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SCHATTEN CLASS HANKEL OPERATORS ON THE SEGAL-BARGMANN SPACE AND THE BERGER-COBURN PHENOMENON

ZHANGJIAN HU AND JANI A. VIRTANEN

ABSTRACT. We give a complete characterization of Schatten class Hankel operators H_f acting on weighted Segal-Bargmann spaces $F^2(\varphi)$ using the notion of integral distance to analytic functions in \mathbb{C}^n and Hörmander's $\bar{\partial}$ -theory. Using our characterization, for $f \in L^\infty$ and $1 < p < \infty$, we prove that H_f is in the Schatten class S_p if and only if $H_{\bar{f}} \in S_p$, which was previously known only for the Hilbert-Schmidt class S_2 of the standard Segal-Bargmann space $F^2(\varphi)$ with $\varphi(z) = \alpha|z|^2$.

1. INTRODUCTION AND MAIN RESULTS

We denote by F^2 the (classical) Segal-Bargmann space of Gaussian square-integrable entire functions on \mathbb{C}^n and let P be the orthogonal projection of L^2 onto F^2 . For a bounded function f , the Hankel operator

$$H_f = (I - P)M_f$$

is a bounded linear operator on F^2 , where $M_f g = fg$ is the multiplication operator and I is the identity operator. In addition to F^2 , which is a central subject in quantum physics, we also consider Hankel operators on weighted Segal-Bargmann spaces. Besides intrinsic interest in operator theory, the treatment of weighted spaces may be useful to understand more complicated quantum phenomena.

A unique feature in the theory of the Segal-Bargmann space is the property that the Hankel operator H_f is compact if and only if $H_{\bar{f}}$ is compact when $f \in L^\infty$. This result of Berger and Coburn [4] is not true for Hankel operators on the Bergman space or the Hardy space. In [27], Zhu conjectured that a possible explanation for this difference is the lack of bounded analytic functions on the entire complex plane, which was indeed confirmed recently in [11]. A natural question then arises as to whether an analogous phenomenon holds true for Hankel operators in the Schatten classes S_p (defined in Section 2.4). This was answered in the affirmative for the Hilbert-Schmidt class S_2 by Bauer [3] while Xia and Zheng [25] stated that the remaining cases “appear to be rather challenging.” In this paper (see Theorem 1.2), we prove that $H_f \in S_p$ if and only if $H_{\bar{f}} \in S_p$ when $1 < p < \infty$, which we refer to as the Berger-Coburn phenomenon on S_p .

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Bauer's proof in [3] is elementary and similar to Stroethoff's characterization of compact Hankel operators in [24]. The proof further depends heavily on the properties of the Hilbert-Schmidt norm, which makes Bauer's techniques unsuitable for the other Schatten classes. Instead, our approach is based on the characterization of the Schatten class membership of single Hankel operators, given in Theorem 1.1, which curiously had not been established before despite analogous results for Hankel operators on the Bergman space due to Luecking [21].

While Theorems 1.1 and 1.2 are new even in F^2 , we prove them for Hankel operators on weighted Segal-Bargmann spaces $F^2(\varphi)$, which include all standard Segal-Bargmann and Fock-Sobolev spaces defined below. Regarding terminology, we note that $F^2(\varphi)$ are also referred to as Fock spaces and Bargmann-Fock spaces.

1.1. Main results. The weighted Segal-Bargmann space $F^2(\varphi)$ consists of all entire functions $f : \mathbb{C}^n \rightarrow \mathbb{C}$ that belong to $L^2(\varphi) = L^2(\mathbb{C}^n, e^{-2\varphi} dv)$, where $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}$ is a suitable weight and dv is the Lebesgue measure on \mathbb{C}^n .

In this paper, we consider real-valued weights $\varphi \in C^2(\mathbb{C}^n)$ that satisfy the property that there are two positive constants m and M such that

$$(1.1) \quad m\omega_0 \leq i\partial\bar{\partial}\varphi \leq M\omega_0$$

in the sense of currents, where $\omega_0 = i\partial\bar{\partial}|z|^2$ is the Euclidean-Kähler form on \mathbb{C}^n . The expression (1.1) is also denoted by $i\partial\bar{\partial}\varphi \simeq \omega_0$ and it simplifies to the form $m \leq \Delta\varphi \leq M$, where $\Delta\varphi$ is the Laplacian of φ , when $n = 1$. Notice that the standard weights $\varphi(z) = \alpha|z|^2$ with $\alpha > 0$ (see, e.g., [27]) satisfy (1.1) and each Fock-Sobolev space $F^{2,m}$ consisting of entire functions f for which $\partial^\alpha f \in F^2$ for all multi-indices $|\alpha| \leq m$ (see [6]) can also be viewed as a $F^2(\varphi)$ with some φ satisfying (1.1).

As above, we denote by P the orthogonal (Bergman) projection of $L^2(\varphi)$ onto $F^2(\varphi)$. Let $\Gamma = \text{span}\{K_z : z \in \mathbb{C}^n\}$, where $K_z = K(\cdot, z)$ is the reproducing kernel of $F^2(\varphi)$ (see Section 2.1), and consider the class of symbols

$$\mathcal{S} = \{f \text{ measurable on } \mathbb{C}^n : fg \in L^2(\varphi) \text{ for } g \in \Gamma\}.$$

Given $f \in \mathcal{S}$ and $g \in \Gamma$, the Hankel operator H_f is well defined, and since Γ is dense in $F^2(\varphi)$, it follows that $H_f = (I - P)M_f$ is densely defined on $F^2(\varphi)$. Notice that clearly $L^\infty \subset \mathcal{S}$.

To state our main results, we define

$$(1.2) \quad G_r(f)(z) = \inf_{h \in H(B(z,r))} \left(\frac{1}{|B(z,r)|} \int_{B(z,r)} |f - h|^2 dv \right)^{\frac{1}{2}} \quad (z \in \mathbb{C}^n)$$

for $f \in L^2_{\text{loc}}$ (the set of all locally square integrable functions on \mathbb{C}^n), where $H(B(z,r))$ is the set of all holomorphic functions on $B(z,r) = \{w \in \mathbb{C}^n : |z - w| < r\}$ and $|B(z,r)| = \int_{B(z,r)} dv$.

For $0 < s \leq \infty$, the space IDA^s (Integral Distance to Analytic Functions) consists of all $f \in L^2_{\text{loc}}$ such that

$$(1.3) \quad \|f\|_{\text{IDA}^s} = \|G_r(f)\|_{L^s} < \infty$$

for some $r > 0$. We write BDA for IDA^∞ . The space VDA consists of all $f \in L^2_{\text{loc}}$ such that $\lim_{z \rightarrow \infty} G_r(f)(z) = 0$ for some $r > 0$.

The spaces IDA^s with $s < \infty$ (and their generalizations $\text{IDA}^{s,p}$ with the convention that $\text{IDA}^s = \text{IDA}^{s,2}$) were introduced in [13], while the notion of bounded

distance to analytic functions (BDA) was introduced by Luecking [21] in the context of the Bergman space.

We can now state our main results on Schatten class Hankel operators.

Theorem 1.1. *Let $0 < p < \infty$ and suppose that $\varphi \in C^2(\mathbb{C}^n)$ is real valued with $i\partial\bar{\partial}\varphi \simeq \omega_0$. Then for $f \in \mathcal{S}$, the following statements are equivalent:*

- (A) $H_f : F^2(\varphi) \rightarrow L^2(\varphi)$ is in S_p .
- (B) $f \in \text{IDA}^p$.
- (C) $\int_{\mathbb{C}^n} \|H_f(k_z)\|^p dv(z) < \infty$, where $k_z(\cdot) = \frac{K(\cdot, z)}{\sqrt{K(z, z)}}$ is the normalized reproducing kernel.

Furthermore,

$$(1.4) \quad \|H_f\|_{S_p} \simeq \|f\|_{\text{IDA}^p} \simeq \left\{ \int_{\mathbb{C}^n} \|H_f(k_z)\|^p dv(z) \right\}^{\frac{1}{p}}.$$

Theorem 1.2. *Suppose $\varphi \in C^2(\mathbb{C}^n)$ is real valued, $i\partial\bar{\partial}\varphi \simeq \omega_0$, and $1 < p < \infty$. Then for $f \in L^\infty$, $H_f \in S_p$ if and only if $H_{\bar{f}} \in S_p$ with the S_p -norm estimate*

$$(1.5) \quad \|H_{\bar{f}}\|_{S_p} \leq C \|H_f\|_{S_p},$$

where the constant C is independent of f .

Notice that Theorem 1.2 fails in general if the symbol f is not bounded—see [3] for an example.

1.2. Outline and further results. In the next section we provide preliminaries on the reproducing kernel, which includes global and local estimates, a consequence of Hörmander's existence theorem, and we also extend the decomposition theorem in [13] for IDA functions. In Section 3, we briefly introduce Toeplitz operators and state a description of their Schatten class properties. Section 4 extends our recent results in [13] on boundedness and compactness of Hankel operators, and we compare them to the results of Stroethoff [24] for *bounded symbols* in the setting of the classical Segal-Bargmann space F^2 .

In Section 5, we prove our characterization of Schatten class Hankel operators using the decomposition theorem and other preliminary results, theory of Schatten class Toeplitz operators, and various estimates together with the general theory of Schatten class operators. As a consequence, when $\varphi(z) = \alpha|z|^2$, we obtain a characterization in a familiar form that agrees with one of the main results in [3] obtained previously only when $p = 2$.

In Section 6, we recall briefly the results in [16, 25] on the simultaneous membership in S_p of the Hankel operators H_f and $H_{\bar{f}}$ acting on F^2 , and extend them to $F^2(\varphi)$ using Theorem 1.1.

In Section 7, we prove our result on the Berger-Coburn phenomenon using the Ahlfors-Beurling operator (to obtain the estimates $\|\partial f\|_{L^p} \lesssim \|\bar{\partial} f\|_{L^p}$) together with our characterization of Schatten class Hankel operators.

On the methodological level, it is worth noting that the previous techniques employed in [3, 4, 11, 25], such as the use of Weyl operators, limit operators or explicit formulas for the Bergman kernel, seem insufficient to obtain our characterizations even in setting of the classical Segal-Bargmann space F^2 . In relation to the Bergman space A^2 , Luecking [21] gave a characterization for H_f on A^2 of the unit disk to be in S_p when $1 \leq p < \infty$, and further indicated that his proofs can be

extended to handle any *bounded* strongly pseudoconvex domain in \mathbb{C}^n . Luecking's work and in particular the concept that he referred to as the "bounded distance to analytic functions" are of fundamental importance to our characterizations (see also [20] for bounded strongly pseudoconvex domains).

While several aspects of the theory of Hankel operators on the Segal-Bargmann space are different from A^2 and H^2 , there are also many similarities, such as the role of BMO-type spaces and their decompositions, which we use to closely track the results for the Bergman space.

2. PRELIMINARIES

2.1. The reproducing kernel function. Let $\varphi \in C^2(\mathbb{C}^n)$ be a real-valued weight such that $i\partial\bar{\partial}\varphi \simeq \omega_0$, see (1.1). Most of the basic properties of $F^2(\varphi)$ can be derived from the following weighted Bergman inequality (see Proposition 2.3 of [23] for further details).

Lemma 2.1. *For each $r > 0$, there is a constant $C > 0$ such that*

$$\left| f(z)e^{-\varphi(z)} \right|^2 \leq C \int_{B(z,r)} \left| f(\xi)e^{-\varphi(\xi)} \right|^2 dv(\xi)$$

for all $f \in F^2(\varphi)$.

It follows from the preceding lemma that for any $z \in \mathbb{C}^n$, the mapping $f \mapsto f(z)$ is a bounded linear functional on $F^2(\varphi)$ and hence there is a unique K_z in $F^2(\varphi)$ which satisfies the reproducing property $f(z) = \langle f, K_z \rangle$ for all $f \in F^2(\varphi)$. The function K_z is referred to as the reproducing kernel of $F^2(\varphi)$. It is often called the Bergman kernel.

Lemma 2.1 also implies that $F^2(\varphi)$ is a closed subspace of $L^2(\varphi)$. We denote by P the orthogonal projection of $L^2(\varphi)$ onto $F^2(\varphi)$. Notice that

$$Pf(z) = \langle Pf, K_z \rangle = \int_{\mathbb{C}^n} f(w)K(z, w)e^{-2\varphi(w)} dv(w)$$

for $f \in L^2(\varphi)$ and $z \in \mathbb{C}^n$.

If $\varphi(z) = \alpha|z|^2$ is a standard weight with $\alpha > 0$, then it is easy to see that

$$|K(z, w)| e^{-\alpha|z|^2 - \alpha|w|^2} = e^{-\alpha|z-w|^2}$$

for $z, w \in \mathbb{C}^n$. For the general weights φ that we consider, this quadratic decay is known not to hold (even in dimension one), and it is, in fact, expected to be very rare (see [7]). However, it turns out that the following estimates for the reproducing kernel will be sufficient for us.

Lemma 2.2. *There exist positive constants θ and C_1 , depending only on n, m and M such that*

$$(2.1) \quad |K(z, w)| \leq C_1 e^{\varphi(z) + \varphi(w)} e^{-\theta|z-w|} \quad \text{for all } z, w \in \mathbb{C}^n,$$

and there exists positive constants C_2 and r_0 such that

$$(2.2) \quad |K(z, w)| \geq C_2 e^{\varphi(z) + \varphi(w)}$$

for $z \in \mathbb{C}^n$ and $w \in B(z, r_0)$.

The estimate (2.1) appeared in [7] for $n = 1$ and in [8] for $n \geq 2$, while the inequality (2.2) can be found in [23]. Notice that Lemma 2.2 implies that

$$(2.3) \quad K(z, z) \simeq e^{2\varphi(z)}, \quad z \in \mathbb{C}^n.$$

For $z \in \mathbb{C}^n$, recall that we write

$$k_z(\cdot) = \frac{K(\cdot, z)}{\sqrt{K(z, z)}}$$

for the normalized reproducing kernel. Then

$$(2.4) \quad \left| k_z(\xi) e^{-\varphi(\xi)} \right| \leq C e^{-\theta|z-\xi|}, \quad \lim_{|z| \rightarrow \infty} k_z(\xi) = 0$$

uniformly in ξ on compact subsets of \mathbb{C}^n , and

$$(2.5) \quad \frac{1}{C} e^{\varphi(z)} \leq \|K_z\|_{p,\varphi} \leq C e^{\varphi(z)}, \quad \frac{1}{C} \leq \|k_z\|_{p,\varphi} \leq C$$

for $z \in \mathbb{C}^n$. Here $\|\cdot\|_{p,\varphi}$ stands for the norm of $L^p(\varphi) = L^p(\mathbb{C}^n, e^{-p\varphi} dv)$ when $1 \leq p < \infty$, $\|f\|_{\infty,\varphi} = \|f e^{-\varphi}\|_{L^\infty}$, and we write $\|\cdot\|_{2,\varphi} = \|\cdot\|$ for simplicity throughout.

We record one more estimate that will be needed for our study of Hankel operators. For this purpose, denote by $L^2_{(0,1)}(\varphi)$ the family of all $(0,1)$ -forms on \mathbb{C}^n with coefficients in $L^2(\varphi)$.

Lemma 2.3 (Hörmander). *Suppose that $\varphi \in C^2(\mathbb{C}^n)$ is real valued and $i\partial\bar{\partial}\varphi \simeq \omega_0$. Then there is a constant $C > 0$ such that for every $\bar{\partial}$ -closed $(0,1)$ -form $\omega \in L^2_{(0,1)}(\varphi)$, there exists a solution u of $\bar{\partial}u = \omega$ for which*

$$\int_{\mathbb{C}^n} |ue^{-\varphi}|^2 dv \leq C \int_{\mathbb{C}^n} |\omega e^{-\varphi}|^2 dv.$$

Proof. Let $\Omega = \mathbb{C}^n$. The assumption $i\partial\bar{\partial}\varphi(z) \geq m i\partial\bar{\partial}|z|^2$ implies that $2m$ is a lower bound for the plurisubharmonicity of 2φ . Now Theorem 2.2.1 of [15] completes the proof. \square

2.2. Lattices and separated sets in \mathbb{C}^n . A sequence $\{w_j\}$ of distinct points in \mathbb{C}^n is called separated if

$$\delta(\{w_j\}) = \inf_{j \neq k} |w_j - w_k| > 0.$$

For $r > 0$, we call a sequence $\{w_j\}$ in \mathbb{C}^n an r -lattice if $\bigcup_j B(w_j, r) = \mathbb{C}^n$ and $B(w_j, \frac{r}{2\sqrt{n}}) \cap B(w_k, \frac{r}{2\sqrt{n}}) = \emptyset$ whenever $j \neq k$.

Given $r > 0$ and $w_1 \in \mathbb{C}^n$, set

$$(2.6) \quad \Lambda = \left\{ w_1 + \frac{r}{\sqrt{n}}(m + is) : m, s \in \mathbb{Z}^n \right\}.$$

It is easy to see that Λ is an r -lattice. For $K \in \mathbb{N}$ fixed, we write

$$\left\{ w_1 + (z_1, \dots, z_n) \in \Lambda : 0 \leq \operatorname{Re} z_j, \operatorname{Im} z_j < K \frac{r}{\sqrt{n}} \right\} = \{w_1, \dots, w_{K^{2n}}\},$$

and for $1 \leq k \leq K^{2n}$,

$$\Lambda_k = \left\{ w_k + K \frac{r}{\sqrt{n}}(m + is) : m, s \in \mathbb{Z}^n \right\}.$$

Then

$$(2.7) \quad \Lambda = \bigcup_{k=1}^{K^{2n}} \Lambda_k, \quad \Lambda_j \cap \Lambda_k = \emptyset \text{ if } j \neq k, \quad |a-b| \geq \frac{Kr}{\sqrt{n}} \text{ if } a, b \in \Lambda_k \text{ and } a \neq b.$$

For $f, e \in L^2(\varphi)$, the tensor product $f \otimes e$ as a rank one operator on $L^2(\varphi)$ is defined by

$$f \otimes e(g) = \langle g, e \rangle f, \quad g \in L^2(\varphi).$$

Lemma 2.4. *Given $r > 0$, there is some constant $C > 0$ such that if Λ is a separated set in \mathbb{C}^n with $\delta(\Lambda) \geq r$ and if $\{e_a : a \in \Lambda\}$ is an orthonormal set in $L^2(\varphi)$, then*

$$\left\| \sum_{a \in \Lambda} k_a \otimes e_a \right\|_{L^2(\varphi) \rightarrow L^2(\varphi)} \leq C.$$

Proof. With the same proof as that of Lemma 2.4 in [12], we get

$$\left\| \sum_{a \in \Lambda} \lambda_a k_a \right\| \leq C \|\{\lambda_a\}_{a \in \Lambda}\|_{l^2},$$

where the constant depends only on the separation constant $\delta(\Lambda)$. In addition, for $g \in L^2(\varphi)$, Parseval's identity implies

$$\sum_{a \in \Lambda} |\langle g, e_a \rangle|^2 \leq \|g\|^2 < \infty.$$

Therefore, we have

$$\left\| \left(\sum_{a \in \Lambda} k_a \otimes e_a \right) (g) \right\|^2 = \left\| \sum_{a \in \Lambda} \langle g, e_a \rangle k_a \right\|^2 \leq C \sum_{a \in \Lambda} |\langle g, e_a \rangle|^2 \leq C \|g\|^2,$$

which completes the proof. \square

2.3. Properties of IDA. The spaces IDA^s were defined above in (1.3) and here we list their basic properties. We start with a remark that follows from Corollary 3.8 of [13] when $s \geq 1$ while the other cases can be proved similarly.

Remark 2.5. Let $0 < s < \infty$. Then the spaces IDA^s , BDA and VDA are independent of r and different values of r give equivalent norms on each space.

For $f \in L^2_{\text{loc}}$, set

$$M_{2,r}(f)(z) = \left\{ \frac{1}{|B(z,r)|} \int_{B(z,r)} |f|^2 dv \right\}^{\frac{1}{2}}.$$

Notice that, for $r > 0$ fixed, by Lemma 3.5 in [13], there is a constant C such that for $f \in L^2_{\text{loc}}$ it holds that $f = f_1 + f_2$ with $f_1 \in C(\mathbb{C}^n)$ and

$$(2.8) \quad |\bar{\partial} f_1(z)| + M_{2,r}(|\bar{\partial} f_1|)(z) + M_{2,r}(f_2)(z) \leq C G_{2r}(f)(z), \quad z \in \mathbb{C}^n.$$

In addition, the following decomposition theorem plays an important role in our analysis.

Theorem 2.6. *Suppose $0 < s \leq \infty$, and $f \in L^2_{\text{loc}}$. Then $f \in \text{IDA}^s$ if and only if f admits a decomposition $f = f_1 + f_2$ such that*

$$(2.9) \quad f_1 \in C^2(\mathbb{C}^n), \quad |\bar{\partial}f_1| + M_{2,r}(\bar{\partial}f_1) + M_{2,r}(f_2) \in L^s$$

for some (or any) $r > 0$. Furthermore,

$$(2.10) \quad \|f\|_{\text{IDA}^s} \simeq \inf \{ \|\bar{\partial}f_1\|_{L^s} + \|M_{2,r}(f_2)\|_{L^s} \}$$

where the infimum is taken over all possible decompositions $f = f_1 + f_2$ that satisfy (2.9) with a fixed r .

Proof. When $1 \leq s < \infty$, the conclusion is only a special case $q = 2$ of Theorem 3.6 and Lemma 3.5 in [13]. A careful check of their proofs shows that the remaining cases $0 < s < 1$ and $s = \infty$ can be proved similarly. \square

2.4. Schatten classes. We finish this section with a brief look at Schatten classes. Recall that for a bounded linear operator $T : H_1 \rightarrow H_2$ between two Hilbert spaces, the singular values $s_j(T)$ of T are defined by

$$(2.11) \quad s_j(T) = \inf \{ \|T - K\| : K : H_1 \rightarrow H_2, \text{rank } K \leq j \},$$

where $\text{rank } K$ denotes the rank of K . The operator T is compact if and only if $s_j(T) \rightarrow 0$. For $0 < p < \infty$, we say that T is in the Schatten class S_p and write $T \in S_p(H_1, H_2)$ if $\|T\|_{S_p}^p = \sum_{j=0}^{\infty} (s_j(T))^p < \infty$, which defines a norm when $1 \leq p < \infty$ and a quasinorm otherwise. Note that S_p are also called the Schatten-von Neumann classes or trace ideals (see [26] for further details).

3. SCHATTEN CLASS TOEPLITZ OPERATORS

Given a Borel measure μ on \mathbb{C}^n , we define the Toeplitz operator T_μ with symbol μ as

$$T_\mu f(z) = \int_{\mathbb{C}^n} K(z, w) f(w) e^{-2\varphi(w)} d\mu(w), \quad f \in F^2(\varphi) \text{ and } z \in \mathbb{C}^n.$$

When $d\mu(z) = g(z) dv(z)$ and g is a complex-valued function, the induced Toeplitz operator is denoted by T_g .

Given an operator $T \in \mathcal{B}(F^2(\varphi), L^2(\varphi))$, we set $\tilde{T}(z) = \langle Tk_z, k_z \rangle$. For a positive Borel measure μ on \mathbb{C}^n and $r > 0$, we define

$$\tilde{\mu}(z) = \int_{\mathbb{C}^n} |k_z|^2 e^{-2\varphi} d\mu$$

and since $|B(z, r)| \simeq r^{2n}$, we simply set

$$\hat{\mu}_r(z) = \int_{B(z, r)} d\mu \quad (z \in \mathbb{C}^n).$$

For a positive Toeplitz operator $T_\mu \in \mathcal{B}(F^2(\varphi), L^2(\varphi))$, it is easy to verify that $\tilde{T}_\mu = \tilde{\mu}$.

The proof of the following lemma can be found in [12] and [17].

Lemma 3.1. *Let μ be a positive Borel measure on \mathbb{C}^n and let $0 < p \leq \infty$. Then the following statements are equivalent:*

- (A) $\tilde{\mu} \in L^p$.
- (B) $\hat{\mu}_r \in L^p$ for some (or any) $r > 0$.

(C) $\{\widehat{\mu}_r(a_j)\}_{j=1}^\infty \in l^p$ for some (or any) r -lattice $\{a_j\}_{j=1}^\infty$.

Furthermore, it holds that

$$\|\widetilde{\mu}\|_{L^p} \simeq \|\widehat{\mu}_r\|_{L^p} \simeq \left\| \{\widehat{\mu}_r(a_j)\}_{j=1}^\infty \right\|_{l^p}.$$

In our analysis we need the following result on Schatten class Toeplitz operators. It was proved in [17] for the generalized weights and in [27] for the standard weights.

Theorem 3.2. *Let $0 < p < \infty$, μ be a positive Borel measure on \mathbb{C}^n and suppose that $\varphi \in C^2(\mathbb{C}^n)$ is real valued with $i\partial\bar{\partial}\varphi \simeq \omega_0$. Then the Toeplitz operator T_μ on $F^2(\varphi)$ belongs to S_p if and only if $\widehat{\mu}_r \in L^p$ for some (or any) $r > 0$. Furthermore,*

$$(3.1) \quad \|T_\mu\|_{S_p} \simeq \|\widehat{\mu}_r\|_{L^p}.$$

4. BOUNDEDNESS AND COMPACTNESS OF HANKEL OPERATORS

While the thrust of our present work is in the Schatten class properties, we also extend some of the recent results in [13] on boundedness and compactness of Hankel operators on $F^2(\varphi)$.

The collection of all bounded (and compact) operators from H_1 to H_2 is denoted by $\mathcal{B}(H_1, H_2)$ (and by $\mathcal{K}(H_1, H_2)$, respectively). The corresponding operator norm is denoted by $\|T\|_{H_1 \rightarrow H_2}$.

Lemma 4.1. *Suppose $0 < p \leq 1$, $0 < s < \infty$ and $r > 0$. There is a constant C such that, for μ a positive Borel measure on \mathbb{C}^n , Ω a domain in \mathbb{C}^n , and $g \in H(\mathbb{C}^n)$, it holds that*

$$\left(\int_{\Omega} |g e^{-\varphi}|^s d\mu \right)^p \leq C \int_{\Omega_r^+} |g e^{-\varphi}|^{sp} \widehat{\mu}_r^p dv,$$

where $\Omega_r^+ = \bigcup_{\{z \in \Omega\}} B(z, r)$.

When $\Omega = \mathbb{C}^n$ and $p = 1$, the preceding result is just Lemma 2.2 in [12]. For general Ω and $0 < p \leq 1$, the proof is similar to that of Lemma 4.2 in [13], although we note that the weights in the present work are slightly more general.

Lemma 4.2. *Suppose $0 < p \leq \infty$, and $f \in \mathcal{S} \cap \text{IDA}^p$ with the decomposition $f = f_1 + f_2$ as in (2.8). Then H_{f_1} and H_{f_2} are well defined on $\Gamma = \text{span}\{K_z : z \in \mathbb{C}^n\}$, and*

$$(4.1) \quad \|H_{f_1}(g)\| \leq C \|g \bar{\partial} f_1\| \quad \text{and} \quad \|H_{f_2}(g)\| \leq C \|g f_2\| \quad \text{for } g \in \Gamma.$$

Proof. For $g \in \Gamma$ and $z \in \mathbb{C}^n$, taking $p = s = 1$ and replacing φ with 2φ in Lemma 4.1, we get

$$\int_{\mathbb{C}^n} |g K_z| e^{-2\varphi} |f_2| dv \leq C \int_{\mathbb{C}^n} |g K_z| e^{-2\varphi} M_{1,r}(f_2) dv.$$

Notice that $M_{t,r}(f_2)$ is increasing with t . If $p \geq 1$, with p' being the conjugate of p , we have

$$\int_{\mathbb{C}^n} |g f_2| |K_z| e^{-2\varphi} dv \leq C \|M_{2,r}(f_2)\|_{L^p} \|g\|_{2p', \varphi} \|K_z\|_{2p', \varphi} < \infty.$$

If $0 < p < 1$, by Lemma 4.1 again,

$$\begin{aligned} \left(\int_{\mathbb{C}^n} |g K_z| e^{-2\varphi} |f_2| dv \right)^p &\leq C \int_{\mathbb{C}^n} |g K_z|^p e^{-2p\varphi} M_{1,r}^p(f_2) dv \\ &\leq C \|M_{2,r}(f_2)\|_{L^p}^p \|g\|_{\infty, \varphi}^p \|K_z\|_{\infty, \varphi}^p < \infty. \end{aligned}$$

This implies that H_{f_2} , and hence also $H_{f_1} = H_f - H_{f_2}$, are both well defined on Γ .

Now for $g \in \Gamma$, if $0 < p \leq 2$, then

$$(4.2) \quad \begin{aligned} \|g\bar{\partial}f_1\|^p &\leq C \int_{\mathbb{C}^n} |ge^{-\varphi}|^p M_{2,r}(|\bar{\partial}f_1|)^p dv \\ &\leq C \|M_{2,r}(|\bar{\partial}f_1|)\|_{L^p}^p \|g\|_{\infty,\varphi}^p < \infty. \end{aligned}$$

If $p > 2$, applying Hölder's inequality with $t = \frac{p}{2}$ and t' we obtain

$$(4.3) \quad \begin{aligned} \|g\bar{\partial}f_1\| &\leq C \left\{ \int_{\mathbb{C}^n} |ge^{-\varphi}|^2 M_{2,r}(|\bar{\partial}f_1|)^2 dv \right\}^{\frac{1}{2}} \\ &\leq C \|M_{2,r}(|\bar{\partial}f_1|)\|_{L^p} \|g\|_{2t',\varphi} < \infty. \end{aligned}$$

Hence $g\bar{\partial}f_1$ is a $\bar{\partial}$ -closed $(0,1)$ -form with $L^2(\varphi)$ coefficients. Notice also that since $\bar{\partial}H_{f_1}(g) = g\bar{\partial}f_1$ and $H_{f_1}(g) = f_1g - P(f_1g) \perp F^2(\varphi)$, $H_{f_1}(g)$ is the canonical solution of the equation $\bar{\partial}u = g\bar{\partial}f_1$. Now, by Lemma 2.3 there is a solution $u_0 \in L^2(\varphi)$ of $\bar{\partial}u = g\bar{\partial}f_1$ such that $\|u_0\| \leq C\|g\bar{\partial}f_1\|$. Hence, $u_0 - H_{f_1}g \in F^2(\varphi)$, which implies that $H_{f_1}g \perp u_0 - H_{f_1}g$. Therefore,

$$(4.4) \quad \|H_{f_1}g\|^2 = \|u_0\|^2 - \|u_0 - H_{f_1}g\|^2 \leq \|u_0\|^2 \leq C \|g\bar{\partial}f_1\|^2.$$

Similarly to (4.2) and (4.3), we can show that $\|gf_2\| < \infty$, and so

$$\|H_{f_2}(g)\| \leq \|gf_2\| < \infty,$$

which completes the proof. \square

Before stating our result on boundedness and compactness, we note that previously Li [19] used the Henkin-Ramirez formula to obtain conditions for boundedness of Hankel operators acting on the classical Segal-Bargmann space. However, some basic estimates involving the Henkin-Ramirez formula in Li's approach are unavailable for the more general weights φ that we consider.

Theorem 4.3. *Suppose that $\varphi \in C^2(\mathbb{C}^n)$ is real valued and $i\bar{\partial}\bar{\varphi} \simeq \omega_0$. Then for $f \in \mathcal{S}$, $H_f \in \mathcal{B}(F^2(\varphi), L^2(\varphi))$ if and only if $f \in \text{BDA}$; and $H_f \in \mathcal{K}(F^2(\varphi), L^2(\varphi))$ if and only if $f \in \text{VDA}$. Furthermore, for $r > 0$,*

$$\|H_f\|_{F^2(\varphi) \rightarrow L^2(\varphi)} \simeq \|f\|_{\text{BDA}}.$$

Proof. Suppose $f \in \text{BDA}$. As in (2.8), we decompose $f = f_1 + f_2$ with

$$f_1 \in C^2(\mathbb{C}^n), \quad |\bar{\partial}f_1| + M_{2,r}(f_2) \leq CG_{2r}(f)(z).$$

For $g \in \Gamma$, by Lemma 4.2,

$$(4.5) \quad \|H_{f_1}(g)\| \leq C \|g\bar{\partial}f_1\| \leq C \|\bar{\partial}f_1\|_{L^\infty} \|g\|.$$

Therefore, $H_{f_1} \in \mathcal{B}(F^2(\varphi), L^2(\varphi))$ with the norm estimate

$$\|H_{f_1}\|_{F^2(\varphi) \rightarrow L^2(\varphi)} \leq C \|\bar{\partial}f_1\|_{L^\infty}.$$

Similarly,

$$(4.6) \quad \|H_{f_2}(g)\| \leq \|gf_2\| \leq C \|M_{2,r}(f_2)\|_{L^\infty} \|g\|.$$

Therefore, $H_f \in \mathcal{B}(F^2(\varphi), L^2(\varphi))$ with the norm estimate

$$\|H_f\|_{F^2(\varphi) \rightarrow L^2(\varphi)} \leq C \|f\|_{\text{BDA}}.$$

Conversely, suppose $H_f \in \mathcal{B}(F^2(\varphi), L^2(\varphi))$. Then for $0 < r \leq r_0$, by Lemma 2.2,

$$(4.7) \quad G_r(f)(z) \leq C \left\{ \int_{B(z,r)} \left| f - \frac{1}{k_z} P(fk_z) \right|^2 dv \right\}^{\frac{1}{2}} \leq C \|H_f(k_z)\|.$$

This and the fact that $\|k_z\| = 1$ implies $f \in \text{BDA}$ with

$$\|f\|_{\text{BDA}} \leq C \|H_f\|_{F^2(\varphi) \rightarrow L^2(\varphi)}.$$

The proof for the compactness can be carried out as that of Theorem 4.1 in [13]. \square

Our next theorem is an analog of Stroethoff's main result in [24] for the generalized weights and unbounded symbols.

Theorem 4.4. *Suppose $\varphi \in C^2(\mathbb{C}^n)$ is real valued and $i\partial\bar{\partial}\varphi \simeq \omega_0$. Then for $f \in \mathcal{S}$, $H_f \in \mathcal{B}(F^2(\varphi), L^2(\varphi))$ if and only if*

$$\sup_{z \in \mathbb{C}^n} \|(I - P)(fk_z)\| < \infty$$

with the norm estimate

$$(4.8) \quad \|H_f\|_{F^2(\varphi) \rightarrow L^2(\varphi)} \simeq \sup_{z \in \mathbb{C}^n} \|(I - P)(fk_z)\|.$$

Further, $H_f \in \mathcal{K}(F^2(\varphi), L^2(\varphi))$ if and only if

$$\lim_{z \rightarrow \infty} \|(I - P)(fk_z)\| = 0.$$

Proof. Suppose $f \in \mathcal{S}$. If $H_f \in \mathcal{B}(F^2(\varphi), L^2(\varphi))$, then trivially

$$\|(I - P)(fk_z)\| \leq \|H_f\|_{F^2(\varphi) \rightarrow L^2(\varphi)} \|k_z\| \leq C \|H_f\|_{F^2(\varphi) \rightarrow L^2(\varphi)}.$$

Conversely, by Theorem 4.3 and the estimate (4.7), we have

$$\|H_f\|_{F^2(\varphi) \rightarrow L^2(\varphi)} \simeq \|G_r(f)\|_{L^\infty} \leq \sup_{z \in \mathbb{C}^n} \|(I - P)(fk_z)\|.$$

The other equivalence can be proved similarly. \square

For a fixed $z \in \mathbb{C}^n$, define the shift $\tau_z : \mathbb{C}^n \rightarrow \mathbb{C}^n$ by $\tau_z(w) = w + z$. Then $f \mapsto f \circ \tau_z(w) = f(w + z)$ for $w \in \mathbb{C}^n$. In the classical Segal-Bargmann space F^2 , we have

$$(4.9) \quad \|H_f(k_z)\| = \|(I - P)(f \circ \tau_z)\|$$

for all $f \in \mathcal{S}$ (see Corollary 1.1 in [3]). Using this observation, we obtain the following corollary.

Corollary 4.5. *Let $\varphi(z) = \frac{\alpha}{2}|z|^2$. For $f \in \mathcal{S}$, $H_f \in \mathcal{B}(F^2(\varphi), L^2(\varphi))$ if and only if $\sup_{z \in \mathbb{C}^n} \|(I - P)(f \circ \tau_z)\| < \infty$ with the norm estimate*

$$(4.10) \quad \|H_f\|_{F^2(\varphi) \rightarrow L^2(\varphi)} \simeq \sup_{z \in \mathbb{C}^n} \|(I - P)(f \circ \tau_z)\|.$$

Further, $H_f \in \mathcal{K}(F^2(\varphi), L^2(\varphi))$ if and only if

$$\lim_{z \rightarrow \infty} \|(I - P)(f \circ \tau_z)\| = 0.$$

Remark 4.6. (i) Notice that the preceding result on compactness implies the analogous results in [11] and [24] where the symbols were assumed to be bounded.

(ii) It should be noted that the norm estimate in (4.10) is similar to the estimate in Luecking's main result for the Bergman space (and also for the standard weights); see Theorem 1(b) of [21].

(iii) It would be interesting to know whether the previous corollary remains true for more general weights.

5. PROOF OF THEOREM 1.1

In this section we prove our characterization of Schatten class Hankel operators. For this purpose we need one more lemma. Given $a \in \mathbb{C}^n$ and $r > 0$, let $L^2(B(a, r), e^{-2\varphi} dv)$ be the Lebesgue space on $B(a, r)$ with respect to measure $e^{-2\varphi} dv$, and let $A^2(B(a, r), e^{-2\varphi} dv)$ be the weighted Bergman space of all holomorphic functions in the space $L^2(B(a, r), e^{-2\varphi} dv)$. We denote by $P_{a,r}$ the orthogonal projection of $L^2(B(a, r), e^{-2\varphi} dv)$ onto $A^2(B(a, r), e^{-2\varphi} dv)$.

Given $f \in L^2(B(a, r), e^{-2\varphi} dv)$, we extend $P_{a,r}(f)$ to \mathbb{C}^n by setting

$$P_{a,r}(f)|_{\mathbb{C}^n \setminus B(a,r)} = 0.$$

It is easy to verify that

$$P_{a,r}^2 f = P_{a,r} f \quad \text{and} \quad \langle f, P_{a,r} g \rangle = \langle P_{a,r} f, g \rangle$$

for $f, g \in L^2(\varphi)$.

Lemma 5.1. *For $f, g \in L^2(\varphi)$, it holds that*

$$(5.1) \quad \langle f - Pf, \chi_{B(a,r)} g - P_{a,r} g \rangle = \langle f - P_{a,r} f, \chi_{B(a,r)} g - P_{a,r} g \rangle.$$

Proof. For $h \in F^2(\varphi)$, it is trivial that $P_{a,r}(h) = \chi_{B(a,r)} h$. Then for $f, g \in L^2(\varphi)$, we have $\langle h, \chi_{B(a,r)} g - P_{a,r} g \rangle = 0$, and hence

$$\begin{aligned} \langle f - Pf, \chi_{B(a,r)} g - P_{a,r} g \rangle &= \langle \chi_{B(a,r)} f, \chi_{B(a,r)} g - P_{a,r} g \rangle \\ &= \langle \chi_{B(a,r)} f - P_{a,r} f, \chi_{B(a,r)} g - P_{a,r} g \rangle. \end{aligned}$$

From this (5.1) follows. □

Proof of Theorem 1.1. **(B) ⇒ (A).** For $f \in \text{IDA}^p$, decompose f as in (2.8). Notice that by Lemma 4.2, both H_{f_1} and H_{f_2} are well defined on $F^2(\varphi)$. We claim that

$$(5.2) \quad \|H_{f_1}\|_{S_p} + \|H_{f_2}\|_{S_p} \leq C \|f\|_{\text{IDA}^p},$$

which gives the desired estimate

$$(5.3) \quad \|H_f\|_{S_p} \leq C \left(\|H_{f_1}\|_{S_p} + \|H_{f_2}\|_{S_p} \right) \leq C \|f\|_{\text{IDA}^p}.$$

To prove (5.2), set $\phi = |\bar{\partial} f_1|$ or $\phi = |f_2|$. By Theorem 3.2, the positive Toeplitz operator $T_{|\phi|^2}$ is in $S_{\frac{p}{2}}$. Now we consider the multiplication operator $M_\phi : F^2(\varphi) \rightarrow L^2(\varphi)$ defined as

$$M_\phi(f) = \phi f.$$

For $\phi = |\bar{\partial}f_1|$, using the inequality in (4.2) with the exponential $\frac{2p}{2+p} < 2$ and Hölder's inequality, for $g \in F^2(\varphi)$ we have

$$\begin{aligned} \|g|\bar{\partial}f_1|\|_{\frac{2p}{2+p}} &\leq C \int_{\mathbb{C}^n} |ge^{-\varphi}|^{\frac{2p}{2+p}} M_{2,r}(|\bar{\partial}f_1|)^{\frac{2p}{2+p}} dv \\ &\leq C \|g\|_{\frac{2p}{2+p}}^{\frac{2p}{2+p}} \left\{ \int_{\mathbb{C}^n} M_{2,r}(|\bar{\partial}f_1|)^p dv \right\}^{\frac{2}{2+p}}. \end{aligned}$$

Similar estimate holds for $\phi = |f_2|$. Thus, M_ϕ is bounded from $F^2(\varphi)$ to $L^2(\varphi)$. Now, for $g, h \in F^2(\varphi)$, it holds that

$$\langle M_\phi^* M_\phi g, h \rangle = \langle M_\phi g, M_\phi h \rangle = \int_{\mathbb{C}^n} g \bar{h} |\phi|^2 dv = \langle T_{|\phi|^2} g, h \rangle,$$

which in turn gives $M_\phi^* M_\phi = T_{|\phi|^2}$. Thus, $M_\phi \in S_p$, and applying Theorem 2.6, we get

$$\|M_\phi\|_{S_p} \leq C \|M_{2,r}(\phi)\|_{L^p} \leq C \|f\|_{\text{IDA}^p}^p.$$

This together with (4.1) give the norm estimate in (5.2).

(A) \Rightarrow (B). Suppose $f \in \mathcal{S}$ and $H_f \in S_p(F^2(\varphi), L^2(\varphi))$. We will prove that

$$(5.4) \quad \|f\|_{\text{IDA}^p} \leq C \|H_f\|_{S_p}.$$

For this purpose, we borrow an idea from the proof of Proposition 6.8 in [9]. By Remark 2.5, it suffices to prove (5.4) for some $r \in (0, r_0)$, where r_0 is as in Lemma 2.2. To do this, let Λ be an r -lattice as in (2.6), and decompose $\Lambda = \cup_k \Lambda_k$ as in (2.7) with $K \geq 2\sqrt{n}$ so that $B(a, r) \cap B(b, r) = \emptyset$ if $a \neq b$ and $a, b \in \Lambda_k$.

We deal with the case $0 < p \leq 1$ first. Since

$$H_f \in S_p(F^2(\varphi), L^2(\varphi)) \subset \mathcal{B}(F^2(\varphi), L^2(\varphi)),$$

Theorem 4.4 shows that $fk_a - P(fk_a) \in L_{\text{loc}}^2$. Clearly $P(fk_a) \in H(\mathbb{C}^n)$, so

$$fk_a \in L^2(B(a, r), e^{-2\varphi} dv) \quad \text{and} \quad P_{a,r}(fk_a) \in A^2(B(a, r), e^{-2\varphi} dv),$$

which implies that $\|\chi_{B(a,r)}fk_a - P_{a,r}(fk_a)\| < \infty$. Now for $a \in \Lambda_k$, set

$$g_a = \begin{cases} \frac{\chi_{B(a,r)}fk_a - P_{a,r}(fk_a)}{\|\chi_{B(a,r)}fk_a - P_{a,r}(fk_a)\|} & \text{if } \|\chi_{B(a,r)}fk_a - P_{a,r}(fk_a)\| \neq 0, \\ 0 & \text{if } \|\chi_{B(a,r)}fk_a - P_{a,r}(fk_a)\| = 0. \end{cases}$$

It is easy to see that $\|g_a\| \leq 1$ and $\langle g_a, g_b \rangle = 0$ if $B(a, r) \cap B(b, r) = \emptyset$. Let J be any finite sub-collection of Λ_k , and let $\{e_a\}_{a \in J}$ be an orthonormal set of $L^2(\varphi)$. Define

$$A = \sum_{a \in J} e_a \otimes g_a : L^2(\varphi) \rightarrow L^2(\varphi).$$

It is trivial to see that A is of finite rank and

$$(5.5) \quad \|A\|_{L^2(\varphi) \rightarrow L^2(\varphi)} \leq 1.$$

We now define another operator $T : L^2(\varphi) \rightarrow F^2(\varphi)$ by

$$T = \sum_{a \in J} k_a \otimes e_a.$$

Since Λ is separated, by Lemma 2.4, there is a constant C depending only on n and r such that

$$(5.6) \quad \|T\|_{L^2(\varphi) \rightarrow F^2(\varphi)} \leq C.$$

It is easy to verify that

$$(5.7) \quad AH_f T = \sum_{a, \tau \in J} \langle H_f k_\tau, g_a \rangle e_a \otimes e_\tau = Y + Z,$$

where

$$(5.8) \quad Y = \sum_{a \in J} \langle H_f k_a, g_a \rangle e_a \otimes e_a, \quad Z = \sum_{a, \tau \in J, a \neq \tau} \langle H_f k_\tau, g_a \rangle e_a \otimes e_\tau.$$

By Lemma 5.1 and Lemma 2.2,

$$\begin{aligned} \langle H_f k_a, g_a \rangle &= \langle f k_a - P(f k_a), g_a \rangle \\ &= \langle \chi_{B(a,r)} f k_a - P_{a,r}(f k_a), g_a \rangle \\ &= \|\chi_{B(a,r)} f k_a - P_{a,r}(f k_a)\| \\ &\geq C \left\| f - \frac{1}{k_a} P_{a,r}(f k_a) \right\|_{L^2(B(a,r), dv)}. \end{aligned}$$

Further, by definition,

$$\langle H_f k_a, g_a \rangle \geq C \left\| f - \frac{1}{k_a} P_{a,r}(f k_a) \right\|_{L^2(B(a,r), dv)} \geq C G_r(f)(a).$$

Thus, there is an N independent of f and J such that

$$(5.9) \quad \|Y\|_{S_p}^p = \sum_{a \in J} \langle H_f k_a, g_a \rangle^p \geq N \sum_{a \in J} G_r(f)(a)^p.$$

On the other hand, for $0 < p \leq 1$, applying Lemma 5 of [22] gives

$$(5.10) \quad \|Z\|_{S_p}^p \leq \sum_{a, \tau \in J, a \neq \tau} |\langle H_f k_\tau, g_a \rangle|^p.$$

Let $Q_{a,r}$ be the Bergman projection of $L^2(B(a,r), dv)$ onto the Bergman space $A^2(B(a,r), dv)$. Then

$$k_\tau Q_{a,r}(f) \in A^2(B(a,r), dv) = A^2(B(a,r), e^{-2\varphi} dv),$$

and further $f k_\tau - P_{a,r}(f k_\tau)$ and $P_{a,r}(f k_\tau) - k_\tau Q_{a,r} f$ are orthogonal in the space $L^2(B(a,r), e^{-2\varphi} dv)$. Thus, for $a, \tau \in \mathbb{C}^n$, by Parseval's identity, we get

$$\|f k_\tau - P_{a,r}(f k_\tau)\|_{L^2(B(a,r), e^{-2\varphi} dv)} \leq \|f k_\tau - k_\tau Q_{a,r} f\|_{L^2(B(a,r), e^{-2\varphi} dv)}.$$

Hence, by Lemma 5.1,

$$\begin{aligned} |\langle H_f k_\tau, g_a \rangle| &= |\langle f k_\tau - P(f k_\tau), g_a \rangle| \\ &= |\langle \chi_{B(a,r)} f k_\tau - P_{a,r}(f k_\tau), g_a \rangle| \\ &\leq \|f k_\tau - P_{a,r}(f k_\tau)\|_{L^2(B(a,r), e^{-2\varphi} dv)} \\ &\leq \|f k_\tau - k_\tau Q_{a,r}(f)\|_{L^2(B(a,r), e^{-2\varphi} dv)} \\ &\leq \sup_{\xi \in B(a,r)} |k_\tau(\xi) e^{-\varphi}| \|f - Q_{a,r}(f)\|_{L^2(B(a,r), dv)} \\ &\leq C e^{-\theta|a-\tau|} \|f - Q_{a,r}(f)\|_{L^2(B(a,r), dv)}. \end{aligned}$$

Notice also that

$$\|f - Q_{a,r}(f)\|_{L^2(B(a,r), dv)} = G_r(f)(a).$$

Since $B(\tau, r) \cap B(a, r) = \emptyset$ for $a, \tau \in J$ with $a \neq \tau$,

$$\begin{aligned} \sum_{\tau \in J, \tau \neq a} e^{-\frac{p\theta}{2}|a-\tau|} &\leq C \sum_{\tau \in J, \tau \neq a} \int_{B(\tau, r)} e^{-\frac{p\theta}{2}|a-\xi|} dv(\xi) \\ &\leq C \int_{\mathbb{C}^n} e^{-\frac{p\theta}{2}|\xi|} dv(\xi) = C. \end{aligned}$$

Therefore, by (2.7) and (2.4),

$$\begin{aligned} \|Z\|_{S_p}^p &\leq \sum_{a, \tau \in J, a \neq \tau} e^{-p\theta|a-\tau|} G_r(f)(a)^p \\ &\leq \sum_{a \in J} G_r(f)(a)^p \sum_{\tau \in J, \tau \neq a} e^{-p\theta|a-\tau|} \\ &\leq e^{-\frac{p\theta}{2\sqrt{n}}Kr} \sum_{a \in J} G_r(f)(a)^p \sum_{\tau \in J, \tau \neq a} e^{-\frac{p\theta}{2}|a-\tau|} \\ &\leq C e^{-\frac{p\theta}{2\sqrt{n}}Kr} \sum_{a \in J} G_r(f)(a)^p, \end{aligned}$$

and hence, we can pick some K sufficiently large so that

$$(5.11) \quad \|Z\|_{S_p}^p \leq \frac{N}{4} \sum_{a \in J} G_r(f)(a)^p.$$

Using the estimate

$$\|Y\|_{S_p}^p \leq 2 \|AH_f T\|_{S_p}^p + 2 \|Z\|_{S_p}^p$$

(see (6.9) in [9] for example), we see that

$$N \sum_{a \in J} G_r(f)(a)^p \leq 2 \|AH_f T\|_{S_p}^p + \frac{N}{2} \sum_{a \in J} G_r(f)(a)^p.$$

Since J is finite, we have

$$(5.12) \quad N \sum_j G_r(f)(a)^p \leq 4 \|AH_f T\|_{S_p}^p,$$

which can be further estimated, using (5.6), as follows

$$(5.13) \quad \|AH_f T\|_{S_p}^p \leq \|A\|_{L^2(\varphi) \rightarrow L^2(\varphi)}^p \|H_f\|_{S_p}^p \|T\|_{L^2(\varphi) \rightarrow F^2(\varphi)}^p \leq C \|H_f\|_{S_p}^p.$$

Putting (5.12) and (5.13) together and applying the duality between l^1 and l^∞ , we obtain

$$\sum_{a \in J} G_r(f)(a)^p \leq C \|H_f\|_{S_p}^p.$$

The constants C above are all independent of f and J . Hence,

$$(5.14) \quad \sum_{a \in \Lambda_k} G_r(f)(a)^p \leq C \|H_f\|_{S_p}^p.$$

Now take Λ to be an $\frac{r}{2}$ -lattice similar to (2.6), which can be viewed as a union of 4^n r -lattice. Then

$$\begin{aligned} \int_{\mathbb{C}^n} G_{\frac{r}{2}}(f)^p dv &\leq \sum_{a \in \Lambda} \int_{B(a, \frac{r}{2})} G_{\frac{r}{2}}(f)^p dv \\ &\leq C \sum_{a \in \Lambda} \sup_{z \in B(a, \frac{r}{2})} G_{\frac{r}{2}}(f)(z)^p \\ &\leq C \sum_{a \in \Lambda} G_r(f)(a)^p \leq C \|H_f\|_{S_p}^p, \end{aligned}$$

and so, for $0 < r \leq r_0$, we have

$$\int_{\mathbb{C}^n} G_{\frac{r}{2}}(f)^p dv \leq C \|H_f\|_{S_p}^p.$$

Therefore, by Theorem 2.6, for each $r > 0$, it holds that

$$(5.15) \quad \int_{\mathbb{C}^n} G_r(f)^p dv \leq C \|H_f\|_{S_p}^p.$$

Now we treat the case $1 \leq p < \infty$. Let $\{e_a : a \in \Lambda_k\}$ be an orthonormal basis of $F^2(\varphi)$ and define linear operators T and B by setting

$$T = \sum_{a \in \Lambda} k_a \otimes e_a : L^2(\varphi) \rightarrow F^2(\varphi),$$

and

$$B = \sum_{a \in \Lambda} g_a \otimes e_a : L^2(\varphi) \rightarrow L^2(\varphi)$$

where

$$g_a = \begin{cases} \frac{\chi_{B(a,r)} H_f(k_a)}{\|\chi_{B(a,r)} H_f(k_a)\|} & \text{if } \|\chi_{B(a,r)} H_f(k_a)\| \neq 0 \\ 0, & \text{if } \|\chi_{B(a,r)} H_f(k_a)\| = 0. \end{cases}$$

Since $\|g_a\| \leq 1$ and $\langle g_a, g_\tau \rangle = 0$ when $a \neq \tau$, it follows that

$$\|B\|_{L^2(\varphi) \rightarrow L^2(\varphi)} \leq 1.$$

For $H_f \in S_p$, by Theorem 4.4, we have $\lim_{z \rightarrow \infty} \|\chi_{B(z,r)} H_f(k_z)\| = 0$. Since

$$\langle B^* M_{\chi_{B(a,r)}} H_f T e_a, e_a \rangle = \langle \chi_{B(a,r)} H_f T(e_a), B(e_a) \rangle = \|\chi_{B(a,r)} H_f(k_a)\|,$$

and

$$\langle B^* M_{\chi_{B(a,r)}} H_f T e_a, e_b \rangle = 0 \quad \text{for } a \neq b,$$

$B^* M_{\chi_{B(a,r)}} H_f T$ is a compact positive operator on $L^2(\varphi)$. Theorem 1.27 of [26] yields

$$\sum_{a \in \Lambda_k} |\langle B^* M_{\chi_{B(a,r)}} H_f T e_a, e_a \rangle|^p \leq \|B^* M_{\chi_{B(a,r)}} H_f T\|_{S_p}^p \leq C \|H_f\|_{S_p}^p.$$

Thus, using (4.7), we have

$$\begin{aligned} \sum_{a \in \Lambda_k} G_r(f)(a)^p &\leq C \sum_{a \in \Lambda_k} \|\chi_{B(a,r)} H_f(k_a)\|^p \\ &= \sum_{a \in \Lambda_k} |\langle B^* M_{\chi_{B(a,r)}} H_f T e_a, e_a \rangle|^p \leq C \|H_f\|_{S_p}^p \end{aligned}$$

which gives (5.14) for $1 \leq p < \infty$. From this, with the same approach as in the other case, we obtain the desired conclusion in (5.15).

(B) \Rightarrow (C). Suppose $f \in \text{IDA}^p$, and decompose $f = f_1 + f_2$ as in the implication (B) \Rightarrow (A).

For $0 < p \leq 2$, applying (4.4) and Lemma 4.1 for $d\mu = |\bar{\partial}f_1|^2 dv$, we get

$$\begin{aligned} & \int_{\mathbb{C}^n} \|H_{f_1}(k_z)\|^p dv \\ & \leq C \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} |k_z(\xi)e^{-\varphi(\xi)}|^2 |\bar{\partial}f_1(\xi)|^2 dv(\xi) \right)^{\frac{p}{2}} dv(z) \\ & \leq C \int_{\mathbb{C}^n} dv(z) \int_{\mathbb{C}^n} |k_z(\xi)e^{-\varphi(\xi)}|^p M_{2,r}(\bar{\partial}f_1)(\xi)^p dv(\xi) \\ & \leq C \int_{\mathbb{C}^n} M_{2,r}(\bar{\partial}f_1)(\xi)^p dv(\xi). \end{aligned}$$

This gives

$$(5.16) \quad \int_{\mathbb{C}^n} \|H_{f_1}(k_z)\|^p dv \leq C \int_{\mathbb{C}^n} M_{2,r}(\bar{\partial}f_1)(\xi)^p dv(\xi) \leq C \|f\|_{\text{IDA}^p}^p.$$

Similarly,

$$\begin{aligned} \|H_{f_2}(k_z)\|^p & \leq C \left(\int_{\mathbb{C}^n} |k_z e^{-\varphi}|^2 M_{2,r}(f_2)^2 dv \right)^{\frac{p}{2}} \\ & \leq C \int_{\mathbb{C}^n} |k_z e^{-\varphi}|^p M_{2,2r}(f_2)^p dv. \end{aligned}$$

Integrating both sides with respect to z over \mathbb{C}^n , we get

$$(5.17) \quad \int_{\mathbb{C}^n} \|H_{f_2}(k_z)\|^p dv(z) \leq C \int_{\mathbb{C}^n} M_{2,2r}(f_2)^p dv \leq C \|f\|_{\text{IDA}^p}^p.$$

For $2 \leq p < \infty$, by (4.4),

$$\begin{aligned} \|H_{f_1}(k_z)\|^p & \leq C \langle |\bar{\partial}f_1|^2 k_z, k_z \rangle^{\frac{p}{2}} \\ & = C \langle T_{|\bar{\partial}f_1|^2} k_z, k_z \rangle^{\frac{p}{2}} \leq C \left\langle \left(T_{|\bar{\partial}f_1|^2} \right)^{\frac{p}{2}} k_z, k_z \right\rangle. \end{aligned}$$

Therefore,

$$(5.18) \quad \begin{aligned} \int_{\mathbb{C}^n} \|H_{f_1}(k_z)\|^p dv(z) & \leq C \int_{\mathbb{C}^n} \left\langle \left(T_{|\bar{\partial}f_1|^2} \right)^{\frac{p}{2}} k_z, k_z \right\rangle dv(z) \\ & \leq C \left\| T_{|\bar{\partial}f_1|^2} \right\|_{S^{\frac{p}{2}}}^{\frac{p}{2}} \leq C \|M_{2,p}(|\bar{\partial}f_1|)\|_{L^p} \\ & \leq C \|f\|_{\text{IDA}^p}^p. \end{aligned}$$

Similarly,

$$\|H_{f_2}(k_z)\|^p \leq \langle |f_2|^2 k_z, k_z \rangle^{\frac{p}{2}} \leq \left\langle \left(T_{|f_2|^2} \right)^{\frac{p}{2}} k_z, k_z \right\rangle,$$

and so

$$(5.19) \quad \begin{aligned} \int_{\mathbb{C}^n} \|H_{f_2}(k_z)\|^p dv(z) & \leq \left\| \left(T_{|f_2|^2} \right)^{\frac{p}{2}} \right\|_{S_1} \\ & \leq C \|M_{2,p}(|f_2|)\|_{L^p} \leq C \|f\|_{\text{IDA}^p}^p. \end{aligned}$$

Combining the estimates (5.16)–(5.19) gives

$$(5.20) \quad \int_{\mathbb{C}^n} \|H_f(k_z)\|^p dv(z) \leq C \|f\|_{\text{IDA}^p}^p.$$

(C) \Rightarrow (B). By (4.7) and (2.10), we have

$$(5.21) \quad \|f\|_{\text{IDA}^p}^p \leq C \int_{\mathbb{C}^n} \|H_f(k_z)\|^p dv(z).$$

Finally, the S_p -norm equivalence in (1.4) follows from (5.3), (5.15), (5.20) and (5.21). The proof is completed. \square

Similarly to Corollary 4.5, restricting to the classical Segel-Bargmann space, we have the following corollary, which was obtained only for $p = 2$ in [3].

Corollary 5.2. *Suppose $\varphi(z) = \frac{\alpha}{2}|z|^2$ and $0 < p < \infty$. Then for $f \in \mathcal{S}$, $H_f \in S_p$ if and only if $\int_{\mathbb{C}^n} \|(I - P)(f \circ \tau_z)\|^p dv(z) < \infty$. Furthermore,*

$$(5.22) \quad \|H_f\|_{S_p} \simeq \left\{ \int_{\mathbb{C}^n} \|(I - P)(f \circ \tau_z)\|^p dv(z) \right\}^{\frac{1}{p}}.$$

6. SIMULTANEOUS MEMBERSHIP OF H_f AND $H_{\bar{f}}$ IN S_p

In the setting of the classical Segel-Bargmann space F^2 , for $0 < p < \infty$ and $f \in \mathcal{S}$, both H_f and $H_{\bar{f}}$ are in S_p if and only if

$$(6.1) \quad \int_{\mathbb{C}^n} (SD(f \circ \tau_z))^p dv(z) < \infty,$$

where $\tau_z(w) = w + z$ and the standard deviation $SD(g)$ of $g \in L^2(\mathbb{C}^n, d\mu)$ with $d\mu(z) = \pi^{-n} e^{-|z|^2} dv(z)$ is defined by

$$SD(g) = \left\{ \int_{\mathbb{C}^n} \left| g - \int_{\mathbb{C}^n} g d\mu \right|^2 d\mu \right\}^{1/2} = \left\{ \int_{\mathbb{C}^n} |g|^2 d\mu - \left| \int_{\mathbb{C}^n} g d\mu \right|^2 \right\}^{1/2}$$

(see [16, 25]). In [10], the result for $1 \leq p < \infty$ was extended to symmetrically-normed ideals S_Φ , that is, it was proved that $H_f, H_{\bar{f}} \in S_\Phi$ if and only if

$$(6.2) \quad \Phi(\{J(f; u)\}_{u \in \mathbb{Z}^{2n}}) < \infty,$$

where Φ is a symmetric norming function,

$$J(f; u) = \left\{ \int_{Q+u} \int_{Q+u} |f(z) - f(w)|^2 dv(w) dv(z) \right\}^{1/2},$$

and $Q = \{(x_1 + iy_1, \dots, x_n + iy_n) \in \mathbb{C}^n : x_1, y_1, \dots, x_n, y_n \in [-1, 2)\}$.

To compare the situation with weighted Segel-Bargmann spaces, we say that $f \in L^2_{\text{loc}}$ is in IMO^s with $0 < s \leq \infty$ if $MO_{2,r}(f) \in L^s$ for some $r > 0$, where

$$MO_{2,r}(f)(z) = \left(\frac{1}{|B(z, r)|} \int_{B(z, r)} |f - \hat{f}_r(z)|^2 dv \right)^{1/2}$$

and \hat{f}_r is the average function defined on \mathbb{C}^n by

$$\hat{f}_r(z) = \frac{1}{|B(z, r)|} \int_{B(z, r)} f dv.$$

For further details on these spaces, see [14].

The following lemma shows the connection between IMO and IDA.

Lemma 6.1. *Suppose $0 < p \leq \infty$. Then for $f \in L^2_{\text{loc}}(\mathbb{C}^n)$, $f \in \text{IDA}^p$ and $\bar{f} \in \text{IDA}^p$ if and only if $f \in \text{IMO}^p$. Furthermore,*

$$\|f\|_{\text{IDA}^p} + \|\bar{f}\|_{\text{IDA}^p} \simeq \|f\|_{\text{IMO}^p}.$$

Proof. The conclusion for $1 < p \leq \infty$ is essentially Proposition 2.5 in [14]. As before, denote by $Q_{z,r}$ the Bergman projection of $L^2(B(z,r), dv)$ onto $A^2(B(z,r), dv)$. If $f \in L^2_{\text{loc}}(\mathbb{C}^n)$, set $h_1 = Q_{z,r}(f)$ and $h_2 = Q_{z,r}(\bar{f})$. Then

$$\frac{1}{|B(z,r)|} \int_{B(z,r)} |f_j - h_j|^2 dv = G_r(f_j)(z)^2,$$

where $f_1 = f$ and $f_2 = \bar{f}$. Set $c(z) = \frac{1}{2}\text{Re}(h_1 + h_2)(z) + \frac{1}{2}i\text{Im}(h_1 - h_2)(z)$. As shown in the proof of Proposition 2.5 of [14],

$$\left\{ \frac{1}{|B(z,r)|} \int_{B(z,r)} |f - c(z)|^2 dv \right\}^{\frac{1}{2}} \leq C \{G_r(f)(z) + G_r(\bar{f})(z)\}.$$

Hence,

$$\begin{aligned} \frac{1}{|B(z,r)|} \int_{B(z,r)} \left| f - \frac{1}{|B(z,r)|} \int_{B(z,r)} f dv \right|^2 dv &\leq \frac{1}{|B(z,r)|} \int_{B(z,r)} |f - c(z)|^2 dv \\ &\leq C (G_r(f)(z) + G_r(\bar{f})(z))^2. \end{aligned}$$

This implies, for $0 < p \leq \infty$, $\|f\|_{\text{IMO}^p} \leq C \{\|f\|_{\text{IDA}^p} + \|\bar{f}\|_{\text{IDA}^p}\}$.

The reverse inequality follows from the fact that $\|f\|_{\text{IDA}^p} \leq \|f\|_{\text{IMO}^p}$ since $\|\bar{f}\|_{\text{IDA}^p} \leq \|f\|_{\text{IMO}^p}$ by definition. \square

Theorem 6.2. *Let $0 < p < \infty$ and suppose $\varphi \in C^2(\mathbb{C}^n)$ is real valued with $i\partial\bar{\partial}\varphi \simeq \omega_0$. Then for $f \in \mathcal{S}$, the following statements are equivalent.*

- (A) Both $H_f, H_{\bar{f}} \in S_p(F^2(\varphi), L^2(\varphi))$.
- (B) $f \in \text{IMO}^p$.

Furthermore,

$$(6.3) \quad \|H_f\|_{S_p} + \|H_{\bar{f}}\|_{S_p} \simeq \|f\|_{\text{IMO}^p}.$$

Proof. Given $f \in \mathcal{S}$ and $0 < p < \infty$, the equivalence between (A) and (B) together with the norm estimates (6.3) follow from Theorem 1.1 and Lemma 6.1. \square

For Hankel operators acting on F^2 , the conditions in (6.1) and (6.2) are of course equivalent to (B), but notice that the latter equivalence can be proved directly.

7. PROOF OF THEOREM 1.2

In this section we prove the Berger-Coburn phenomenon for Schatten p -class Hankel operators when $1 < p < \infty$. For this purpose we employ the Ahlfors-Beurling operator which is a well-known Calderón-Zygmund operator on $L^p(\mathbb{C})$, $1 < p < \infty$, defined as follows

$$\mathfrak{T}(f)(z) = p.v. - \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\xi)}{(\xi - z)^2} dv(\xi),$$

where *p.v.* means the Cauchy principal value. See [1] and [2] for further details.

Lemma 7.1. *Suppose $1 < p < \infty$. Then there is a constant C depending only on p such that, for $f \in C^2(\mathbb{C}^n) \cap L^\infty$ and $j = 1, 2, \dots, n$,*

$$(7.1) \quad \left\| \frac{\partial f}{\partial z_j} \right\|_{L^p} \leq C \left\| \frac{\partial f}{\partial \bar{z}_j} \right\|_{L^p}.$$

Proof. We take $n = 1$ temporarily. Let $f \in C^2(\mathbb{C}) \cap L^\infty$. If $\left\| \frac{\partial f}{\partial \bar{z}_j} \right\|_{L^p} = 0$, then $f \in H(\mathbb{C}) \cap L^\infty$, which implies f is constant and the estimates in (7.1) follow. So we suppose $\left\| \frac{\partial f}{\partial \bar{z}_j} \right\|_{L^p} > 0$. Take $\psi(r) \in C^\infty(\mathbb{R})$ to be decreasing such that $\psi(x) = 1$ for $x \leq 0$, $\psi(x) = 0$ for $x \geq 1$, and $0 \leq -\psi'(x) \leq 2$ for $x \in \mathbb{R}$. For $R > 0$ fixed, set $\psi_R(x) = \psi(x - R)$. Now for $f \in C^2(\mathbb{C}^n) \cap L^\infty$, define $f_R(z) = f(z)\psi_R(|z|)$. It is trivial that $f_R(z) \in C_c^2(\mathbb{C})$, the set of C^2 functions on \mathbb{R}^2 with compact support. From Theorem 2.1.1 in [5] we have

$$f_R(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial f_R}{\partial \bar{z}} d\xi \wedge d\bar{\xi}.$$

Notice that $\frac{\partial f_R}{\partial \bar{z}} = \psi_R \frac{\partial f}{\partial \bar{z}} + f \frac{\partial \psi_R}{\partial \bar{z}}$. By Lemma 2 on page 52 in [1], we get

$$(7.2) \quad \frac{\partial f_R}{\partial z}(z) = \mathfrak{T} \left(\frac{\partial f_R}{\partial \bar{z}} \right) (z) = \mathfrak{T} \left(\psi_R \frac{\partial f}{\partial \bar{z}} \right) (z) + \mathfrak{T} \left(f \frac{\partial \psi_R}{\partial \bar{z}} \right) (z).$$

Now for $r > 0$ and $|z| < r$, when R is sufficiently large, it holds that

$$\mathfrak{T} \left(f \frac{\partial \psi_R}{\partial \bar{z}} \right) (z) \leq \frac{\|f\|_{L^\infty}}{\pi(R-r)^2} \int_{R \leq |\xi| \leq R+1} dv(\xi) \leq \frac{3R\|f\|_{L^\infty}}{(R-r)^2},$$

and hence

$$(7.3) \quad \left\| \mathfrak{T} \left(f \frac{\partial \psi_R}{\partial \bar{z}} \right) \right\|_{L^p(D(0,r),dv)} \leq \left\| \frac{\partial f}{\partial \bar{z}} \right\|_{L^p}.$$

On the other hand, by the boundness of \mathfrak{T} on L^p (see for example, Theorem 4.5.3 in [2], or the estimate (11) on page 53 in [1]), we get

$$(7.4) \quad \left\| \mathfrak{T} \left(\psi_R \frac{\partial f}{\partial \bar{z}} \right) \right\|_{L^p} \leq C \left\| \psi_R \frac{\partial f}{\partial \bar{z}} \right\|_{L^p} \leq C \left\| \frac{\partial f}{\partial \bar{z}} \right\|_{L^p}.$$

From (7.2), (7.3) and (7.4) we obtain

$$\left\| \frac{\partial f}{\partial z} \right\|_{L^p(D(0,r),dv)} = \left\| \frac{\partial f_R}{\partial z} \right\|_{L^p(D(0,r),dv)} \leq C \left\| \frac{\partial f}{\partial \bar{z}} \right\|_{L^p}.$$

Therefore,

$$(7.5) \quad \left\| \frac{\partial f}{\partial z} \right\|_{L^p} \leq C \left\| \frac{\partial f}{\partial \bar{z}} \right\|_{L^p}.$$

Now for $n \geq 2$ and $f \in L^\infty \cap C^2(\mathbb{C}^n)$, from (7.5)

$$\begin{aligned} \int_{\mathbb{C}^n} \left| \frac{\partial f}{\partial z_1}(\xi) \right|^p dv(\xi) &= \int_{\mathbb{C}^{n-1}} dv(\xi') \int_{\mathbb{C}} \left| \frac{\partial f}{\partial z_1}(\xi_1, \xi') \right|^p dv(\xi_1) \\ &\leq C \int_{\mathbb{C}^{n-1}} dv(\xi') \int_{\mathbb{C}} \left| \frac{\partial f}{\partial \bar{z}_1}(\xi_1, \xi') \right|^p dv(\xi_1). \end{aligned}$$

This implies (7.1) for $j = 1$. Similarly, we have (7.1) for $j = 2, \dots, n$, which completes the proof. \square

Proof of Theorem 1.2. Suppose $1 < p < \infty$ and $H_f \in S_p$. By Theorem 1.1, we have

$$\|f\|_{\text{IDA}^p} \simeq \|H_f\|_{S_p} < \infty.$$

We decompose $f = f_1 + f_2$ as in (2.8). Then, since $M_{2,r}(\overline{f_2}) = M_{2,r}(f_2) \in L^p$, we have $H_{\overline{f_2}} \in S_p$ and

$$(7.6) \quad \|H_{\overline{f_2}}\|_{S_p} \leq C\|M_{2,r}(f_2)\|_{L^p} \leq C\|f\|_{\text{IDA}^p}.$$

In addition, since $f \in L^\infty$, as in (5.3) of [13], we may assume

$$\|f_1\|_{L^\infty} \leq C\|f\|_{L^\infty},$$

where the constant C is independent of f . We now apply Lemma 7.1 to obtain

$$\left\| \frac{\partial \overline{f_1}}{\partial \overline{z_j}} \right\|_{L^p} = \left\| \frac{\partial f_1}{\partial z_j} \right\|_{L^p} \leq C \left\| \frac{\partial f_1}{\partial \overline{z_j}} \right\|_{L^p}.$$

This and (5.2) yield

$$(7.7) \quad \|H_{\overline{f_1}}\|_{S_p} \leq C\|\overline{\partial f_1}\|_{L^p} \leq C\|\overline{\partial f_1}\|_{L^p} \leq C\|f\|_{\text{IDA}^p}.$$

It follows from (7.6), (7.7) and Theorem 1.1 that

$$\|H_{\overline{f}}\|_{S_p} \leq C\|f\|_{\text{IDA}^p} \leq C\|H_f\|_{S_p},$$

which completes the proof. \square

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