consistent with aforementioned studies. AJD models allow finite-activity jumps, and they can detect only several jumps per year (usually large jumps); by contrast, Lévy jump models have more flexible jump structures with infinite jump arrival rates and the ability to capture both large and small jumps.

[6] It is worth mentioning that the rough volatility model has recently been found to have a better fit to the

3.2 Change of Measure

Section 3.1 introduces the model specifications under the physical probability measure $\mathbb{P}$. However, option pricing requires the underlying dynamics under the risk-neutral measure $\mathbb{Q}$. This section discusses the change of measure between $\mathbb{P}$ and $\mathbb{Q}$ as well as the closed-form solution to the characteristic function of the log stock price.

The corresponding Brownian motions under $\mathbb{Q}$ are written as $dW_t^Y(\mathbb{Q}) = dW_t^Y(\mathbb{P}) + \gamma_t^Y dt$ and $dW_t^V(\mathbb{Q}) = dW_t^V(\mathbb{P}) + \gamma_t^V dt$, respectively, where γ_t^Y and γ_t^V are defined as in Pan (2002): $\gamma_t^Y = \eta_s\sqrt{V_t}$ and $\gamma_t^V = -\frac{1}{\sqrt{1-\rho^2}}\left(\rho\eta_s + \frac{\eta_v}{\sigma_V}\right)\sqrt{V_t}$.

Following Bates (2000), Pan (2002) and Broadie et al. (2007), we restrict $\kappa^{\mathbb{P}}\theta^{\mathbb{P}} = \kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}$; $\rho^{\mathbb{P}} = \rho^{\mathbb{Q}}$; and $\sigma_V^{\mathbb{P}} = \sigma_V^{\mathbb{Q}}$.[7] Also, as explained in Pan (2002), Eraker (2004) and Yu et al. (2011), some of the jump parameters are common to both measures, and the sets of parameters in SVJ, SVCJ, SVVG and SVLS are given by $(\lambda, \mu_J^{\mathbb{P}}, \sigma_J, \mu_J^{\mathbb{Q}})$, $(\lambda, \mu_J^{\mathbb{P}}, \sigma_J, \mu_J^{\mathbb{Q}}, \mu_V, \rho_J)$, $(\nu, \gamma^{\mathbb{P}}, \sigma^{\mathbb{P}}, \gamma^{\mathbb{Q}}, \sigma^{\mathbb{Q}})$ and (α, σ), respectively.[8]

Given the settings above, the Radon-Nikodym derivatives are:[9]

$$\begin{aligned}\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_t = \exp\Bigg\{&-\int_0^t \gamma_s^Y dW_s^Y(\mathbb{P}) - \int_0^t \gamma_s^V dW_s^V(\mathbb{P}) \\ &-\frac{1}{2}\left[\int_0^t (\gamma_s^Y)^2 ds + \int_0^t (\gamma_s^V)^2 ds\right]\Bigg\}\exp(U_t),\end{aligned} \tag{14}$$

where the expression of U_t can be found in Sato et al. (1999). The dynamics of Y_t and V_t

market (Gatheral et al., 2018). Moreover, the semiclosed formula for the characteristic function of the rough Heston models has also been studied in El Euch and Rosenbaum (2019), which can speed up the process of pricing European options. There is also a related study that estimates the parameters jointly with the S&P 500 index options and VIX options (Gatheral et al., 2020) with rough volatility models under the risk-neutral probability measure. Nevertheless, we have to leave the model risk of rough volatility models for further research for two reasons: firstly, such an analysis would require us to study the joint estimation of parameters under both physical and risk-neutral probability measures with a Bayesian approach; secondly, the closed-form formulae for pricing exotic options need to be derived; otherwise, pricing with Monte Carlo methods can be quite intricate because of the non-Markovian nature of rough volatility models.

[7] $\kappa^{\mathbb{Q}} = \kappa^{\mathbb{P}} - \eta_v$ and $\theta^{\mathbb{Q}} = \frac{\kappa^{\mathbb{P}}\theta^{\mathbb{P}}}{\kappa^{\mathbb{Q}}}$, we use κ and θ to represent $\kappa^{\mathbb{P}}$ and $\theta^{\mathbb{P}}$ in this paper. For simplicity, we use ρ to denote $\rho^{\mathbb{P}}$ and $\rho^{\mathbb{Q}}$, and use σ_V to represent $\sigma_V^{\mathbb{P}}$ and $\sigma_V^{\mathbb{Q}}$.

[8] Parameters with no superscripts are parameters common to both measures while parameters with the superscript $\mathbb{Q}$ denote that the parameters are unique to $\mathbb{Q}$.

[9] Both e and exp denote the exponential function.

under $\mathbb{Q}$ are expressed as:

$$
\begin{aligned}
dY_t &= \left(r_t - \frac{1}{2}V_t + \Phi_J(-i)\right) dt + \sqrt{V_t} dW_t^Y(\mathbb{Q}) + dJ_t^Y(\mathbb{Q}), \\
dV_t &= [\kappa(\theta - V_t) + \eta_v V_t]\, dt + \sigma_V \sqrt{V_t} dW_t^V(\mathbb{Q}) + dJ_t^V(\mathbb{Q}),
\end{aligned}
\tag{15}
$$

where r_t represents the risk-free rate and $\Phi_J(\centerdot)$ is the jump component.[10] The drift term of Y_t under $\mathbb{P}$ can be derived as $\mu = r_t - \frac{1}{2}V_t + \Phi_J(-i) + \eta_s V_t$ (Yu et al., 2011).

When the interest rate is constant, the risk-neutral dynamics in (15) lead to the closed-form solution to the characteristic function of the log stock price under $\mathbb{Q}$ given by:

$$
\phi(t,u) = \exp\left[iuY_0 + iu(r + \Phi_J(-i))t\right] \exp\left(-t\Phi_J(u)\right) \exp\left(-b(t,u)V_0 - c(t,u)\right), \tag{16}
$$

where $\kappa^M(u) = \kappa - \eta_v - iu\sigma_V\rho$; $\delta(u) = \sqrt{(\kappa^M(u))^2 + (iu + u^2)\sigma_V^2}$; $Y_0 = \ln(S_0)$ denotes the log-spot price; $c(t,u) = \frac{\kappa\theta}{\sigma_V^2}\left[2\ln\frac{2\delta(u) - (\delta(u) - \kappa^M(u))(1 - e^{-\delta(u)t})}{2\delta(u)} + (\delta(u) - \kappa^M(u))t\right]$; $b(t,u) = \frac{(iu+u^2)(1-e^{-\delta(u)t})}{(\delta(u)+\kappa^M(u)) + (\delta(u) - \kappa^M(u))e^{(-\delta(u)t)}}$ and V_0 represents the initial variance. The fast Fourier transform methods can be applied for vanilla option pricing using the characteristic function (Carr and Madan, 1999).

3.3 Variance Swap

This section provides closed-form solutions to computing VS rates (fair variance strikes) using candidate models in this paper. The VS, usually traded in the OTC market, is a forward contract written on the realized variance of an asset's price. In a VS contract, one counterparty will pay the other the difference between a fixed value and the realized variance $RV_{[0,T]}$ during a reference period, multiplied by the contract's notional amount N.[11] The fixed amount is the variance strike K_{var}, and it is typically set as the fair variance strike at the initiation, which makes the net present value of the VS zero, that is $N \times \left(RV_{[0,T]} - K_{var}\right) = 0$. As such, pricing VS's reduces to the calculation of the fair variance strikes.

Given the number of observations during the time interval $[0, T]$, $\varkappa$, and the annualization factor, AF, the realized variance can be calculated as:

$$
RV_{[0,T]} = \frac{AF}{\varkappa - 1} \sum_{i=0}^{\varkappa-1} \left(\ln\left(\frac{S_{i+1}}{S_i}\right)\right)^2. \tag{17}
$$

[10] For SVCJ, $\Phi_J(u) = \lambda\left(1 - \frac{\exp\left(iu\mu_J^{\mathbb{Q}} - \frac{1}{2}\sigma_J^2 u^2\right)}{1 - iu\mu_V\rho_J - iu\mu_V}\right)$. One can further derive the expression of $\Phi_J(u)$ for SVJ by setting $\mu_V = 0$. For SVVG, $\Phi_J(u) = \frac{\ln\left(1 - iu\gamma^{\mathbb{Q}}\nu + \frac{(\sigma^{\mathbb{Q}})^2\nu u^2}{2}\right)}{\nu}$. For SVLS, $\Phi_J(u) = (\sigma|u|)^\alpha\left(1 + i \times \mathrm{sign}(u) \times \tan\left(\frac{\pi\alpha}{2}\right)\right)$, where $\mathrm{sign}(u)$ is the sign of u.

[11] $N = 1,000$ in this paper.

Here we assume daily sampling and $AF = 252$. Also, we consider VS's to be continuously monitored in this paper.[12] In the case of the stochastic volatility model, the continuously realized variance during $[0,\, T]$ is $\frac{1}{T}\int_0^T \sigma_t^2 dt$. At inception, the fair variance strike is:

$$K^*_{var} = \mathrm{E}_0^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T \sigma_t^2 dt\right]. \tag{18}$$

The closed-form expressions of the fair variance strike for stochastic volatility models have been studied in many works. Broadie and Jain (2008) investigate the fair continuous variance strike under the MJD model, which is computed as $K^*_{var} = \sigma^2_{MJD} + \lambda\left(\left(\mu_J^{\mathbb{Q}}\right)^2 + \sigma_J^2\right)$. Broadie and Jain (2008) and Sepp (2008) further study the fair variance strike under SV and SVJ: under the SV model, the fair variance strike is given by $K^*_{var} = \theta^{\mathbb{Q}} + \frac{V_0-\theta^{\mathbb{Q}}}{\kappa^{\mathbb{Q}}T}\left(1-e^{-\kappa^{\mathbb{Q}}T}\right)$; while $K^*_{var} = \theta^{\mathbb{Q}} + \frac{V_0-\theta^{\mathbb{Q}}}{\kappa^{\mathbb{Q}}T}\left(1-e^{-\kappa^{\mathbb{Q}}T}\right) + \lambda\left(\left(\mu_J^{\mathbb{Q}}\right)^2 + \sigma_J^2\right)$ under SVJ. Sepp (2008) proposes the fair variance strike for the SVCJ model, with $K^*_{var} = \theta^{\mathbb{Q}} + \frac{(V_0-\theta^{\mathbb{Q}})\kappa^{\mathbb{Q}}-\lambda\mu_V}{(\kappa^{\mathbb{Q}})^2 T}\left(1-e^{-\kappa^{\mathbb{Q}}T}\right) + \lambda\left(\left(\mu_J^{\mathbb{Q}}\right)^2 + \sigma_J^2\right) + \frac{\lambda\mu_V}{\kappa^{\mathbb{Q}}}$. Ruan et al. (2016) derive the fair strike of the continuously sampled VS in the SVVG model based on the moment-generating approach, given by $K^*_{var} = \theta^{\mathbb{Q}} + \frac{V_0-\theta^{\mathbb{Q}}}{\kappa^{\mathbb{Q}}T}\left(1-e^{-\kappa^{\mathbb{Q}}T}\right) + \nu\left(\left(\sigma^{\mathbb{Q}}\gamma^{\mathbb{Q}}\right)^2 + \frac{1}{\nu}\right)$. The variance of the log-stable distribution is undefined for $\alpha < 2$; thus, the SVLS model is excluded from our analysis in estimating the VS rates.

One purpose of the paper is to study the model risk in estimating the fair variance strikes of VS's. We also further study the effect of this model risk on estimating the variance risk premium (VRP). VRP is defined as the difference between the fair variance strike and the realized variance (Carr and Wu, 2009). Let $VRP_{[t,\,t+T_0]}$ and $K^*_{var,\,[t,\,t+T_0]}$ denote the VRP and the fair variance strike, respectively, estimated at time t for the period $[t,\, t+T_0]$. The VRP is computed as:

$$VRP_{[t,\,t+T_0]} \equiv K^*_{var,\,[t,\,t+T_0]} - RV_{[t,\,t+T_0]}, \tag{19}$$

where $RV_{[t,\,t+T_0]}$, given in Equation (17), is the realized variance at time t during $[t,\, t+T_0]$.

3.4 Forward Starting Options

An FS option is an exotic option, with strike price set at the determination time of the strike, T_0, which is before the maturity of the option, T. The period $T - T_0$ is the tenor

[12] We apply the same assumption as Coqueret and Tavin (2016); that is the differences between continuously monitored VS's and actual contracts are small. Broadie and Jain (2008) show that the fair continuous variance strike converges to the discrete variance strike linearly with the number of sampling observations $\varkappa$. See Badescu et al. (2019) and Pun et al. (2015) for a more detailed discussion of continuously and discretely monitored VS's.

of the FS option. During the period $\Delta T = [0, T_0]$, the FS option is a path-dependent derivative; and it can be considered as a vanilla option on the percentage returns during $[T_0, T]$, with S_{T_0} revealed at T_0. The payoff of an FS call option is:

$$N \times \left(\frac{S_T}{S_{T_0}} - k\right)^+, \tag{20}$$

where N represents the notional amount of the contract and k is the percentage strike.

We consider the general solution to the forward characteristic function (FCF) for Heston type stochastic volatility (Cox–Ingersoll–Ross process) models with jumps in the return process. When the interest rate is constant, the FCF of $\ln \frac{S_T}{S_{T_0}}$ under the SV model, ϕ_{FSV}, is given by:

$$\begin{aligned}\phi_{FSV}(\Delta T, T_0, u) = &\exp\left[iur\Delta T - c(\Delta T, u)\right] \times \left[1 + b(\Delta T, u)B(T_0)^{-\frac{2\kappa\theta}{\sigma_V^2}}\right] \\ &\times \exp\left[\frac{-b(\Delta T, u)e^{-(\kappa-\eta_v)T_0}}{1 + b(\Delta T, u)B(T_0)} V_0\right],\end{aligned} \tag{21}$$

where $B(T_0) = \frac{\sigma_V^2}{2\kappa}\left(1 - e^{-(\kappa-\eta_v)T_0}\right)$ and the other components are introduced in (16); see Kruse and Nögel (2005) for detailed derivations. When the jump component is independent of the diffusion process, the FCF of stochastic volatility models with jumps in the return process is (Zhang and Geng, 2017):

$$\phi_F(\Delta T, T_0, u) = \phi_{FSV}(\Delta T, T_0, u) \times \exp\left[\Delta T iu\Phi_J(-i) - \Delta T\Phi_J(u)\right]. \tag{22}$$

Given the FCF, the forward starting options can be priced using fast Fourier transform methods, see Carr and Madan (1999). Pricing FS options with SVCJ can be more complicated due to the jumps in the variance process, and we exclude the SVCJ model from our analysis of the model risk of pricing FS options.

4 Empirical Analysis

We explore the main aspects of model risk for VS's and FS options. For VS's, we study the model risk in computing fair variance strikes associated with various pricing model specifications, and find that the model risk shows large variations depending on market conditions. Given the estimated fair variance strikes, one important extension we consider is the investigation of how model risk affects the significance of VRP. We then turn our focus to the model risk in pricing FS options.

4.1 Data

We consider the S&P 500 index as the underlying asset, which is one of the most prevalent underlying indices for derivatives transactions. The data spans from January 3, 1996 to June 28, 2019, and consists of daily S&P 500 index spot prices and the corresponding S&P 500 index option prices. We obtain both spot and option prices from WRDS Option Metrics. For VS, our focus is on one-month VS's, and we use ATM-forward call options with 30 days expiry as the options data to estimate fair variance strikes. For FS options, we investigate six-month ATM FS call options with a tenor of three months. We use the ATM-forward call options with 91 and 182 days expiry to estimate FS option prices.[13]

The ATM-forward option dataset is obtained from the Std_Option_Price file in Option Metrics. This file contains information on ATM-forward options with expiration ranges from 30 to 730 calendar days. It includes the forward price of underlying on the expiration date of the option, which is calculated with the zero-coupon yield curve and projected dividends; the strike price of the option is set the same as the forward price. The implied volatility (IV) and premium on these standardized options are calculated daily from the volatility surface using linear interpolation, specifically a kernel smoothing technique. Compared with picking options in the real market, standardized options have constant duration, and they are all ATM, which reduces the measurement error caused by the difference in maturities and moneyness in the actual market data. The daily one-month, three-month, and six-month risk-free rates are downloaded from the U.S. Department of the Treasury.[14]

4.2 Empirical Results for Variance Swaps

This section discusses the model risk of VS's. Using S&P 500 index spot prices and one-month ATM-forward call options, we estimate the one-month VS rates. And then the model risk of each model is computed based on Section 2 with $\eta = 5\%$. Because the swap price is zero at initiation, what we actually compare is the difference between the estimated fair variance strike and the realized variance. In this case, in Formula (5), $\widetilde{F}_t(\mathcal{H}; \mathcal{M}_\vartheta(\Theta_\vartheta), \mathcal{D}, \mathcal{K})$ refers to the estimated posterior distribution of the fair variance strike at time t, and $P_{t-q}(\mathcal{H}^q)$ is the realized variance at time $t-q$. Moreover, the observation period, $\mathcal{T}^*$, is set

[13]Our option dataset only contains ATM-forward option prices. This is consistent with Yu et al. (2011), and the ATM options are usually the most liquid.

[14]The three-month yield curve is used in line with the one-month risk-free rates before July 31, 2001, as one-month Treasury yield curve rates are not available before this date.

as one month with a fixed number of 21 trading days. Considering the one-month duration of VS's and the one-month model risk observation period, we actually measure the model risk of VS's from March 05, 1996 to June 28, 2019.

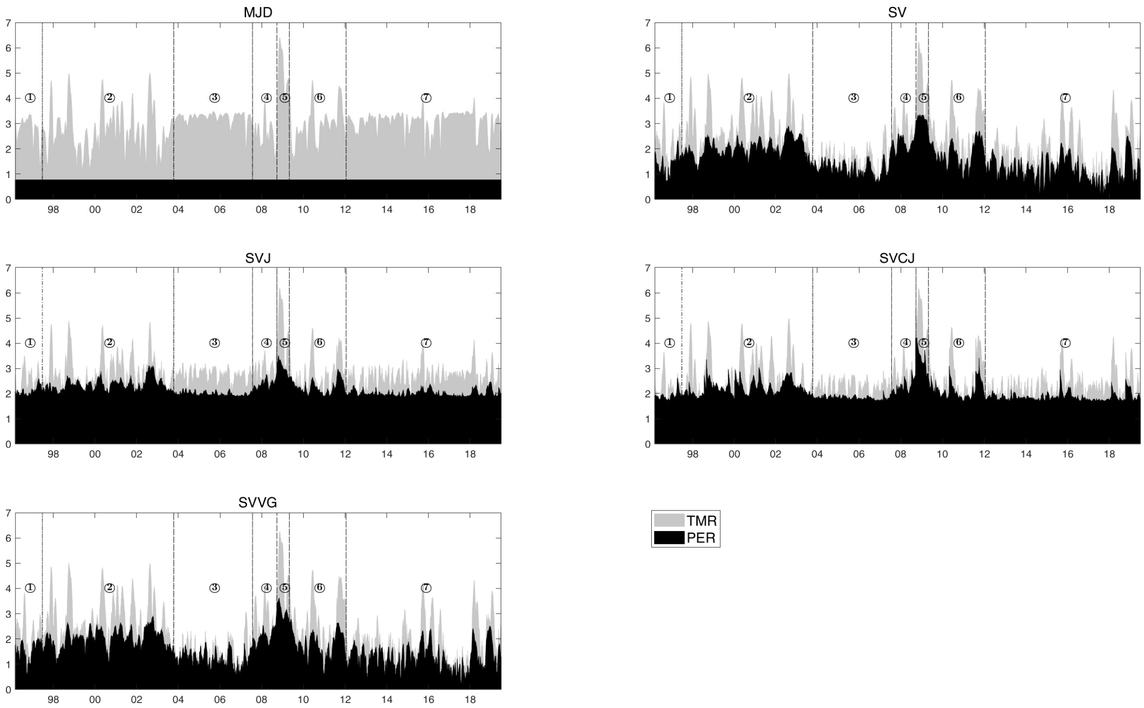

Figure 3: Model Risk in Estimating Fair Variance Strikes for Variance Swaps.
This figure presents the log values of TMR (grey columns) and PER (black columns) of models used to estimate the one-month fair variance strikes of VS's. The sizes of the grey columns measure the MSpR. The results are based on daily spot prices on the S&P 500 and daily prices of standardized at-the-money-forward call options with 30 days to expiration between January 3, 1996 and June 28, 2019. Estimated fair variance strikes of VS's are updated within the estimation process at the end of each iteration after burn-in. The total iterations are 30,000 and the first 10,000 runs are discarded as burn-in. The details of the estimation methods can be found in the Supplementary Appendix.

Structural breaks indicate time points where there is a change in the characteristics of the observations, and at these points the fit of the models can change. As such, we use the method of Killick et al. (2012) and find that the sample can be split into seven time windows by detecting abrupt changes in the average values of IV of the one-month standardized ATM call options. The identified periods are as follows: period ①: March 05, 1996 to June 24, 1997; period ②: June 25, 1997 to October 13, 2003; period ③: October 14, 2003 to July 24, 2007; period ④: July 25, 2007 to September 23, 2008; period ⑤: September 24, 2008 to April 29, 2009; ⑥: April 30, 2009 to January 19, 2012; ⑦: January 20, 2012 to June 28, 2019. The average values of the implied variance, which is computed as the square of the IV for the standardized ATM options (1,000's), during different time windows are reported in Panel A of Table 1. The mean implied variance for the whole sample period is about 38. Periods ①, ③ and ⑦ are tranquil periods, with mean implied variance values of 26, 15 and 18, respectively. The market is turbulent during periods ②, ④ and ⑥, when the mean implied variance values are all above 48.[15] Period ⑤ marks the global financial crisis of 2008, when the mean IV shoots up to 212.

Figure 3 plots the model risk (on a log scale) of estimating one-month fair variance strikes. It is straightforward to see that the TMR peaks during period ⑤, the global financial crisis, for all models. In volatile periods, ②, ④ and ⑥, all models are affected by high TMR. By contrast, during periods ①, ③ and ⑦, when the market is stable, the model risk of models, especially SV and SVVG, is low.

Table 1: Model Risk in Estimating Fair Variance Strikes for VS's

Periods	①	②	③	④	⑤	⑥	⑦	Whole
Panel A. Implied Variance								
	26.83	53.78	15.89	48.83	211.91	49.16	18.13	38.02
Panel B. MJD								
PER	2.16	2.16	2.16	2.16	2.16	2.16	2.16	2.16
MSpR	15.73	21.74	23.95	13.35	218.56	23.57	21.50	26.53
TMR	17.89	23.90	26.12	15.51	220.72	25.73	23.66	28.69
L-S	-0.21	-0.21	-0.21	-0.21	-0.21	-0.21	-0.21	-0.21
Panel C. SV								
PER	4.30	8.08	3.53	9.51	24.56	6.95	3.72	6.10
MSpR	6.22	16.49	1.35	9.71	148.59	14.99	5.64	12.87
TMR	10.52	24.57	4.88	19.22	173.14	21.94	9.36	18.97
L-S	-1.28	-2.12	-1.24	-2.66	-2.81	-1.70	-1.24	-1.64
Panel D. SVJ								
PER	8.29	10.84	7.37	11.03	21.52	9.98	7.59	9.28
MSpR	5.15	12.43	7.99	9.18	142.50	10.93	7.86	12.84
TMR	13.44	23.26	15.36	20.21	164.02	20.92	15.45	22.12
L-S	-1.69	-2.15	-1.51	-1.93	-3.23	-1.50	-1.57	-1.78
Panel E. SVCJ								
PER	6.95	9.74	6.28	11.20	32.75	9.04	6.56	8.58
MSpR	3.51	13.56	3.37	9.92	140.96	11.66	5.07	11.49
TMR	10.46	23.30	9.65	21.12	173.71	20.70	11.63	20.07
L-S	-2.09	-2.71	-2.04	-2.33	-3.91	-2.29	-2.00	-2.30
Panel F. SVVG								
PER	5.00	8.13	3.55	8.70	23.06	6.89	4.01	6.16
MSpR	5.77	16.65	1.29	10.33	151.90	15.61	5.41	13.00
TMR	10.77	24.78	4.84	19.03	174.95	22.50	9.42	19.16
L-S	-1.78	-2.32	-1.26	-1.80	-2.59	-1.67	-1.44	-1.74

NOTE: Panels B to F report the mean values of model risk in estimating one-month fair variance strikes for VS's, for all models considered. L-S denotes the difference between the TMR for long and short positions. The whole sample period is from March 05, 1996 to June 28, 2019. For comparison, the mean values of implied variance in different periods are reported in Panel A. The implied variance is calculated as the square of the IV for the standardized ATM options multiplied by 1,000.

Panels B to F of Table 1 report the mean values of model risk in estimating one-month VS rates. The PER of MJD is constant, as this model assumes constant volatility. For

[15]Period ② covers the Asian-Russia long-term capital management crisis between 1997 and 1998, the currency crisis in Brazil from 1998 to 1999, the dot-com crash between 2000 and 2002, the Argentine depression from 1998 to 2002, the 911 (September 11, 2001) and the WorldCom accounting scandal in 2002; period ④ involves the subprime crisis; whilst the European credit crisis happened at the end of 2009 during period ⑥.

the whole period, the MJD model has the smallest PER, 2.16.[16] The PER of stochastic volatility models is much higher than that of MJD. Typically models with more parameters have higher PER, but it is not always the case. The MSpR is higher than PER for all models in estimating fair variance strikes of VS's. Not surprisingly, MJD is most affected by MSpR stemming from its poor specification; by contrast, the values of MSpR of stochastic volatility models are much lower than the MSpR of MJD. The MSpR of SV is 12.87. Adding Poisson jumps in the return process of SV (which gives the SVJ model) slightly reduces the MSpR, while adding variance-gamma jumps in the return process of SV (which is SVVG) increases it. Further adding Poisson jumps in the variance process of SVJ (SVCJ) yields a much lower MSpR, 11.49, in estimating VS rates. Kaeck et al. (2017) also find that jump diffusion models yield more reliable estimates of the VS rates.

The magnitudes of TMR for all models are lower than the implied variance except for the MJD model in period ⑤. The MJD model has the largest TMR. Although the SVJ and SVCJ have lower MSpR compared with SV and SVVG, their TMR values are still large due to their large PER. It is the SV model that is least affected by TMR (18.97) over the whole period, followed by SVVG. For MJD, the sizes of MSpR are all at least five times the values of the PER in all seven time windows. It is clear that both MSpR and PER rise when the market becomes volatile. The MSpR tends to dominate during turbulent during periods ②, ④ and ⑥ for SV and SVVG. By contrast, the magnitude of MSpR and PER for SVJ and SVCJ are still very close during these three periods of turmoil; and the values of MSpR in period ④ are smaller than the PER values. The model risk peaks during the global financial crisis (period ⑤), when the PER almost doubled and the MSpR soars more than tenfold. SVCJ is the least affected by MSpR during the crash, followed by SVJ; while the SVJ model bears the lowest TMR in this period. The signs of L-S are all negative, implying that the short position bears higher model risk. Additionally, the size of model risk can be compared with the average model estimated VS rates of different sub-periods (reported in Table D2, in the Supplementary Appendix), and we find that the model risk is generally high when the model estimated VS rates are far from the implied variance.

Overall, the presence of model risk is evident in estimating fair variance strikes of VS's, especially when the market is turbulent. Allowing for stochastic volatility obviously reduces the MSpR. Moreover, all jump models considered in this paper show signs of misspecifica-

[16]The MJD model is a constant-volatility model, and the parameters are estimated with the whole sample. Thus the PER of MJD in estimating fair variance strikes does not vary over time.

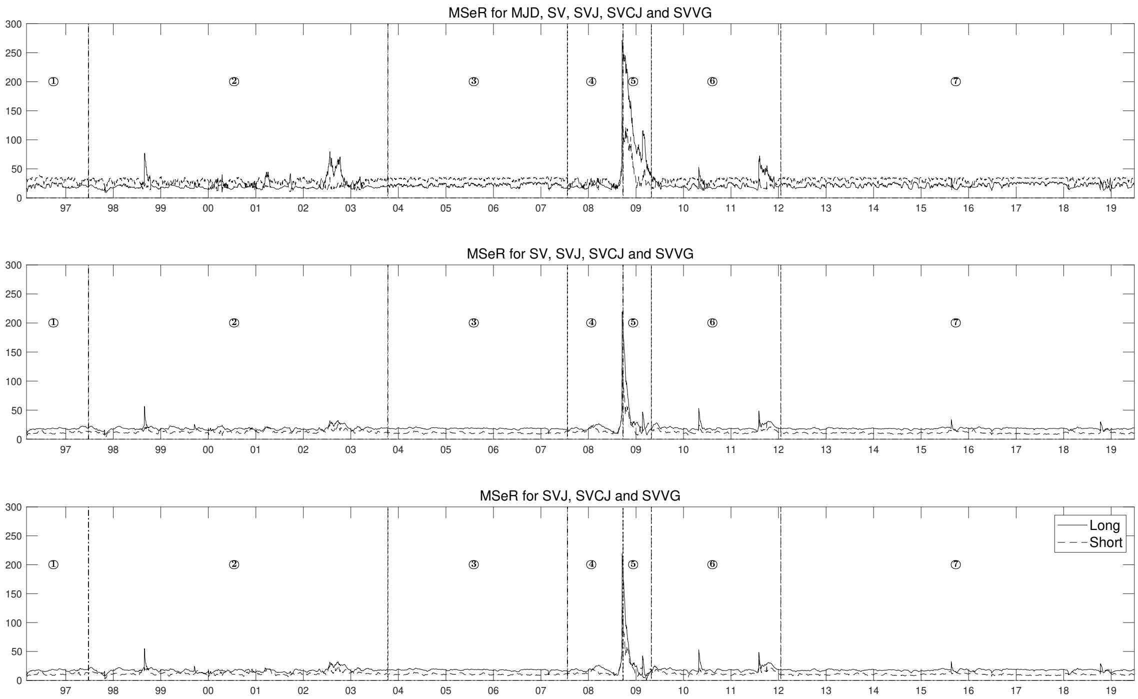

Figure 4: MSeR for Long and Short Positions in Estimating Fair Variance Strikes for VS's. The top panel plots the MSeR for the model class of all models used to estimate VS rates in the paper; the middle panel shows the MSeR for the model class of stochastic volatility models (SV, SVJ, SVCJ and SVVG); the bottom panel is for the model class of stochastic volatility models with jump specifications (SVJ, SVCJ and SVVG).

tion, and modelling Poisson jumps slightly decreases MSpR.

Table 2: MSeR in Estimating VS Rates

Periods	①	②	③	④	⑤	⑥	⑦	Whole
Panel A. Long Position								
Υ_1^{VS}	20.48	21.94	21.94	26.15	118.05	23.88	22.01	24.75
Υ_2^{VS}	18.13	18.63	18.32	22.37	43.29	20.09	18.17	19.37
Υ_3^{VS}	17.47	17.65	18.15	22.12	39.38	19.64	17.77	18.75
Panel B. Short Position								
Υ_1^{VS}	31.67	26.40	32.88	27.44	53.52	27.63	32.09	30.44
Υ_2^{VS}	11.22	13.00	10.64	16.68	28.69	12.93	10.67	12.34
Υ_3^{VS}	11.05	12.39	10.52	15.58	25.68	12.74	10.51	11.94
Panel C. Estimated Prices								
Υ_1^{VS}	27.72	23.08	30.57	24.96	79.23	26.11	29.64	28.70
Υ_2^{VS}	14.35	15.42	14.51	19.03	32.80	16.28	14.39	15.59
Υ_3^{VS}	14.13	14.77	14.46	18.80	30.63	15.99	14.19	15.23

NOTE: This table reports the average values of MSeR in estimating one-month fair variance strikes of VS's; these are presented for different model classes during different periods and the whole research period as well. $\Upsilon_1^{VS} = \{\text{MJD}, \text{SV}, \text{SVJ}, \text{SVCJ}, \text{SVVG}\}$, $\Upsilon_2^{VS} = \{\text{SV}, \text{SVJ}, \text{SVCJ}, \text{SVVG}\}$ and $\Upsilon_3^{VS} = \{\text{SVJ}, \text{SVCJ}, \text{SVVG}\}$. Panel A reports the MSeR for long positions; Panel B reports the MSeR for short positions; and Panel C presents the MSeR calculated with model estimated prices.

We then study the MSeR with three model classes: $\Upsilon_1^{VS} = \{\text{MJD}, \text{SV}, \text{SVJ}, \text{SVCJ}, \text{SVVG}\}$

includes all five models used in estimating VS rates; $\Upsilon_2^{VS} = \{\mathrm{SV}, \mathrm{SVJ}, \mathrm{SVCJ}, \mathrm{SVVG}\}$ contains all four stochastic volatility models; and $\Upsilon_3^{VS} = \{\mathrm{SVJ}, \mathrm{SVCJ}, \mathrm{SVVG}\}$ comprises all three stochastic volatility models with jumps. The time series of MSeR for long and short positions with different model classes in estimating VS rates are illustrated in Figure 4. Intuitively, the MSeR flattens during stable periods, fluctuates in volatile periods and peaks during the financial crisis. According to the top panel, the short position tends to bear higher MSeR when the model class is Υ_1^{VS} under most periods except for the crisis period. This is because during stable markets MJD tends to overestimate VS rates, which makes the short position bear higher MSeR for the model class Υ_1^{VS}; by contrast, when the market is turbulent, MJD tends to underestimate VS rates, thus leading to a smaller MSeR for the short position. By contrast, for model classes Υ_2^{VS} and Υ_3^{VS}, the levels of MSeR for a long position are always higher than those for a short position during the whole sample period. Table 2 reports the mean values of the MSeR for these three model classes during different periods and the whole sample period. It is evident that the MSeR can be reduced by removing models from the model class. The size of MSeR is high for model class Υ_1^{VS}, in which both the constant volatility and stochastic volatility models are included. By contrast, the MSeR of the model class Υ_2^{VS} is only slightly larger than that of Υ_3^{VS} during most periods, indicating that the range of prices produced by SV is largely within the range of prices produced by the more complex models, which are different extensions of the SV model. The only exception is period ⑤, the global financial crisis, when the MSeR of Υ_2^{VS} is about 10% larger than that of Υ_3^{VS}. Additionally, Panel C of Table 2 reports the MSeR calculated using Equation 9 with model estimated prices; the sizes of the MSeR computed with estimated prices are generally between the MSeR for long and short positions. Overall, the size of the MSeR in estimating VS rates is considerable and sensitive to market conditions.

4.3 Variance Risk Premium

Given the estimated fair variance strikes of VS's, an immediate yet nontrivial extension is to investigate the significance of the estimated daily variance risk premiums. Figure 5 plots the time series of the estimated daily one-month VRP from March 05, 1996 to June 28, 2019. The trends of estimated VRP are very similar across models. The sizes of VRP are very small during periods ①, ③ and ⑦ with a flat trend; by contrast, VRP becomes volatile with several jumps during periods ②, ④ and ⑥; it is very high at the beginning of the financial crisis and then stabilizes after about four months.

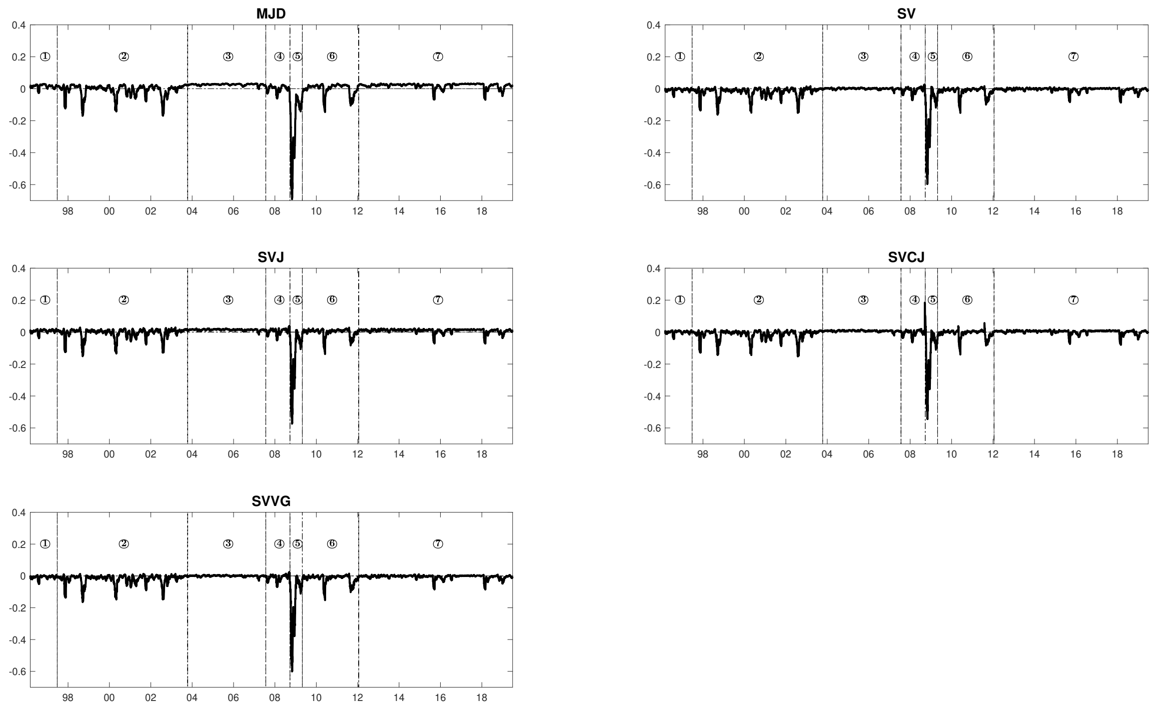

Figure 5: Estimated VRP.

This figure presents the estimated daily one-month VRP from March 05, 1996 to June 28, 2019. VRP is calculated using Equation (19).

Table 3: VRP Statistics

Periods	①	②	③	④	⑤	⑥	⑦	Whole
Panel A. Average values of estimated mean VRP								
MJD	14.89	-8.25	25.98	-0.26	-224.62	-0.78	19.24	3.09
SV	-7.99	-20.82	0.41	-8.87	-167.91	-16.25	-5.41	-14.39
SVJ	5.85	-6.68	14.14	4.64	-154.74	-1.79	8.45	-0.44
SVCJ	-1.20	-15.26	7.33	-1.17	-143.82	-8.60	1.61	-7.27
SVVG	-8.29	-21.10	-0.32	-10.55	-169.69	-17.06	-5.68	-14.92
Panel B. Percentage of days that VRP is significantly positive								
MJD	90.94%	61.03%	99.58%	61.82%	0.00%	67.15%	90.60%	77.58%
SV	19.94%	15.89%	61.30%	39.53%	5.30%	18.75%	38.76%	32.04%
SVJ	81.57%	61.03%	97.90%	71.28%	7.95%	68.17%	86.28%	76.20%
SVCJ	52.27%	36.32%	94.85%	54.05%	9.93%	48.26%	75.01%	60.63%
SVVG	17.52%	15.51%	55.31%	35.14%	0.66%	17.59%	35.02%	29.17%
Panel C. Percentage of days that VRP is insignificant								
MJD	0.00%	0.00%	0.00%	0.00%	0.00%	0.15%	0.00%	0.02%
SV	71.00%	45.33%	38.28%	24.66%	0.00%	48.69%	51.84%	45.88%
SVJ	10.27%	5.93%	1.68%	2.70%	0.00%	5.23%	5.18%	4.86%
SVCJ	38.67%	25.35%	4.73%	10.47%	0.00%	20.20%	15.70%	17.68%
SVVG	73.41%	45.84%	44.27%	29.39%	0.00%	50.44%	55.58%	48.76%
Panel D. Percentage of days that VRP is significantly negative								
MJD	9.06%	38.97%	0.42%	38.18%	100.00%	32.70%	9.40%	22.40%
SV	9.06%	38.78%	0.42%	35.81%	94.70%	32.56%	9.40%	22.08%
SVJ	8.16%	33.04%	0.42%	26.01%	92.05%	26.60%	8.54%	18.94%
SVCJ	9.06%	38.34%	0.42%	35.47%	90.07%	31.54%	9.29%	21.69%
SVVG	9.06%	38.65%	0.42%	35.47%	99.34%	31.98%	9.40%	22.08%

NOTE: Panel A reports the average values of estimated mean VRP multiplied by 1,000. Panel B, C and D present the percentage of days that VRP is significantly positive, insignificant and significantly negative at 5% level, respectively. Results are displayed for all models during different periods and the whole research period.

VRP statistics are reported in Table 3. Although the average values of VRP estimated by stochastic volatility models are all negative for the whole period as reported in Panel

A, they still tend to be significantly positive by comparing the percentage of days in Panel B and D. Since VRP is taken as the premium that a market participant is willing to pay to hedge against variations in future realized variance, it is expected to be positive. The average values of VRP during period ⑤ are all significant, VRP during this period is most likely to be significantly negative. The percentage of days that VRP is significantly negative are very close across the models in all periods.

4.4 Empirical Results for Forward Starting Options

We also investigate the model risk in estimating prices of six-month ATM FS call options with a tenor of three months. The observation period is one month with 21 trading days. The model risk estimation period for FS option models is from August 2, 1996 to June 28, 2019 by considering the one-month model risk observation period and the six-month duration of the FS option.

As in Section 4.2, we divide the sample period by finding abrupt changes in the average values of IV of the one-month standardized ATM call options using the method in Killick et al. (2012), and we identify six sub-periods. Except for the first breakpoint, the remaining four breakpoints are exactly the same as the last four breakpoints identified in Section 4.2 on the model risk of VS's; the end date of the first time window is October 14, 2003, while the end date of period ② in the VS case is October 13, 2003 with only one day difference. For comparison, here we follow the numbering of the sub-periods in Section 4.2 for periods ③ to ⑦. The first period we denote as period ⊛, ranging from August 2, 1996 to October 13, 2003 with an average implied variance of 50.66, ignoring the one day difference.[17]

We compute model risk using the method in Section 2 with $\eta = 5\%$, where the realized payoff in Formula (5) is the discounted value of Equation (20). The time series of the model risk of estimating ATM FS call option prices for different models are illustrated in Figure 6. According to the figure, the MSpR accounts for the largest proportion of TMR at almost every point on the timeline for all models in estimating FS call option prices. The PER appears flat for all models; the PER actually fluctuates through time slightly, as can be seen in Table 4. The size of TMR is typically below 50 except for some sharp spikes in periods ⊛ and ⑥. The most obvious spike is in the early part of period ⑥. The majority of the spikes in period ⊛ stem from the sudden changes in the ratio of the index forward price to spot price, which affects FS option payoffs and consequently model risk; while the

[17] This should not affect our conclusions.

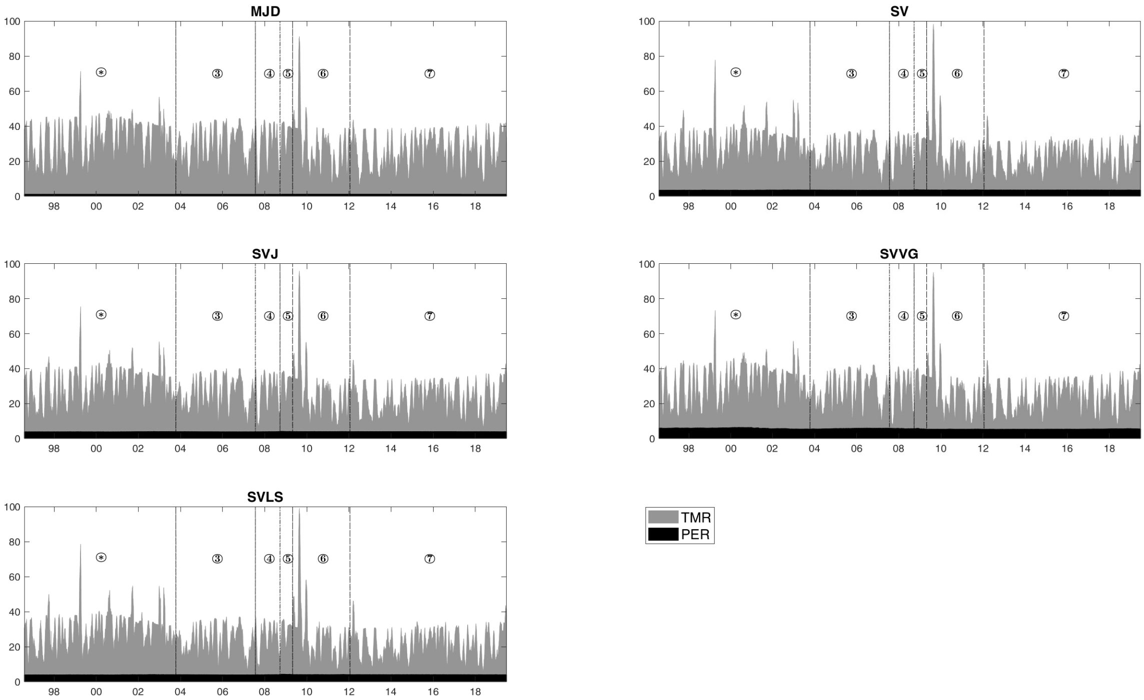

Figure 6: Model Risk in Estimating ATM FS Call Option Prices.

This figure presents the TMR (grey columns) and PER (black columns) of models used to estimate the six-month ATM FS call option prices with a tenor of three months. The sizes of the grey columns measure the MSpR. The results are based on daily spot prices on the S&P 500 and daily prices of standardized at-the-money-forward call options with 30 days to expiration between January 3, 1996 and June 28, 2019. Estimated prices are updated within the estimation process at the end of each iteration after burn-in. The total iterations are 30,000 and the first 10,000 runs are discarded as burn-in. The details of the estimation methods can be found in the Supplementary Appendix.

spikes in period ⑥ are mainly caused by large realized payoffs of ATM FS call option. The model risk in estimating FS call options prices does not seem to be affected by market conditions. The model risk actually stays at a comparatively low level during the financial crisis.

Table 4 shows the mean values of model risk in estimating prices of six-month ATM FS call options with a tenor of three months. The last column reports the model risk for the whole sample period; this shows that MJD has the lowest PER, but the largest MSpR and TMR; SVLS bears the least MSpR, followed by SV and SVVG. Furthermore, the SVLS model also bears the lowest TMR during all sub-periods. SV reveals the lowest PER compared with other stochastic volatility models while SVVG has the largest PER. The sizes of MSpR of SVJ are greater than those of SVVG, but the TMR of SVVG is higher due to the large PER. The signs of L-S for all models are negative in all time windows, indicating that a short position tends to have a higher model risk.

We then study the MSeR of three model classes: $\Upsilon_1^{FS} = \{\text{MJD}, \text{SV}, \text{SVJ}, \text{SVVG}, \text{SVLS}\}$ includes all five models used in estimating FS option prices; $\Upsilon_2^{FS} = \{\text{SV}, \text{SVJ}, \text{SVVG}, \text{SVLS}\}$

Table 4: Model Risk in Estimating FS option Prices

Periods	⊛	③	④	⑤	⑥	⑦	Whole
Panel A. MJD							
PER	1.13	1.14	1.14	1.15	1.15	1.15	1.14
MSR	31.77	28.46	28.43	34.40	27.71	25.54	28.62
TMR	32.91	29.59	29.57	35.55	28.86	26.69	29.76
L-S	-0.05	-0.04	-0.05	-0.06	-0.05	-0.05	-0.05
Panel B. SV							
PER	3.50	3.50	3.54	3.70	3.59	3.56	3.54
MSR	26.39	21.05	22.01	26.35	23.05	18.90	22.46
TMR	29.90	24.55	25.55	30.05	26.65	22.47	26.00
L-S	-0.63	-0.63	-0.64	-0.67	-0.65	-0.64	-0.64
Panel C. SVJ							
PER	3.91	3.91	3.95	4.07	3.99	3.97	3.94
MSR	27.00	22.33	22.94	27.89	23.33	19.85	23.29
TMR	30.91	26.24	26.89	31.96	27.33	23.82	27.23
L-S	-0.66	-0.68	-0.66	-0.63	-0.69	-0.69	-0.68
Panel D. SVVG							
PER	5.94	5.77	5.76	5.62	5.37	5.43	5.66
MSR	26.07	21.96	22.32	27.41	22.40	19.17	22.56
TMR	32.02	27.73	28.08	33.03	27.77	24.60	28.22
L-S	-0.76	-0.77	-0.76	-0.72	-0.77	-0.77	-0.77
Panel E. SVLS							
ER	4.02	4.01	4.07	4.26	4.12	4.08	4.06
MSR	25.55	19.98	21.06	25.15	22.40	17.97	21.55
TMR	29.57	23.99	25.13	29.41	26.51	22.05	25.61
L-S	-0.35	-0.46	-0.48	-0.62	-0.74	-0.70	-0.54

NOTE: This table reports the mean values of model risk in estimating prices of six-month ATM FS call options with a tenor of three months. The model risk estimates for all models during different periods and the whole research period are presented.

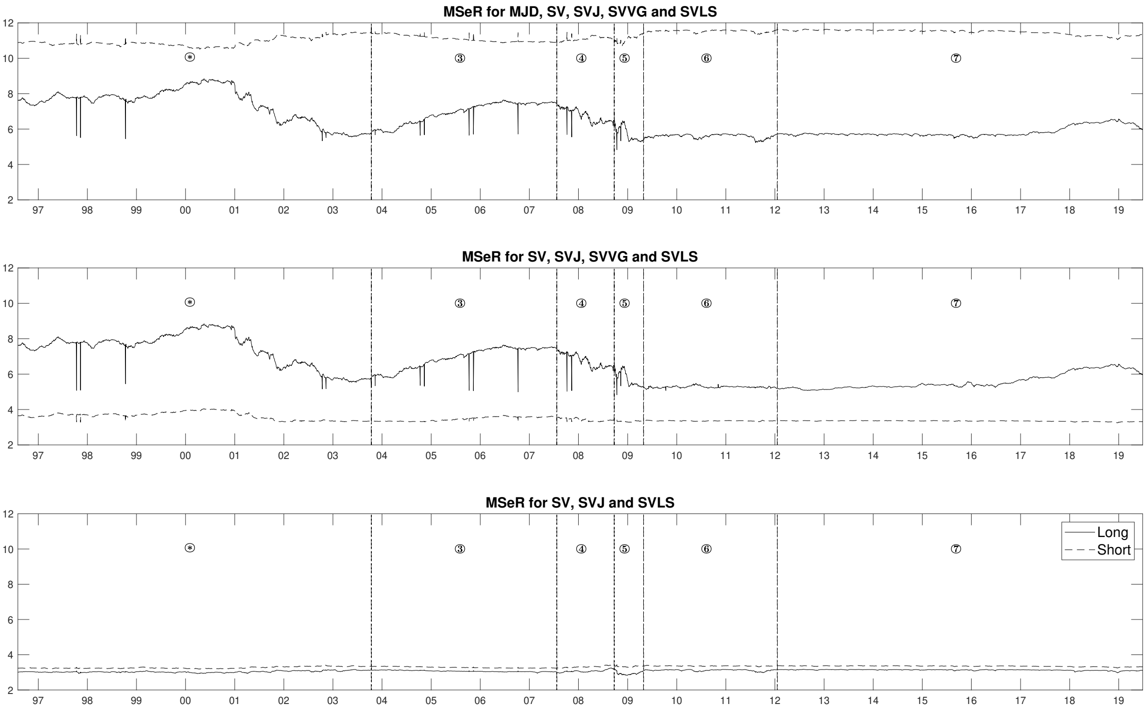

Figure 7: MSeR for Long and Short Positions in Estimating ATM FS Call Option Prices. The top panel plots the MSeR for the model class of all models used to estimate FS option prices in the paper; the middle panel shows the MSeR for the model class of stochastic volatility models (SV, SVJ, SVVG and SVLS); the bottom panel is for the model class composed of SV, SVJ and SVLS.

Table 5: MSeR Statistics in Estimating FS Options Prices

Periods	⊛	③	④	⑤	⑥	⑦	Whole
Panel A. Long Position							
Υ_1^{FS}	7.42	6.92	6.76	5.72	5.60	5.84	6.53
Υ_2^{FS}	7.41	6.92	6.75	5.71	5.27	5.59	6.41
Υ_3^{FS}	3.04	3.07	3.07	2.95	3.12	3.13	3.08
Panel B. Short Position							
Υ_1^{FS}	10.93	11.10	11.10	11.09	11.53	11.49	11.22
Υ_2^{FS}	3.64	3.47	3.43	3.33	3.36	3.35	3.46
Υ_3^{FS}	3.27	3.28	3.32	3.33	3.36	3.35	3.31
Panel C. Estimated Prices							
Υ_1^{FS}	7.69	7.87	7.82	7.64	8.21	8.19	7.95
Υ_2^{FS}	5.29	5.01	4.92	4.31	4.00	4.20	4.69
Υ_3^{FS}	3.11	3.13	3.14	3.09	3.20	3.20	3.15

NOTE: This table reports the mean values of MSeR in estimating prices of six-month ATM FS call options with a tenor of three months. The MSeR during different periods and the whole research period are presented. $\Upsilon_1^{FS} = \{\text{MJD}, \text{SV}, \text{SVJ}, \text{SVVG}, \text{SVLS}\}$, $\Upsilon_2^{FS} = \{\text{SV}, \text{SVJ}, \text{SVVG}, \text{SVLS}\}$ and $\Upsilon_3^{FS} = \{\text{SV}, \text{SVJ}, \text{SVLS}\}$. Panel A reports the MSeR for long positions; Panel B reports the MSeR for short positions; and Panel C presents the MSeR calculated with model estimated prices.

contains all four stochastic volatility models; and $\Upsilon_3^{FS} = \{\text{SV}, \text{SVJ}, \text{SVLS}\}$ composed of SV, SVJ and SVLS. Figure 7 plots the time series of MSeR for long and short positions in estimating the six-month ATM FS call option prices. The corresponding mean values of MSeR are reported in Table 5. Comparing the MSeR of Υ_1^{FS} and Υ_2^{FS}, the exclusion of MJD greatly reduces the MSeR of the short position and merely reduces that of the long position. By contrast, excluding SVVG from the model class Υ_2^{FS} leads to a much smaller MSeR for the long position and slightly lower MSeR for the short position. Although the MSeR of model classes Υ_1^{FS} and Υ_2^{FS} for the long position are very close, the sizes of MSeR calculated with estimated prices (reported in Panel C of Table 5) are decreasing gradually from Υ_1^{FS} to Υ_3^{FS}. This implies that it is necessary to measure the MSeR for long and short positions separately.

4.5 Robustness Check

The length of the observation period will only affect the values of MSpR. Results of MSpR above are all based on a 21-day observation period. We also calculate the MSpR with different lengths of observation periods containing 42 (two months), 63 (three months), 126 (six months), and 252 (one year) trading days, respectively. The results are shown in Table 6. We notice a small effect of the length of the observation periods on the estimates of MSpR when estimating VS rates. According to the results in Section 4.2, the model risk in estimating VS rates responds to the market conditions. As such, a shorter observation period, which can better reflect the current market condition, should lead to lower MSpR; by contrast, a longer observation period dilutes the effect of recent data and results in higher MSpR. We find that the length of the observation period does not affect the estimates of MSpR of the six-month FS options significantly as reported in Panel B; the values fluctuate within a small range.

Table 6: The MSpR Computed with Different Lengths of the Observation Period

Panel A. One-month VS's

	21	42	63	126	252
MJD	26.53	26.57	26.60	26.70	26.88
SV	12.87	13.99	14.84	16.46	17.63
SVJ	12.84	13.99	14.90	16.58	17.98
SVCJ	11.49	12.81	13.75	15.48	16.81
SVVG	13.00	14.16	14.98	16.51	17.65

Panel B. Six-month ATM FS Options with a Tenor of Three Months

	21	42	63	126	252
MJD	28.62	28.58	28.55	28.50	28.50
SV	22.46	22.41	22.39	22.34	22.36
SVJ	23.29	23.25	23.22	23.17	23.18
SVVG	22.56	22.52	22.49	22.44	22.44
SVLS	21.55	21.51	21.48	21.44	21.45

Panel C. One-month Call Options with a Delta of 40, Computed with Realized Payoffs

	21	42	63	126	252
MJD	19.46	19.54	19.61	19.72	19.80
SV	15.84	15.56	15.47	15.18	15.03
SVJ	15.26	15.14	15.05	14.95	14.85
SVCJ	16.90	16.81	16.83	16.92	16.93
SVLS	16.12	15.89	15.69	15.42	15.23
SVVG	17.29	17.15	17.09	17.06	17.00

Panel D. One-month Call Options with a Delta of 40, Computed with Prices

MJD	SV	SVJ	SVCJ	SVLS	SVVG
8.54	6.92	2.35	4.03	4.64	3.62

NOTE: This table reports the mean values of MSpR for the whole sample period computed with different lengths of the observation periods. The observation periods contain 21 (one month), 42 (two months), 63 (three months), 126 (six months), and 252 (one year) trading days, respectively. Panel A presents the results for the one-month VS; Panel B reports the results for the six-month ATM FS options with a tenor of three months; Panel C gives the results for the one-month call option with a Delta of 40; Panel D provides the MSpR of different models when pricing the one-month call option with a Delta of 40 computed with the method in Lazar et al. (2020).

The MSpR in this paper is estimated with realized payoffs of options in the OTC market, where option holders typically follow a "buy and hold" strategy. By contrast, options in the exchange are liquidly traded by active traders; thus, option market prices should be used to compute the MSpR. We compare the MSpR borne by option traders following different strategies, the "buy and hold" and "active trading" strategies, through estimating the MSpR for European options with the approach in the current paper and the approach of Lazar et al. (2020). Panel C of Table 6 reports the MSpR, computed with the methods in this paper, of estimating one-month European call options with a Delta of

40. According to the results, the length of the observation period has no apparent effects on the values of MSpR, and the sizes of the differences among the MSpR calculated with different observation days are all within one. Panel D provides the MSpR of different models when pricing the one-month call option with a Delta of 40 computed with the method in Lazar et al. (2020).[18] Obviously, the "buy and hold" option holders are exposed to much larger MSpR than the active traders. Nevertheless, both approaches suggest that the MJD model has the largest MSpR, and SVJ bears the lowest MSpR when pricing the one-month European call options with a Delta of 40.

5 Conclusions

In this paper, we propose a model risk measure based on realized payoffs that is able to quantify PER and MSpR of pricing models, respectively, for options in the OTC market, where the market prices of products are not available, and option holders tend to follow the "buy and hold" strategy. Our model risk measurement also considers MSeR of a model class for long and short positions. We then apply this measurement to AJD models and Lévy jump models to investigate the model risk when estimating fair variance strikes of VS's and FS option prices.

Our empirical findings are striking. We find that stochastic volatility is a crucial feature in estimating both fair variance strikes of VS's and FS option prices. Also, modeling Poisson jumps decreases MSpR in estimating fair variance strikes of VS's; the introduction of variance jumps further reduces it. By contrast, Poisson jumps and variance-gamma jumps lead to higher MSpR in estimating FS option prices; while the addition of log-stable jumps on the SV model reduces the MSpR. Additionally, we compare the model risk of short and long positions. We find that PER and MSpR are both main sources of TMR in estimating VS rates, while MSpR accounts for a larger proportion when estimating FS option prices. We also find that PER and MSpR tend to go against each other; a well-specified model often bears less MSpR but carries larger PER. This is consistent with Chernov et al. (2003); they also find tradeoffs among various model specifications.

Additionally, the MSeR is also considerable according to our results; this can be substan-

[18]The data is downloaded in the volatility surface file of Option Metrics, which contains daily options data with constant duration and Delta values. We compute MSpR for call options with a Delta of 40, which are out-of-the-money options. As an additional check, we also compute MSpR for call options with Delta values of 45, 50, 55, and 60; the results are very similar and are available on request.

tial when the model class contains both constant volatility and stochastic volatility models; the exclusion of constant volatility models from the model class can lower the MSeR significantly. Furthermore, we investigate the significance of the estimated daily VRP; we find that there is consistency in the significantly negative VRP estimated by models.

Our research highlights some interesting puzzles for further study. A natural extension of this research would be to study the model risk in pricing path-dependent options; however, closed-form valuation formulae for path-dependent options are typically not available and the model prices are obtained using time-consuming numerical methods. Our method computes prices in each iteration during the estimation, which consumes exponentially increasing time. A possible solution is to combine the fast computational method for pricing options with jump-diffusion models proposed in Feng and Linetsky (2008) with our risk measures. Additionally, several recent studies derive semi-analytical formulae for various path-dependent options under different model specifications. For example, see the results for path-dependent options provided in Aquino and Bernard (2019), Carr et al. (2020) and Carr and Itkin (2020). One of the future research directions could be to compare the model risk of different models in pricing barrier options using the semi-closed formulae derived in the aforementioned three papers.

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