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Vectorial variational problems in L^∞ constrained by the Navier–Stokes equations*

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Abstract
We study a minimisation problem in L^p and L^∞ for certain cost functionals, where the class of admissible mappings is constrained by the Navier–Stokes equations. Problems of this type are motivated by variational data assimilation for atmospheric flows arising in weather forecasting. Herein we establish the existence of PDE-constrained minimisers for all p, and also that L^p minimisers converge to L^∞ minimisers as $p \to \infty$. We further show that L^p minimisers solve an Euler–Lagrange system. Finally, all special L^∞ minimisers constructed via approximation by L^p minimisers are shown to solve a divergence PDE system involving measure coefficients, which is a divergence-form counterpart of the corresponding non-divergence Aronsson–Euler system.

Keywords: Navier–Stokes equations, calculus of variations in L^∞, PDE-constrained optimisation, Euler–Lagrange equations, Aronsson–Euler systems, data assimilation

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1. Introduction and main results

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set and let also $n \geqslant 2$ and $\nu, T > 0$. Consider the Navier–Stokes equations

$$\begin{cases} \partial_t u - \nu\Delta u + (u \cdot \mathrm{D})u + \mathrm{Dp} - f = y, & \text{in } \Omega \times (0, T), \\ \operatorname{div} u = 0, & \text{in } \Omega \times (0, T), \\ u(\cdot, 0) = u_0, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega \times (0, T), \end{cases} \tag{1.1}$$

and for brevity let us henceforth symbolise $\nabla u := (\mathrm{D}u, \partial_t u)$ and $\Omega_T := \Omega \times (0, T)$, where $\mathrm{D}u = (\partial_{x_1} u, \ldots, \partial_{x_n} u) \in \mathbb{R}^{n\times n}$ symbolises the spatial gradient. The system of PDEs (1.1) describes the velocity $u : \Omega_T \longrightarrow \mathbb{R}^n$ and the pressure $\mathrm{p} : \Omega_T \longrightarrow \mathbb{R}$ of a flow, for some given initial data $u_0 : \Omega \longrightarrow \mathbb{R}^n$ with source $f : \Omega_T \longrightarrow \mathbb{R}^n$. Here the map $y : \Omega_T \longrightarrow \mathbb{R}^n$ is a parameter and should be understood as a (deterministic) noise or error. Let also $N \in \mathbb{N}$ and suppose we are given a mapping

$$\mathrm{Q} \; : \quad \Omega_T \times \left(\mathbb{R}^n \times \mathbb{R}^{(n+1)\times n} \times \mathbb{R}\right) \longrightarrow \mathbb{R}^N. \tag{1.2}$$

A problem of interest in the geosciences, in particular in data assimilation for atmospheric flows in relation to weather forecasting (see e.g. [16–18]), can be formulated as follows: find solutions (u, p) to (1.1) such that, in an appropriate sense,

$$\begin{cases} \qquad\qquad\qquad y \approx 0, \\ \mathrm{Q}(\cdot, \cdot, u, \nabla u, \mathrm{p}) - q \approx 0, \end{cases} \tag{1.3}$$

where $q : \Omega_T \longrightarrow \mathbb{R}^N$ is a vector of given measurable 'data' arising from some specific measurements, taken through the 'observation operator' Q of (1.2). In (1.1) and (1.3), y represents an error in the measurements which forces the Navier–Stokes equations to be satisfied only approximately for solenoidal (divergence-free) vector fields. Namely, we are looking for solutions to (1.1) such that simultaneously the error y vanishes, and also the measurements q match the prediction of the solution (u, p) through the observation operator (1.2). In application, Q is typically some component (e.g. linear projection or nonlinear submersion) of the atmospheric flow that we can observe. Unfortunately, the data fitting problem (1.3) is severely ill-posed; an exact solution may well not exist, and even if it does, it may not be unique.

In this paper, inspired by the methodology of data assimilation, especially variational data assimilation in continuous time (for relevant works we refer e.g. to [13, 19, 26, 28, 29, 43, 45, 50]), we seek to minimise the misfit functional

$$(u, \mathrm{p}, y) \mapsto (1 - \lambda) \left\|\mathrm{Q}(\cdot, \cdot, u, \nabla u, \mathrm{p}) - q\right\| + \lambda \left\|y\right\|$$

over all admissible triplets (u, p, y) which satisfy (1.1), for a fixed weight $\lambda \in (0, 1)$. The role of this weight is to obtain essentially a Pareto family of extremals, one for each value λ, even though in this paper we do not pursue further this viewpoint of vector-valued minimisation (the interested reader may e.g. consult [21]). The standard approach to data assimilation is to use Hilbert space methods (or least squares in the discrete case), hence minimising in L^2. The novelty of our approach, which is also justified from the viewpoint of applications, is to consider instead *minimisation in* L^∞, namely by interpreting the norms above as L^∞ ones (or maxima in the discrete case). There is a significant advantage of considering a min-max problem instead of minimising the squared averages: the misfit becomes uniformly small throughout the space-time domain Ω_T and not just on average, hence large 'spikes' of deviations from zero are at the outset excluded.

When one passes from a variational problem for an integral norm to one for the supremum norm, even though this is justified from the viewpoint of desired outputs, it poses some serious challenges. The L^∞ norm is neither differentiable nor strictly convex, and the space L^∞ is neither reflexive nor separable. Additionally, with respect to the domain argument, the L^∞ norm is not additive but only sub-additive. Further, one would also need estimates for (1.1) in appropriate subspaces of L^∞ for weakly differentiable functions, which, to the best of our knowledge, do not exist even for linear strongly elliptic systems (see e.g. [32]). Even then, if one somehow solves the L^∞ minimisation problem (by using, for instance, the direct method of the calculus of variations as in [24], under the appropriate quasiconvexity assumptions for $|\mathrm{Q} - q| + |y|$ as in [12]), the analogue of the Euler–Lagrange equations for the L^∞ problem cannot be derived directly by perturbation/sensitivity methods due to the lack of smoothness of the L^∞ norm.

In this paper, to overcome the difficulties described above, we follow the methodology of the relatively new field of calculus of variations in L^∞ (see e.g. [22, 35] for a general introduction to the scalar-valued theory), and in particular the ideas from [37–40] involving higher order and vectorial problems, as well as problems involving PDE-constraints, which have only recently started being investigated. To this end, we follow the approach of solving the desired L^∞ variational problem by solving respective approximating L^p variational problems for all p, and obtain appropriate compactness estimates which allow to pass to the limit as $p \to \infty$. The case of finite $p > 2$ studied herein is also of independent interest, especially for numerical discretisation schemes in L^∞ (see e.g. [41, 42]), but in this paper we treat it mostly as an approximation device to solve efficiently the L^∞ problem. The idea of this approach is based on the observation that, for a fixed essentially bounded function on a domain of finite measure, the L^p norm tends to the L^∞ norm of the function as $p \to \infty$.

In order to state our hypotheses and main results, let us set

$$\mathrm{K}\,(x, t, \eta, A, a, r) := \mathrm{Q}\,(x, t, \eta, A, a, r) - q(x, t) \tag{1.4}$$

(note that in (1.4) $(x, t) \in \Omega_T$ is treated as two arguments and the two arguments (A, a) are for $\nabla u = (\mathrm{D}u, \partial_t u)$, which we conveniently display abbreviated as one) and, by considering the (strong) divergence operator $\mathrm{div} : W^{1,1}(\Omega; \mathbb{R}^n) \longrightarrow L^1(\Omega)$, we henceforth assume that

$$\begin{cases} \text{(a)} \quad \Omega \text{ is bounded and has } C^2 \text{ boundary } \partial\Omega, \\ \text{(b)} \quad u_0 \in \left(W^{2,\infty} \cap W^{1,\infty}_0\right)(\Omega; \mathbb{R}^n) \cap \ker(\mathrm{div}), \\ \text{(c)} \quad f \in L^\infty(\Omega_T; \mathbb{R}^n) \quad \& \quad q \in L^\infty(\Omega_T; \mathbb{R}^N), \\ \text{(d)} \quad \mathrm{K}(x, t, \cdot, \cdot, \cdot, \cdot) \text{ is } C^1 \text{ for almost every } (x, t), \\ \text{(e)} \quad \mathrm{K}(\cdot, \cdot, \eta, A, a, r) \text{ is } L^\infty \quad \text{for all } (\eta, A, a, r), \\ \text{(f)} \quad |\mathrm{K}(x, t, \eta, A, \cdot, \cdot)|^2 \text{ is convex for all } (x, t, \eta, A). \end{cases} \tag{1.5}$$

Then, for any $p \in (1, \infty)$, we define the L^p misfit $\mathrm{E}_p : \mathfrak{X}^p(\Omega_T) \longrightarrow \mathbb{R}$ by setting

$$\mathrm{E}_p\,(u, \mathrm{p}, y) := (1 - \lambda)\|\mathrm{K}(\cdot, u, \nabla u, \mathrm{p})\|_{\dot{L}^p(\Omega_T)} + \lambda\|y\|_{\dot{L}^p(\Omega_T)}. \tag{1.6}$$

We note that in (1.6) and subsequently, the dotted $\dot{L}^p$ quantities are regularisations of the respective norms at the origin, obtained by regularising the Euclidean norm in the respective target space:

$$\|h\|_{\dot{L}^p(\Omega_T)} := \||h|_{(p)}\|_{L^p(\Omega_T)}, \qquad |\cdot|_{(p)} := \sqrt{|\cdot|^2 + p^{-2}}. \tag{1.7}$$

Further, since we will only be dealing with finite measures, we will always be using the normalised L^p norms in which we replace the integral over the domain with the respective average, for example for $L^p(\Omega_T)$ with the $(n+1)$-Lebesgue measure, the norm will be

$$\|h\|_{L^p(\Omega_T)} := \left(⨍_{\Omega_T} |h|^p \, \mathrm{d}\mathcal{L}^{n+1} \right)^{1/p}.$$

The admissible minimisation class $\mathfrak{X}^p(\Omega_T)$ over which E_p is considered, is defined as follows:

$$\mathfrak{X}^p(\Omega_T) := \{(u, \mathrm{p}, y) \in \mathcal{W}^p(\Omega_T) \ : \ (u, \mathrm{p}, y) \text{ satisfies weakly } (1.1)\}, \tag{1.8}$$

where

$$\mathcal{W}^p(\Omega_T) := W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T; \mathbb{R}^n) \times W^{1,0;p}_{\sharp}(\Omega_T) \times L^p(\Omega_T; \mathbb{R}^n). \tag{1.9}$$

The rather complicated functional spaces appearing in (1.9) are defined as follows. The space $W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T; \mathbb{R}^n)$ consists of solenoidal maps which are $W^{2,p}$ in space and $W^{1,p}$ in time, and also laterally vanishing on $\partial\Omega \times (0, T)$:

$$\begin{cases} W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T; \mathbb{R}^n) := L^p\left((0,T); W^{2,p}_{0,\sigma}(\Omega; \mathbb{R}^n)\right) \bigcap W^{1,p}\left((0,T); L^p(\Omega; \mathbb{R}^n)\right), \\ W^{2,p}_{0,\sigma}(\Omega; \mathbb{R}^n) := \left(W^{2,p} \cap W^{1,p}_0\right)(\Omega; \mathbb{R}^n) \cap \ker(\mathrm{div}). \end{cases} \tag{1.10}$$

The space $W^{1,0;p}_{\sharp}(\Omega_T)$ consists of scalar-valued functions which are $W^{1,p}$ in space with zero average, and L^p in time:

$$\begin{cases} W^{1,0;p}_{\sharp}(\Omega_T) := L^p\left((0,T); W^{1,p}_{\sharp}(\Omega)\right), \\ W^{1,p}_{\sharp}(\Omega) := \left\{ g \in W^{1,p}(\Omega) \ : \ \int_\Omega g \, \mathrm{d}\mathcal{L}^n = 0 \right\}. \end{cases} \tag{1.11}$$

The associated norms in these spaces are the expected ones, namely

$$\begin{cases} \|v\|_{W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T)} := \|v\|_{L^p(\Omega_T)} + \|\nabla v\|_{L^p(\Omega_T)} + \|\mathrm{D}^2 v\|_{L^p(\Omega_T)}, \\ \|g\|_{W^{1,0;p}_{\sharp}(\Omega_T)} := \|g\|_{L^p(\Omega_T)} + \|\mathrm{D}g\|_{L^p(\Omega_T)}. \end{cases} \tag{1.12}$$

Note also that the divergence-free condition for u in (1.1) has now been incorporated in the functional space $W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T)$. Finally, the L^∞ misfit $\mathrm{E}_\infty : \mathfrak{X}^\infty(\Omega_T) \longrightarrow \mathbb{R}$ is defined by setting

$$\mathrm{E}_\infty(u, \mathrm{p}, y) := (1-\lambda)\|\mathrm{K}(\cdot, \cdot, u, \nabla u, \mathrm{p})\|_{L^\infty(\Omega_T)} + \lambda\|y\|_{L^\infty(\Omega_T)}, \tag{1.13}$$

where the admissible class $\mathfrak{X}^\infty(\Omega_T)$ is given by

$$\mathfrak{X}^\infty(\Omega_T) := \bigcap_{1<p<\infty} \mathfrak{X}^p(\Omega_T). \tag{1.14}$$

Note that the natural topology of $\mathfrak{X}^\infty(\Omega_T)$ is not induced by a complete norm in a Banach space, but instead its topology is defined as the locally convex topology induced from the ambient

Frechét space $\bigcap_{1<p<\infty}\mathcal{W}^p(\Omega_T)$. Notwithstanding, no difficulties will arise from this fact, which is a necessity that stems from the lack of $W^{2,\infty}$–$W^{1,\infty}$ estimates for (1.1). In particular, $\mathfrak{X}^\infty(\Omega_T)$ is *not* obtained by considering the strictly smaller Cartesian product space

$$\mathcal{W}^\infty(\Omega_T) = W^{2,1;\infty}_{\mathrm{L},\sigma}(\Omega_T) \times W^{1,0;\infty}_{\sharp}(\Omega_T) \times L^\infty(\Omega_T;\mathbb{R}^n).$$

However, we will assume that the solution (u,p) to (1.1) is strong and satisfies $W^{2,p}$–$W^{1,p}$ estimates for any finite p. This is deducible under assumption (1.5) in the case of $n=2$ (see e.g. [31, 51]), and also under smallness conditions on u_0 in any dimension $n \geqslant 3$ (see e.g. [2, 33, 52]). Hence, our additional hypothesis is

$$\begin{cases} \text{Either} \\ \qquad\qquad n=2, \\ \text{or, } n\geqslant 3 \text{ but for any } p\in(1,\infty), \text{ exists } C>0 \text{ depending only on } p \text{ and} \\ \quad \text{on } \partial\Omega, T, \|u_0\|_{L^2(\Omega)}, \|f\|_{L^2(\Omega_T)}, \text{ such that} \\ \|u\|_{W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T)} + \|\mathrm{p}\|_{W^{1,0;p}_{\sharp}(\Omega_T)} \leqslant C\left(\|u_0\|_{W^{2-\frac{2}{p},p}(\Omega)} + \|f\|_{L^p(\Omega_T)}\right), \\ \text{when } (u,\mathrm{p}) \text{ solves weakly (1.1) with } y\equiv 0. \end{cases} \tag{1.15}$$

Assumption (1.15), albeit restrictive, is compatible with situations of interest in weather forecasting (see e.g. [16–18]). Our first main result concerns the existence of E_p-minimisers in $\mathfrak{X}^p(\Omega_T)$, the existence of E_∞-minimisers in $\mathfrak{X}^\infty(\Omega_T)$ and the approximability of the latter by the former as $p\to\infty$.

Theorem 1 (E_∞-minimisers, E_p-minimisers & convergence as $p\to\infty$). *Suppose that (1.5) and (1.15) hold true. Then, for any $p\in(n+2,\infty]$, the functional E_p (given by (1.6) for $p<\infty$ and by (1.13) for $p=\infty$) has a constrained minimiser (u_p,p_p,y_p) in the admissible class $\mathfrak{X}^p(\Omega_T)$:*

$$\mathrm{E}_p\left(u_p,\mathrm{p}_p,y_p\right) = \inf\left\{\mathrm{E}_p(u,\mathrm{p},y) :\ (u,\mathrm{p},y)\in\mathfrak{X}^p(\Omega_T)\right\}. \tag{1.16}$$

Additionally, there exists a subsequence of indices $(p_j)_1^\infty$ such that the sequence of respective E_{p_j}-minimisers $(u_{p_j},\mathrm{p}_{p_j},y_{p_j})$ satisfies $(u_p,\mathrm{p}_p,y_p)\rightharpoonup(u_\infty,\mathrm{p}_\infty,y_\infty)$ in $\mathcal{W}^q(\Omega_T)$ for any $q\in(1,\infty)$, as $p_j\to\infty$. Additionally,

$$\begin{cases} u_p \rightharpoonup u_\infty, & \text{in } W^{2,1;q}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n), \\ u_p \longrightarrow u_\infty, & \text{in } C\left(\overline{\Omega_T};\mathbb{R}^n\right), \\ \mathrm{D}u_p \longrightarrow \mathrm{D}u_\infty, & \text{in } C\left(\overline{\Omega_T};\mathbb{R}^{n\times n}\right), \\ \mathrm{p}_p \rightharpoonup \mathrm{p}_\infty, & \text{in } W^{1,0;q}_{\#}(\Omega_T;\mathbb{R}^n), \\ y_p \rightharpoonup y_\infty, & \text{in } L^q(\Omega_T), \end{cases} \tag{1.17}$$

for any $q\in(1,\infty)$, and also

$$\mathrm{E}_p(u_p,\mathrm{p}_p,y_p) \longrightarrow \mathrm{E}_\infty(u_\infty,\mathrm{p}_\infty,y_\infty) \tag{1.18}$$

as $p_j\to\infty$.

Given the existence of constrained minimisers established by theorem 1 above, the next natural question concerns the existence of necessary conditions in the form of PDEs governing the constrained minimisers. We first consider the case of $p < \infty$. Unsurprisingly, the PDE constraint of (1.1) used in defining (1.8) gives rise to a generalised Lagrange multiplier in the Euler–Lagrange equations, obtained by utilising well-known results on the Kuhn–Tucker theory from [55]. Interestingly, however, the incorporation of the solenoidality constraint into the functional space (recall (1.10)), allows us to have only one generalised multiplier corresponding only to the parabolic system in (1.1), instead of two.

To state our second main result, we first need to introduce some notation. For any $M \in \mathbb{N}$ and $p \in (1, \infty)$, we define the operator

$$\mathfrak{M}_p \;:\; L^p(\Omega_T;\mathbb{R}^M) \longrightarrow L^{p'}(\Omega_T;\mathbb{R}^M),$$

where $p' := p/(p-1)$, by setting

$$\mathfrak{M}_p(V) := \frac{|V|_{(p)}^{p-2} V}{\left(\|V\|_{\dot{L}^p(\Omega_T)}\right)^{p-1}}. \tag{1.19}$$

Here $|\cdot|_{(p)}$ is the regularisation of the Euclidean norm of $\mathbb{R}^M$, as defined in (1.7). By Hölder's inequality it is immediate to verify that (for the normalised $L^{p'}$ norm) we actually have

$$\|\mathfrak{M}_p(V)\|_{L^{p'}(\Omega_T)} \leqslant 1,$$

and therefore $\mathfrak{M}_p$ is valued in the unit ball of $L^{p'}(\Omega_T;\mathbb{R}^M)$. Further, for brevity we will use the notation

$$\begin{cases} \mathrm{K}[u,\mathrm{p}] := \mathrm{K}\left(\cdot,\cdot,u,\nabla u,\mathrm{p}\right), \\ \mathrm{K}_\eta[u,\mathrm{p}] := \mathrm{K}_\eta\left(\cdot,\cdot,u,\nabla u,\mathrm{p}\right), \\ \mathrm{K}_{(A,a)}[u,\mathrm{p}] := \mathrm{K}_{(A,a)}\left(\cdot,\cdot,u,\nabla u,\mathrm{p}\right), \\ \mathrm{K}_r[u,\mathrm{p}] := \mathrm{K}_r\left(\cdot,\cdot,u,\nabla u,\mathrm{p}\right), \end{cases} \tag{1.20}$$

for K and its partial derivatives K_η, $\mathrm{K}_{(A,a)}$, K_r with respect to the arguments for u, ∇u and p respectively.

Theorem 2 (Variational equations in L^p). *Suppose that (1.5) and (1.15) hold true. Then, for any $p \in (n+2, \infty)$, there exists a Lagrange multiplier*

$$\Psi_p \in \left(W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega;\mathbb{R}^n)\right)^* \tag{1.21}$$

associated with the constrained minimisation problem (1.16), such that the minimising triplet $(u_p, \mathrm{p}_p, y_p) \in \mathfrak{X}^p(\Omega_T)$ satisfies the relations

$$\begin{cases} (1-\lambda)\displaystyle\int_{\Omega_T}\left(\mathrm{K}_\eta[u_p,\mathrm{p}_p]\cdot u + \mathrm{K}_{(A,a)}[u_p,\mathrm{p}_p] : \nabla u\right)\cdot \mathfrak{M}_p\left(\mathrm{K}[u_p,\mathrm{p}_p]\right)\mathrm{d}\mathcal{L}^{n+1} \\ = -\lambda\displaystyle\int_{\Omega_T}\left(\partial_t u - \nu\Delta u + (u\cdot \mathrm{D})u_p + (u_p\cdot \mathrm{D})u\right)\cdot\mathfrak{M}_p(y_p)\mathrm{d}\mathcal{L}^{n+1} + \langle \Psi_p, u(\cdot,0)\rangle, \end{cases} \tag{1.22}$$

$$(1-\lambda)\int_{\Omega_T}\mathrm{K}_r[u_p,\mathrm{p}_p]\mathrm{p}\cdot\mathfrak{M}_p\left(\mathrm{K}[u_p,\mathrm{p}_p]\right)\mathrm{d}\mathcal{L}^{n+1} = -\lambda\int_{\Omega_T}\mathrm{D}\mathrm{p}\cdot\mathfrak{M}_p(y_p)\mathrm{d}\mathcal{L}^{n+1} \tag{1.23}$$

for all test mappings

$$(u, \mathrm{p}) \in W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T; \mathbb{R}^n) \times W^{1,0;p}_{\sharp}(\Omega_T),$$

where the operators $\mathrm{K},\ \mathrm{K}_\eta,\ \mathrm{K}_{(A,a)},\ \mathrm{K}_r$ *are given by* (1.20).

Now we consider the case of $p = \infty$. For this extreme case, which is obtained by an appropriate passage to limits as $p \to \infty$ in theorem 2, we need to assume additionally that the operator $\mathrm{K}[u, \mathrm{p}]$ does not depend on $(\partial_t u, \mathrm{p})$, hence in this case we will symbolise

$$\begin{cases} \mathrm{K}[u] := \mathrm{K}\left(\cdot, \cdot, u, \mathrm{D}u\right), \\ \mathrm{K}_\eta[u] := \mathrm{K}_\eta\left(\cdot, \cdot, u, \mathrm{D}u\right), \\ \mathrm{K}_A[u] := \mathrm{K}_A\left(\cdot, \cdot, u, \mathrm{D}u\right), \end{cases} \tag{1.24}$$

for K and its partial derivatives $\mathrm{K}_\eta,\ \mathrm{K}_A$ with respect to the arguments for $u,\ \mathrm{D}u$ respectively, all of which will also need to be assumed to be continuous. We note that, when $p = \infty$, there is no direct analogue of the divergence structure Euler Lagrange equations. Instead, one of the central points of calculus of variations in L^∞ is that Aronsson–Euler PDE systems may be derived, under appropriate (stringent) assumptions. Even in the unconstrained case, these PDE systems are always non-divergence and even fully nonlinear and with discontinuous coefficients (see e.g. [7, 8, 23, 36, 42]). The case of L^∞ problems involving only first order derivative of scalar-valued functions is nowadays a well established field which originated from the work of Aronsson in the 1960 [4, 5], today largely interconnected to the theory of viscosity solution to nonlinear elliptic PDE (for a general pedagogical introduction see e.g. [22, 35]). However, vectorial and higher L^∞ variational problems involving constraints, have only recently began being explored (see [38, 39], but also the relevant earlier contributions [3, 6, 10]). For several interesting developments on L^∞ variational problems we refer the interested reader to [9, 11, 14, 15, 20, 26, 30, 44, 47–49].

In this paper, motivated by recent progress on higher order and on constrained L^∞ variational problems made in [40] by the author jointly with Moser and by the author in [38, 39] (inspired by earlier contributions by Moser and Schwetlick deployed in a geometric setting in [46]), we follow a slightly different approach which does not lead an Aronsson–Euler type system; instead, it leads to a *divergence structure* PDE system. However, there is a toll to be paid, as the divergence PDEs arising as necessary conditions involve measures as auxiliary parameters whose determination becomes part of the problem. Notwithstanding, the central point of this idea is to use a scaling in the Euler–Lagrange equations before letting $p \to \infty$, which is different from the scaling used to (formally) derive the Aronsson–Euler equations as $p \to \infty$.

In the light of the above comments, our final main result concerns the satisfaction of necessary PDE conditions for the PDE-constrained minimisers in L^∞ constructed in theorem 1, and reads as follows.

Theorem 3 (Variational equations in L^∞). *Suppose that* (1.5) *and* (1.15) *hold true, and that additionally* K *does not depend on* $(\partial_t u, \mathrm{p})$ *with* $\mathrm{K},\ \mathrm{K}_\eta,\ \mathrm{K}_A$ *in* (1.24) *being continuous on* $\overline{\Omega_T} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$. *Then, there exists a linear functional*

$$\Psi_\infty \in \bigcap_{r>n+2} \left(W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega; \mathbb{R}^n) \right)^* \tag{1.25}$$

which is a Lagrange multiplier associated with the constrained minimisation problem (1.16) for p $= \infty$*. There also exist vector measures*

$$\Sigma_\infty \in \mathcal{M}\left(\overline{\Omega_T};\mathbb{R}^N\right), \quad \sigma_\infty \in \mathcal{M}\left(\overline{\Omega_T};\mathbb{R}^n\right) \tag{1.26}$$

such that the minimising triplet $(u_\infty, \mathrm{p}_\infty, y_\infty) \in \mathfrak{X}^\infty(\Omega_T)$ *satisfies the relations*

$$\begin{cases} (1-\lambda)\displaystyle\int_{\overline{\Omega_T}} \big(\mathrm{K}_\eta[u_\infty]\cdot u + \mathrm{K}_A[u_\infty] : \mathrm{D}u\big)\cdot \mathrm{d}\Sigma_\infty \\ \quad = -\lambda \displaystyle\int_{\overline{\Omega_T}} (\partial_t u - \nu\Delta u + (u\cdot \mathrm{D})u_\infty + (u_\infty \cdot \mathrm{D})u)\cdot \mathrm{d}\sigma_\infty + \langle \Psi_\infty, u(\cdot,0)\rangle, \end{cases} \tag{1.27}$$

$$\int_{\overline{\Omega_T}} \mathrm{D}\mathrm{p}\cdot \mathrm{d}\sigma_\infty = 0, \tag{1.28}$$

for all test mappings

$$(u,\mathrm{p}) \in \left(W^{2,1;\infty}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n)\cap C^2\left(\overline{\Omega_T};\mathbb{R}^n\right)\right)\times \left(W^{1,0;\infty}_{\sharp}(\Omega_T)\cap C^1\left(\overline{\Omega_T}\right)\right).$$

Further, the multiplier Ψ_∞ *and the measures* $\Sigma_\infty, \sigma_\infty$ *can be approximated as follows:*

$$\begin{cases} \Psi_p \xrightarrow{\ *\ } \Psi_\infty, & \text{in } \left(W^{2-2/r,r}_{0,\sigma}(\Omega;\mathbb{R}^n)\right)^*, \quad \text{for all } r > n+2, \\ \Sigma_p \xrightarrow{\ *\ } \Sigma_\infty, & \text{in } \mathcal{M}\left(\overline{\Omega_T};\mathbb{R}^N\right), \\ \sigma_p \xrightarrow{\ *\ } \sigma_\infty, & \text{in } \mathcal{M}\left(\overline{\Omega_T};\mathbb{R}^n\right), \end{cases} \tag{1.29}$$

along a subsequence $p_j \to \infty$*, where*

$$\begin{cases} \Sigma_p := \mathfrak{M}_p\left(\mathrm{K}[u_p]\right)\mathcal{L}^{n+1}\llcorner_{\Omega_T}, \\ \sigma_p := \mathfrak{M}_p(y_p)\mathcal{L}^{n+1}\llcorner_{\Omega_T}. \end{cases} \tag{1.30}$$

Finally, Σ_∞ *concentrates on the set whereon* $|\mathrm{K}[u_\infty]|$ *is maximised over* $\overline{\Omega_T}$

$$\Sigma_\infty\left(\left\{|\mathrm{K}[u_\infty]| < \|\mathrm{K}[u_\infty]\|_{L^\infty(\Omega_T)}\right\}\right) = 0, \tag{1.31}$$

and σ_∞ *asymptotically concentrates on the set whereon* $|y_\infty|$ *is approximately maximised over* $\overline{\Omega_T}$*, in the sense that for any* $\varepsilon > 0$ *small,*

$$\lim_{p\to\infty} \sigma_p\left(\{|y_p| < \|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon\}\right) = 0. \tag{1.32}$$

Even though the weak interpretation of the equations (1.22) and (1.23) is relatively obvious, this is not the case for (1.27) and (1.28) despite having a simpler form. The reason is that the limiting measures $(\Sigma_\infty, \sigma_\infty)$ are not product measures on $\overline{\Omega_T} = \overline{\Omega}\times[0,T]$ in order to use the Fubini theorem, therefore due to the temporal dependence, (1.28) cannot be simply interpreted as 'div$(\sigma_\infty) = 0$'. Similar arguments can be made for (1.27) as well. Since this point is not utilised any further in this paper, we only provide a brief discussion in the next section.

We conclude this introduction with some remarks regarding the organisation of this paper. This introduction is followed by section 2, in which we discuss some preliminaries and also establish some basic estimates which are utilised subsequently to establish our main results. In section 3 we prove theorem 1 by establishing the existence of constrained minimisers for all p including $p = \infty$, as well as the convergence of minimiser of the former problems to those of the latter. In section 4 we prove theorem 2, deriving the necessary PDE conditions which constrained minimisers in L^p satisfy. Finally, in section 5 prove theorem 3, deriving the necessary PDE conditions that constrained minimisers in L^∞ satisfy, as well as the additional properties that the measures arising in these PDEs satisfy. A key ingredient here is that we establish appropriate weak* compactness for the Lagrange multipliers arising in the L^p problems in order to pass to the limit as $p \to \infty$.

2. Preliminaries and the main estimates

We begin by recording for later use the following modified Hölder inequality for the dotted $\dot{L}^p$ regularised 'norms' defined in (1.7): for any $1 \leqslant q \leqslant p < \infty$ and $h \in L^p(\Omega_T)$, we have the inequality

$$\|h\|_{\dot{L}^q(\Omega_T)} \leqslant \|h\|_{\dot{L}^p(\Omega_T)} + \sqrt{q^{-2} - p^{-2}},$$

which can be very easily confirmed by a direct computation. Next, we continue with a brief discussion regarding the weak interpretation of the equations (1.27) and (1.28). As already noted in the introduction, since $(\Sigma_\infty, \sigma_\infty)$ are not in general neither product measures nor absolutely continuous with respect to the $(n+1)$-Lebesgue measure on $\overline{\Omega_T} = \overline{\Omega} \times [0, T]$, one needs to use the disintegration 'slicing' theorem for Young measures in order to express them appropriately, as follows. Since σ_∞ is a vector measure in $\mathcal{M}(\overline{\Omega_T}; \mathbb{R}^n)$, by the Radon–Nikodym theorem, we may decompose

$$\sigma_\infty = \frac{\mathrm{d}\sigma_\infty}{\mathrm{d}\|\sigma_\infty\|}\|\sigma_\infty\|,$$

where $\|\sigma_\infty\| \in \mathcal{M}(\overline{\Omega_T})$ is the scalar total variation measure and $\mathrm{d}\sigma_\infty/\mathrm{d}\|\sigma_\infty\|$ is the vector-valued Radon–Nikodym derivative of σ_∞ with respect to $\|\sigma_\infty\|$. Fix now any $h \in L^1(\overline{\Omega_T}, \|\sigma_\infty\|)$. By the disintegration 'slicing' theorem for Young measure (see e.g. [27, theorem 3.2, p 179]), we have the representation formula

$$\int_{\overline{\Omega_T}} h\mathrm{d}\|\sigma_\infty\| = \int_{[0,T]} \left(\int_{\overline{\Omega}} h(x,t)\mathrm{d}\|\sigma_\infty\|_t(x) \right) \mathrm{d}\|\sigma_\infty\|^o(t)$$

where the measure $\|\sigma_\infty\|^o \in \mathcal{M}([0,T])$ and the family of measures $(\|\sigma_\infty\|_t)_{t\in[0,T]} \subseteq \mathcal{M}(\overline{\Omega})$ are defined as follows:

$$\|\sigma_\infty\|^o := \|\sigma_\infty\| \left(\overline{\Omega} \times \cdot\right), \qquad \|\sigma_\infty\|_t(A) := \frac{\mathrm{d}\|\sigma_\infty\| \,(A \times \cdot)}{\mathrm{d}\|\sigma_\infty\| \left(\overline{\Omega} \times \cdot\right)}(t), \quad \text{for } A \subseteq \overline{\Omega} \text{ Borel.}$$

Namely, $\|\sigma_\infty\|^o$ is one of the marginals of σ_∞ and for $\|\sigma_\infty\|^o$-a.e. $t \in [0, T]$, the measure $\|\sigma_\infty\|_t$ evaluated at A is defined as the Radon–Nikodym derivative of the measure $\|\sigma_\infty\| \,(A \times \cdot)$ with respect to $\|\sigma_\infty\| \left(\overline{\Omega} \times \cdot\right)$ at the point $t \in [0, T]$. Then, in view of (1.28), by choosing p in the form $\mathrm{p}(x,t)=\pi(x)\tau(t)$, we have

$$\begin{aligned}0&=\int_{\Omega_T}\mathrm{Dp}\cdot\mathrm{d}\sigma_\infty\\&=\int_{\Omega_T}\left(\mathrm{Dp}\cdot\frac{\mathrm{d}\sigma_\infty}{\mathrm{d}\|\sigma_\infty\|}\right)\mathrm{d}\|\sigma_\infty\|\\&=\int_{[0,T]}\left(\int_\Omega\left(\mathrm{Dp}\cdot\frac{\mathrm{d}\sigma_\infty}{\mathrm{d}\|\sigma_\infty\|}\right)(x,t)\mathrm{d}\|\sigma_\infty\|_t(x)\right)\mathrm{d}\|\sigma_\infty\|^o(t)\\&=\int_{[0,T]}\left(\int_\Omega\left(\mathrm{D}\pi(x)\cdot\frac{\mathrm{d}\sigma_\infty}{\mathrm{d}\|\sigma_\infty\|}(x,t)\right)\mathrm{d}\|\sigma_\infty\|_t(x)\right)\tau(t)\mathrm{d}\|\sigma_\infty\|^o(t).\end{aligned}$$

The arbitrariness of τ implies that for $\|\sigma_\infty\|^o$-a.e. $t\in[0,T]$, we have

$$\int_\Omega\left(\mathrm{D}\pi(x)\cdot\frac{\mathrm{d}\sigma_\infty}{\mathrm{d}\|\sigma_\infty\|}(x,t)\right)\mathrm{d}\|\sigma_\infty\|_t(x)=0.$$

When restricting our attention to those test function for which $\pi|_{\partial\Omega}\equiv0$, we obtain the next weak interpretation of (1.28):

$$\operatorname{div}\left(\frac{\mathrm{d}\sigma_\infty}{\mathrm{d}\|\sigma_\infty\|}(\cdot,t)\|\sigma_\infty\|_t\right)=0,\quad\text{in }\Omega,$$

for $\|\sigma_\infty\|^o$-a.e. $t\in[0,T]$. Similar considerations apply also to equation (1.27), but the arguments are considerably more complicated.

Next we prove a general compact embedding lemma by means of interpolation theory.

Lemma 4. *Suppose that $p>n+2$. Then, there exists $\alpha\in(0,1)$ such that the space*

$$W^{2,1;p}(\Omega_T):=L^p\left((0,T);W^{2,p}(\Omega)\right)\bigcap W^{1,p}\left((0,T);L^p(\Omega)\right)$$

is compactly embedded in the space $C^{0,\alpha}\left([0,T];C^{1,\alpha}(\overline{\Omega})\right)$.

Proof of Lemma 4. Let us use the abbreviated space notation

$$\mathrm{X}_1:=W^{2,p}(\Omega),\quad \mathrm{X}_0:=L^p(\Omega)$$

and select θ such that

$$\frac{p+n}{2p}<\theta<\frac{p-1}{p},$$

which is possible since

$$\frac{p-1}{p}-\frac{p+n}{2p}=\frac{p-(n+2)}{2p}>0.$$

Since $1-\theta>1/p$, direct application of the interpolation result in [1, theorem 5.2] for the exponents $s_0:=1, s_1:=0$ and $p_0\equiv p_1:=p$ yields that space $W^{2,1;p}(\Omega_T)$ is compactly embedded in the space $C^{0,\alpha}([0,T];\mathrm{X})$, where $0<\alpha<1-\theta-1/p$ and $\mathrm{X}=(\mathrm{X}_0,\mathrm{X}_1)_{\theta,p}$ symbolises the real interpolation between the Banach spaces X_0 and X_1. Now it remains to identify the space

X. By using standard results in interpolation theory (see e.g. [53, theorem 4.3.1.1 and formula (2.4.2/9)] or [54] for Lipschitz domains) we get:

$$\left(L^p(\Omega), W^{2,p}(\Omega)\right)_{\theta,p} = \mathrm{B}^{2\theta}_{pp}(\Omega) = W^{2\theta,p}(\Omega).$$

Since $2\theta > 1 + n/p$, by the standard Sobolev embedding theorem for fractional spaces (e.g. [25, theorem 8.2], we have that $W^{2\theta,p}(\Omega)$ is continuously embedded in the space $C^{1,\alpha}(\overline{\Omega})$, where $0 < \alpha \leqslant 2\theta - 1 - n/p$. The conclusion ensues. □

Remark 5. Let us now record for later use the following simple inclusion of space (which is in fact a continuous embedding):

$$C^{0,\alpha}\left([0,T]; C^{0,\alpha}(\overline{\Omega})\right) \subseteq C^{0,\alpha}\left(\overline{\Omega_T}\right).$$

Indeed, for any $h \in C^{0,\alpha}\left([0,T]; C^{0,\alpha}(\overline{\Omega})\right)$, we compute

$$\begin{aligned} |h(t_1,x_1) - h(t_2,x_2)| &\leqslant |h(t_1,x_1) - h(t_2,x_1)| + |h(t_2,x_1) - h(t_2,x_2)| \\ &\leqslant \|h(t_1,\cdot) - h(t_2,\cdot)\|_{C(\overline{\Omega})} + \|h(t_2,\cdot)\|_{C^{0,\alpha}(\overline{\Omega})}|x_1 - x_2|^\alpha \\ &\leqslant \left(|t_1 - t_2|^\alpha + |x_1 - x_2|^\alpha\right)\|h\|_{C^{0,\alpha}([0,T];C^{0,\alpha}(\overline{\Omega}))} \end{aligned}$$

which establishes the claim.

Lemma 6. *Suppose that assumptions* (1.5) *and* (1.15) *are satisfied. We have that*

$$\mathfrak{X}^\infty(\Omega_T) \neq \emptyset$$

(*and consequently we have* $\mathfrak{X}^p(\Omega_T) \neq \emptyset$ *for all* $p > 1$). *Further, if* $(u, \mathrm{p}, y) \in \mathfrak{X}^p(\Omega_T)$ *for some* $p > 1$ *which satisfies*

$$\mathrm{E}_p(u, \mathrm{p}, y) \leqslant M$$

for some $M > 0$, *then for any* $q \leqslant p$ *there exists* $C(q, M) > 0$ *such that*

$$\|u\|_{W^{2,1;q}_{\mathrm{L},\sigma}(\Omega_T)} + \|\mathrm{p}\|_{W^{1,0;q}_{\sharp}(\Omega_T)} + \|y\|_{L^q(\Omega_T)} \leqslant C(q, M).$$

Further, if $p > n + 2$ *and* $q \in (n + 2, p]$, *then there exists* $\alpha \in (0, 1)$ *and a constant* $C(M, q) > 0$ *such that additionally*

$$\|u\|_{C^{0,\alpha}(\Omega_T)} + \|\mathrm{D}u\|_{C^{0,\alpha}(\Omega_T)} \leqslant C(q, M).$$

We note that the constants above also depend on $n, \partial\Omega, T, f, u_0, \lambda$, but as all these are fixed throughout this paper, we suppress denoting the explicit dependence on them.

Proof of Lemma 6. By assumptions (1.5)(b) and (1.5)(c), we have that the triplet $(u_0, 0, y_0)$, where

$$y_0 := -\nu\Delta u_0 + (u_0 \cdot \mathrm{D})u_0 - f$$

satisfies that $(u_0, 0, y_0) \in \mathfrak{X}^\infty(\Omega_T)$, and in fact lies also in the smaller space

$$W^{2,1;\infty}_{\mathrm{L},\sigma}(\Omega_T) \times W^{1,0;\infty}_{\sharp}(\Omega_T) \times L^\infty(\Omega_T; \mathbb{R}^n).$$

Next, if $(u, \mathrm{p}, y) \in \mathfrak{X}^p(\Omega_T)$ with $\mathrm{E}_p(u, \mathrm{p}, y) \leqslant M$, then we readily have that

$$\|y\|_{L^q(\Omega_T)} \leqslant \|y\|_{\dot{L}^p(\Omega_T)} \leqslant \frac{M}{\lambda},$$

whilst by assumptions (1.15) and (1.5)(c) we have that

$$\|u\|_{W^{2,1;q}_{\mathrm{L},\sigma}(\Omega_T)} + \|\mathrm{p}\|_{W^{1,0;q}_{\sharp}(\Omega_T)} \leqslant C(q)\left(1 + \frac{M}{\lambda}\right).$$

for some q-dependent constant $C(q)$, for any $q \in (n, p]$. Further, suppose $p > n + 2$ and $n + 2 < q \leqslant p$. Then, the above estimate in particular implies

$$\|u\|_{L^q(\Omega_T)} + \|\nabla u\|_{L^q(\Omega_T)} \leqslant C(q, M),$$

where at application of the Morrey imbedding theorem yields

$$\|u\|_{C^{0,\alpha'}(\Omega_T)} \leqslant C(q, M),$$

for a new constant $C(q, M)$ and some $\alpha' \in (0, 1)$. Next, by lemma 4, remark 5 and the established estimate for $q > n + 2$, we have

$$C(q)\|\mathrm{D}u\|_{C^{0,\alpha''}(\Omega_T)} \leqslant \|u\|_{W^{2,1;q}(\Omega_T)} \leqslant C(q, M),$$

for some $\alpha'' \in (0, 1)$ and some constant $C(q) > 0$. By choosing $\alpha := \min\{\alpha', \alpha''\}$, the conclusion ensues. $\square$

3. Minimisers of L^p problems and convergence as $p \to \infty$

In this section we establish theorem 1, by utilising the results of section 2.

Proof of Theorem 1. Fix $p \in (n + 2, \infty)$. By lemma 4, $\mathfrak{X}^p(\Omega_T) \neq \emptyset$, therefore $0 \leqslant \inf_{\mathfrak{X}^p(\Omega_T)} \mathrm{E}_p < \infty$. By lemma 6, it follows that $\mathfrak{X}^p(\Omega_T)$ is sequentially weakly compact. Note now that $y \mapsto \|y\|^p_{\dot{L}^p(\Omega_T)}$ is trivially convex, and by the identity

$$\|\mathrm{K}(\cdot, u, \mathrm{D}u, \partial_t u, \mathrm{p})\|^p_{\dot{L}^p(\Omega_T)} = \int_{\Omega_T} \left(|\mathrm{K}(\cdot, u, \mathrm{D}u, \partial_t u, \mathrm{p})|^2 + p^{-2}\right)^{\frac{p}{2}} \mathrm{d}\mathcal{L}^{n+1},$$

assumption (1.5)(f) yields that

$$(\partial_t u, \mathrm{p}) \mapsto \|\mathrm{K}(\cdot, u, \mathrm{D}u, \partial_t u, \mathrm{p})\|^p_{\dot{L}^p(\Omega_T)}$$

is also convex. Again by lemma 6 and by standard results in the calculus of variations (see e.g. [24]), it follows that E_p is weakly lower semicontinuous in $\mathfrak{X}^p(\Omega_T)$ as the convex combination of the pth roots of two weakly lower semicontinuous functionals. Hence, E_p attains its infimum at some $(u_p, \mathrm{p}_p, y_p) \in \mathfrak{X}^p(\Omega_T)$.

Consider now the family of minimisers $(u_p, \mathrm{p}_p, y_p)_{p>n+2}$. For any $(u, \mathrm{p}, y) \in \mathfrak{X}^\infty(\Omega_T)$ and any $q \leqslant p$, minimality and the Hölder inequality for the dotted $\dot{L}^p$ functionals yield

$$\mathrm{E}_p(u_p, \mathrm{p}_p, y_p) \leqslant \mathrm{E}_p(u, \mathrm{p}, y) \leqslant \mathrm{E}_\infty(u, \mathrm{p}, y) + p^{-1}.$$

By choosing $(u, \mathrm{p}, y) = (u_0, 0, y_0)$, by lemma 6 and a standard diagonal argument, we have that the family of minimisers is weakly precompact in $\mathcal{W}^q(\Omega_T)$ for all $q \in (n + 2, \infty)$. Further, by lemma 4 and remark 5, $W^{2,1;q}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n)$ is compactly embedded in $C^{0,\alpha}\big([0,T];C^{1,\alpha}(\overline{\Omega};\mathbb{R}^n)\big)$. Hence, for any sequence of indices $p_j\to\infty$, there exists $(u_\infty,\mathrm{p}_\infty,y_\infty)\in\cap_{q\in(n+2,\infty)}\mathcal{W}^q(\Omega_T)$ and a subsequence denoted again as $(p_j)_1^\infty$ such that (1.17) holds true. Additionally, due to these modes of convergence, it follows that $(u_\infty,\mathrm{p}_\infty,y_\infty)$ solves (1.1), therefore in fact $(u_\infty,\mathrm{p}_\infty,y_\infty)\in\mathfrak{X}^\infty(\Omega_T)$. Again now by minimality and the Hölder inequality for the dotted $\dot{L}^p$ functionals, for any $(u,\mathrm{p},y)\in\mathfrak{X}^\infty(\Omega_T)$ we have

$$\mathrm{E}_q(u_p,\mathrm{p}_p,y_p)-\sqrt{q^{-2}-p^{-2}}\leqslant\mathrm{E}_p(u_p,\mathrm{p}_p,y_p)\leqslant\mathrm{E}_\infty(u,\mathrm{p},y)+p^{-1}.$$

Since as we have already shown, E_q is weakly lower semicontinuous in $\mathfrak{X}^q(\Omega_T)$, by letting $p\to\infty$ along the subsequence in the above inequality yields

$$\begin{aligned}\mathrm{E}_q(u_\infty,\mathrm{p}_\infty,y_\infty)-\sqrt{q^{-2}}&\leqslant\liminf_{p_j\to\infty}\mathrm{E}_p(u_p,\mathrm{p}_p,y_p)\\&\leqslant\limsup_{p_j\to\infty}\mathrm{E}_p(u_p,\mathrm{p}_p,y_p)\\&\leqslant\mathrm{E}_\infty(u,\mathrm{p},y).\end{aligned}$$

By further letting $q\to\infty$, we obtain

$$\begin{aligned}\mathrm{E}_\infty(u_\infty,\mathrm{p}_\infty,y_\infty)&\leqslant\liminf_{p_j\to\infty}\mathrm{E}_p(u_p,\mathrm{p}_p,y_p)\\&\leqslant\limsup_{p_j\to\infty}\mathrm{E}_p(u_p,\mathrm{p}_p,y_p)\\&\leqslant\mathrm{E}_\infty(u,\mathrm{p},y),\end{aligned}$$

for any $(u,\mathrm{p},y)\in\mathfrak{X}^\infty(\Omega_T)$. The above inequality establishes on the one hand that $(u_\infty,\mathrm{p}_\infty,y_\infty)$ minimises E_∞ over $\mathfrak{X}^\infty(\Omega_T)$, and on the other hand by choosing $(u,\mathrm{p},y):=(u_\infty,\mathrm{p}_\infty,y_\infty)$ that (1.18) holds true. Hence, theorem 1 has been established. □

4. The equations for L^p PDE-constrained minimisers

In this section we establish the proof of theorem 2. We begin with some preparation. Firstly, it will be convenient to consider the functional E_p of (1.6) as being defined in the wider Banach space $\mathcal{W}^p(\Omega_T)$ defined in (1.9):

$$\mathrm{E}_p\ :\ \mathcal{W}^p(\Omega_T)\longrightarrow\mathbb{R}.$$

Next, we introduce a mapping on $\mathcal{W}^p(\Omega_T)$ which incorporates the PDE constraint (1.1) appearing in (1.8) as follows. We define

$$\mathrm{G}=\begin{bmatrix}\mathrm{G}_1\\\mathrm{G}_2\end{bmatrix}\ :\ \mathcal{W}^p(\Omega_T)\longrightarrow L^p(\Omega_T;\mathbb{R}^n)\times W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega;\mathbb{R}^n)$$

by setting

$$\begin{cases}\mathrm{G}_1(u,\mathrm{p},y):=\partial_t u-\nu\Delta u+(u\cdot\mathrm{D})u+\mathrm{Dp}-(y+f),\\\mathrm{G}_2(u,\mathrm{p},y):=u(\cdot,0)-u_0.\end{cases}$$

Then, we may express (1.8) as

$$\mathfrak{X}^p(\Omega_T) = \mathcal{W}^p(\Omega_T) \cap \{\mathrm{G} = 0\}.$$

We are now ready to prove our second main result.

Proof of Theorem 2. By assumption (1.5), for any $p \in (n+2, \infty)$ the functional $\mathrm{E}_p : \mathcal{W}^p(\Omega_T) \longrightarrow \mathbb{R}$ is Frechét differentiable and its derivative

$$\mathrm{dE}_p \ : \ \mathcal{W}^p(\Omega_T) \longrightarrow (\mathcal{W}^p(\Omega_T))^*,$$
$$(\mathrm{dE}_p)_{(\bar{u},\bar{\mathrm{p}},\bar{y})}(u,\mathrm{p},y) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathrm{E}_p\left(\bar{u}+\varepsilon u, \bar{\mathrm{p}}+\varepsilon \mathrm{p}, \bar{y}+\varepsilon y\right)$$

can be easily computed and is given by the formula

$$(\mathrm{dE}_p)_{(\bar{u},\bar{\mathrm{p}},\bar{y})}(u,\mathrm{p},y) = p(1-\lambda) \fint_{\Omega_T} \big(\mathrm{K}_\eta[\bar{u},\bar{\mathrm{p}}]\cdot u + \mathrm{K}_{(A,a)}[\bar{u},\bar{\mathrm{p}}] : \nabla u + \mathrm{K}_r[\bar{u},\bar{\mathrm{p}}]\mathrm{p}\big)\cdot$$
$$\cdot\, \mathfrak{M}_p\left(\mathrm{K}[\bar{u},\bar{\mathrm{p}}]\right)\mathrm{d}\mathcal{L}^{n+1} + p\lambda \fint_{\Omega_T} \mathfrak{M}_p(\bar{y})\cdot y \,\mathrm{d}\mathcal{L}^{n+1},$$

where the operator $\mathfrak{M}_p : L^p(\Omega_T;\mathbb{R}^M) \longrightarrow L^{p'}(\Omega_T;\mathbb{R}^M)$ (for $M \in \{N, n\}$) is given by (1.19) and we have used the notation introduced in (1.20). Next, we note that the mapping G which incorporates the PDE constraint is also Fréchet differentiable and it can be easily confirmed that its derivative

$$\mathrm{dG} \ : \ \mathcal{W}^p(\Omega_T) \longrightarrow \mathcal{B}\left(\mathcal{W}^p(\Omega_T), L^p(\Omega_T;\mathbb{R}^n) \times W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega;\mathbb{R}^n)\right),$$
$$(\mathrm{dG})_{(\bar{u},\bar{\mathrm{p}},\bar{y})}(u,\mathrm{p},y) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathrm{G}\left(\bar{u}+\varepsilon u, \bar{\mathrm{p}}+\varepsilon \mathrm{p}, \bar{y}+\varepsilon y\right)$$

is given by the formula

$$(\mathrm{dG})_{(\bar{u},\bar{\mathrm{p}},\bar{y})}(u,\mathrm{p},y) = \begin{bmatrix} \partial_t u - \nu\Delta u + (u\cdot \mathrm{D})\bar{u} + (\bar{u}\cdot\mathrm{D})u + \mathrm{Dp} - y \\ u(\cdot,0) \end{bmatrix}.$$

We now claim that the differential

$$(\mathrm{dG})_{(\bar{u},\bar{\mathrm{p}},\bar{y})} \ : \ \mathcal{W}^p(\Omega_T) \longrightarrow L^p(\Omega_T;\mathbb{R}^n) \times W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega;\mathbb{R}^n)$$

is a surjective map, for any $(\bar{u},\bar{\mathrm{p}},\bar{y}) \in \mathcal{W}^p(\Omega)$. This is equivalent to the statement that for any $p > n+2$, the linearised Navier–Stokes problem

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\mathrm{D})\bar{u} + (\bar{u}\cdot\mathrm{D})u + \mathrm{Dp} = F, & \text{in } \Omega_T, \\ \operatorname{div} u = 0, & \text{in } \Omega_T, \\ u(\cdot,0) = v, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega\times(0,T), \end{cases}$$

has a solution $(u, \mathrm{p}) \in W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n) \times W^{1,0;p}_{\sharp}(\Omega_T)$, for any $\bar{u} \in W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n)$ and any data

$$(F, \upsilon) \in L^p(\Omega_T;\mathbb{R}^n) \times W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega;\mathbb{R}^n).$$

This is indeed the case, and it is a consequence of a classical result of Solonnikov [51, theorem 4.2] for $n = 3$ and of Giga-Sohr [34, theorem 2.8] for $n > 3$, as a perturbation of the Stokes problem. As a consequence, the assumptions of the generalised Kuhn–Tucker theorem hold true (see e.g. Zeidler [55, corollary 48.10 & theorem 48B]). Hence, there exists a Lagrange multiplier

$$\Lambda_p \in \left(L^p(\Omega_T;\mathbb{R}^n) \times W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega_T;\mathbb{R}^n)\right)^*$$

such that

$$\left(\mathrm{dE}_p\right)_{(u_p,\mathrm{p}_p,y_p)}(u,\mathrm{p},y) = \left\langle (\mathrm{dG})_{(u_p,\mathrm{p}_p,y_p)}(u,\mathrm{p},y), \Lambda_p \right\rangle,$$

for any $(u, \mathrm{p}, y) \in \mathcal{W}^p(\Omega)$. By standard duality arguments, the Riesz representation theorem and by taking into account the form of the differentials dE_p and dG, we may identify Λ_p with a pair of Lagrange multipliers

$$(\phi_p, \Psi_p) \in L^{p'}(\Omega_T;\mathbb{R}^n) \times \left(W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega_T;\mathbb{R}^n)\right)^*$$

such that, the constrained minimiser $\left(u_p, p_p, y_p\right) \in \mathfrak{X}^p(\Omega_T)$ satisfies the equation

$$\begin{aligned}(1-\lambda)\int_{\Omega_T} &\left(\mathrm{K}_\eta[u_p,\mathrm{p}_p]\cdot u + \mathrm{K}_{(A,a)}[u_p,\mathrm{p}_p] : \nabla u + \mathrm{K}_r[u_p,\mathrm{p}_p]\mathrm{p}\right)\\ &\cdot \mathfrak{M}_p\left(\mathrm{K}[u_p,\mathrm{p}_p]\right)\mathrm{d}\mathcal{L}^{n+1} + \lambda\int_{\Omega_T}\mathfrak{M}_p(y_p)\cdot y\,\mathrm{d}\mathcal{L}^{n+1}\\ &= \int_{\Omega_T}\left(\partial_t u - \nu\Delta u + (u\cdot \mathrm{D})u_p + (u_p\cdot\mathrm{D})u + \mathrm{Dp} - y\right)\cdot\phi_p\,\mathrm{d}\mathcal{L}^{n+1} + \langle \Psi_p, u(\cdot,0)\rangle,\end{aligned}$$

for any $(u, \mathrm{p}, y) \in \mathcal{W}^p(\Omega_T)$. We note that here we have tacitly rescaled (ϕ_p, Ψ_p) by multiplying them with the factor $p(\mathcal{L}^{n+1}(\Omega_T))^{-1}$, in order to remove the averages arising from E_p on the left-hand side and to be able to obtain non-trivial limits as $p \to \infty$ of the multipliers themselves later on. By using linear independence, the above equation actually decouples to the triplet of relations

$$\left\{\begin{aligned}&(1-\lambda)\int_{\Omega_T}\left(\mathrm{K}_\eta[u_p,\mathrm{p}_p]\cdot u + \mathrm{K}_{(A,a)}[u_p,\mathrm{p}_p] : \nabla u\right)\cdot\mathfrak{M}_p\left(\mathrm{K}[u_p,\mathrm{p}_p]\right)\mathrm{d}\mathcal{L}^{n+1}\\ &= \int_{\Omega_T}\left(\partial_t u - \nu\Delta u + (u\cdot\mathrm{D})u_p + (u_p\cdot\mathrm{D})u\right)\cdot\phi_p\,\mathrm{d}\mathcal{L}^{n+1} + \langle\Psi_p, u(\cdot,0)\rangle,\end{aligned}\right.$$

$$(1-\lambda)\int_{\Omega_T}\left(\mathrm{K}_r[u_p,\mathrm{p}_p]\mathrm{p}\right)\cdot\mathfrak{M}_p\left(\mathrm{K}[u_p,\mathrm{p}_p]\right)\mathrm{d}\mathcal{L}^{n+1} = \int_{\Omega_T}\mathrm{Dp}\cdot\phi_p\,\mathrm{d}\mathcal{L}^{n+1},$$

$$\lambda\int_{\Omega_T}\mathfrak{M}_p(y_p)\cdot y\,\mathrm{d}\mathcal{L}^{n+1} = -\int_{\Omega_T} y\cdot\phi_p\,\mathrm{d}\mathcal{L}^{n+1}.$$

The arbitrariness of $y \in L^p(\Omega_T;\mathbb{R}^n)$ in the third equation readily yields that the multiplier ϕ_p equals

$$\phi_p = -\lambda \mathfrak{M}_p(y_p).$$

By substituting this into the first two equations, we see that the theorem has been established. □

5. The equations for L^∞ PDE-constrained minimisers

In this section we establish our final main result. In this case we need to assume that K does not depend on $(\partial_t u, \mathrm{p})$ and we will invoke the symbolisations (1.24) for K and its derivatives $\mathrm{K}_\eta,\ \mathrm{K}_A$, all of which are additionally assumed to be continuous.

Proof of Theorem 3. By theorem 2 it follows that for any $p \in (n+2,\infty)$, the minimising triplet $(u_p, \mathrm{p}_p, y_p) \in \mathfrak{X}^p(\Omega_T)$ satisfies

$$\begin{cases} (1-\lambda)\displaystyle\int_{\Omega_T} \big(\mathrm{K}_\eta[u_p]\cdot u + \mathrm{K}_A[u_p] : \mathrm{D}u\big)\cdot \mathfrak{M}_p\left(\mathrm{K}[u_p]\right) \mathrm{d}\mathcal{L}^{n+1} \\ \quad = -\lambda \displaystyle\int_{\Omega_T} \big(\partial_t u - \nu\Delta u + (u\cdot \mathrm{D})u_p + (u_p\cdot \mathrm{D})u\big)\cdot \mathfrak{M}_p(y_p)\mathrm{d}\mathcal{L}^{n+1} + \langle \Psi_p, u(\cdot,0)\rangle, \end{cases}$$

and also

$$\int_{\Omega_T} \mathrm{Dp}\cdot \mathfrak{M}_p(y_p)\mathrm{d}\mathcal{L}^{n+1} = 0,$$

for all test mappings $(u,\mathrm{p}) \in W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n) \times W^{1,0;p}_{\sharp}(\Omega_T)$. The first goal is to pass to the limit as $p\to\infty$ in these equations in order to obtain (1.27) and (1.28). Since by (1.19) we readily have that $\mathfrak{M}_p(y_p)$ and $\mathfrak{M}_p\left(\mathrm{K}[u_p]\right)$ are valued in the unit balls of $L^{p'}(\Omega_T;\mathbb{R}^n)$ and of $L^{p'}(\Omega_T;\mathbb{R}^N)$ respectively, by defining Σ_p and σ_p as in (1.30), the existence of limiting measures $\Sigma_\infty, \sigma_\infty$ is guaranteed along perhaps a further subsequence such that

$$\Sigma_p \xrightarrow{*} \Sigma_\infty \quad \text{in } \mathcal{M}\left(\overline{\Omega_T};\mathbb{R}^N\right) \quad \text{and} \quad \sigma_p \xrightarrow{*} \sigma_\infty \quad \text{in } \mathcal{M}\left(\overline{\Omega_T};\mathbb{R}^n\right),$$

as $p_j\to\infty$. Further, by lemma 6 we have that $u_p \to u_\infty$ and $\mathrm{D}u_p \to \mathrm{D}u_\infty$, both uniformly on $\overline{\Omega_T}$ as $p_j\to\infty$. Also, by the continuity assumption on $\mathrm{K},\ \mathrm{K}_\eta,\ \mathrm{K}_A$ on $\overline{\Omega_T}\times\mathbb{R}^n\times\mathbb{R}^{n\times n}$ and again lemma 6, it follows that

$$\mathrm{K}[u_p] \longrightarrow \mathrm{K}[u_\infty],\ \mathrm{K}_\eta[u_p] \longrightarrow \mathrm{K}_\eta[u_\infty] \quad \text{and} \quad \mathrm{K}_A[u_p] \longrightarrow \mathrm{K}_A[u_\infty],$$

all uniformly on $\overline{\Omega_T}$ as $p_j\to\infty$. Putting all this together, we see that the remaining main point is to obtain a uniform estimate on the family of Lagrange multipliers $(\Psi_p)_{p>n+2}$ in order to deduce that

$$\Psi_p \xrightarrow{*} \Psi_\infty \quad \text{in } \left(W^{2-2/r,r}_{0,\sigma}(\Omega;\mathbb{R}^n)\right)^*, \quad \text{for all } r > n+2,$$

which would allow to pass to the limit as $p_j\to\infty$. Once this has been achieved, passing to the limit in the equations follows by standard duality pairing arguments, which are made possible by restricting the class of test functions (u,p) to those which are continuous together with those derivatives appearing in the relations.

In order to derive the desired estimate on $(\Psi_p)_{p>n+2}$, we argue as follows. Consider (1.22) for $\mathrm{K}_a \equiv 0$ (the first equation appearing in this proof) and let us fix the initial value on $\Omega \times \{0\}$

$$u(\cdot, 0) \equiv \hat{u} \in W^{2,\infty}_{0,\sigma}(\Omega; \mathbb{R}^n)$$

of the arbitrary test function u, but we will select u on Ω_T such that the term in the bracket in the integral on the right-hand-side becomes a gradient. Then, this term will vanish identically as a consequence of (1.23) when $\mathrm{K}_r \equiv 0$ (the second equation appearing in this proof). Indeed, let $p > n + 2$ and let also $(\tilde{u}, \tilde{\mathrm{p}})$ be the (unique) solution to

$$\begin{cases} \partial_t \tilde{u} - \nu \Delta \tilde{u} + (\tilde{u} \cdot \mathrm{D}) u_p + (u_p \cdot \mathrm{D}) \tilde{u} + \mathrm{D}\tilde{\mathrm{p}} = 0, & \text{in } \Omega_T, \\ \operatorname{div} \tilde{u} = 0, & \text{in } \Omega_T, \\ \tilde{u}(\cdot, 0) = \hat{u}, & \text{on } \Omega, \\ \tilde{u} = 0, & \text{on } \partial\Omega \times (0, T), \end{cases}$$

The solvability of the above problem is a consequence of the classical result of Solonnikov [51, theorem 4.2] for $n = 3$ and of Giga–Sohr [34, theorem 2.8] for $n > 3$, as a perturbation of the Stokes problem: by choosing $q > n + 2$ in Solonnikov's assumption (4.14), a solution as claimed does exist. Further, since $\hat{u}$ is in $W^{2,\infty}_{0,\sigma}(\Omega; \mathbb{R}^n)$, by [51, corollary 2, p 489] we have the uniform estimate

$$\|\tilde{u}\|_{W^{2,1;r}_{\mathrm{L},\sigma}(\Omega_T)} + \|\tilde{\mathrm{p}}\|_{W^{1,0;r}_{\sharp}(\Omega_T)} \leqslant C(r) \|\hat{u}\|_{W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega)},$$

for any $r \in (1, \infty)$. By lemmas 4 and 6 and remark 5, if we restrict our attention to $r \in (n + 2, \infty)$, we additionally have the bound

$$\|\tilde{u}\|_{L^\infty(\Omega_T)} + \|\mathrm{D}\tilde{u}\|_{L^\infty(\Omega_T)} \leqslant C(r) \|\hat{u}\|_{W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega)},$$

for some new constant $C(r)$ (which is unbounded as $r \searrow n + 2$). By setting

$$\begin{cases} K_\infty := \sup \left\{ |\mathrm{K}_\eta| + |\mathrm{K}_A| : \ \Omega_T \times \mathbb{B}^n_{R_\infty}(0) \times \mathbb{B}^{n\times n}_{R_\infty}(0) \right\}, \\ R_\infty := \sup\limits_{p>n+2} \left(\|u_p\|_{L^\infty(\Omega_T)} + \|\mathrm{D}u_p\|_{L^\infty(\Omega_T)} \right), \end{cases}$$

where $\mathbb{B}^n_{R_\infty}(0)$ and $\mathbb{B}^{n\times n}_{R_\infty}(0)$ denote the balls of radius R_∞ centred at the origin of $\mathbb{R}^n$ and of $\mathbb{R}^{n\times n}$ respectively, we estimate by using (1.22) and (1.23) (for $\mathrm{K}_a \equiv 0$, $\mathrm{K}_r \equiv 0$) and that by (1.19) we have $\left\| \mathfrak{M}_p \left(\mathrm{K}[u_p] \right) \right\|_{L^1(\Omega_T)} \leqslant 1$ (for the normalised L^1 norm):

$$\begin{aligned} |\langle \Psi_p, \hat{u} \rangle| \leqslant\ & \lambda \left| \int_{\Omega_T} \mathrm{D}\tilde{\mathrm{p}} \cdot \mathfrak{M}_p(y_p) \mathrm{d}\mathcal{L}^{n+1} \right| \\ & + (1 - \lambda) \left| \int_{\Omega_T} \left(\mathrm{K}_\eta[u_p] \cdot \tilde{u} + \mathrm{K}_A[u_p] : \mathrm{D}\tilde{u} \right) \cdot \mathfrak{M}_p \left(\mathrm{K}[u_p] \right) \mathrm{d}\mathcal{L}^{n+1} \right| \\ =\ & (1 - \lambda) T \mathcal{L}^n(\Omega) \left| \fint_{\Omega_T} \left(\mathrm{K}_\eta[u_p] \cdot \tilde{u} + \mathrm{K}_A[u_p] : \mathrm{D}\tilde{u} \right) \cdot \mathfrak{M}_p \left(\mathrm{K}[u_p] \right) \mathrm{d}\mathcal{L}^{n+1} \right| \\ \leqslant\ & (1 - \lambda) T \mathcal{L}^n(\Omega) K_\infty \left(\|\tilde{u}\|_{L^\infty(\Omega_T)} + \|\mathrm{D}\tilde{u}\|_{L^\infty(\Omega_T)} \right) \\ \leqslant\ & (1 - \lambda) T \mathcal{L}^n(\Omega) K_\infty C(r) \|\hat{u}\|_{W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega)}, \end{aligned}$$

for any r fixed. Therefore, for any $r \in (n+2, \infty)$ and any $\hat{u} \in W^{2,\infty}_{0,\sigma}(\Omega; \mathbb{R}^n)$, we have the estimate

$$|\langle \Psi_p, \hat{u} \rangle| \leqslant (T\mathcal{L}^n(\Omega) K_\infty)\, C(r) \|\hat{u}\|_{W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega)}.$$

Since $W^{2,\infty}_{0,\sigma}(\Omega; \mathbb{R}^n)$ is dense in $W^{2-2/r,r}_{0,\sigma}(\Omega; \mathbb{R}^n)$, by the Hahn–Banach theorem, the above estimate implies that for any fixed $p > n+2$, the bounded linear functional

$$\Psi_p \ : \ W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega; \mathbb{R}^n) \longrightarrow \mathbb{R}$$

can be (uniquely) extended to a functional $\Psi_p : W^{2-2/r,r}_{0,\sigma}(\Omega; \mathbb{R}^n) \longrightarrow \mathbb{R}$ for all $r \in (n+2, p]$, whose extension we denote again by Ψ_p. Therefore, Ψ_p can be extended to a unique continuous linear functional

$$\Psi_p \ : \ \bigcup_{r>n+2} W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega; \mathbb{R}^n) \longrightarrow \mathbb{R}$$

on the above Fréchet space, whose topology can be defined in the standard locally convex sense by the family of seminorms

$$\left\{ \| \cdot \|_{W^{2-2/r,r}(\Omega)} : \ r > n+2 \right\}.$$

Additionally, the uniformity of the estimate with respect to p implies that

$$(\Psi_p)_{p>n+2} \text{ is bounded in } \left(\bigcup_{r>n+2} W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega; \mathbb{R}^n) \right)^*$$

(in the locally convex sense). Hence, as it can be seen by a customary diagonal argument in the scale of Banach spaces $\left\{ W^{2-2/r,r}_{0,\sigma}(\Omega_T; \mathbb{R}^n) : r > n+2 \right\}$ comprising the Fréchet space, there exists a continuous linear functional

$$\Psi_\infty \ : \ \bigcup_{r>n+2} W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega; \mathbb{R}^n) \longrightarrow \mathbb{R}$$

and a further subsequence as $p \to \infty$ such that along which we have $\Psi_p \xrightarrow{*} \Psi_\infty$ in the locally convex sense. Additionally, since

$$\Psi_\infty \in \bigcap_{r>n+2} \left(W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega; \mathbb{R}^n) \right)^*$$

the convergence $\Psi_p \xrightarrow{*} \Psi_\infty$ is equivalent to weak* convergence in the Banach space $W^{2-2/r,r}_{0,\sigma}(\Omega; \mathbb{R}^n)$ for any fixed $r > n+2$. In conclusion, we see that (1.27) and (1.28) have now been established.

Now we complete the proof of theorem 3 by establishing (1.31) and (1.32). Since $\mathrm{K}[u_p] \to \mathrm{K}[u_\infty]$ in $C\left(\overline{\Omega_T}; \mathbb{R}^N\right)$, by applying [37, proposition 10], we immediately obtain that Σ_∞ concentrates on the set whereon $|\mathrm{K}[u_\infty]|$ is maximised over $\overline{\Omega_T}$:

$$\Sigma_\infty \left(\left\{ |\mathrm{K}[u_\infty]| < \max_{\overline{\Omega_T}} |\mathrm{K}[u_\infty]| \right\} \right) = 0.$$

This proves (1.31). For (1.32), we argue as follows. We first note that

$$\|y_p\|_{\dot{L}^p(\Omega_T)} \longrightarrow \|y_\infty\|_{L^\infty(\Omega_T)}$$

as $p \to \infty$, along a subsequence. In view of (1.6) and (1.13), this is a consequence of (1.18) and the fact that $\mathrm{K}[u_p] \to \mathrm{K}[u_\infty]$ uniformly on $\overline{\Omega_T}$, which implies

$$\|\mathrm{K}[u_p]\|_{\dot{L}^p(\Omega_T)} \longrightarrow \|\mathrm{K}[u_\infty]\|_{L^\infty(\Omega_T)}.$$

As a consequence of the convergence of $\|y_p\|_{\dot{L}^p(\Omega_T)}$ to $\|y_\infty\|_{L^\infty(\Omega_T)}$, for any $\varepsilon > 0$ we may choose p large so that

$$\|y_p\|_{\dot{L}^p(\Omega_T)} \geqslant \|y_\infty\|_{L^\infty(\Omega_T)} - \frac{\varepsilon}{2}.$$

Let us define now the following subset of Ω_T, which without loss of generality we may assume it has positive $\mathcal{L}^{n+1}$-measure:

$$A_{p,\varepsilon} := \left\{|y_p| \leqslant \|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon\right\}.$$

In particular, if $\mathcal{L}^{n+1}(A_{p,\varepsilon}) > 0$, then necessarily $\|y_\infty\|_{L^\infty(\Omega_T)} > 0$. For any Borel set $B \subseteq \Omega_T$ such that $\mathcal{L}^{n+1}(\Omega_T \cap B) > 0$, we estimate by using (1.30), (1.19) and (1.7) and the above:

$$\begin{aligned}
\sigma_p(A_{p,\varepsilon} \cap B) &\leqslant \frac{\mathcal{L}^{n+1}(A_{p,\varepsilon} \cap B)}{\|y_p\|^{p-1}_{\dot{L}^p(\Omega_T)}} \fint_{A_{p,\varepsilon}\cap B} \left(|y_p|_{(p)}\right)^{p-1} \mathrm{d}\mathcal{L}^{n+1} \\
&\leqslant \frac{\mathcal{L}^{n+1}(A_{p,\varepsilon} \cap B)}{\|y_p\|^{p-1}_{\dot{L}^p(\Omega_T)}} \fint_{A_{p,\varepsilon}\cap B} \left(\|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon\right)^{p-1} \mathrm{d}\mathcal{L}^{n+1} \\
&\leqslant \frac{\mathcal{L}^{n+1}(A_{p,\varepsilon} \cap B)}{\|y_p\|^{p-1}_{\dot{L}^p(\Omega_T)}} \left(\|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon\right)^{p-1} \\
&\leqslant \mathcal{L}^{n+1}(A_{p,\varepsilon} \cap B) \left(\frac{\|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon}{\|y_\infty\|_{L^\infty(\Omega_T)} - \frac{\varepsilon}{2}}\right)^{p-1}.
\end{aligned}$$

As a result, for any $\varepsilon > 0$ small, any p large enough and any Borel set $B \subseteq \Omega_T$ with $\mathcal{L}^{n+1}(\Omega_T \cap B) > 0$, we have obtained the density estimate

$$\frac{\sigma_p(A_{p,\varepsilon} \cap B)}{\mathcal{L}^{n+1}(A_{p,\varepsilon} \cap B)} \leqslant \left(1 - \frac{\varepsilon}{2\|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon}\right)^{p-1}.$$

The above estimate in particular implies that $\sigma_p(A_{p,\varepsilon}) \to 0$ as $p \to \infty$ for any $\varepsilon > 0$ fixed, therefore establishing (1.32). The proof of theorem 3 is now complete. □

Remark 7. It is perhaps worth noting (in relation to the preceding arguments in the proof of (1.32)) that the modes of convergence

$$\|y_p\|_{L^p(\Omega_T)} \longrightarrow \|y_\infty\|_{L^\infty(\Omega_T)} \quad \text{and} \quad y_p \overset{*}{\longrightarrow} y_\infty \quad \text{in } L^\infty(\Omega_T; \mathbb{R}^n)$$

as $p \to \infty$, in general by themselves do not suffice to obtain $y_p \to y_\infty$ in any strong sense, hence precluding the derivation of a stronger property than (1.32), along the lines of (1.31). A simple counter-example, even in one dimension, is the following: let $p \in 2\mathbb{N}$ and set

$$y_p := \sum_{j=0}^{(p-2)/2} \left[\chi_{\left(\frac{2j}{p},\frac{2j+1}{p}\right)} - \chi_{\left(\frac{2j+1}{p},\frac{2j+2}{p}\right)} \right] + \chi_{(1,2)},$$

and also $y_\infty := \chi_{(1,2)}$. Then, we have $|y_p| = 1$ $\mathcal{L}^1$-a.e. on $(0, 2)$ for all p, hence we deduce that $\|y_p\|_{L^p(0,2)} \longrightarrow \|y_\infty\|_{L^\infty(0,2)}$, whilst we also have $y_p \overset{*}{\longrightarrow} y_\infty$ in $L^\infty(0, 2)$ as $p \to \infty$, but $y_p \not\longrightarrow y_\infty$ neither a.e., nor in L^1 or in measure.

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