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THE SPECTRUM OF THE NEUMANN POINCARÉ OPERATOR ON BOW-TIE CURVES

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Declaration

I declare that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

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Finally, to my late Father, to whom's memory this work is dedicated. You were not just a brilliant Father, but a great mentor, and a great friend. I will always love you Dad, and always miss you.

Abstract

The study of the spectral properties of those operators which output potential functions has historically been of great value, particularly in the resolution of certain problems in potential theory and the mathematical study of gravitational and electromagnetic fields. The initial inspiration for this work was to build on the existing body of results that describe the spectrum of the Neumann Poincaré operator on different surfaces, particularly those with corners and edges.

The particular spectral properties of this integral operator align nicely with a number of scenarios discussed in physics, most notably in the study of plasmonics, where there is a noted coincidence between the elements of the spectrum of the Neumann Poincaré operator when acting on specific function spaces and the phenomena of plasmon resonances.

This work is a study of the spectral properties of the Neumann Poincaré operator when considered over sets bounded by bow-tie curves, formed of two tear drop shapes or 'wings', each with a corner that coincides, a scenario that distinguishes itself from prior studies in that the surface being acted on is neither completely smooth nor can it be characterized as a Lipschitz domain in the region of the curve's singular point.

Summary

Whilst a study of the Neumann Poincaré operator on bow-tie curves has been made, the existing work has been done in regard to the aforementioned relation to plasmonics, restricting itself to the most congenial space of functions for that study, known as the 'energy space'.

We shall consider the properties of our operators spectrum on more broadly defined spaces, beginning with the L^2 space on our bow-tie. Initially we consider the scenario of two infinitely large wedges with coincident vertices, formulating a matrix operator with entries given by the Neumann Poincaré adjoint operator as it acts over the combinations of pairs of the edges that form the boundary of the two wedges.

We then determine a unitarily equivalent operator matrix in the manner of Perfekt-Putinar where the entries are given in terms of multiplication operators. Together these two properties facilitate a much simpler study of the spectrum of our operator matrix.

Before proceeding with this however we divert to discuss the localizations of these results to finite neighbourhoods of these infinite wedges, and go on to compare them with those bow-tie curves which are similar to said localizations within a sufficiently small neighbourhood of the corner. We determine that the essential spectrum in both scenarios is identical, before moving to calculate the essential norm of our operator matrix on these localizations.

In the process we demonstrate how, on L^2, the radius of the spectrum of our operator matrix exceeds that observed in previous studies done over more restricted spaces of functions. In particular we can compare this value with the spectral radius determined in the aforementioned study of plasmonics.

Returning to our original formulation, we generate an explicit formula for the spectrum of our operator matrix on infinite wedges. Unifying this with our prior localization results, taking advantage of the specific form of our unitarily equivalent operator matrix to identify the spectrum in the infinite scenario with the essential spectrum in the case of the bow-tie curve, using techniques developed by Mitrea .

Given the previous difference observed in the spectral radii of the Neumann Poincaré operator on L^2 and the energy space, we are then motivated to consider what difference is made by further restricting our operator matrix to act on the space of continuous functions.

Calculating directly from the essential norm of our operator matrix using the techniques discussed by Kress on curvilinear polygons, we formulate an upper bound to said norm, before using this result to determine that not only is complete symmetry

between the angles defining our bow-tie curve a requirement to ensure a spectral radius within the expected range, but, based on the manner in which we parameterize our bow-tie curve, it is equivalent to having our operator matrix be bounded on the space of continuous functions on the curve.

Contents

1 Introduction

The following work comprises a study of the properties of the adjoint of the Neumann Poincaré operator as it acts over different function-spaces on domains given by bow-tie curves, specifically developing a number of results with regards to its spectral properties on such spaces.

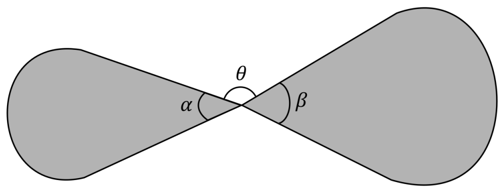


Figure 1: A bow-tie curve, defined by the angles α, β and θ

In particular, the original results developed in this work are focused on determining the spectral radius of the operator as it acts over different bow-tie curves and, where possible, determining the exact form of the spectrum on such curves. The spectrum of an operator K is given by the set

$$\sigma(K) := \{\lambda \in \mathbb{C} : K - \lambda I \text{ is not invertible}\} \subset \mathbb{C}.$$

We also take this opportunity to introduce the following recurring notion and its notation.

Definition 1.1. The *spectral radius* of an operator F is given by

$$\|\sigma(F)\| := \max_{\lambda \in \sigma(F)} |\lambda|$$

that is, the supremum of the magnitudes of the spectrum of F

Given the varied scenarios in which an operator may prove to lack invertibility, as well as the varied types of invertibility, the spectrum can be subdivided on this basis into distinct subsets, some of which we will discuss in the following sections.

In order to construct rigorous results for our surface potential, it is neccesary to have a rigorously defined notion of the domain we are working on. Specifically, we must introduce the concept of a hypersurface, in $\mathbb{R}^n$. Analogous to the concept a surface in three dimensions, a hypersurface takes the form of a $n-1$ dimensional embedding in $\mathbb{R}^n$.

Of particular interest to us, at least initialy, is the use of ’smooth’ hypersurfaces. Indeed, the reader may note that in some of the following results, one of the common conditions on the domain is that it has a ’sufficiently smooth’ surface.

To this end we introduce the following concept from ([28], Chp.2, pg. 30) that, given a bounded domain $D \subset \mathbb{R}^n$, then D is of class C^m if its closure $\overline{D}$ admits a finite open covering, and that for each element of the covering that intersects with ∂D the intersection with the closure $\overline{D}$ is bijective to the half-ball $H := \{x \in \mathbb{R}^n : |x| < 1, x_n \geq 0\}$, where both it and its inverse are n times continuously differentiable, and the intersection of each element with ∂D maps onto $H \cap \{x \in \mathbb{R}^n : x_n = 0\}$. Indeed, if a boundary describes such a space, we may also say that said boundary,∂D , is of class C^m.

Such a space, is also known as a *hypersurface of class C^m*. When discussing smooth surfaces, it is typically intended to mean surfaces of class C^1 or greater.

The Neumann Poincaré operator is an integral operator, the study of which begins in the field of potential theory. Given a domain $D \in \mathbb{R}^n$, for $n \geq 2$ and taking $\phi \in C(\partial D)$ we have that the *double layer potential* with density ϕ is given by the function u with the following representation,

$$u(x) := \int_{\partial D} \phi(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} d\sigma(y), \quad x \in \mathbb{R}^n \setminus \partial D$$

where Φ denotes the *fundamental solution of Laplace's equation*, given by

$$\Phi(x,y) := \begin{cases} \frac{1}{2\pi} \ln(\frac{1}{|x-y|}), & n = 2, \\ \frac{1}{n(n-2)\alpha_n} \frac{1}{|x-y|}, & n \geq 3, \end{cases}, \quad x \neq y$$

where α_n is the volume of the unit ball on $\mathbb{R}^n$, and where $\nu(y)$ represents the unit normal to ∂D at y, oriented to the exterior of D.

The notation $d\sigma(y)$ here represents the *surface element* at $y \in \partial D$, given in the 2-Dimensional and 3-dimensional cases by arc length and surface area respectively. This notion can be further extended to $\mathbb{R}^n$ by using the *volume element* of the $n-1$-dimensional subspace ∂D. (See [28], Chp. 2, pg. 30, or [11], Chp. 5, pg. 83.)

Explicitly, if we take ∂D to be given by the Lipschitz graph

$$\left\{(x, \phi(x)) \in \mathbb{R}^n : x \in \mathbb{R}^{n-1}\right\}$$

then we get

$$d\sigma := \sqrt{1 + |\nabla \phi|^2} dx.$$

Localizing this result, we get the definition of the surface measure for a *Lipschitz Domain* (see Definition 1.8.)

Given this differential is well defined almost everywhere, it is sufficient for the purposes of our study as the non-Lipschitz element of a bow-tie curve has zero measure.

Given $\frac{\partial \Phi(x,y)}{\partial \nu(y)}$ is a directional and specifically a normal derivative, we can decompose it into an inner product and thus gain the following representation of the double layer potential

$$u(x) := \frac{1}{\omega_n} \int_{\partial D} \phi(y) \frac{\langle \nu(y), (x-y) \rangle}{|x-y|^n} d\sigma(y), \quad x \in \mathbb{R}^n \setminus \partial D \tag{1.1}$$

where

$$\omega_n := \begin{cases} 2\pi, & n = 2, \\ n(n-2)\alpha_n, & n \geq 3, \end{cases}.$$

The interest in the analysis of the double layer potential emerged in the 1800s, when it was proposed as a solution to the Dirichlet boundary value problem, a recurring problem in mathematical physics that required a harmonic function as a solution which took the form of some previously specified continuous function on ∂D, more explicit details of which are discussed in the next chapter.

The Dirichlet problem is typically split up in terms of a solution for the interior and exterior of the domain being acted on, the interior problem requiring continuity over $\overline{D}$, and the exterior requiring continuity over $\mathbb{R}^n \setminus D$.

Examining the inner-product description of the double layer potential (1.1), it is natural that one might question whether, for all $x \in \partial D$, the function u is well defined, let alone continuous. Specifically, one might question whether we have that, for $x = y$ the kernel defining the double layer potential is singular.

It is however possible to demonstrate that for $n \geq 2$, if we have that ∂D is sufficiently smooth then the kernel is actually *weakly singular*, a specific subcategory of singularity dependent on the boundedness of the kernel, which, upon further analysis, the specifics of which shall be given in the next chapter, one can use to determine that the integral is not only well defined, that is, it is finite for all $x \in \partial D$, but the resultant function u is also continuous on ∂D.

So, we have continuous behaviour on the boundary, as well as on the interior and the exterior spaces separately. It then remains to consider, does this continuity extend to the inclusion of the boundary? This proved a more difficult matter to resolve.

Whilst the double layer potential can be shown to meet the majority of the criteria of the problem, it was already clear when it was first considered that when the double layer potential was evaluated for inputs on the boundary there was a distinct discontinuity in the function as it passed across the boundary, either from the interior or exterior.

In order to resolve this issue, it was neccesary to determine a comparable function, or functions which, when acting over the boundary did so continuously with respect to the behaviour of the function on either the inside or outside of the domain. To this end, an examination was made of the limits of the potential as its variable approached the boundary from within and without. This led to the formulation of the *jump relations*, given below.

Theorem 1.2. *Let D be a class C^2 domain and take $\phi \in C(\partial D)$ to be the density of the double layer potential u. Then for $x \in \partial D$,*

$$u_{\pm}(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu} \phi(y) d\sigma(y) \pm \frac{1}{2}\phi(x)$$

where $u_{\pm}(x)$ represents the limits (from outside and inside ∂D, respectively) as the argument tends to x along the normal vector $\nu(x)$.

So, if one defines a function to act as the double layer potential on the interior or exterior and for $x \in \partial D$ to take on the value of this limit from the interior or exterior, respectively, then we not only carry over the initial properties that made the double layer potential so appealing, but we also have continuity as we pass into or out of the boundary for both the internal and external Dirichlet problem, that is, we have a solution to both problems.

In order to better understand such solutions, it is neccesary to reframe them from the perspective of functional analysis. Indeed, for $\phi \in C(\partial D)$, the Neumann Poincaré operator is given by

$$K\phi(x) := \frac{2}{\omega_n} \text{p.v.} \int_{\partial D} \phi(y) \frac{\langle \nu(y), (x-y) \rangle}{|x-y|^n} d\sigma(y), \quad x \in \partial D, \tag{1.2}$$

with adjoint given by

$$K^*\phi(x) := \frac{2}{\omega_n} \text{p.v.} \int_{\partial D} \phi(y) \frac{\langle \nu(x), (x-y) \rangle}{|x-y|^n} d\sigma(y), \quad x \in \partial D, \tag{1.3}$$

that is, for the given density ϕ the N.P. operator returns a function formulated in the same manner as the double layer potential corresponding to ϕ, acting over the boundary ∂D.

We observe that the operators are given in terms of *principle value* integrals. This is to account for those scenarios where the surface integral is an improper integral.

It was the study of the properties of this operator by Henri Poicaré(1854-1912) in relation to the Dirichlet problem, as well as the earlier work on the double layer potential and its jump relations by Carl Neumann (1832-1925) which led to its modern appellation.

In tandem with the jump relations given above, it can be seen that for a suitable choice of ϕ, the operators $K \pm I$ generate solutions to the interior and exterior dirichlet boundary problem.

It is in fact possible to show that not only can the jump relations generated be used to help solve the Dirichlet boundary value problem, but the solutions generated are unique, furthermore, as stated above, a distinguishing requirement of the Dirichlet problem is that, for a given $f \in C(\partial D)$, our solution identifies with f on ∂D. We can write this in terms of the N.P. operator as

$$K\phi \pm \phi = f.$$

As a result of this identity we see value in the study of the invertibility of $K \pm I$ or in other words, whether or not $\pm 1 \in \sigma(K)$. This gives us a direct link between the spectrum of K and its use in solving the Dirichlet problem.

From here we see the origin of the study of the spectral theory of the N.P. operator, however, the study the spectrum of both K and K^* does not end at the Dirichlet problem. Indeed, the analysis of the spectrum is instrumental in the study of *plasmonics*, a subfield of the study of electromagnetism, and in particular *plasmon resonances*.

Similar to the Dirichlet problem, the relation between plasmonics and the spectrum of the N.P. operator begins with a boundary value problem. Specifically, in reducing Maxwell's equations for electromagnetism to the scenario given by plasmonics, there emerges a boundary value problem the solution of which not only exists, but, when substituted back in, allows us to rewrite and rearrange the related partial differential equation into the form

$$K\phi - \lambda\phi = f,$$

so the boundary value problem for a fixed function f has a unique solution given a suitable ϕ exists, that is, if $K - \lambda I$ is invertible. In particular it is when we do not have invertibility that we encounter the previously mentioned plasmon resonances.

Again we have a clear relation, this time between the entirety of the spectrum and the solution to the plasmonic boundary value problem.

These results have already been much expanded on, in particular with regards to how they vary on different spaces of functions, as well as over different types of domain boundaries.

1.1 Some historical context

We here give a brief overview of the historical development of potential theory, further details of which can be found in [18].

The study of potential theory originates in mathematical physics in the late 18$^{\text{th}}$ to early 19$^{\text{th}}$ century, the term 'potential' deriving specifically from the earliest attempts at describing both electromagnetic and gravitational potentials. The modern idea of a *potential function*, and indeed the original usage of the term *potential* can be traced to Daniel Bernoulli (1700-1782), in his 1738 work *Hydrodynamica* [19], in which he demonstrated that the forces involved in the equations of fluid motion could be determined by taking the partial derivatives of a scalar valued function.

The notion of a potential in its more recognizeable context is first attributed to Joseph-Louis Lagrange (1736-1813), in his paper *On the secular equation of the Moon* (1773), determining the form taken by the gravitational force from a spheroid in a vacuum, with varying density and continuously distributed mass, as the *gradient* of its velocity vector.

Mathematically speaking, the more abstract notion of a potential has the following definition,

Definition 1.3. For a given vector field V, that is, an assignment of a vector to each point in a given space, a *scalar potential* of V is any function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that

$$V = \nabla\phi(x_1, x_2, ..., x_n) = \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, ..., \frac{\partial\phi}{\partial x_n}\right)$$

where ∇ is the gradient operator with vector notation $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right)$.

Any vector field with a representation as a potential gradient is described as being *conservative.*

We observe here briefly that the concept of potentials can be extended to vector-valued functions as a *vector potential*, where, in place of the gradient, the vector field is generated by the degree of rotation (formally, the *curl*) present in the vector field generated by the image of our vector-valued function. Given the context of this work, we shall continue to focus exclusively on the study scalar potentials. To this end, from this point, when we use the term 'potential function', we will be referring to a scalar potential.

As a result of the definition of a potential, in combination with the Gauss-Green Theorems, we can characterize a potential ϕ for a given vector field V as the line integral of V

$$\begin{aligned}\phi(x) &= \int_\gamma V(y)dy \\ &= \int_\gamma \nabla\phi(y)dy\end{aligned}$$

where γ is a parameterised path from some predetermined point to the point x.

At the time, the key question being asked by Lagrange and his contemporaries was how one might determine a potential function for the gravitational forces exerted by a given spheroid. It was almost a decade later, in 1783, when Pierre-Simon Laplace (1749-1827) published his memoir[30], later reprinted in 1784, in which he determined that, given the presupposed conditions on the spheroid, this gravitational potential function would have to satisfy what, for obvious reasons, became known as a *Laplace equation.*

Definition 1.4. A scalar-valued function $\phi : \mathbb{R}^n \to \mathbb{R}$, with well defined, second order partial derivatives, satisfies Laplace's equation, if

$$\Delta\phi(x_1, ..., x_n) = \nabla \cdot \nabla\phi(x_1, ...x_n) = \sum_{i=1}^{n} \frac{\partial^2\phi}{\partial {x_i}^2} = 0$$

where Δ is the Laplacian or Laplace operator. Such a function ϕ is described as being *Harmonic.*

The use of the notation $\nabla\cdot$, or taking the dot product of the gradient vector-operator with our vector field $\nabla\phi$ is typically referred to as its *divergence*, that being the degree of outward flow in a vector field from any singular point. As such, one might equivalently say that a function ϕ satisfies Laplace's equation if the field generated by its gradient has zero divergence.

It is likely Laplace's early development of solutions in potential theory that has lead to the field being typified as the study of Harmonic functions.

However, the work of Coulomb (1736-1806)[12] established that, similar to Newton's law of gravitation, the electrostatic force between charged particles could be described using a similar inverse-square relation. As a result, applying the framework of the existing potential theory, Siméon Denis Poisson (1781-1840) determined in 1813 [39] that Laplace's result for potential functions could be generalised and in fact took the form

$$\Delta\phi = f,$$

in particular where f represented a scalar multiple of a function known as the *mass distribution*, describing the density throughout the body being acted upon. Indeed, *Poisson's equation*, as given above, reduced to Laplace's equation, the homogeneous case, precisely when said distribution was equal to zero.

It may be more accurate then to describe potential theory as the study of solutions to Poisson's equation, with Laplace's equation being the homogeneous case. Of relevance to our study is the eventual development of the subset of Poisson's equations, Helmholtz equations, in which we have

$$\Delta\phi = \lambda\phi$$

where λ is an eigenvalue corresponding to the potential function ϕ. We will see a more specific example of the Helmholtz equation, more directly relevant to this study, in the following section.

Subsequent to Poisson, we see work by Carl Friederich Gauss (1777-1855), developing on and providing proofs for the work of his contemporaries on harmonic functions, in particular it is Gauss who is attributed with first proposing the double layer potential as a solution to the Dirichlet problem.

Gauss was also the first to recognise and go on to prove the issues of the discontinuity of the double layer potential over such bounded regions, as mentioned above, and the originator of the development of the neccesary jump relations.

In addition Gauss's other notable contribution to potential theory is his work on the proof of the *Divergence Theorem*.

Theorem 1.5. *Let $D \subset \mathbb{R}^3$ be a solid region bounded by a closed surface ∂D and set ν to be the unit normal vector to the surface, oriented to its exterior. Taking V to be a vector field which is componentwise partially differentiable throughout D, then we have:*

$$\int_D \nabla \cdot V dD = \int_{\partial D} V \cdot \nu d(\partial D).$$

Originally formulated by Joseph-Louis Lagrange (1736-1813) [29] and first proved by Mikhail Ostrogradsky (1801-1862) (1831), in the intervening period, Gauss's attempts to determine an overall proof led to him making significant advances in the theorem, most notably in 1813, when he formulated, though did not prove, an equivalent result for the two-dimensional case. This result would come to be known as *Green's Theorem*.

Theorem 1.6. *Green's Theorem*

Take D to be a bounded domain of class C^1 (See Section 2.1), and ν the exterior-oriented unit normal vector to ∂D, then for $u \in C^1(D)$ and $v \in C^2(D)$

$$\int_D (u\Delta v + \nabla u \cdot \nabla v)dx = \int_{\partial D} u\frac{\partial v}{\partial \nu}ds,$$

and for $u, v \in C^2(D)$

$$\int_D (u\Delta v - v\Delta u)dx = \int_{\partial D} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) ds.$$

This attribution to George Green (1793-1841) originates with his own work on the Divergence Theorem in 1828, separate from both Lagrange and Gauss, in which he again wrote on the two-dimensional case of the Divergence Theorem. Green's Theorem was not in fact proved until 1831 by Augustin-Louis Cauchy (1789-1857).

This result is instrumental in the current understanding of harmonic functions as well as being the origin of many of its more fundamental results, in particular, it is Green's Theorem and the results that can subsequently be derived from it that enabled the further analysis of *layer potentials.*

The issue of discontinuity for the double layer potentials over smooth boundaries was resolved as a result of the work in 1868 by Julian Sokhotski (1842-1927) and later, rediscovered in 1908 by Josip Plemelj (1873-1967).

Theorem 1.7 (Plemelj Formula)**.** *Let ∂D be the boundary of a two-dimensional, class C^2 domain, and let φ satisfy the Hölder condition on ∂D, that is, there exist real constants $C_o > 0$ and $t > 0$ such that*

$$|\varphi(x) - \varphi(y)| \leq C_0\|x - y\|^t \quad x, y \in \partial D.$$

Then the Cauchy integral

$$\phi(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{\varphi(\xi)}{\xi - z}d\xi$$

can be extended uniformly Hölder continuously onto ∂D from either the interior or exterior of D by the limiting values

$$\lim_{\omega \to z} \phi_\pm(\omega) = \frac{1}{2\pi i}\int_{\partial D} \frac{\varphi(\xi)}{\xi - z}d\xi \pm \frac{1}{2}\varphi(z), \quad z \in \partial D$$

where the subcripts $\pm$ indicate the limit is being taken in ω from the interior and exterior of C respectively.

This gives a precise way of approximating such boundary integrals on smooth continuous curves, from both the interior and exterior of the boundary. Considering the Plemelj Formula and the double layer potential together, it is possible to show that the jump formulas

$$u_{\pm}(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu} \phi(y) d\sigma(y) \pm \frac{1}{2}\phi(x) \quad x \in \partial D$$

hold in the two-dimensional case. Extending this result to three dimensions requires further arguments, which we will dicuss separately in the next chapter. This work by Plemelj then gives justification to the continuous perturbation of the double layer potential function which could be used to solve the Dirichlet problem in two dimensions.

As has been discussed above, the study of the double layer potential and its subsequent extension to a solution of the Dirichlet problem, led naturally to the study of the spectral properties of the N.P. operator and its adjoint. Whilst the study of the spectrum on such domains may vary depending on the specifics of their boundary, there are a number of results which are consistent for all sufficiently smooth boundaries. This however does not account for those scenarios where the domain in question has a boundary featuring corners, or, in the three-dimensional case, edges or conical points.

A suggested introductory approach can be found discussed by Rainer Kress (1941-?) in [28](Chp.6), in which he expounds on techniques similar to the usage of the jump relations on vertices of curvilinear polygons, more specifics of which can be seen in the following chapter.

A key factor in the decision to study bow-tie spaces was the seeming dearth of existing material, particularly with regards to a space with such a singularly determined pair of vertices.

Due to its initial development, one might expect the interest in the N.P. operator to be a result of its potential applications in resolving certain problems in physics and engineering. Indeed, one of the key inspirations for the following work has been its ongoing use with regards to the study of plasmonics.

1.2 Plasmonics and their relation to the double layer potential

The original inspiration for this piece of research was to make a study of and develop on the existing results in the spectral and operator theory of plasmon resonances on surfaces with corners and edges.

To this end, the following section will discuss the relevant theory of plasmonics, with the intent of giving a contextual background of the broader physical concepts and the development of the mathematical model on which our research is based, as well as discussing the most relevant recent results.

The field of plasmonics is the study in physics of the properties of those strongly localized electromagnetic fields resultant from the oscillations of surface electrons of metallic particles, otherwise known as 'plasmons'. Of particular interest is the scenario when these oscillations become sustained, generating a 'plasmon resonance'.

These plasmon resonances are typically the result of incident light, the electromagnetic field being generated by the interaction between the light and the surface electrons.

Indeed, as described in [6] resonances will typically only occur on the surfaces of metallic particles when 'the real parts of the dielectric coefficients of the particle are negative' and 'their size comparable to or smaller than the wavelength of the excitation'.

In terms of studying these plasmons mathematically, a great deal of the existing work has focussed on the case where the system of Maxwell's equations modelling this problem can be reduced to a Helmholtz equation, as well as the 'asymptotic limit' when the particle diameter is small when compared with the frequency ω of the incident light. This is otherwise known as the electrostatic case. This study can be described using the conduction equation

$$\mathrm{div}(\epsilon(\omega)^{-1}\nabla u(x)) = 0,$$

with suitable boundary conditions, where $\epsilon(\omega)$ represents the permittivity of the dielectric ambient medium, which can be represented by the Drude-Lorentz Law:

$$\epsilon(\omega) = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2 + i\omega\gamma}\right)$$

where ω_p is the frequency of the plasma and γ the conductivity of the medium. In this electrostatic approximation, the plasmon resonances of a particle in a homogeneous medium can be described in terms of the outside permitivity ϵ_1 and the permitivity ϵ_2 inside particle. For sufficiently smooth particle, the solution u of the conduction equation may be derived from layer potentials, where the plasmon resonances on the surface of the particle are given by the 'contrast':

$$\frac{\epsilon_1 + \epsilon_2}{2(\epsilon_1 - \epsilon_2)}$$

which in this scenario equals the eigenvalues of N.P. integral operator.

Following the procedure described in [24] we model our particle as a domain D embedded in some infinite space Ω. To the interior and exterior of said domain do we asign permittivities, specifically $\epsilon_1 \in \Omega \setminus \overline{D}$ and $\epsilon_2 \in \text{int}D$. The electrostatic equation, given above can be better understood by writing it out explicitly. We must find a potential function U that satisfies the properties of being continuous across the boundary of D, and that

$$\Delta U(x) = 0, x \in \Omega \setminus \partial D,$$

with boundary conditions, and behaviour at infinity given by,

$$\epsilon_1 \frac{\partial}{\partial v_x} U_+(x) = \epsilon_2 \frac{\partial}{\partial v_x} U_-(x)$$

and

$$\lim_{x \to \infty} \nabla U(x) = e,$$

respectively, where v_x represents the exterior unit normal of ∂D at x, the subscripts $(+/-)$ indicate the limits as being taken from the exterior and interior of ∂D, respectively and where e is an applied unit field.

Given the desired harmonic property of our potential, we make use of the fundamental solutions to the Laplace equation in two and three dimensions,

$$\Phi(x,y) = \begin{cases} -\frac{1}{2\pi} \log|x-y|, & \dim(D) = 2 \\ \frac{1}{4\pi} \frac{1}{x-y}, & \dim(D) = 3. \end{cases}$$

We may then interpret $U(x)$ in terms of layer potentials. Indeed, if we consider the following ansatz

$$U(x) = \int_{\partial D} \Phi(x,y)\phi(y)d\sigma(y) + e \cdot x,$$

a perturbation of the single layer potential, with density ϕ on ∂D, and $d\sigma$ an element of surface area, then we satisfy both the conditions of harmonicity and behaviour at infinity, following from the harmonic nature the single layer potential as well as its limiting values. Finally, by substituting this into our boundary conditions we get, by Plemelj's theorem (see Section 2.3, Theorem 2.20), that

$$\begin{aligned} \epsilon_1 \frac{\partial_+}{\partial \nu_x}\left(\int_{\partial D} \Phi(x,y)\phi(y)d\sigma(y) + e \cdot x\right) &= \epsilon_2 \frac{\partial_-}{\partial \nu_x}\left(\int_{\partial D} \Phi(x,y)\phi(y)d\sigma(y) + e \cdot x\right) \\ \Rightarrow \epsilon_1 \frac{\partial_+}{\partial \nu_x}(S\phi(x) + e \cdot x) &= \epsilon_2 \frac{\partial_-}{\partial \nu_x}(S\phi(x)d\sigma(y) + e \cdot x) \\ \Rightarrow \epsilon_1\left(\frac{1}{2}(K\phi(x) - \phi(x)) + e \cdot \nu_x\right) &= \epsilon_2\left(\frac{1}{2}(K\phi(x) + \phi(x)) + e \cdot \nu_x\right) \\ \Rightarrow (\epsilon_1 - \epsilon_2)\frac{1}{2}(K\phi(x) + e \cdot \nu_x) &= \frac{1}{2}(\epsilon_1 + \epsilon_2)\phi(x) \end{aligned}$$

$$\Rightarrow \frac{(\epsilon_2 - \epsilon_1)}{(\epsilon_1 + \epsilon_2)} \frac{1}{2} K\phi(x) + \frac{1}{2}\phi(x) = -\frac{(\epsilon_2 - \epsilon_1)}{(\epsilon_1 + \epsilon_2)} e \cdot \nu_x$$
$$\Rightarrow -\frac{1}{\lambda} K\phi(x) + \phi(x) = \frac{2}{\lambda} e \cdot \nu_x$$

where we have that
$$\lambda = \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 - \epsilon_2}.$$
We observe how the result is given as a perturbation of the double layer potential. Rearranging, we can clearly represent this equation as an operator problem, in terms of the Neumann Poincaré operator K,

$$(\lambda I - K)\phi(x) = \lambda g(x).$$

This is a second order Fredholm integral problem (See section 3.1), the presence of which suggests a consideration of the essential spectrum of K and K^*. Indeed, whilst we know plasmon resonances occur for eigenvalues λ the plasmonic interpretation of the essential spectrum is more subtle, though of great interest, see the discussion in [22].

It is natural to ask whether this is always the case, and if not, under what circumstances, and in what specific ways our results may differ.

We show in Theorem.2.4 that the N.P. operator and its adjoint are compact operators when acting on sufficiently smooth surfaces. Combined with Atkinson's Theorem (See Theorem 3.5) , taking our regularizer operator to be the identity, we then clearly must have that

$$(I - \frac{1}{\lambda}K)I = I(I - \frac{1}{\lambda}K) = (I - \frac{1}{\lambda}K).$$

Thus on a sufficienty smooth surface our operator $\lambda I - K$ is always Fredholm. This then means that there are no none-zero elements in its essential spectrum and that the spectrum of the determining (N.P.) operator itself is discrete and accumulates at 0.

So, for sufficiently smooth surfaced particles, we must also have that the potential plasmon resonances on its surface are also discrete in terms of distribution, and again must accumulate as they tend closer to zero.

Continuing in our previous line of questioning we might ask if, given, the Fredholm property holds specifically because we are acting on a smooth surface, to what degree do we have that the Fredholm property will hold on a non-smooth surface?

In fact, it has been shown that in a number of settings where the the surface being acted on has clearly defined edges and/or conical points ([23],[40],[6]) the operator has a non-empty *essential spectrum* (See Definition 3.12), that is, $K - \lambda I$ is not Fredholm for all λ on such a surface

Furthermore we have as a result that in these scenarios the overall image of the spectrum is distinct from that seen on smooth surfaces, in particular, the discreteness seen prior is no longer de facto scenario for all elements of the spectrum.

Over all, this has lead to a shift in the focus of the spectral theory of the N.P. operator from a study of its eigenvalues to a focus on its Fredholm properties and its essential spectrum. The study of the spectral properties of the N.P. operator and its adjoint when examined over surfaces with corners, edges and conical points is of great interest to both mathematicians and physicists.

Much of the existing potential theory related to plasmonics is discussed in terms of *Lipschitz Domains* and the *Energy space*, $H^{-1/2}(\partial D)$.

Definition 1.8. Lipschitz domains take the form of an open subset $D \subset \mathbb{R}^n$ where for each element $z \in \partial D$ there is a rectangular coordinate system (x, s), $x \in \mathbb{R}^{n-1}, s \in \mathbb{R}$ and Lipschitz continuous function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ dependent on a such that, for some neighbourhood $X \subset \mathbb{R}^n$ of z we have,

$$X \cap D = \{(x, s) : s > \phi(x)\} \cap X.$$

The definition of Lipschitz domains permits the existence of corners and edges on otherwise smooth surfaces, making them an ideal foundation for the study of the N.P. operator on non-smooth surfaces.

Definition 1.9. The Sobolev Slobodeckij space $H^{1/2}(\partial D)$ of functions on ∂D, the smooth closed boundary of the domain $D \subset \mathbb{R}^n$, is the Hilbert space of functions $u \in L^2(\partial D)$ equipped with the norm

$$||u||^2_{H_{1/2}(\partial D)} = ||u||^2_{L^2(\partial D)} + ||u||^2_{0,1/2}.$$

determined by the following inner products on functions u and v,

$$\begin{aligned}\langle u, v\rangle_{H_{1/2}(\partial D)} =& \langle u, v\rangle_{L^2(\partial D)} + \langle u, v\rangle_{0,1/2}\\ \langle u, v\rangle_{0,1/2} =& \int_{\partial D}\int_{\partial D} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+1}} dxdy.\end{aligned}$$

We denote the dual space of $H^{1/2}(\partial D)$ by $H^{-1/2}(\partial D)$.

The use of the $H^{1/2}(\partial D)$ can be seen to go back as far as Poincaré and his work on the Dirichlet problem, however its dual can be shown to be considerably more useful to work over.

More precisely, it was shown by Dmitry Khavinson [27], having studied Poincaré's work over $H^{1/2}(\partial D)$ and using further results by Plemelj, that by taking a new inner product on $H^{-1/2}(\partial D)$ given in terms of layer potentials, generating an equivalent norm to that defined above, and completing the space with respect to the resultant norm, one achieved the aforementioned energy space, over which the operator K is self-adjoint with respect to the new inner product.

This is especially notable as, under its original consideration over the space of continuous functions, the N.P. operator was distinctly not self-adjoint.

The property of having K and K^* coincide on $H^{-1/2}(\partial D)$ is inherently advantageous, not least for the fact that one need not distinguish when discussing the spectral properties of one or the other.

It is for this reason that many of the existing studies take place over the energy space. In particular, there exists a treatment of the spectral properties of the N.P. (adjoint) operator as it acts on $H^{-1/2}(\partial D)$ where D is a bow-tie space [6].

This work diverges from the preexisting material in its study of the spectral properties of the N.P. adjoint, with our operator acting on a broader range of functions. Specifically, we will be considering those results produced when the existing techniques are applied over the L^2 space, as well as the space of continuous functions.

A number of the techniques used in the process of this work build upon and make use of those discussed by [33] in particular with regard to the spectral properties of the operator K^* over L^p spaces.

More explicitly, our work will begin by focussing on the essential spectral properties of the N.P. adjoint operator as it acts on those L^2 functions with domain given by the boundary of the space formed by the union of a pair of infinite wedges in $\mathbb{R}^2$ which coincide only at their vertices.

Given our operators output is unaffected by translation, such a space will only be defined by the angles α, β of our wedge interiors, and the angle θ that separates them. We will denote such a boundary $\Gamma_{\alpha,\beta,\theta}$. As with preceding work in this area, the parameterisation of $\Gamma_{\alpha,\beta,\theta}$ neccesitates us extend our original integral operator into a square operator matrix with entries given in terms of our integral operator acting on each permutation of pairs of the curves that will form our parameterisation. We observe here that this scenario leads to us dealing with a previously little considered non-Lipschitz singularity. Indeed, such singularities have only previously been considered in [6], and that specifically in the setting of energy spaces. Our research shall be considering this case over $L^2(\Gamma_{\alpha,\beta,\theta})$ and, eventually, $C(\Gamma_{\alpha,\beta,\theta})$.

With this formulation in mind, we will then examine how we may compare the essential spectral properties of our operator when over localizations of $\Gamma_{\alpha,\beta,\theta}$ (which we will denote $\widetilde{\Gamma}_{\alpha,\beta,\theta}$) to some neighbourhood of the shared vertex, with their compact perturbations, thereby giving us an inroad into the spectral properties on bow-tie curves.

Having done this we then move on to determining an explicit form for the spectrum over infinte wedges, before going on to relate this result back to our localizations and get an overall result from which we can express said localizations essential spectral properties.

Examples

Case 1

We will begin by considering the case where $\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$, forming a pair of infinite wedges, bounded by the parametric curves, for $t \in \mathbb{R}_+$

$$
\begin{aligned}
l_1(t) &= (t, 0),\\
l_2(t) &= (t\cos(\pi/2), t\sin(\pi/2)) = (0, t),\\
l_3(t) &= (t\cos(\pi), t\sin(\pi)) = (-t, 0),\\
l_4(t) &= (t\cos(3\pi/2), t\sin(3\pi/2)) = (0, -t),
\end{aligned}
$$

and evaluating each element of our multiplication operator we get the following image of its spectrum

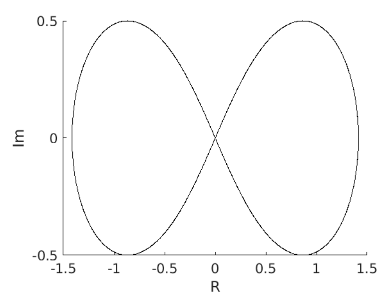


Figure 2: The spectrum of the N.P. operator on $L^2(\Gamma_{\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}})$

Case 2

We will now consider the case where $\alpha = \frac{\pi}{6}$, $\beta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{2}$, an asymmetric structure.

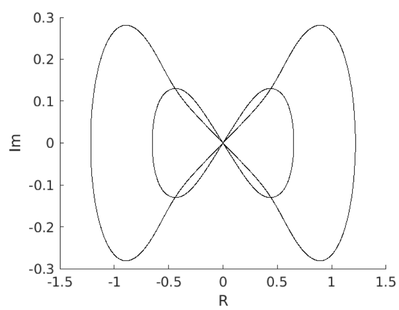


Figure 3: The spectrum of the N.P. operator on $L^2(\Gamma_{\frac{\pi}{6},\frac{\pi}{2},\frac{\pi}{4}})$

Upon examining the resultant curves in $\mathbb{C}$ that form this spectrum, we go on to observe a distinction between the previously established results on Sobolev spaces and Lipschitz domains. In particular the determined spectral radius is clearly distinct from the *Spectral Radius Conjecture* [26] given below:

Conjecture 1.10. *For D an arbitrary bounded Lipschitz domain in $\mathbb{R}^n$, consider the function space*

$$L_0^2(\partial D) = \left\{ f \in L^2(\partial D) : \int_{\partial D} f = 0 \right\}.$$

Then on $L_0^2(\partial D)$ we have

$$\|\sigma(K^*)\| < 1/2.$$

This may not seem immediately remarkable but the ongoing study of the subcases of the conjecture have consistently born out results that support it, see, for example,[16].

One such study [10] has been able to show that on the energy space $H^{-1/2}(\partial D)$, the above conjecture holds true and in particular, in said scenario the spectrum may be determined as being composed of real-line elements of the interval $[\frac{1}{2}, \frac{1}{2})$. As mentioned previously, this result can also be extended to K, as well as giving further conditions on the possible solutions of the Dirichlet problem for functions in $H^{-1/2}(\partial D)$.

We note that, by our definition of $\widetilde{\Gamma}_{\alpha,\beta,\theta}$, the singular point at the joint vertex of our bow-tie curve prohibits our domain from being Lipschitz in neighbourhoods of this point. Thus the conjecture does not apply to $\widetilde{\Gamma}_{\alpha,\beta,\theta}$.

What is significant is the degree of distinction demonstrated, particularly in comparison with the energy space $H^{-1/2}(\partial D)$.

Indeed we may observe two clear distinctions between the spectrum measured over $L^2(\partial D)$ and that on $H^{-1/2}(\partial D)$. The first distinction being that the spectrum takes the form of a closed curve in $\mathbb{C}$, as opposed to a line segment, or elements of said segment, and the second being that the spectral radius in the L^2 case will typically exceed $\frac{1}{2}$, the radius seen when studying the energy space $H^{-1/2}(\partial D)$ and proposed in the above conjecture.

Upon determining a formula for the spectral radius on $\widetilde{\Gamma}_{\alpha,\beta,\theta}$, we analyse the subcase where the angles that form $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ are completely symmetric, from which we go on to observe how, for this subcase we may explicitly show that the spectral radius must exceed those typical of other results. With this in mind, we then go on to consider the boundedness properties over a smaller space of functions by shifting to the space of continuous functions on $\widetilde{\Gamma}_{\alpha,\beta,\theta}$.

Determining a formula for the spectral radius in this scenario in terms of α, β and θ. We then go on to consider the magnitude of the angles defining $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ on a case-by-case basis, and from there we are able to prove that not only is complete symmetry neccesary to ensure our spectral radius on $C(\widetilde{\Gamma}_{\alpha,\beta,\theta})$ remains within the more typical parameters, but that our operator on such a space is only well defined when we have this complete symmetry holds.

1.3 Results

To give a general concept of our technique we lay out our method as the following. More specifics of the results and operators used can be found below. Taking $\Gamma_{\alpha,\beta,\theta}$ to be the union of two the wedges W_α and W_β, defined by angles α and β respectively, where the vertices of both wedges coincide at the origin and there exists an angle θ separating them. We can parameterize each edge forming the boundary of $\Gamma_{\alpha,\beta,\theta} = W_\alpha \cup W_\beta$ on $\mathbb{R}_+$ as,

$$\begin{aligned}
l_1(t) &= (t, 0),\\
l_2(t) &= (t\cos(\alpha), t\sin(\alpha)),\\
l_3(t) &= (t\cos(\alpha+\theta), t\sin(\alpha+\theta)),\\
l_4(t) &= (t\cos(\alpha+\theta+\beta), t\sin(\alpha+\theta+\beta))
\end{aligned}$$

Using this we formulate a 4×4 operator matrix to represent the N.P. adjoint operator as it acts on points of different curves in the manner described in [36]. On the weighted space $L^2(\Gamma_{\alpha,\beta,\theta}, t^a dt)$ we derive our matrix operator

$$K^a_{\Gamma_{\alpha,\beta,\theta}} = \begin{bmatrix} 0 & K^a_\alpha & K^a_{\alpha+\theta} & K^a_{\alpha+\theta+\beta} \\ K^a_\alpha & 0 & -K^a_\theta & -K^a_{\theta+\beta} \\ -K^a_{\alpha+\theta} & -K^a_\theta & 0 & K^a_\beta \\ K^a_{\alpha+\theta+\beta} & K^a_{\theta+\beta} & K^a_\beta & 0 \end{bmatrix}$$

Where each of the above non-zero entries represents the N.P. adjoint operator (1.3) (or its negative) acting on the wedge defined by the angle given in the entry. Establishing a general form for each operator entry, we then proceed to apply a suitable unitary operator to establish equivalence between our operator and a Mellin convolution.

Lemma 1.11. *The operator*

$$V_{\frac{a+1}{2}} : L^2(\mathbb{R}_+, t^a f(t)) \to L^2(\mathbb{R}_+, dt/t)$$
$$f(t) \to t^{\frac{a+1}{2}} dt$$

is unitary.

This results in the Mellin convolution (see Section 4) of f with the function

$$j_{\omega,a}(t) = t^{\frac{a+1}{2}} \frac{\sin(\omega)}{\pi(t^2 + 1 - 2t\cos(\omega))}$$

Applying the Mellin transform, we then use the convolution property of this transform to achieve similarity between our N.P. adjoint operator and a multiplication operator on $L^2(\mathbb{R})$, the explicit form of which, as a complex integral, we calculate using basic branch-cutting techniques.

Lemma 1.12. *Calculating the explicit form of the multiplication operator, we have that*

$$\mathcal{M}(j_{\omega,a})(\xi) = \frac{\sin(\left(\frac{a-1}{2} + i\xi\right)(\pi - \alpha))}{\sin(\pi\left(\frac{a-1}{2} + i\xi\right))}.$$

We can then extend this result to the entire operator matrix

$$V_{\frac{a+1}{2}} \otimes Id \cdot K^a_{\Gamma_{\alpha,\beta,\theta}} \cdot V_{-\frac{a+1}{2}} \otimes Id = \begin{bmatrix} 0 & \mathcal{M}(j_{\alpha,a}) & \mathcal{M}(j_{\alpha+\theta,a}) & \mathcal{M}(j_{\alpha+\theta+\beta,a}) \\ \mathcal{M}(j_{\alpha,a}) & 0 & -\mathcal{M}(j_{\theta,a}) & -\mathcal{M}(j_{\theta+\beta,a}) \\ -\mathcal{M}(j_{\alpha+\theta,a}) & -\mathcal{M}(j_{\theta,a}) & 0 & \mathcal{M}(j_{\beta,a}) \\ \mathcal{M}(j_{\alpha+\theta+\beta,a}) & \mathcal{M}(j_{\theta+\beta,a}) & \mathcal{M}(j_{\theta+\beta,a}) & 0 \end{bmatrix}$$

and using the commutativity of our multiplication operator, we may then determine the spectrum of our operator matrix using its determinant, similarly defined as a sum of compositions of multiplication operators.(See below)

$$\begin{aligned}\sigma(K^a_{\Gamma_{\alpha,\beta,\theta}}) &= \sigma(V_{\frac{a+1}{2}} \otimes Id \cdot K^a_{\Gamma_{\alpha,\beta,\theta}} \cdot V_{-\frac{a+1}{2}} \otimes Id)\\ &= \Big\{\lambda \in \mathbb{C} : \det(V_{\frac{a+1}{2}} Id \cdot K^a_{\Gamma_{\alpha,\beta,\theta}} \cdot V_{-\frac{a+1}{2}} Id - \lambda Id) \text{ is invertible } \Big\}\end{aligned}$$

From there we go on to consider localizations of our result, and in particular compare and determine compactness properties of the difference between said localizations and comparable bow-tie curves.

By applying cut-off functions of compact support on $[0, 1)$ to each operator entry of $K^a_{\Gamma_{\alpha,\beta,\theta}}$ we restrict our matrix operator, to act only on those segments of the boundary of $\Gamma_{\alpha,\beta,\theta}$ within a small neighbourbood of the shared vertex. We denote this cut-off operator $K^{loc}_{\Gamma_{\alpha,\beta,\theta}}$, acting on $\tilde{\Gamma}_{\alpha,\beta,\theta}$. (For simplicities sake, at this point we ignore the notion of weightedness.)

We then compare said cut-off operator $K^{loc}_{\Gamma_{\alpha,\beta,\theta}}$ with $K_{\tilde{\Gamma}_{\alpha,\beta,\theta}}$, that is, our original operator acting on the localization of $\Gamma_{\alpha,\beta,\theta}$ to the same neighbourhood $\tilde{\Gamma}_{\alpha,\beta,\theta}$ as above. This was done by taking the difference of their respective kernels, and attempting to evaluate the square integral of the difference.

Separating out the square integral of the new kernel and making use of a preexisting Lemma discussed in the following chapter to account for an arising weak singularity, we can determine piecewise that said square integral must be finitely bounded, from which we gain that the difference of our two operators is Hilbert Schmidt, and therefore must be compact, from which it follows

Theorem 1.13. *For $K^{loc}_{\Gamma_{\alpha,\beta,\theta}}$ the cut-off of the N.P. adjoint operator and $K_{\tilde{\Gamma}_{\alpha,\beta,\theta}}$ the unaltered operator, acting on $\widetilde{\Gamma}_{\alpha,\beta,\theta}$*

$$\sigma_{ess}(K^{loc}_{\Gamma_{\alpha,\beta,\theta}}) = \sigma_{ess}(K_{\tilde{\Gamma}_{\alpha,\beta,\theta}}).$$

This result gives us a direct link between the essential spectrum of our localized operator matrix $K^{loc}_{\tilde{\Gamma}_{\alpha,\beta,\theta}}$ and any of its compact perturbations. Continuing in our study of the localized N.P. adoint matrix operator on $\tilde{\Gamma}_{\alpha,\beta,\theta}$, from there we directly calculate an upper bound for the essential norm by means of the ∞-norm, using the unitarily equivalent mutiplication operator form of $K_{\Gamma_{\alpha,\beta,\theta}}$,

$$\|K_{\tilde{\Gamma}_{\alpha,\beta,\theta}}\|_{ess} = \left\| \begin{bmatrix} 0 & \mathcal{M}(j_\alpha)(\xi) & \mathcal{M}(j_{\alpha+\theta})(\xi) & \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) \\ \mathcal{M}(j_\alpha)(\xi) & 0 & -\mathcal{M}(j_\theta)(\xi) & -\mathcal{M}(j_{\theta+\beta})(\xi) \\ -\mathcal{M}(j_{\alpha+\theta})(\xi) & -\mathcal{M}(j_\theta)(\xi) & 0 & \mathcal{M}(j_\beta)(\xi) \\ \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) & \mathcal{M}(j_{\theta+\beta})(\xi) & \mathcal{M}(j_\beta)(\xi) & 0 \end{bmatrix} \right\|_{ess}$$

and in particular, by substituting in a suitable algebra for the determining angles, explicitly prove that in the case where the angles α, β and γ are completely symmetric, the essential norm must be strictly greater than one.

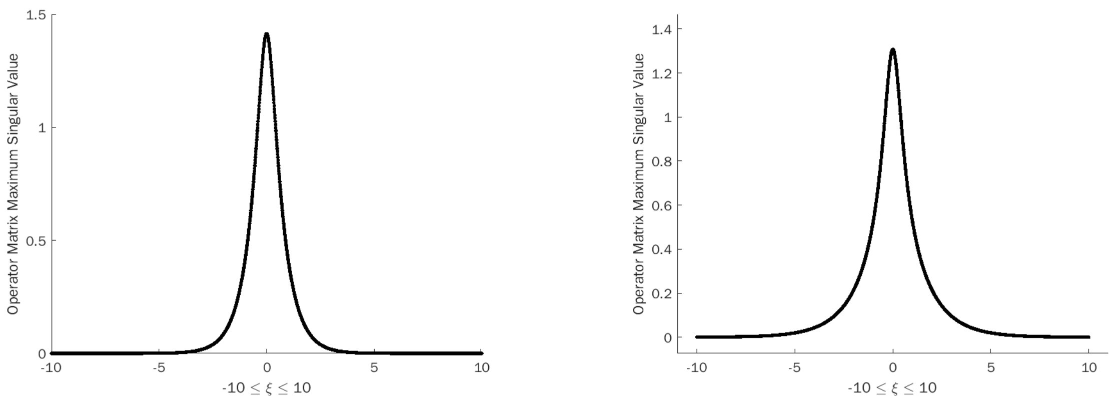


Figure 4: Max Singular values of matrix operator on $\widetilde{\Gamma}_{\frac{\pi}{2}\frac{\pi}{2}\frac{\pi}{2}}$ and $\widetilde{\Gamma}_{\frac{\pi}{4}\frac{3\pi}{4}\frac{\pi}{4}}$

From there we are further able to surmise by analysis the following result:

Theorem 1.14. *The essential norm of $K_{\tilde{\Gamma}_{\alpha,\beta,\theta}}$ on $L^2(\widetilde{\Gamma}_{\alpha,\beta,\theta})$ is given by*

$$\sup_{\xi\in\mathbb{R}} \left\| \begin{bmatrix} 0 & \mathcal{M}(j_\alpha)(\xi) & \mathcal{M}(j_{\alpha+\theta})(\xi) & \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) \\ \mathcal{M}(j_\alpha)(\xi) & 0 & -\mathcal{M}(j_\theta)(\xi) & -\mathcal{M}(j_{\theta+\beta})(\xi) \\ -\mathcal{M}(j_{\alpha+\theta})(\xi) & -\mathcal{M}(j_\theta)(\xi) & 0 & \mathcal{M}(j_\beta)(\xi) \\ \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) & \mathcal{M}(j_{\theta+\beta})(\xi) & \mathcal{M}(j_\beta)(\xi) & 0 \end{bmatrix} \right\|_{B(\ell^2(\mathbb{C}^4))}.$$

Using the result developed by [33] as discussed in Chapter 4 and applied in a similar manner to that discussed in [38], we evaluate our multiplication operator entrywise to determine that it has the properties required (Definition.4.5) for said result and in doing so, we are able directly determine the equality of the spectrum on $\Gamma_{\alpha,\beta,\theta}$ and the essential spectrum on $\widetilde{\Gamma}_{\alpha,\beta,\theta}$, as well as its compact perturbations. This, in combination with our previous explicit derivation of the spectrum of $K^a_{\Gamma_{\alpha,\beta,\theta}}$ gives us that

Theorem 1.15. *For $K_{\tilde{\Gamma}_{\alpha,\beta,\theta}}$ the localization of the N.P. adjoint operator matrix on $\widetilde{\Gamma}_{\alpha,\beta,\theta}$. Then we have that the operator and its compact perturbations have essential spectrum given by*

$$\sigma_{ess}(K_{\tilde{\Gamma}_{\alpha,\beta,\theta}}) = \sigma_{ess}(K_{\tilde{\Gamma}_{\alpha,\beta,\theta}} + Comp) = \sigma(K_{\Gamma_{\alpha,\beta,\theta}}),$$

the formula for which is given by,

$$\sigma(K_{\Gamma_{\alpha,\beta,\theta}}) = \left\{ m(\xi) = \pm\sqrt{\frac{-S(\xi) \pm \sqrt{S(\xi)^2 - 4T(\xi)}}{2}} : \xi \in \mathbb{R} \right\} \cup \{0\},$$

Where, S and T are given by

$$\begin{aligned} S(\xi) = -\mathcal{M}(j_\alpha)(\xi)^2 &+ \mathcal{M}(j_{\alpha+\theta})(\xi)^2 \\ -\mathcal{M}(j_{\alpha+\theta+\beta})(\xi)^2 &- \mathcal{M}(j_\theta)(\xi)^2 \\ +\mathcal{M}(j_{\theta+\beta})(\xi)^2 &- \mathcal{M}(j_\beta)(\xi)^2 \end{aligned}$$

$$\begin{aligned} T(\xi) = {} & \mathcal{M}(j_{\theta+\beta})(\xi)^2\mathcal{M}(j_{\alpha+\theta})(\xi)^2 \\ & +\mathcal{M}(j_{\alpha+\theta+\beta})(\xi)^2\mathcal{M}(j_\theta)(\xi)^2 \\ & +\mathcal{M}(j_\alpha)(\xi)^2\mathcal{M}(j_\beta)(\xi)^2 \\ & -2\mathcal{M}(j_{\theta+\beta})(\xi)\mathcal{M}(j_{\alpha+\theta})(\xi) \\ & \times\mathcal{M}(j_{\alpha+\theta+\beta})(\xi) \\ & -2\mathcal{M}(j_\alpha)(\xi)\mathcal{M}(j_{\theta+\beta})(\xi) \\ & \times\mathcal{M}(j_{\alpha+\theta})(\xi)\mathcal{M}(j_\beta)(\xi) \\ & +2\mathcal{M}(j_\alpha)(\xi)\mathcal{M}(j_{\alpha+\theta+\beta})(\xi) \\ & \times\mathcal{M}(j_\theta)(\xi)\mathcal{M}(j_\beta)(\xi). \end{aligned}$$

and for a given angle γ we have,

$$\mathcal{M}(j_\gamma)(\xi) = \frac{\sin((\pi-\gamma)(i\xi - 1/2))}{\sin(\pi(i\xi - 1/2))}$$

.

Our results concerning the spectral radius on $L^2(\tilde{\Gamma}_{\alpha,\beta,\theta})$ motivate us to consider the possibilities for said radius when studied over the smaller field of $C(\tilde{\Gamma}_{\alpha,\beta,\theta})$ functions.

We then go on to extend Kress's technique [28] (Section 2.4) on the corners of curvilinear polygons, to construct an equivalent operator matrix to the compact perturbation of $K_{\tilde{\Gamma}_{\alpha,\beta,\theta}}$ that is defined over a closed bow-tie curve, where each entry represents such a curvilinear corner, with defining angle and sign determined for each entry by our original matrix. We will denote this operator $\widetilde{K}$, and parameterize said bow-tie curve with four distinct continuous curves on $[0, 1]$, determined so as to ensure our operator remains continuos when acting on them. As a result we have that the target space of $\widetilde{K}$ is $C([0,1]^4)$

By applying the ∞-norm, we are once again able to determine an upper bound on the essential norm of $\widetilde{K}$ and, by considering the possible 'legal' combinations of angles α, β and θ we eliminate those combinations that lead in some way to a contradiction and from their we obtain

Theorem 1.16. *Let $\widetilde{K} : C(\widetilde{\Gamma}_{\alpha,\beta,\theta}) \to C[0,1]^4$ be the matrix operator as mentioned above. Then we have that there exists an upper bound $\|\widetilde{K}\|_{ess} \leq M$, where*

$$M = \max \left\{ \begin{array}{l} \left|1 - \frac{\alpha}{\pi}\right| + \left|1 - \frac{\alpha+\theta}{\pi}\right| + \left|1 - \frac{\alpha+\theta+\beta}{\pi}\right|, \\ \left|1 - \frac{\alpha}{\pi}\right| + \left|1 - \frac{\theta}{\pi}\right| + \left|1 - \frac{\theta+\beta}{\pi}\right|, \\ \left|1 - \frac{\alpha+\theta}{\pi}\right| + \left|1 - \frac{\theta}{\pi}\right| + \left|1 - \frac{\beta}{\pi}\right|, \\ \left|1 - \frac{\alpha+\theta+\beta}{\pi}\right| + \left|1 - \frac{\theta+\beta}{\pi}\right| + \left|1 - \frac{\beta}{\pi}\right| \end{array} \right\}$$

and furthermore we have that $M \leq 1$ if and only if

$$\alpha = \beta$$
$$\alpha + \theta = \pi,$$

that is, the angles that determine $\Gamma_{\alpha,\beta,\theta}$ are completely symmetric.

As mentioned previously, in order to ensure the parameterisation of our bow-tie curve generates a continuous image under $\widetilde{K}$, we must have that the four vectors which form said parameterisation coincide at their intersection, in particular, at the shared vertex. With this condition imposed, we can demonstrate algebraically how

Theorem 1.17. *$\widetilde{K}$ is an operator on $C(\widetilde{\Gamma}_{\alpha,\beta,\theta})$ if and only if the angles that define $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ are completely symmetric, that is $\alpha = \beta$ and $\alpha + 2\theta + \beta = 2\pi$.*

2 Introduction to the double layer potential

This chapter is intended to provide the reader with a more in depth look at the theory behind the double layer potential, begining with its more general properties, before going on to discuss its use with regard to the Dirichlet problem. From there we will discuss some of the practical applications of the problem in physics, as well as some applications of the operator in general, before proceeding to key results regarding the spectrum of K^*, both in general as well as specific recent work on non-smooth surfaces.

To clarify, the results and proofs in this section are not my own and an extensive effort has been made to reference each while providing a sufficient coverage of the relevant techniques. In particular, the initial sections make heavy reference to results from [28] and [17].

2.1 The operator and its basic properties

Given our description of the double layer potential (1.1), and specifically considering the expansion of its kernel into an inner product,

$$u(x) := \frac{1}{\omega_n} \int_{\partial D} \phi(y) \frac{\langle \nu(y), (x-y) \rangle}{|x-y|^n} d\sigma(y), \quad x \in \partial D \tag{2.1}$$

we recall our prior concerns about the double layer potentials well definedness, as well as its continuity over ∂D. We thus take this opportunity to introduce the aforementioned notion of weak singularity, as well as the results that show how, as a result, our potential is well-behaved on ∂D.

Definition 2.1. For any integral operator defined by a kernel $k(x, y)$, continuous for all $x, y \in D \subset \mathbb{R}^n$ save when $x = y$ and given the existence of positive constants C and $\alpha \in (0, n]$ such that,

$$|k(x, y)| \leq C|x-y|^{\alpha-n} \quad x, y \in D, x \neq y,$$

we then say that such an integral operator is *weakly singular*.

We observe here that for our purposes, we are considering the activity of the N.P. operator on the $n-1$ dimensional boundary ∂D, and so we transpose the above definition to the $n-1$ dimensional equivalent. With this is in mind, we have the following result.

Lemma 2.2. *Let ∂D be of class C^2. Then there exists $L > 0$ such that*

$$|\langle x - y, \nu(x)\rangle| \leq L|x - y|^2.$$

Proof. By applying the Cauchy Schwarz inequality, in tandem with the unit-norm property of $\nu(x)$ we get for all $x, y \in D$,

$$\begin{aligned}|\langle x - y, \nu(x)\rangle| \leq &|x - y||\nu(x)| \\ = &|x - y| \cdot 1 \\ = &|x - y|\end{aligned}$$

so we can assume that

$$|x - y| \leq 1.$$

Given the invariance of our results under translation and rotation, we take $x = 0$, giving us $\nu(x) = (0, ..., 0, 1)$ and, as a result, $|\langle x - y, \nu(x)\rangle| = |y_n|$. Given ∂D is class C^2 we have that in a neighbourhood of x, ∂D can be represented by the C^2 function defined by $f(y_1, ..., y_{n-1}) = y_n$ where $f(0) = 0$ and $\nabla f(0) = 0$. Then, by Taylor's theorem, in combination with the definition a C^2 space there exists a bound L for the second partial derivative of f such that, by applying Taylor's theorem,

$$\begin{aligned}|\langle x - y, \nu(x)\rangle| = &|f(y_1, ..., y_{n-1})| \\ \leq &L|(y_1, ..., y_{n-1})|^2 \\ \leq &L|y|^2 = L|x - y|^2,\end{aligned}$$

giving us the desired inequality. $\square$

By examination, we can see that, as a composite of functions that are continuous for $x \neq y$ and combined with the above lemma, we have,

$$\begin{aligned}\left|\frac{\partial \Phi(x, y)}{\partial \nu(y)}\right| &= \frac{|\langle x - y, \nu(x)\rangle|}{\omega_n |x - y|^n} \\ &\leq L \frac{|x - y|^2}{\omega_n |x - y|^n} \\ &\leq \frac{L}{\omega_n} |x - y|^{2-n},\end{aligned}$$

and thus our N.P. adjoint operator is weakly singular. In fact, this result is instrumental in determining the conditions for which u is well defined. The following theorem, the proof of which is given in ([28], Chp.2, Thm.2.30, pg.31-32), makes this more explicit.

Theorem 2.3. *Setting D to be a class C^1 domain, for integral operator K, we assume that it's kernel $f(x,y)$ is weakly singular. Then we have that K is well defined. That is f(x,y)u(y) is integrable, or*

$$\int_{\partial D} |f(x,y)||u(y)|d\sigma(y) < \infty$$

for all $x \in \partial D$ and $u \in C(\partial D)$.

Proof. Given ∂D is of class C^1, we have that the mapping from each element of ∂D to the normal, exterior oriented vector v is continuous. As such, we can take $R \in (0,1]$, sufficiently small that for all $x \in \partial D$ the set $S[x;R] = \{y \in \partial D : |x-y| < R\} \subset \partial D$ is connected, and furthermore that

$$\nu(x) \cdot \nu(y) \geq \frac{1}{2}.$$

This inequality then implies that we can project $S[x;R] \subset \partial D$ bijectively onto the tangent plane to ∂D at x. We can now use the fact that $f(x,y)$ is assumed to be weakly continuous and estimate our integral on $S[x;R] \subset \partial D$, in terms of polar coordinates, in the tangent plane, centred at x.

$$\begin{aligned}\int_{S[x;R]} |f(x,y)|\,|u(y)|\,d\sigma(y) &\leq C||u||_\infty \int_{S[x;R]} |x-y|^{\alpha-n} d\sigma(y) \\ &\leq C||u||_\infty \omega_n \int_0^R \rho^{\alpha-n}\rho^{n-1} d\rho.\end{aligned}$$

Then using the fact that $|x-y| \geq \rho$ and the estimate

$$d\sigma(y) = \frac{\rho^{n-1} d\rho d\omega}{\nu(x)\cdot\nu(y)} \leq 2\rho^{n-1} d\rho d\omega,$$

we determine

$$C||u||_\infty \omega_n \int_0^R \rho^{\alpha-n}\rho^{n-1} d\rho = C||u||_\infty \omega_n \frac{R^\alpha}{\alpha}.$$

So K is integrable on $S[x;R]$.
Finally, given $f(x,y)$ is continuous for $x \neq y$, it must be integrable on $\partial D \setminus S[x;R]$. □

Naturally, the weakly singular property derived previously gives us that this boundedness property holds equally for the integrals that define the N.P. operator and its adjoint, thereby giving us that both operators are equally well defined on spaces of continuous functions acting on domains of boundary class C^2.

The proof used in the above result may also be used to determine the following.

Theorem 2.4. *Integral operators with continuous or weakly singular kernel are compact linear operators on $C(\partial D)$ for ∂D of class C^1.*

Proof. Given ∂D is bounded we then have

$$|K\phi(x)| \leq |\partial D| \|\phi\|_\infty \max_{x,y\in D} |f(x,y)|,$$

from which we can use the uniform continuity of $f(x,y)$ to show the equicontinuity of $K\phi$. Thus the compactness in the continuous case follows from the Arzella-Ascolli theorem ([28], pg.10, Theorem 1.18) which gives us that the image of the unit ball under K is weakly compact.

The weakly singular case follows by using a sequence of cut-off functions $h(n|x-y|)$, for $n \in \mathbb{N}$ and $h \in C([0,\infty))$, where, for $t \in [0,1/2]$, $h(t) = 0$ and, for $t \in [1,\infty)$, $h(t) = 1$. With this is constructed a sequence of integral operators K_n, with kernels respectively given by

$$f_n(x,y) := \begin{cases} h(n|x-y|)f(x,y) & x \neq y, \\ 0, \quad x = y. \end{cases}$$

each compact as a result of their continuous kernel, and thus being Hilbert Schmidt operators. Using the bound derived in Theorem 2.3, setting $R = \frac{1}{n}$ we may determine that

$$|K\phi(x) - K_n\phi(x)| \leq M\|\phi\|_\infty \frac{\omega_n}{\alpha n^\alpha}$$

From which it follows that $K_n\phi \to K\phi$ uniformly, $K\phi \in C(\partial D)$ and

$$\|K - K_n\|_\infty \leq M\frac{\omega_n}{\alpha n^\alpha} \to 0, \quad n \to \infty.$$

We then have that the limit of a sequence of compact operators is compact ([28], Theorem 2.22, pg.26.) which gives us the desired end result. □

We once again can apply this result to both the N.P. operator and its adjoint. Furthermore, we have the following result, the proof of which comes from ([17], pg. 122, Proposition 3.12),

Theorem 2.5. *For f(x,y) a continuous kernel of order $\alpha \in [0, n-1)$, i.e.*

$$f(x,y) = \frac{A(x,y)}{|x-y|^{-\alpha}},$$

for $A(x,y)$ bounded and $f(x,y)$ continuous for $x \neq y$. Then the integral operator determined by this kernel transforms bounded functions into continuous functions.

Proof. Assuming $\alpha > 0$, then for $x \in \partial D$ and $\delta > 0$ we will set $S[x;\delta] = \{y \in \partial D : |x-y| < \delta\} \subset \partial D$, and so for $y \in S[x;R]$

$$\begin{aligned}
|K\phi(x) - K\phi(y)| &= \left| \int_{\partial D} (f(x,z) - f(y,z))\phi(z) d\sigma(z) \right| \\
&\leq \int_{S[x;2\delta]} (|f(x,z)| - |f(y,z)|)|\phi(z)| d\sigma(z) \\
&+ \int_{\partial D \setminus S[x;2\delta]} |f(x,z) - f(y,z)||\phi(z)| d\sigma(z)
\end{aligned}$$

where we have

$$\begin{aligned}
\int_{S[x;2\delta]} (|f(x,z)| - |f(y,z)|)|\phi(z)| d\sigma(z) &\leq \|A\|_\infty \|\phi\|_\infty \int_{S[x;2\delta]} (|x-z|^{-\alpha} + |y-z|^{-\alpha}) d\sigma(z) \\
&\leq 2\|A\|_\infty \|\phi\|_\infty (2\delta)^{-\alpha} \int_{S[x;2\delta]} d\sigma(z) \\
&\leq 2^{n+1-\alpha} \|A\|_\infty \|\phi\|_\infty \omega_n (\delta)^{n-\alpha}
\end{aligned}$$

so for an appropriate choice of δ the integral over $S[x;2\delta]$ is less than $\epsilon/2$.

For $y \in S[x;\delta]$ and $z \in \partial D \setminus S[x;2\delta]$ we have $|x-z| \geq 2\delta$ and $|y-z| \geq \delta$. Thus the continuity of $K(x,y)$ for $x \neq y$ gives us, for $z \in \partial D \setminus S[x;2\delta]$ that

$$\begin{aligned}
f(x,z) - f(y,z) &\to 0 \\
\text{as } y &\to x
\end{aligned}$$

and so the integral over $\partial D \setminus S[x;2\delta]$ will be less than $\epsilon/2$ for x sufficiently close to y. Thus we have $K\phi$ is continuous. $\square$

Considering our kernel, we can rewrite $\frac{\partial\Phi(x,y)}{\partial\nu(y)}$ as below,

$$\frac{\partial\Phi(x,y)}{\partial\nu(y)} = \frac{\langle x-y, \nu(x)\rangle}{\omega_n |x-y|^n}$$

$$= \frac{\langle x-y, \nu(x)\rangle}{\omega_n |x-y|^2} \cdot \frac{1}{\omega_n |x-y|^{n-2}}$$

and, applying Lemma 2.2 we have for x and y in ∂D, a C^2 boundary, that

$$\frac{\langle x-y, \nu(x)\rangle}{\omega_n |x-y|^2} \leq \frac{|x-y|^2}{\omega_n |x-y|^2} = \frac{L}{\omega_n}.$$

In addition to the above demonstrated boundedness, by inspection we have that the necessary weak singularity conditions hold for our kernel.

Thence we have that the double layer potential or, equivalently, the outputs of the N.P. operator and its adjoint are continuous functions on both $D \setminus \partial D$ and D.

We can also demonstrate that the double layer potential is harmonic, one of the key properties that led to it's study in relation to the Dirichlet problem.

Theorem 2.6. *Let D be a C^2 domain in $\mathbb{R}^n$, as in the previous theorem. For $\phi \in C(\partial D)$, the double layer potential with density ϕ is harmonic for $x \notin \partial D$.*

Proof. Firstly, by evaluating the x component of the gradient of $\frac{\partial \Phi(x,y)}{\partial \nu(y)}$,

$$\nabla_x \frac{\partial \Phi(x,y)}{\partial \nu(y)} = \frac{1}{\omega_n} \frac{\nu(y)}{|x-y|^n} - n \frac{x-y}{|x-y|^2} \frac{\partial \Phi(x,y)}{\partial \nu(y)}$$

The divergence of which will equal $\Delta_x \frac{\partial \Phi(x,y)}{\partial \nu(y)}$, the x component of the Laplacian. Explicitly,

$$\begin{aligned}
\Delta_x \frac{\partial \Phi(x,y)}{\partial \nu(y)} =& div \left(\nabla_x \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right) \\
=& div \left(\frac{1}{\omega_n} \frac{\nu(y)}{|x-y|^n} - n \frac{x-y}{|x-y|^2} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right) \\
=& \frac{-n}{|x-y|^2} \frac{\partial \Phi(x,y)}{\partial \nu(y)} - n \left(\frac{n-2}{|x-y|^2} + \frac{1-n}{|x-y|^2} \right) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \\
=& 0.
\end{aligned}$$

We then have that using our previous result for the existence of the integral, there exist the conditions for Lebesgue's Dominated Convergence Theorem to hold, allowing us to pass partial derivatives inside the integral. Combining this with our first result we get

$$\begin{aligned}
\Delta u(x) =& \Delta \left(\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \phi(y) d\sigma(y) \right) \\
=& \int_{\partial D} \Delta_x \frac{\partial \Phi(x,y)}{\partial \nu(y)} \phi(y) d\sigma(y)
\end{aligned}$$

$$=0.$$

$\square$

However, as was discussed in the previous chapter, continuity does not hold as we pass from the domain, or its exterior to the boundary. What follows is a proof of the result by Gauss, in which he demonstrated that for the double layer potential there is a definitive 'jump' between interior, boundary and exterior.

Lemma 2.7 (Gauss' Lemma)**.** *Let D be a class C^2 domain in $\mathbb{R}^n$, then considering the double layer potential $u(x)$ defined on ∂D with constant, unitary density, we have that,*

$$u(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} d\sigma(y) = \begin{cases} 0, & x \in \overline{D}^c, \\ -1, & x \in D, \\ -\frac{1}{2}, & x \in \partial D. \end{cases} .$$

Key to this result is the application of the following corollary to Green's identities ([28], Chp.6, Cor.6.4, pg.77),

Corollary 2.8. *For $u \in C^2(\overline{D})$ and harmonic on D, we have*

$$\int_{\partial D} \frac{\partial u}{\partial \nu} d\sigma = 0.$$

Proof of Lemma 2.7. We can prove by induction that $\Phi(x,y) \in C^\infty(\overline{D})$ in y, and furthermore by partial derivatives demonstrate that $\Phi(x,y)$ is harmonic w.r.t. $y \in D$ for$x \in D^c$. Our first equality $u(x) = 0$ for $x \in D^c$ is then immediate from Corollary 2.8. For $x \in D$, take $\epsilon > 0$ sufficiently small that $\overline{B_\epsilon(x)} \subset D$. We then apply the above corollary to $\Phi(x,y)$ on $D \setminus \overline{B_\epsilon(x)}$, and evaluate (keeping in mind the orientation of v) to get,

$$\begin{aligned} 0 &= \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} d\sigma(y) + \frac{\epsilon^{1-n}}{\omega_n} \int_{B_\epsilon(x)} d\sigma \\ &= \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} d\sigma(y) + 1. \end{aligned}$$

Finally, for $x \in \partial D$ we similarly take a ball $B_\epsilon(x)$ centred at x, of radius ϵ, our unit normal oriented to the interior on $B_\epsilon(x)$, and apply our corollary, to get

$$0 = \int_{\{y \in \partial D : |y-x| \geq r\}} \frac{\partial \Phi(x,y)}{\partial \nu(y)} d\sigma(y) + \int_{B_\epsilon(x) \cap D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} d\sigma(y).$$

Combined with the following,

$$\lim_{r\to 0} 2\int_{B_\epsilon(x)\cap D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} d\sigma(y) = \lim_{r\to 0} \frac{2}{\omega_n r^{n-1}} \int_{B_\epsilon(x)\cap D} d\sigma(y)$$
$$=1,$$

we get that

$$\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} d\sigma(y) = -\frac{1}{2}.$$

□

Gauss' Lemma explores the jump relation specifically in the scenario where the density $u(x) = 1$, though it is not difficult to see how one could expand the result for discontinuity at the boundary for a wider variety of functions u.

As discussed previously, this discrepancy in the continuity prohibits us from using u directly to resolve problems where such boundary conditions hold. However, as has been mentioned in the previous chapter, it is possible to prove that the Jump formulas (See Theorem 1.2) hold, providing way of defining a suitable substitute fun for which we have the normal properties of the double layer potential hold on the curve ∂D and we also have continuity as we pass into the interior or exterior of D.

We omit the formal proof of Theorem 1.2 due to its length, however further reference to the jump formulae and their proof can be found in ([28], Chp.6, Thm.6.18 ,pg.89) in addition to many introductory texts on the subject of layer potentials.

From here we will discuss further properties of the double layer potential, in particular those relevant to its use inresolving the interior and exterior Dirichlet boundary-value problems, before proceeding to results specific to discuss the problems themselves. We take this opportunity to introduce big-O and little-o notation.

Given a function f, we can use big-O and little-o notation to describe its behaviour in the neighbourhood of some argument a in terms of another function g. Specifically, we write,

$$f(x) = O(g(x)), x \to a \iff |f(x)| \leq C|g(x)| \text{ for some } C \in \mathbb{R}_+$$
$$f(x) = o(g(x)), x \to a \iff \frac{f(x)}{g(x)} \to 0.$$

Here, we also introduce the following notion.

Definition 2.9. We say that u is harmonic at infinity if its *Kelvin transform*

$$\tilde{u}(x) = |x|^{2-n} u(|x^{-2}x|)$$

has a removable singularity at 0, that is, there exists harmonic U such that $U = \tilde{u}$ on $\mathbb{R}^n \setminus \{0\}$.

Making use of the above notation and definition, we have the following equivalence result from [17](Chapter 2, pg. 114, Prop. 2.74).

Theorem 2.10. *For u harmonic on the complement of a bounded set in $\mathbb{R}^n$, the following are equivalent:*

- $|u(x)| = O(|x|^{2-n})$ *as* $x \to \infty$
- *As* $x \to \infty$, $u(x) \to 0$ *for* $n > 2$ *and* $|u(x)| = o(\log|x|)$ *for* $n = 2$
- u *is harmonic at* ∞

We can demonstrate directly that the first of these criteria holds for the double layer potential with density $\phi \in L(\mathbb{R}^n)$,

$$\begin{aligned}
|u(x)| &= \left| \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu_y} \phi(y) d\sigma(y) \right| \\
&\leq \int_{\partial D} \left| \frac{\partial \Phi(x,y)}{\partial \nu_y} \right| |\phi(y)| \, d\sigma(y) \\
&\leq \|\phi\|_\infty \int_{\partial D} \left| \frac{\langle \nu_y, x-y \rangle}{|x-y|^n} \right| d\sigma(y) \\
&\leq \|\phi\|_\infty \int_{\partial D} \frac{|\nu_y| \, |x-y|}{|x-y|^n} d\sigma(y),
\end{aligned}$$

by the Cauchy-Schwartz inequality. Further, given that ν_y is unitary, we can further simplify this to,

$$\|\phi\|_\infty \int_{\partial D} \frac{1}{|x-y|^{n-1}} d\sigma(y),$$

and, given we are taking $|x|$ to ∞, we can assume for our purposes that it is sufficiently large that $|y| < \frac{|x|}{2}$. Thus, by the reverse triangle inequality

$$\|\phi\|_\infty \int_{\partial D} \frac{1}{|x-y|^{n-1}} d\sigma(y) \leq \|\phi\|_\infty \int_{\partial D} \frac{1}{||x|-|y||^{n-1}} d\sigma(y)$$

$$
\begin{aligned}
&\leq \|\phi\|_\infty \int_{\partial D} \frac{1}{\left||x| - \frac{|x|}{2}\right|^{n-1}} d\sigma(y) \\
&\leq \|\phi\|_\infty \int_{\partial D} d\sigma(y) \left|\frac{|x|}{2}\right|^{1-n} \\
&\leq \|\phi\|_\infty \int_{\partial D} d\sigma(y) 2^{n-1} |x|^{1-n}
\end{aligned}
$$

Finally we have that $\int_{\partial D} d\sigma(y)$ is just the arc lengh of ∂D which must be finite and positive and, as we are taking $|x|$ to ∞, we must have that, for $|x|$ sufficiently large

$$
\begin{aligned}
\|\phi\|_\infty \int_{\partial D} d\sigma(y) 2^{n-1} |x|^{1-n} &\leq \|\phi\|_\infty \int_{\partial D} d\sigma(y) 2^{n-1} \frac{1}{|x|} |x|^{2-n} \\
&\leq C|x|^{2-n} \quad \text{for } C = \|\phi\|_\infty \int_{\partial D} d\sigma(y) 2^{n-1}
\end{aligned}
$$

We can then combine this result with Theorem 2.6 in order to show that all three conditions of Theorem 2.10 hold for the double layer potential u. Indeed, calculating the limits explicitly, as $|x| \to \infty$ we get that

$$
u(x) = \begin{cases} O(1), & n = 2, \\ o(1), & n > 2. \end{cases}
$$

2.2 The Dirichlet problem and related results

As was discussed in the preceding chapter, the analysis of the double layer potential is typicaly traced to its suggested use with regards to the Dirichlet problem. Here follows a more explicit treatment of the problem, its solutions and their properties.

A solution to the *Interior Dirichlet boundary value problem* on a bounded domain $D \subset \mathbb{R}^n$ of class C^2 is a function u that is harmonic in D, continuous in $\overline{D}$ and satisfies

$$
u = f \text{ on } \partial D
$$

for a given continuous function f.

We have that any solution u to the *Exterior Dirichlet boundary value problem* is harmonic on $\mathbb{R}^n \setminus \overline{D}$, continuous on $\mathbb{R}^n \setminus D$ and satisfies the boundary condition

$$
u = f \text{ on } \partial D
$$

for f a given continuous function. Furthermore, we must have that for sufficiently large $|x|$,

$$u(x) = O(1), n = 2 \text{ and } u(x) = o(1), n = 3$$

uniformly, for all directions.

We take this opportunity to introduce the single layer potential. The following integral operator outputs a function in the form of a single layer potential,

$$Sf(x) := \int_{\partial D} f(y)\Phi(x,y)d\sigma(y), x \in \mathbb{R}^n \setminus \partial D.$$

Developed alongside the double layer potential, particularly in the resolution of specific boundary value problems, the theory of both has been strongly related throughout their development, and we see both in the following theorem as well as further results how the interrelation between the operators determined by these layer potentials is relevant to our study of spectral theory. In particular, this result gives us a relation between the limits of the gradient of $S\phi$ as x approaches ∂D from the interior or exterior, and the jump relations described previously on the double layer potential.

Theorem 2.11. *Take ∂D to be a C^2 boundary and $\phi \in C(\partial D)$. Then*

$$\frac{\partial S\phi(x)_\pm}{\partial \nu} = \int_{\partial D} \phi(y)\frac{\partial \Phi(x,y)}{\partial \nu(x)} ds(y) \mp \frac{1}{2}\phi(x) \quad x \in \partial D,$$

where the integral exists as an improper integral and

$$\frac{\partial S\phi(x)_\pm}{\partial \nu} := \lim_{t \to +0} \nu(x) \cdot \nabla S\phi(x \pm t\nu(x))$$

is to be understood as uniform convergence on ∂D.

Of importance to our later results concerning potential solutions to the above problems we have the following three theorems.

Theorem 2.12. *Take ∂D to be a boundary of class C^2, and $\phi \in C(\partial D)$. Then we have that for the double layer potential u, the limits $\frac{\partial u_+}{\partial \nu}$ and $\frac{\partial u_-}{\partial \nu}$ exist as in the prior result and are equal, for all $x \in \partial D$.*

Theorem 2.13. *Both the interior and exterior Dirichlet problem have at most one solution on $C(\partial D)$.*

Proof. The proof of the above result relies on the comparison of two solutions u_1 and u_2 to the Dirichlet problem, and demonstrating that they are equivalent, by showing their difference $u_1 - u_2$ to be zero. This follows in the interior case, as we must have that our difference function meets the requirements of the homogeneous Dirichlet problem.

We then use the following corollary of the Maximum-minimum principle ([28], Chp.6, Thm.6.9, pg.80):

Corollary 2.14. *If u is harmonic on the bounded domain D and continuous on $\overline{D}$, then u achieves both its maximum and minimum on ∂D.*

This gives us $u_1 - u_2 = 0$ in D since we have that our function is both harmonic and zero on ∂D. The $\mathbb{R}^3 \setminus \overline{D}$ case follows directly from $(u_1 - u_2)(x) = o(1)$ as $|x| \to \infty$.

The 2-Dimensional case follows via a combination of the Maximum-minimum principle and the Exterior form of Green's Formula ([28], Chp. 6, Thm. 6.11, pg.81), from which we get that either the maximum and minimum values of $u_1 - u_2$ are obtained on the boundary, in which case we immediately get the desired result from the homogeneous boundary condition, or one of them will take the value

$$(u_1 - u_2)_\infty = \frac{1}{2\pi r} \int_{|y|=r} (u_1 - u_2)(y) d\sigma(y),$$

the mean value property at ∞ for sufficiently large r.

Assuming $(u_1 - u_2)_\infty$ to be our supremum, the inequality

$$(u_1 - u_2)(x) \leq (u_1 - u_2)_\infty, \quad x \in \mathbb{R}^2 \setminus D,$$

in addition to the Mean value theorem ([28], Chp.6, Thm.6.8, pg.80) we get the equality $(u_1 - u_2) = (u_1 - u_2)_\infty$ on the exterior of some circle, which we can extend to the whole of $\mathbb{R}^2 \setminus D$ by the maximum principle and once again apply the homogeneous boundary condition to get the desired result. We argue similarly for the case where $(u_1 - u_2)_\infty$ is the infimum. $\square$

It is important to note that while the above result gives us that any solution of the boundary value problem is unique, we do not necessarily have that such a solution exists. It is however instrumental in the proof of the following statement from which existence of a solution can be deduced. From this point on we shall use K to refer to the Neuman Poincarré operator, with adjoint K^*.

Theorem 2.15. *Acting on $C(\partial D)$, the operator $I - K$ has a trivial nullspace, and the nullspace of $I + K$ is one-dimensional with*

$$N(I + K) = Span\,\{1\}\,.$$

Proof. Assume ϕ to be some element of the nullspace of $I - K$, and take u to be the double layer potential with ϕ as its density.

From Theorem 1.2, we have $2u_- = K\phi - \phi = 0$, combined with the uniqueness of the solutions to the interior Dirichlet problem, we get that $u = 0$ on D. As a result of Theorem 2.12, we have that

$$\frac{\partial u_+}{\partial \nu} = 0$$

and furthermore we have $u(x) = o(1)$ as $|x| \to \infty$, thus by [28], Chp.6, Thm 6.13, pg 86, we also have $u = 0$ on $\mathbb{R}^n \setminus \overline{D}$. Again, as a result of Theorem 1.2 we get that $\phi = u_+ - u_- = 0$, giving us the first result.

In a similar way, for ϕ an element of $N(I + K)$, we can establish that $\phi + K\phi$ is constant. Combined with Theorem 1.2, and by establishing that $N(I+K)$ is not empty, the second result follows. $\square$

The injectivity implied by the above result, combined with the compactness property of K gives us by ([28], Chp.3, Thm.3.4, pg.38) that $I - K$ must be invertible on $C(\partial D)$.

As discussed previously, there is a strong relationship between the potential solutions of the Dirichlet problem and the case when ± 1 belongs to $\sigma(K)$. Indeed, the above result will be used initially in regards to solving the Dirichlet problem, but we will return later to discuss its spectral ramifications.

Returning to our discussion of the Dirichlet problem, by a comparison of the harmonic and boundary conditions of the interior and exterior Dirichlet problem and the established properties of the double layer potential, we observe that, with the exception of the required continuity property across the boundary, the double layer potential would otherwise form an appropriate solution of both cases, including the required behaviour as $|x| \to \infty$.

As such, the following result is apparent by applying Theorem 1.2 in the interior and exterior cases.

Theorem 2.16. *In the interior case the double layer potential on $D \subset \mathbb{R}^n$ with density $\phi \in C(\partial D)$ and in the exterior case the modified double layer potential*

$$u(x) = \int_{\partial D} \phi(y) \left(\frac{\partial \Phi(x, y)}{\partial \nu(y)} + \frac{1}{|x|^{n-2}} \right) d\sigma(y), \quad x \in \mathbb{R}^n \setminus \overline{D}$$

solve the Interior and Exterior Dirichlet problems with boundary data f, respectively, if we have that ϕ solves the below integral equations, again, respective of the Interior and

Exterior cases,

$$\begin{aligned} \phi(x) - 2\int_{\partial D} \phi(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}d\sigma(y) &= -2f(x), \\ \phi(x) + 2\int_{\partial D} \phi(y)\left(\frac{\partial\Phi(x,y)}{\partial\nu(y)} + \frac{1}{|x|^{n-2}}\right)d\sigma(y) &= 2f(x), \end{aligned} \qquad x \in \partial D.$$

Finally, we establish the following property of solutions to the Dirichlet problem.

Theorem 2.17. *The interior and exterior Dirichlet problems have unique solutions on D.*

Proof. The existence of the solution to the integral equation $\phi - K\phi = -2f$ follows from Theorem 2.15 and the compactness of K as a result of the aforementioned implied invertibility.

In the exterior case, examining the difference between the modified double layer potential and the original,

$$\int_{\partial D} \phi(y)\left(\frac{\partial\Phi(x,y)}{\partial\nu(y)} + \frac{1}{|x|^{n-2}}\right)d\sigma(y) - \int_{\partial D} \phi(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}d\sigma(y) = \int_{\partial D} \phi(y)\frac{1}{|x|^{n-2}}d\sigma(y)$$

given the resultant kernel is continuous we then must have that the integral operator which outputs the difference between the double layer potential and its modified form is compact difference. Thus we also have the operator

$$\overline{K}\phi(x) := \int_{\partial D} \phi(y)\left(\frac{\partial\Phi(x,y)}{\partial\nu(y)} + \frac{1}{|x|^{n-2}}\right)d\sigma(y) \quad x \in \partial D$$

that outputs the modified double layer potential is compact. Taking ϕ to be such that

$$\phi + \int_{\partial D} \phi(y)\left(\frac{\partial\Phi(x,y)}{\partial\nu(y)} + \frac{1}{|x|^{n-2}}\right)d\sigma(y) = 0,$$

and, setting u to take the value of the modified layer potential at $x \in \mathbb{R}^n \setminus \overline{D}$, then $u = 0$ on ∂D, which we can extend to $\mathbb{R}^n \setminus \overline{D}$ by the uniqueness of the solutions for the exterior Dirichlet problem.

Using the asymptotic properties of the fundamental Laplace solution and its partial derivatives, we can determine that

$$|x|^{n-2}u(x) = \int_{\partial D} \phi(y)ds(y) + O\left(\frac{1}{|x|}\right) \quad |x| \to \infty,$$

uniformly, in all directions. Given $u = 0$ in $\mathbb{R}^n\setminus\overline{D}$ we then must have $\int_{\partial D} \phi(y)ds(y) = 0$.

Thus,

$$\phi(x) + \int_{\partial D} \phi(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} d\sigma(y) = 0$$

and by Theorem 2.15 we get that ϕ is constant on ∂D, which, given $\int_{\partial D} \phi(y) ds(y) = 0$ gives us $\phi = 0$ on ∂D.

We then can use the fact that $\overline{K} - I$ is a Fredholm operator, of index 0 (See Chapter3) to get the required bijectivity which gives us the existence of a unique solution. □

Thus it follows that, in the event we can determine the existence of a solution such as that described in Theorem 2.16, said solution is unique. That is, the only existing solution is given in terms of a double layer potential.

2.3 Spectral theory of the double layer potential

Of particular relevance to our research inspiration, we have the example of [3] which explicitly studies the spectral properties of the Neumann Poincaré operator in the analysis of *plasmon resonances.*

Plasmons can be considered the fluctuations in the plasma formed of the collective conduction electrons typical to the surfaces of conductive materials such as metals. Plasmon resonances take place when resonant oscillations of these conduction electrons occur as a response to the stimulus of incident light.

As detailed in [3], these plasmon resonances can be determined as the point spectrum of the Neumann Poincaré operator acting on the surfaces of their respective materials. In particular [3] is focused on formulating and analysing the spectrum of the N.P. operator, and thereby the plasmon resonances on smooth surfaces, particularly ellipses and three dimensional balls.

The original inspiration of our research being a more abstract study of the spectral and operator theory of plasmon resonances on surfaces with corners and edges, there is a significant overlap with regard to the current relevant theory.

To this end we return to our overview of said material, particularly with a focus on results relevant to the spectral theory of the operator and its adjoint as well as an overview of the spaces they act on.

Thus far we have focussed on the properties of the N.P. operator and its adjoint as it acts on $C(\partial D)$ on sufficiently smooth domains. From this point onwards, a broader domain of well-definedness would be preferable in our study of the integral operator.

Indeed, such a domain specifically for our choice of integral operator is established in [15] wherein we see sufficient proof of the well-definedness of the N.P. operator and its adjoint pointwise almost everywhere on $L^p(\partial D)$ spaces of functions specifically as a result of showing in both cases the existence of and some uniform estimates of the pointwise limit from a dense class, i.e. $C^1(\partial D)$ in $L^p(\partial D)$.
This, along with two other significant results are summarised below from [15]

Theorem 2.18. *For ∂D of class C^1 our N.P. operator and its adjoint are both well-defined on $L^p(\partial D)$, $p \in (1, \infty)$ and pointwise almost everywhere on ∂D. Additionally, the N.P. operator is bounded on $L^p(\partial D)$, with upper bound determined by p (the index of the L^p space), ∂D and the dimension of the space n and furthermore, both the N.P. and N.P. adjoint operators are compact.*

Establishing these properties for $L^p(\partial D)$ enables us to further extend a number our previous results, as well as justifying our choice of function space in the results to follow.

Returning to our discussion of explicitly smooth domains, the following results give an overview of some of the more significant spectral properties of the N.P. operator and its adjoint, whilst also leading us to more recent approaches and results in their spectral theory.

What follows is a formal statement of the result determining the form of the spectrum of our operator on smooth surfaces.

Theorem 2.19 (Riesz-Schauder Theorem)**.** *For T a compact operator on a Banach space, we have that any non-zero element of $\sigma(T)$ is an eigenvalue of finite multiplicity. Furthermore $\sigma(T) \setminus \{0\}$ is an at most countably infinite discrete set, such that if any accumulation point of $\sigma(T) \setminus \{0\}$ exists, it must be 0.*

This gives us a much clearer image of the structure of the spectrum of the N.P. operator and its adjoint on sufficiently smooth surfaces, in addition to an initial indication of its spectral decomposition in the point spectrum. The following result has the analytical advantage of providing a bound on the spectrum of our operators.

Theorem 2.20 (Plemelj's Theorem)**.** *For D a C^2 class domain, the spectrum of the N.P. operator and its adjoint on $L^2(\partial D)$ are both contained in the interval* $[-1, 1)$

Proof. The first element of the proof makes use of the equations below given in terms of the single layer potential $S\phi$, following as a result of the boundedness of $S\phi$ [28] (Chapter 6, Thm 6.15) and as a further result of previously discussed jump relations on the double layer potential combined with the formal definition of $S\phi$. For λ an eigenvalue of K^*, with eigenfunction ϕ, then we have:

$$\begin{aligned} S\phi_+ =& S\phi_- \\ \frac{\partial S\phi_\pm}{\partial \nu} =& \frac{1}{2}(K^*\phi \mp \phi) \quad \text{ on } \partial D \\ =& \frac{1}{2}(\lambda \mp 1)\phi \end{aligned}$$

where $S\phi_\pm$ represents the limit from above and below along the normal vector of the single layer potential. By rearrangement and an application of Green's theorem [28] we get

$$(1+\lambda)\int_{\mathbb{R}^n \setminus D} |\nabla S\phi|^2 dx = (1-\lambda)\int_D |\nabla S\phi|^2 dx.$$

Defining the two quantities,

$$J(S\phi) := \int_D |\nabla S\phi|^2, \text{ and } J'(S\phi) = \int_{\mathbb{R}^3 \setminus D} |\nabla S\phi|^2 dx$$

Taking the difference of the above exterior and interior directional derivatives, we get

$$\frac{\partial S\phi_+}{\partial \nu} - \frac{\partial S\phi_-}{\partial \nu} = -\phi \text{ on } \partial D.$$

This implies that $J(S\phi)$ and $J'(S\phi)$ are not identically 0, else, we would have $\phi = 0$, contradicting the assumption that it is an eigenfunction. Thus we can rearrange our previous identity to get:

$$\lambda = \frac{J(S\phi) - J'(S\phi)}{J(S\phi) + J'(S\phi)},$$

implying then that $\lambda \in [-1, 1]$. The final element of the proof, eliminating 1 as an eigenvalue, follows directly from Theorem 0.12. □

We here introduce the *Plemelj Symmetrisation principle.* As described and proven in [27], this theorem is invaluable in its use in the results following it, being key both to our study of the properties of the N.P.-spectrum, and furthermore in the development of yet more significant function space properties.

Theorem 2.21. *For D a class C^2 domain or Lipschitz domain, the following identity,*

$$KS = SK^*$$

holds on $L^2(\partial D)$.

Proof. By use of Green's formula, we get

$$\begin{aligned}
\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \left(\int_{\partial D} \Phi(y,z)\phi(z) d\sigma(z) \right) d\sigma(y) = \\
&= \int_{\partial D} \phi(z) \left(\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \Phi(z,y) d\sigma(y) \right) d\sigma(z) \\
&= \int_{\partial D} \phi(z) \left(\int_{\partial D} \frac{\partial \Phi(z,y)}{\partial \nu(y)} \Phi(x,y) d\sigma(y) \right) d\sigma(z) \\
&= \int_{\partial D} \Phi(x,y) \left(\frac{\partial}{\partial \nu(y)} \int_{\partial D} \Phi(z,y)\phi(z) d\sigma(z) \right) d\sigma(y)
\end{aligned}$$

That is, the composition of the double layer potential with the single layer potential with density ϕ is equal to the composition of the single layer potential with the normal derivative of the single layer potential with density ϕ. Taking x through to a point on the boundary, we then get from the jump relations on u and the directional derivatives of $S\phi$ that

$$\frac{1}{2}(S\phi - KS\phi)(x) = S(\frac{1}{2}(\phi - K^*\phi))(x),$$

which we rearrange to get the desired equation. $\square$

This result is an essential element of the proof of the previously discussed self-adjointness of the N.P. operator on the energy space $H^{-\frac{1}{2}}(\partial D)$. Indeed, for a sufficiently smooth domain D, making use of this result, in combination with the aforementioned self adjointness and further Min-Max results from [27], [35] gives us the following, potentially enabling us to determine further individual eigenvalues based on initial information.

Theorem 2.22. *Let $(\lambda_i^+)_{i\in\mathbb{N}}$ and $(\lambda_i^-)_{i\in\mathbb{N}_0}$ respectively represent the monotone decreasing/increasing sequences of positive/negative eigenvalues of K^* acting on a class C^1 domain D, repeated according to their multiplicity, with $\lambda_0^- = -1$ and each eigenvalue corresponding to some eigenfunction $\phi_i^\pm$. Then we have that,*

$$\lambda_k^+ = \max_{f \perp \{\phi_1^+, \phi_2^+, \dots, \phi_{k-1}^+\}} \frac{\langle SK^*f, f\rangle_{L^2}}{\langle Sf, f\rangle_{L^2}},$$
$$\lambda_k^- = \min_{f \perp \{\phi_0^-, \phi_1^-, \dots, \phi_{k-1}^-\}} \frac{\langle SK^*f, f\rangle_{L^2}}{\langle Sf, f\rangle_{L^2}},$$

where $f \perp \phi$ is taken to mean $\langle f, S\phi\rangle_{L^2} = 0$.

Observe how this result is given in terms of the inner product with respect to the image of S, in addition to the composition with K^*. This is an example of one potential for results on $H^{-\frac{1}{2}}(\partial D)$. We will discuss in more explicit detail later in this chapter the nature of the inner product which generates the desired self-adjointness.

Before proceeding further, we collect our previous results on the N.P. operator spectrum on $L^2(\partial D)$ with their equivalent on its adjoint, using the following.

Theorem 2.23. *Let T be a bounded operator on a Banach space $\mathcal{B}$, then for T^* the adjoint operator of T, we have $\sigma(T) = \sigma(T^*)$.*

With these results in mind, we then have a clear image of the general form of the spectrum of the N.P. operator and its adjoint on L^2 functions defined upon smooth domains.

A number of papers exist discussing the precise form of the spectra of the N.P. operator and its adjoint upon specific space boundaries in $\mathbb{R}^n$, [25] and [2] for example giving the explicit forms of said spectra on spaces in $\mathbb{R}^2$ formed by intersecting disks, and on Tori respectively. Whilst this information may prove useful in those specific scenarios, a broader more general system of description would naturally be preferable for a greater range of application.

It may also be noted that in these two scenarios, the spectrum on the domain containing corners in actuality comprises a closed symmetric interval, as opposed to the more conventional countable set described for the spectrum on Tori. This clearly indicates significant shift between the general form of spectra on smooth spaces and those with edges, again, a more detailed, general theory of which would be preferable.

Before continuing on to the next section we take this opportunity to diverge briefly to discuss spectral decomposition, and in particular focus on the properties of the *Essential spectrum*.

The Essential spectrum of an operator T, commonly denoted $\sigma_{ess}(T)$ is defined to be the set of complex values λ, such that $T - \lambda I$ is not a *Fredholm operator*. That is, $ker(T) and Coker(T)$ are finite dimensional. This property may be considered a weaker form of the standard invertibility/uninvertibility of $T - \lambda I$. We give a more in depth treatment of both the essential specctrum and its properties in the following chapter.

2.4 Kress's technique on curvilinear polygons and the essential spectral radius

Given the specific focus of our research on curves with corners, and in particular the spectral properties of the N.P. operator and its adjoint, it is worth determining how we may apply those results for smooth surfaces and use them to extend the theory accordingly. Following the path described in [28] we consider the two dimensional domain D,

with boundary ∂D defined by a finite number of closed, class C^2 arcs $\Gamma_1, ..., \Gamma_m$ that intersect at the corners $x_1, ..., x_m$, of angles $\gamma_1, ..., \gamma_m$ respectively, where $\gamma_i \in (0, 2\pi)$. From this proviso, it is clear we do not include cusps in our model. In particular, we shall assume that, within sufficiently small neighbourhoods of each corner, our boundaries consist of straight lines. Essentially, we have constructed a domain bounded by some curvilinear polygon.
Whilst the continuity of the image of the N.P. operator and its adjoint with constant density carries over immediately from the jump relations on the double layer potential, in order to compensate for the normal vector becoming discontinuous at each corner, we must modify our operator further as we reach each corner,

$$u_{\pm}(x_i) = \int_{\partial D} \frac{\partial \Phi(x_i, y)}{\partial \nu} \phi(y) d\sigma(y) \pm \frac{1}{2} \delta_i^{\pm} \phi(x_i), i = 1, ..., m.$$

where we have $\delta_i^+ = \frac{\gamma_i}{\pi}$ and $\delta_i^- = 2 - \frac{\gamma_i}{\pi}$.

We can see that these outer and inner limits hold for a continuous density ϕ by applying the jump relations in the C^2 case to the superposition of two double layer potentials onto a pair of C^2 curves which intersect at the i^{th} corner. The limits from the exterior and interior of the superposition respectively are given by the exterior and interior limits of the angle γ_i in addition to the limits on those segments of the curve outside of ∂D.

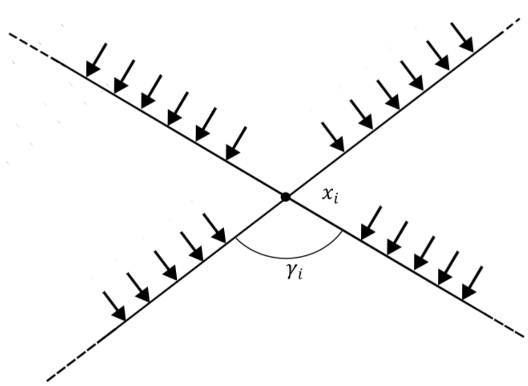


Figure 5: The exterior limit of the superposition of two arcs at γ_i

These extraneous limits are then deleted by setting the density ϕ to be zero on those parts of each curve which don't coincide with points in ∂D. Evaluating the remaining limits gives us the desired jump relation at the corner.

Indeed, in order to better accommodate these corners, we may redefine our integral operator thus, for $x \in \partial D$,

$$\tilde{K}u(x) = \begin{cases} K\phi(x), & x \neq x_i, i = 1, ..., m \\ K\phi(x) + (\frac{\gamma_i}{\pi} - 1)\phi(x_i), & x = x_i, i = 1, ..., m. \end{cases}$$

giving us an operator that maintains continuity at each corner, as at each corner it equals the sum of the interior and exterior limits, i.e. $\tilde{K}u = u_+ + u_-$, the boundary values that lend continuity to the double layer potential. Now, whilst our kernel remains for the most part continuous on our boundary, we still have the issue of singularities as x and y approach a corner. To this end, for each $n \in \mathbb{N}$ we define the operator $K_n : C(\partial D) \to C(\partial D)$ by,

$$K_n(\phi(x)) := \frac{1}{\pi} \int_{\partial D} h(n|x-y|) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \phi(y) ds(y), x \in \partial D,$$

where h is a continuous function on $[0, \infty)$, such that $h(x) = 0$ for $x \in [0, \frac{1}{2}]$ and $h(x) = 1$ for $x > 1$. The cut-off induced by h as the argument approaches each corner keeps the kernel of K_n continuous on each $\partial D \times \Gamma_i$, indicating that the Hilbert Schmidt property holds and thus that for each $n \in \mathbb{N}$, K_n is compact. Taking n sufficiently large that for each $x \in \partial D$ the disc $B[x, 1/n]$ intersects only one or, sufficiently close to a corner, two of the arcs that make up the boundary, and defining $\tilde{K}_n = \tilde{K} - K_n$ we can show, using the techniques in [28] that $\tilde{K}_n$ is bounded on those ϕ continuous on $\partial D \cap B(x, \frac{1}{n})$ in both these cases. Specifically, for each $x \in \partial D$ we take the disk $B[x, 1/n]$ where n is sufficiently large to ensure one of two scenarios, those being that said disc intersects precisely one or two arcs in the vicinity of some cornerγ_i. Our assumptions about the nature of our corners gives us that for sufficiently large n our intersection takes the form of straight-line segments.

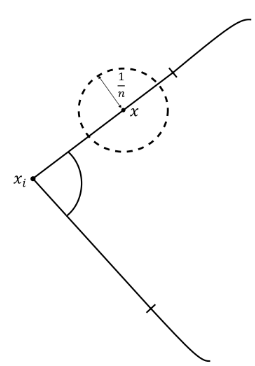


Figure 6: The corner x_i for $B[x, 1/n]$ intersect with a singular edge

In our first scenario, setting

$$M := \max_{i=1,\dots,m} \max_{x,y\in\gamma_i} |\frac{\partial\Phi(x,y)}{\partial\nu(y)}|$$

then by the smoothness property on the single line segment, we can project our integral onto the tangent line and using that get the following upper bound

$$\begin{aligned}
|\widetilde{K}_n\phi(x)| &= \frac{1}{\pi}|\int_{\partial D\cap B[x,1/n]} \frac{\partial\Phi(x,y)}{\pi\partial\nu(y)}\phi(y)ds(y) - \int_{\partial D} h(n|x-y|)\frac{\partial\Phi(x,y)}{\pi\partial\nu(y)}\phi(y)ds(y)| \\
&\leq \frac{1}{\pi}|\int_{\partial D\cap B[x,1/n]} \frac{\partial\Phi(x,y)}{\pi\partial\nu(y)}\phi(y)ds(y)| + \frac{1}{\pi}|\int_{\partial D} h(n|x-y|)\frac{\partial\Phi(x,y)}{\pi\partial\nu(y)}\phi(y)ds(y)| \\
&\leq \frac{1}{\pi}\int_{\partial D\cap B[x,1/n]} |\frac{\partial\Phi(x,y)}{\pi\partial\nu(y)}||\phi(y)|ds(y) + \frac{1}{\pi}\int_{\partial D} |h(n|x-y|)||\frac{\partial\Phi(x,y)}{\pi\partial\nu(y)}||\phi(y)|ds(y) \\
&\leq \frac{2}{\pi}M||\phi||_\infty \int_{\partial D\cap B[x,1/n]} ds(y) \\
&\leq \frac{2}{\pi}M||\phi||_\infty 2\pi\frac{1}{n} \\
&= M||\phi||_\infty\frac{4}{n},
\end{aligned}$$

on each edge.

Now to determine $\widetilde{K}_n$ for x and y on separate edges, in the case of two line segments being our intersection with the disc, for $x \neq x_i$ on one such segment we may form a triangle from the edges given by the segment from x_i to x, the entire line segment on the secondary curve and the line segment between the point x and non-x_i endpoint of the secondary curve segment.

We now want to try and evaluate the N.P. operator on this line segment and use it to determine an upper bound for $|\widetilde{K}_n\phi(x)|$.

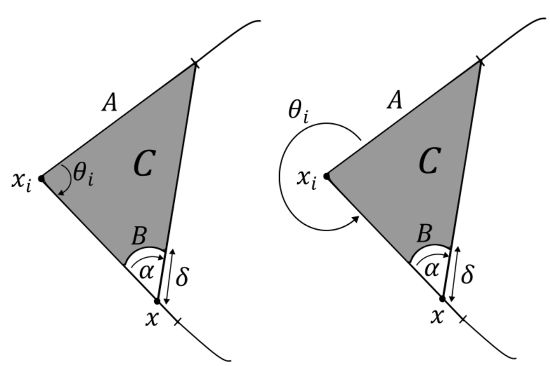


Figure 7: The corner x_i for θ_i acute and obtuse

Denoting this segment A we see that by taking some circular arc B, centred at x of radius $\delta < \frac{1}{n}$ of angle α between the two remaining edges of the triangle, we form a bounded subset C of our triangle on the boundary of which we have that Φ is harmonic, since we have excluded the case where $y = x$ so, on this boundary we get

$$\int_{\partial C} \frac{\partial \Phi(x,y)}{\partial \nu(y)} = 0,$$

by Green's Theorem, see Corollary 2.8.
Additionally, since we have that the two remaining straight edges of our triangle segment radiate directly from the point x, we must have that

$$\frac{\partial \Phi(x,y)}{\partial \nu(y)} = 0,$$

for y belonging to either of those segments. This then implies that

$$\left| \int_A \frac{\partial \Phi(x,y)}{\partial \nu(y)} d\sigma(y) - \int_B \frac{\partial \Phi(x,y)}{\partial \nu(y)} d\sigma(y) \right| = 0.$$

Furthermore, given the positivity of $\frac{\partial \Phi(x,y)}{\partial \nu(y)}$ on A we also must have,

$$\int_A \left| \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right| d\sigma(y) = \left| \int_A \frac{\partial \Phi(x,y)}{\partial \nu(y)} d\sigma(y) \right|$$

Thus we need only calculate $\left| \int_B \frac{\partial \Phi(x,y)}{\partial \nu(y)} d\sigma(y) \right|$, which, given the invariance of our operator under rotation and translation, we can determine by a simple parameterization of the equivalent curve.
We set this parametrization to be the closed curve given by $B : [0, \alpha] \to \mathbb{R}^2$, where

$$B(t) := (\delta \cos(t), \delta \sin(t))$$

and given x must coincide here with the origin, $(0, 0)$ we can calculate our operator on B directly,

$$\begin{aligned}\left| \int_B \frac{\partial \Phi(x,y)}{\partial \nu(y)} d\sigma(y) \right| = \Bigg| \frac{1}{2\pi} \int_{[0,\alpha]} & \frac{\langle (-\delta \cos(t), -\delta \sin(t)), (-\cos(t), -\sin(t)) \rangle}{\langle (-\delta \cos(t), -\delta \sin(t)), (-\delta \cos(t), -\delta \sin(t)) \rangle} \\ & \times |(-\delta \sin(t), \delta \cos(t))| \, dt \Bigg|\end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{1}{2\pi} \int_{[0,\alpha]} \frac{\delta}{\delta^2} \cdot \delta dt \right| \\
&= \frac{1}{2\pi} \int_{[0,\alpha]} dt \\
&= \frac{\alpha}{2\pi}.
\end{aligned}
$$

Thus, using standard triangle geometry and given that within the disc $B[x, 1/n]$ our kernel is non-zero only for $y \in A$, we must have

$$
\begin{aligned}
|(\widetilde{K}_n\phi)(x)| \leq 2 \int_A & \left| \frac{\partial \Phi(x,y)}{\partial \nu(y)} \phi(y) \right| d\sigma(y) \\
& \leq 2\|\phi\|_\infty \left| \int_A \frac{\partial \Phi(x,y)}{\partial \nu(y)} d\sigma(y) \right| \leq \frac{\alpha}{\pi} \|\phi\|_\infty \leq \left| 1 - \frac{\gamma_i}{\pi} \right| \|\phi\|_\infty.
\end{aligned}
$$

Furthermore, given $k(x_i, y) = 0$ for all $y \neq x_i$ and using our earlier determination of the upper and lower limits of the N.P. operator as x tends to x_i, we then must have

$$
(\widetilde{K}_n\phi)(x_i) = \left(\frac{\gamma_i}{\pi} - 1 \right) \phi(x_i).
$$

Given this result must hold for all corners, then, for sufficiently large n we have that

$$
\|\widetilde{K}_n\|_\infty \leq \max_{1,\ldots,m} \left| \frac{\gamma_i}{\pi} - 1 \right|.
$$

What's more, given K_n is weakly singular, we also must have that it is compact, acting on the continuous parameterization of the boundary.
This gives us that the *Essential Norm* of $\tilde{K}$, that is

$$
||\tilde{K}||_{ess} = \inf\{||\tilde{K} - T|| : \text{Where } T \text{ is compact}\}
$$

is also bounded by $\max_{i=1,\ldots,m} |(\frac{\gamma_i}{\pi} - 1)|$.

Thus the above result and the *Gelfand formula* (See Theorem 3.14) for the essential spectral radius, which we shall henceforce denote $||\sigma(\cdot)||_{ess}$, gives us

$$
||\sigma(\tilde{K})||_{ess} \leq \max_{i=1,\ldots,n} |(\frac{\gamma_i}{\pi} - 1)|,
$$

that is, for said curvilinear polygon with edges determined by continuous curves, we may determine an upper bound for the essential spectrum of the modified N.P. adjoint operator in terms of the polygon's internal angles. We will consider this approach in our own research and in applying it to further scenarios.

If we return to consider Theorem 2.15, for D a C^1 domain, we have that the operator $I-K$ is invertible on $L^p(\partial D)$, indicating that 1 is not an element of the spectrum of the N.P. operator, or its adjoint. Indeed, we have multiple similar results in [15], in which it is proven this invertibility holds equally for p-integrable functions upon a broader range of surfaces, with a particular focus upon *Lipschitz domains*, which we first discussed in the previous chapter (Definition 1.8).

Whilst being locally Lipschitz continuous does not discount being n-times locally continuously differentiable, and in fact we also must have that any class C^1 domain boundary is also locally Lipschitz as a result of [34], results dependent upon a higher degree of 'smoothness' may not apply. That is, every smooth surface is Lipchitz, but not every Lipschitz surface is smooth.

There are a variety of examples of Lipschitz spaces with singular points and edges, such as the cone or cylinder in $\mathbb{R}^3$ [14]. Given the nature of our research is to study the spectral properties of the N.P. adjoint operator on such non-smooth surfaces, the extension of the following results from the case of the operator acting on a C^n class space will be useful in developing the theory of our operators on a broader class of spaces.

We give the following existence result, along with the spectrally relevant results, as referenced here [42], below.

Theorem 2.24. *For D a Lipschitz domain, both the Neumann Poincaré operator and its adjoint exist in $L^p(\partial D)$ pointwise for almost every input from ∂D. Furthermore, for $D \in \mathbb{R}^2$, $I - K^*$ and $I - K$ are invertible on $L^2(\partial D)$*

The above result is particularly worth noting as we do not have the compactness property of either operator upon a general Lipschitz domain to give us the desired invertibility.

Indeed a number of results determining the properties, such as compactness, of the N.P. operator and its adjoint on C^n class domains may not hold in general on Lipschitz domains.

2.5 Sobolev spaces and recent results on non-smooth domains

In order to extend our study beyond the scope of domains with explicitly smooth boundaries, as well as giving us a foothold in the most recent developments in the spectral theory of the N.P. operator/adjoint operator, we must refine our focus yet further, whilst retaining the properties already discussed. To this end we return to, and give a proper introduction to the briefly aforementioned concept of *Sobolev spaces*.

Definition 2.25. For $p \in [1, \infty]$ the Sobolev space is the set $W^{m,p}(\mathbb{R}^n)$ of functions u on $\mathbb{R}^n$ given by,

$$W^{m,p}(\mathbb{R}^n) := \{u \in L^p(\mathbb{R}^n) : D^\alpha u \in L^p(\mathbb{R}^n), |\alpha| \leq m\},$$

where D^α represents the α-th *weak* partial derivative determined by,

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x^{a_1} ... \delta x^{a_n}} \text{ for } |\alpha| = \sum_{i=1}^{n} a_i.$$

and a function v is considered the α-th *weak* partial derivative if for all smooth, compactly supported functions ϕ on $\mathbb{R}^n$, we have,

$$\int_{\mathbb{R}^n} u D^\alpha \phi = (-1)^{|\alpha|} \int_{\mathbb{R}^n} v\phi.$$

We can equip the Sobolev space $W^{m,p}(\mathbb{R}^n)$ with the following norm,

$$||u||_{m,p,\mathbb{R}^n} := \left(\sum_{|\alpha| \leq m} ||D^\alpha u||^p_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{p}}.$$

Furthermore, for p=2, we can determine an equivalent inner product,

$$\langle u, v \rangle_{m,\mathbb{R}^n} := \sum_{|\alpha| \leq m} \int_{\mathbb{R}^n} D^\alpha u D^\alpha v,$$

making $W^{m,2}(\mathbb{R}^n)$ a Hilbert space, for which reason we use the more specific notation $H^m(\mathbb{R}^n)$. We can further extend our definition of the $H^m(\mathbb{R}^n)$ to accommodate indiscrete indices, in the form of the *Sobolev-Slobodeckij space*, or the $H^s(\mathbb{R}^n)$ Sobolev space, identical to our previous definition for $s \in \mathbb{N}$, and further for $s > 0$ determined as $s = m + \epsilon$, $m \in \mathbb{N} \cup \{0\}$, $\epsilon \in (0, 1)$, given by the space of those functions u for which we have the norm given below is finite:

$$||u||^2_{H_s(\mathbb{R}^n)} = ||u||^2_{H_m(\mathbb{R}^n)} + |u|^2_{m,s} < \infty.$$

The above norms are determined by the following inner product on functions u and v,

$$\begin{aligned}\langle u, v\rangle_{H_s(\mathbb{R}^n)} =& \langle u, v\rangle_{H_m(\mathbb{R}^n)} + \langle u, v\rangle_{m,\epsilon} \\ \langle u, v\rangle_{m,s} =& \sum_{|\alpha|=k} \int_{\mathbb{R}^n} \int \mathbb{R}^n \frac{(D^\alpha u(x) - D^\alpha u(y))(D^\alpha v(x) - D^\alpha v(y))}{|x-y|_2^{n+2s}} dxdy.\end{aligned}$$

Note, we use a negative index $H^{-s}(\mathbb{R}^n)$ to represent the dual space of $H^s(\mathbb{R}^n)$. Additionally, through all this we can see that for $s = 0$, $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

Indeed, when acting over a surface or curve ∂D we define $H^s(\partial D)$ in terms of local coordinates, and the dual space $H^{-s}(\partial D)$ via duality. Of particular significance is the case where we have $s = \frac{1}{2}$.

Using Theorem 2.21, and equipping $H^{-\frac{1}{2}}(\partial D)$ with the inner product,

$$(\phi, \psi)_{H^{-\frac{1}{2}}(\partial D)} = -\langle \phi, S\psi\rangle_{L^2},$$

[27] gives us that the N.P. operator is self-adjoint on $H^{-\frac{1}{2}}(\partial D)$. Indeed, from Theorem 2.21 we have

$$\begin{aligned}\langle K^*\phi, S\psi\rangle_{L^2} &= \langle \phi, KS\psi\rangle_{L^2} \\ &= \langle \phi, SK^*\psi\rangle_{L^2},\end{aligned}$$

from which follows the previously discussed self-adjointness of K with respect to this inner product, and in addition [27] provides a proof of how this inner product generates an equivalent norm to the standard one used on $H^{-\frac{1}{2}}(\partial D)$.

Thus we are able, when acting on the energy space $H^{-\frac{1}{2}}(\partial D)$, to apply the spectral theory of self adjoint operators to K.

Originally derived by Poincaré in his work on the N.P. Operator, this norm equivalence and its significance were not initially recognised due to the form in which they were determined. Indeed, it was many years later, thanks to the work of Kavinson, Putinar, Costabel and Shapiro [27]that this property of the inner product on $H^{-1/2}$ was made clear.

The importance of this result is further emphasized by how the N.P. operator is typically distinguished for its not being self adjoint when studied over some of the more widely used function spaces, such as $L^2(\partial D)$ and $C(\partial D)$.

Much of the subsequent research regarding the N.P. operator or now, equivalently, its adjoint, is done within the setting of the energy space, taking advantage of the self-adjointness property.

A number of the more significant results developed in [38] and [37] with regard to N.P. operator spectral theory make use of the following operator. The Beurling-Ahlfors transform determined by a domain Ω in $\mathbb{R}^2$ is the integral operator given by,

$$T_\Omega f(z) = p.v.\frac{1}{\pi}\int_\Omega \frac{f(\xi)}{(\bar{\xi}-\bar{z})^2}dA(\xi), f \in L_a^2(\Omega), z \in \mathbb{R}.$$

Note that this operator is defined on the Bergman space $L_a^2(\Omega)$, meaning that f is analytic and it is square integrable with respect to the *Area measure*. Further properties of the transform are discussed in [20].

As is explained below, the usefulness of this much more easily analysed integral operator is given by the following results from [37].

Theorem 2.26. *The N.P. adjoint operator K on $H_0^{-\frac{1}{2}}(\partial\Omega)$ is similar to $\bar{T_\Omega}$ on $L_a^2(\Omega)$, for $\Omega \subset \mathbb{R}^2$ an open, Lipschitz domain. That is, there exists an operator $U : H_0^{-\frac{1}{2}}(\partial\Omega) \to L_a^2(\Omega)$ such that $U^{-1}TU = K$.*

Corollary 2.27. *Let Ω be as in the previous theorem, then we have that, on $H^{-\frac{1}{2}}(\partial\Omega)$, $\sigma(K) \setminus \{-1\}$ is symmetric.*

Proof. The above result follows as a result of the above Theorem 2.26 which gives us

$$\sigma(K) = \sigma(T_\Omega)$$

and from the antilinearity of T_Ω, that is,

$$T_\Omega(if) = -iT_\Omega(f).$$

For eigenvalue $\lambda \neq -1$ with eigenfunction f we have

$$T_\Omega f - \lambda f$$

is invertible. Thus we also must have

$$\begin{aligned} -i(T_\Omega f - \lambda f) &= T_\Omega if + \lambda if \\ &= T_\Omega if - (-\lambda)if \end{aligned}$$

is also invertible. That is, for eigenfunction if, $-\lambda$ is an eigenvalue of T_Ω. □

The following results from [38] give some of the latest, most relevant developments in the spectral theory of the N.P. operator and its adjoint, in particular with regard to our study of our operators spectral properties on surfaces with corners and edges, with a focus on those results in the complex plane. In the following results, W_α represents the wedge determined by the angle α in the complex plane, and J denotes the antilinear conjugation operator on $L_a^2(W_a)$ given by $Jf(z) = \overline{f(\overline{z})}$.

Lemma 2.28. *For $\alpha \in (0, \pi)$, $T_{W_a}J$ is positive with spectrum*

$$\sigma(T_{W_a}J) = \left\{x \in \mathbb{R} : x \in \left(0, 1 - \frac{\alpha}{\pi}\right)\right\}$$

with each element of multiplicity 2.

Theorem 2.29. *Take L to be a linear fractional transform from $W_\alpha \cup \{0\}$ onto a bounded domain. We then have that for K acting on $H^{-\frac{1}{2}}(\partial L(W_\alpha))$, its spectrum is given by*

$$\sigma(K) = \left\{x \in \mathbb{R} : |x| \leq \left|1 - \frac{\alpha}{\pi}\right|\right\} \cup \{-1\},$$

where $\sigma(K) \setminus \{-1\} = \sigma_{ess}(K^*)$, each element of multiplicity 2.

Proof. The proof given in [38] makes use of the Beurling-Ahlfors transform on W_α, in combination with J. Combining the self-adjointness of $T_{W_a}J$ [20] with its being J-symmetric, that is, $T_{W_a}J = J(T_{W_a}J)J = JT_{W_a}$ [20], to give us $JT_{W_a}J = T_{W_a}$ and thus $(T_{W_a}J)^2 = (T_{W_a})^2$. This is followed by the use of Lemma 2.28 which gives us the magnitude and multiplicity of our spectrum, and the result is complete via the symmetric spectral properties of T_{W_a} from Corollary 2.27. $\square$

The above result gives a complete description of the spectrum of our operator in the complex plane in the most basic scenario of a 'curve with a corner'. The following, also given by [38] extends this result significantly.

Theorem 2.30. *For Ω the C^2 class curvilinear polygon with angles $\alpha_1, ..., \alpha_n$ in the complex plane, the Essential spectrum of K^* on $H^{\frac{1}{2}}(\partial\Omega)$ is given by,*

$$\sigma_{ess}(K^*) = \left\{x \in \mathbb{R} : |x| \leq \max_{i \in \{1,...,n\}} \left|1 - \frac{\alpha_i}{\pi}\right|\right\}.$$

Most notably, in both cases, we see not only how the introduction of corners has led to the spectrum having an at least partially continuous component, as opposed to the countability property on purely smooth spaces.

As we discussed previously, the aim of our work will be to make a study of the N.P. adjoint operator spectrum when considered over $L^2(\partial D)$ and $C(\partial D)$, for ∂D a bow-tie curve. This is a departure from the existing work and its particular focus on the energy space. We have previously discussed the existing results on boundaries with corners, explicitly, on curvilinear polygons, both here, in the context of $H^{-1/2}$ and in Kress's work on the space of continuous functions. For this reason, we temporarily diverge to consider a similar result from [33] for curvilinear polygons, where our operator is acting over the L^2 space.

Given a curvilinear polygon Γ, consistiing of $n \in \mathbb{N}$ curves, [33] shows us that the $L^2(\Gamma)$ spectrum of the N.P. operator is composed of n bow-tie curves. We observe that in our own calculations where the bow-tie curve is actiing as a domain, it also appears to hold that the resultant spectrum consists of bow-tie curves as well, though whether this in general is true of our results is a matter of further analysis.

Additionally, for a given $\lambda \in \mathbb{C}$, [33] gives a formula for the Fredholm index of $K - \lambda I$,

$$index(K - zI) = \dim ker(K - zI) - \dim coker(K - zI),$$

the finiteness of which is sufficient to determine the Fredholm property described previously, when $K - \lambda I$ is acting over the curvilinear polygon.

In fact, this index is given explicitly by the sum of the winding numbers of the n bow-tie curves that form the spectrum of K over $L^2(\Gamma)$, about λ, that is, the number of times each curve travels anticlockwise around λ in the complex plane. We shall discuss in more detail a similar, less signicant result in the following chapter.

Using similar techniques to those discussed in the proof of Theorem 2.30, [24] goes on to explore the spectrum of N.P. operator and its adjoint when acting on the surfaces of three-dimensional domains, and in particular we see a similar result to that developed in [33], discussed above, giving a formula for the essential spectrum and a further formula for the Fredholm index in terms of the winding number of a conica point.

Specifically from [24] we see an exact determination of the formula for the spectrum when our domain is a conical point formed by revolving a C^5 curve where the opening angle and orientation of our conical point form parameters of our curves set-definition.

From there, taking Γ_n to be the set of points defining our curve at a countably determined point in its revolution, indexed by $n \in \mathbb{Z}$, we are given that the essential spectrum of our operator, acting on L^2 functions with our conical point as their domain is given by

$$\sigma_{ess}(K) = \bigcup_{n \in \mathbb{Z}} \Gamma_n,$$

and using the winding number of about any complex point $\lambda \notin \sigma_{ess}(K)$ with respect to Γ_n, to determine the index of $K - \lambda I$, once again as the sum of said winding numbers for each curve Γ_n. In this manner we can also determine any remaining isolated eigenvalues to get the complete spectrum of K, in this particular case showing how any such values must be real.

Given the focus of our research on the spectral properties of the N.P. adjoint operator on bow-tie curves, we will conclude this section by discussing the results of the paper [6] on the spectrum of the operator on bow-ties ∂D, in particular determining the properties of the spectrum when the N.P. operator is acting over the energy space $H^{-\frac{1}{2}}(\partial D)$.

A particular difficulty addressed in [6] is the fact that, unlike most curvilinear polygons, for which there already exists a description of the essential spectrum (Theorem 2.30), a bow-tie curve is defined by its coincident corners, meaning that in the vicinity of these corners, it cannot be characterized as Lipschitz.

This difficulty is addressed by considering the relation between the essential spectrum of K^* and the Poincaré variational operator $T_D : H_0^1(\Omega) \to H_0^1(\Omega)$, for D a connected, smoothly-bounded subset of the bounded, open set Ω with smooth open boundary. Here $H_0^1(\Omega)$ represents an alternate *energy space*, given by the space of functions

$$\left\{ u \in H^1(\Omega) : u|_{\partial\Omega} = 0 \right\},$$

equipped with the following inner product and resultant norm:

$$\langle u, v \rangle_{H_0^1(\Omega)} := \int_\Omega \nabla u(x) \cdot \nabla v(x) dx$$

$$\|u\|_{H_0^1(\Omega)} := \left(\int_\Omega |\nabla u(x)|^2 \right)^{\frac{1}{2}} dx$$

We define T_D for $u \in H_0^1(D)$ by

$$\int_\Omega \nabla T_D u \cdot \nabla v dx = \int_D \nabla u \cdot \nabla v dx \quad \forall v \in H_0^1(\Omega).$$

Indeed, for D a Lipschitz domain, it is established in [7] that the spectrum of K^* relates to that of T_D by

$$\sigma(K^*) = \frac{1}{2} - \sigma(T_D) \text{ and } \sigma_{ess}(K^*) = \frac{1}{2} - \sigma_{ess}(T_D).$$

Instead of modifying K^* to accomodate the fact that ∂D is not Lipschitz, [6] considers the spectrum of T_D directly on the bow-tie curve, determining the equality

$$\sigma(T_D) = \sigma_{ess}(T_D) = [0, 1] \quad u \in H_0^1(D),$$

before going on to compare the result on D to a domain D_δ comprised of two Lipschitz 'wings', with corners separated by a line of length δ, such that, as δ tends to 0, D_δ tends to D.

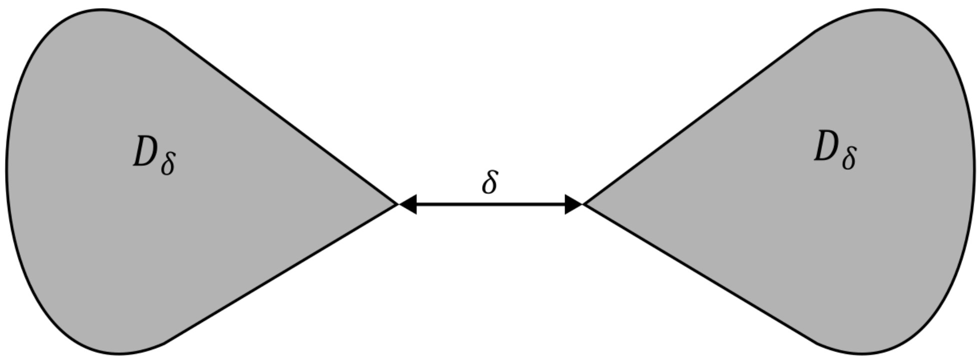


Figure 8: Bow-tie curve, separated into two Lipschitz 'wings'

The separation of the two 'wings' permits the evaluation of the essential spectrum of K^* in terms of that of T_{D_δ}. Taking δ to 0, [6] goes on to show how

$$\lim_{\delta \to 0} \sigma_{ess}(T_{D_\delta}) = \sigma(T_D) = [0, 1]$$

. The relation between the K^* and the T_D essential spectrum for Lipschitz domain, in combination with this limiting property and the result given in Theorem 2.30, allows us to describe the essential spectrum of K^* on D as a closed, symmetric interval subset of $[-\frac{1}{2}, \frac{1}{2}]$.

As stated prior, this result is in line with the Spectral Radius Conjecture (1.10). However, there is a marked difference between the description given here acting on the energy space and the properties determined in our study of the N.P. adjoint operator spectrum on $L^2(\partial D)$ and $C(\partial D)$, specifically, the determination of the spectrum on such spaces as closed curves in the complex plane.

3 Relevant operator theory

The core premise of this work being to expand upon specific areas in the theory of operators, specifically the spectral theory of the Neumann Poincaré Adjoint operator, the following section is provided to both discuss and define those most relevant elements of operator theory to the research being performed. Again, the results and proofs in this section are not my own, a number of which being drawn from [21] in addition to other works cited below.

As discussed in our introduction, the purpose of this work is to build on those existing results on the spectral properties of the Neumann Poincaré adjoint integral operator K^*. More precisely, we wish to examine the properties inherent to the operator,

$$K^* - \lambda I, \text{ for } \lambda \in \mathbb{C} \setminus \{0\}$$

specifically, those circumstances under which we have said operator is invertible. To this end, we begin our examination of the underlying theory by considering the work of Swedish mathematician Eric Ivar Fredholm (1866-1927), and specifically his work on solutions to a sub-type of integral equations. A *Fredholm Integral equation* is an integral equation of the form

$$\int_a^b k(x,y)f(y)dy - g(x) = \lambda f(x) \tag{3.1}$$

for fixed functions $k(x,y) \in L^2([a,b]^2)$ and g and fixed constant $\lambda \in \mathbb{C} \setminus \{0\}$. Rearranging, and setting the limits and kernel of the integral to those used in the N.P. adjoint operator, we see that our problem and that of the Fredholm Integral equation are the same, as we are trying to determine both the possible existence of, and the possible uniqueness of a solution f to the problem

$$(K^* - \lambda)(f) = g,$$

for any fixed g. Indeed, Fredholm was able to prove the following result, with regards to solutions to this particular brand of integral equation:

Theorem 3.1 (Fredholm Alternative)**.** *For any integral equation of the Fredholm type, such that the integral with kernel $k(x,y)$ represents a compact integral operator on some Banach space X of functions, either*

- *There exists a non-trivial solution f when $g = 0$, or*
- *There exists a unique solution f for given g.*

This can be expanded into a broader result on the space of compact operators $\mathcal{K}(X)$, where X is a Banach space ([4], Chap 3, pg 87, Thm 3.2.2),

Theorem 3.2. *Take $K \in \mathcal{K}(X)$ a compact operator on the Banach space X. The we have that either $K - \lambda I$ is invertible or ker$(K - \lambda I)$ is non-trivial for $\lambda \in \mathbb{C} \setminus \{0\}$.*

Furthermore, for such λ we have that

- *dim(ker$(K - \lambda I)) < \infty$,*
- *Ran$(K - \lambda I) \subset X$ is closed,*
- *Coker$(K - \lambda I) < \infty$,*
- *dim(ker$(K - \lambda I)) =$ dim(Coker$(K - \lambda I)$).*

The first two of these further statements are also known as *Riesz's First and Second Theorems* ([28], Chp 3, Thm3.1, Thm 3.2).

Given the properties of the N.P. adjoint discussed in our introduction, specifically regarding compactness, we can see that the Fredholm alternative holds when determining the existence and uniqueness of resolvents for $K - \lambda I$. In the context of our problem, either

$$K^* - \lambda I$$

is an invertible operator, or for some $f \neq 0$

$$(K^* - \lambda)f = 0.$$

This identification between Fredholm's work and our own is reason enough for a more in-depth analysis of his work, though with a strict focus on those elements most relevant to our own research. To this end, we consider the theory of those operators which carry the *Fredholm Property.*

A bounded linear operator F between two Banach spaces A and B is considered to be a *Fredholm Operator* if both Ran(F) is closed and

$$\dim(\ker(F)) < \infty$$
$$\dim(\mathrm{Coker}(F)) < \infty.$$

The *Index* of a Fredholm operator F is the finite number given by

$$\mathrm{ind}(F) := \dim(\ker(F)) - \dim(\mathrm{Coker}(F)).$$

We observe here that it is possible to show that the requirement that the $\mathrm{Ran}(F)$ be a closed set is in fact superfluous given the other two assumptions. Indeed, by [1] we have

Theorem 3.3. *An operator F between Banach space X and Y has closed range if and only if there exists closed subspace Z of Y such that the intersection of the range of F with Z is the set $\{0\}$ and the direct sum of the range and Z is closed.*

Proof. If we assume $\mathrm{Ran}(F)$ to be closed, then it is sufficient to take $Z = \{0\}$. In proving the converse, given the assumptions above we must have that for $W = \mathrm{Ran}(F) \oplus Z$ the quotient space W/Z is a Banach space when equipped with its quotient norm. Additionally, the linear mapping

$$\begin{aligned} J : \mathrm{Ran}(F) &\to W/Z \\ x &\longrightarrow x + Z \end{aligned}$$

can be shown to be a surjective isomorphism. Thus, by using the Open Mapping Theorem we can prove that J must have a continuous inverse J^{-1} and from the boundedness of continuous operators, we can immediately deduce the existence of $c > 0$ such that

$$c||x||_F \leq ||J(x)||_{W/Z}.$$

By the properties of the quotient norm, we have

$$||J(x)||_{W/Z} \leq ||x + Z||_{W/Z} \leq ||x||_{\mathrm{Ran}(F)}$$

Hence

$$c||x||_{\mathrm{Ran}(F)} \leq ||x + Z||_{W/Z} \leq ||x||_{\mathrm{Ran}(F)}$$

and so the norms $||x||_{\mathrm{Ran}(F)}$ and $||x||_{W/Z}$ are equivalent on $\mathrm{Ran}(F)$, and so $\mathrm{Ran}(F)$ is a closed subspace in Y. $\square$

The corollary below follows as a result of the above theorem, and the closedness of finite dimensional Banach spaces (a consequence of Riesz's Theorem),

Corollary 3.4. *Let F be a bounded operator between Banach spaces X and Y. Then if $Y/Ran(F)$ is finite dimensional, $Ran(F)$ must be closed.*

From which it is clear that the closed-image criterion is pre-empted by the remaining Fredholm criteria. Having firmly established the essentials of the properties of a Fredholm operator, we proceed to discuss an equivalent definition for such an operator, from which we can extract further properties of Fredholm operators, by means of *Atkinson's Theorem.*

Theorem 3.5 (Atkinson's Theorem)**.** *Let F be a linear operator between Banach spaces X and Y, then we have F is Fredholm if and only if there exist linear regularizer operators $G : Y \to X$ in tandem with compact operators $\mathcal{K}_1 \in \mathcal{K}(Y)$ and $\mathcal{K}_2 \in \mathcal{K}(X)$ such that*

$$FG = I + \mathcal{K}_1 \text{ and } GF = I + \mathcal{K}_2$$

Proof. In order to prove the first result, we must show that F being Fredholm implies the existance of operators which fit the above identities. If we take F to be Fredholm, then we must have $\ker(F)$ and $\mathrm{Ran}(F)$ are finite dimensional, implying the existence of closed subspaces Z_1 and Z_2 of X and Y respectively, such that $X = \mathrm{Ker}(F) \oplus Z_1$ and $Y = \mathrm{Ran}(F) \oplus Z_2$. As in [21] we then define the operators

$$\begin{aligned} F_0 : Z_1 &\to \mathrm{Ran}(F), & \qquad G_0 : Y &\to \mathrm{Ran}(F), \\ x &\to Fx, & y &\to y_0, \end{aligned}$$

Where $y_0 \in Y$ is the component of y that belongs to $\mathrm{Ran}(F)$.

By the Bounded Inverse Theorem, given Z_1 and $\mathrm{Ran}(F)$ are Banach spaces, we must have that the operator F_0 is invertible and by the closedness of Z_2 and $\mathrm{Ran}(F)$ we get that G_0 is bounded. Furthermore, by our definition of G_0, we must have $\mathrm{Ran}(I - S) = Z_2$, a finite dimensional subspace, hence the operator $(I - G_0)$ must be compact.

Applying the aforementioned invertibility of F_0 we observe that

$$\begin{aligned} FF_0^{-1}G_0y &= F_0F_0^{-1}G_0y \\ &= G_0y \qquad\qquad y \in Y. \\ &= y - (I - G_0)y \end{aligned}$$

As such, for $G = F_0^{-1}G_0$ and $\mathcal{K}_1 = G_0 - I$ we get the first of the desired identities,

$$FGy = (I + \mathcal{K}_1)y$$

We argue similarly to get $GF = I + \mathcal{K}_2$.

It remains to prove that in the opposite scenario we have

$$\dim(\ker(F)) - \dim(\mathrm{Coker}(F)) < \infty.$$

This can be proven by applying the expansion of the *Fredholm Alternative* to the space of compact operators.

Assuming the existence of $G \in \mathcal{L}(Y, X)$, with $\mathcal{K}_1 \in \mathcal{K}(Y)$ and $\mathcal{K}_2 \in \mathcal{K}(X)$ such that $FG = I + \mathcal{K}_1$ and $GF = I + \mathcal{K}_2$, then by Theorem 3.2 $\dim(\ker(I + \mathcal{K}_2))$ is finite and

$$\dim(\ker(F)) \leq \dim(\ker(GF)) = \dim(\ker(I + \mathcal{K}_2)) < \infty.$$

Now, we must also have

$$\operatorname{Ran}(I + \mathcal{K}_1) = \operatorname{Ran}(FG) \subset \operatorname{Ran}(F)$$

and again by Theorem 3.2 we have that $\operatorname{Coker}(I + \mathcal{K}_1)$ is finite dimensional.

This then implies that $\operatorname{Coker}(FG)$ is finite dimensional, and since $\operatorname{Ran}(FG) \subseteq \operatorname{Ran}(F)$ we then must have that $\operatorname{Coker}(F)$ is finite dimensional. Thus we must have F is a Fredholm operator. This completes the proof. $\square$

From this result we can determine the following properties of a Fredholm operator in terms of its regularizer.

Corollary 3.6. *The compact perturbation of any regularizer for a Fredholm operator, is also a regularizer of that operator.*

Proof. Let G be the regularizer of F, equipped with compact operators $\mathcal{K}_1$ and $\mathcal{K}_2$, then taking $\mathcal{K}_3$ to be another compact operator, then we also must have,

$$F(G + \mathcal{K}_3) = FG + F\mathcal{K}_3 = I + \mathcal{K}_1 + F\mathcal{K}_3 \text{ and}$$
$$(G + \mathcal{K}_3)F = GF + \mathcal{K}_3 F = I + \mathcal{K}_2 + \mathcal{K}_3 F.$$

Given the composition of compact and bounded operators in both directions is compact, and the sum of compact operators is also compact, we then also have that $G + \mathcal{K}_3$ meats the definition of a regularizer of F. $\square$

By similar argument we also have that

Corollary 3.7. *Any compact perturbation of a Fredholm operator, is also a Fredholm operator.*

Additionally, we have

Corollary 3.8. *For F a Fredholm linear operator on the Banach space X, the adjoint operator F^* is also Fredholm.*

Proof. Let G be the regularizer of F, equipped with compact operators $\mathcal{K}_1$ and $\mathcal{K}_2$.

$$\begin{aligned}
\langle G^*F^*x, x\rangle &= \langle F^*x, Gx\rangle \\
&= \langle x, FGx\rangle \\
&= \langle x, (I+\mathcal{K}_1)x\rangle \\
&= \langle x, x\rangle + \langle x, \mathcal{K}_1 x\rangle \\
&= \langle x, x\rangle + \langle \mathcal{K}_1^* x, x\rangle \\
&= \langle (I+\mathcal{K}_1^*)x, x\rangle.
\end{aligned}$$

By Schauder's theorem, $\mathcal{K}_1^*$ is a compact operator. Applying the same argument for F^*G^* we get an analogous result for $\mathcal{K}_2^*$. Thus taking G^* to be the regularizer of F, we have that F meets the criteria of Atkinson's theorem. □

Corollary 3.9. *For X, Y and Z Banach spaces and $F : X \to Y$ and $H : Y \to Z$ Fredholm operators, the composite $HF : X \to Z$ is also a Fredholm operator.*

Proof. Given F and H are Fredholm, Let G_1 and G_2 be respective regularizers, with compact operators, $\mathcal{K}_{1F}$, $\mathcal{K}_{2F}$, $\mathcal{K}_{1H}$ and $\mathcal{K}_{2H}$, again respective of F and H. Then,

$$\begin{aligned}
(HF)(G_1G_2) &= H(I + K_{1F})G_2 \\
&= (H + HK_{1F})G_2 \\
&= I + K_{1H} + HK_{1F}G_2
\end{aligned}$$

Again, the sum of compact operators is compact and the composition of compact and bounded operators is compact, giving us that $K_{1HF} = K_{1H} + HK_{1F}G_2$ is a compact operator such that

$$(HF)(G_1G_2) = I + K_{1HF}$$

and similarly we have $K_{2HF} = K_{2F} + G_1K_{2H}F$ a compact operator such that

$$(G_1G_2)(HF) = I + K_{2HF}.$$

Thus G_1G_2 is a regularizer of HF, and, by Atkinson's theorem HF must be a Fredholm operator. □

Given our focus on the spectrum of an operator which can be used to represent a Fredholm integral equation, particularly the invertibility of $K - \lambda I$, determining a clear criterion for invertibility (or lack thereof) is essential. Indeed, by restricting Atkinson's Theorem to acting on a single Banach Space X, we get the explicit result,

Corollary 3.10. *F is a Fredholm linear operator on the Banach space X if and only if it can be mapped to an invertible element of the space $\mathcal{L}(X) \setminus \mathcal{K}(X)$ also known as the Calkin Algebra.*

More generally speaking, such a Fredholm operator on X may be considered 'nearly invertible' or 'invertible modulo $\mathcal{K}(X)$'.

Indeed, by modifying our notion of invertibility to coincide with that exercised in the Calkin algebra, we can similarly reconsider our study of the N.P. adjoint spectrum, restricting our analysis to those spectral elements for which we have this form of invertibility. Additionally, the above result also enables us to apply a number of more general Banach-space related results.

To this end, we offer the following definitions.

Definition 3.11. For an operator F on the Banach space A, the *essential norm* of F is given by

$$\|F\|_{ess} := \inf\{||F - G|| : G \in \mathcal{K}(A)\},$$

or, equivalently, the distance to the space of compact operators $\mathcal{K}(A)$.

The essential norm of an operator defines the norm of the Calkin algebra of that operator.

Definition 3.12. The *essential spectrum* of the Operator $F \in \mathcal{L}(X)$ for Banach space X is given by

$$\sigma_{ess} := \{\lambda \in \mathbb{C} : F - \lambda I \text{ is not a Fredholm operator}\}.$$

Each element of the essential spectrum of an operator belongs to its respective Calkin algebra.

Definition 3.13. A Banach algebra is a non-zero banach space $\mathcal{A}$ equipped with a product

$$\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A},$$
$$(x, y) \to x \cdot y$$

where $\|xy\| \leq \|x\|\|y\|$ and $\cdot$ is associative and linear in both arguments.

We observe that, for a non-zero Banach space X the space of linear operators $\mathcal{L}(X)$ is a Banach algebra, with product defined as the composition of operators.

We can extend the concept of the spectrum from spaces of operators to Banach algebras. In particular for $a \in \mathcal{A}$ and $e \in \mathcal{A}$ the identity with respect to the product on $\mathcal{A}$, then we can define the spectrum of a to be

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda e \text{ is not invertible in } \mathcal{A}\}.$$

The proof of the following useful result can be found in [41]

Theorem 3.14 (Gelfand's Formula)**.** *Given a Banach algebra $\mathcal{A}$, then for $a \in \mathcal{A}$, we have the following:*

$$\|\sigma(a)\| := \lim_{k\to\infty} ||a^k||^{\frac{1}{k}}$$

.

Given the fact that the Calkin algebra is also a Banach algebra and that on the Calkin algebra we have that the Fredholm property gives invertibility, we may apply Gelfand's theorem to the essential spectrum, and thus determine a bound on the magnitude of its values. Specifically,

Definition 3.15. The essential spectral radius of the operator $F \in \mathcal{L}(X)$, for X a Banach space, is the spectral radius of the essential spectrum of F

$$\|\sigma_{ess}(F)\| := \sup\{|\lambda| : \lambda \in \sigma_{ess}(F)\} = \lim_{n\to\infty} \|F^n\|_{ess}^{1/n}.$$

The *winding number*, $\omega(x,\gamma)$, of a closed path γ in a plane about a point x is the integer given by the total number of times γ travels anticlockwise about x. Note that if γ is oriented clockwise we have $\omega(x,\gamma) \le 0$, else $\omega(x,\gamma) \ge 0$. In the complex plane, for a closed path γ, we have that the winding number about the origin is given by

$$\int_\gamma \frac{1}{z} dz.$$

Definition 3.16. Given an operator F on the Banach space X, the *point spectrum* of F is the subset $\sigma_p(F) \in \mathbb{C}$ composed of eigenvalues of F.

Definition 3.17. Given an operator F on the Banach space X, the *approximate point spectrum* of F is the subset of $\sigma_{ap}(F) \subset \mathbb{C}$ such that there exists a sequence $(x_n)_{n\in\mathbb{N}}$, where $\|x_n\| = 1$ and $\|Fx_n - \lambda x_n\| \to 0$.

Applying the Bounded Inverse Theorem to the operator $F - \lambda I : X \to \ker(F - \lambda I)$, we can show that any $\lambda \in \sigma_{ap}(F)$ is either an eigenvalue, or the range of the operator $F - \lambda I$ is also closed. Indeed, we have the following is an equivalent definition of the *approximate point spectrum*:

$$\sigma_{ap}(F) := \{\lambda \in \mathbb{C} : \mathrm{Ker}(F - \lambda I) \neq \{0\} \text{ or } \mathrm{Ran}(F - \lambda I) \text{ is not closed}\}.$$

The following extemely useful result and its proof are drawn from ([21], Thm 3.13).

Theorem 3.18 (Index Formula)**.** *Let $F \in \mathcal{L}(X, Y)$ and $G \in \mathcal{L}(Y, Z)$ be Fredholm operators, where X, Y and Z are Banach spaces, then we have*

$$ind(GF) = \ ind(G) + \ ind(F).$$

Proof. Considering the following sequence of operators,

$$\begin{aligned} 0 : \{0\} &\to \ker(F) \text{a 0 operator} \\ x &\to 0, \\ i : \ker(F) &\to \ker(GF) \text{ the inclusion operator} \\ x &\to x, \\ F : \ker(GF) &\to \ker(G) \\ x &\to F(x), \\ \pi : \ker(G) &\to \mathrm{Coker}(F) \\ y &\to y + \mathrm{Ran}(F), \end{aligned}$$

$$\begin{aligned} \overline{G} : Y/\mathrm{Ran}(F) &\to \mathrm{Coker}(GF) \\ y + \mathrm{Ran}(F) &\to Gy + \mathrm{Ran}(GF), \\ \bar{i} : Z/\mathrm{Ran}(GF) &\to \mathrm{Coker}(G) \\ z + \mathrm{Ran}(GF) &\to z + \mathrm{Ran}(G), \\ 0 : Z/\mathrm{Ran}(G) &\to \{0\} \text{ a 0 operator} \\ x &\to 0. \end{aligned}$$

We observe that in this sequence, the image of each operator is the kernel of the operator following it, and furthermore each domain and target space in the above sequence must be finite dimensional, as a consequence of F and G being Fredholm. This allows us to apply the Rank-Nullity theorem ([9] Thm 7.4) to each operator, which enables us to use the following chain of reasoning:

$$\begin{aligned} 0 =& \dim(\mathrm{Ran}(0)) \\ =& \dim(\ker(i)) \end{aligned}$$

Which is immediate from the previously observed equality between the image of each operator and the kernel of the operator following it. We then have

$$\begin{aligned}0 =&\dim(\ker(F)) - \dim(\mathrm{Ran}(i))\\ =&\dim(\ker(F)) - \dim(\ker(F)),\end{aligned}$$

which follows by first applying Rank-Nullity Theorem to split $\dim(\ker(i))$ since i has domain $\ker(F)$ and then uses the aforementioned property of this sequence of operators to get $\mathrm{Ran}(i) = \ker(F)$. We split $\dim(\ker(F))$ and, given F in this sequence of operators has domain $\ker(GF)$ and target space $\ker(G)$, we get,

$$\begin{aligned}0 =&\dim(\ker(F)) - \dim(\ker(GF)) + \dim(\mathrm{Ran}(F) \cap \ker(G))\\ =&\dim(\ker(F)) - \dim(\ker(GF)) + \dim(\ker(\pi))\end{aligned}$$

again using the property of our operators to get that $\mathrm{Ran}(F) = \ker(\pi)$. Continuing in this manner we split and rewrite $\dim(\ker(\pi))$, using the fact that π has domain $\ker(G)$ and the equality $\mathrm{Ran}(\pi) = \ker(\overline{G})$ which gives us

$$\begin{aligned}0 =&\dim(\ker(F)) - \dim(\ker(GF)) + \dim(\ker(G)) - \dim(\mathrm{Ran}(\pi))\\ =&\dim(\ker(F)) - \dim(\ker(GF)) + \dim(\ker(G)) - \dim(\ker(\overline{G})),\end{aligned}$$

and as we carry on splitting and rewriting terms in this fashion we get

$$\begin{aligned}0 =&\dim(\ker(F)) - \dim(\ker(GF)) + \dim(\ker(G)) - \dim(\mathrm{Coker}(F))\\ &+ \dim(\mathrm{Ran}(\overline{B}))\\ =&\dim(\ker(F)) - \dim(\ker(GF)) + \dim(\ker(G)) - \dim(\mathrm{Coker}(F))\\ &+ \dim(\ker(\bar{i}))\\ =&\dim(\ker(F)) - \dim(\ker(GF)) + \dim(\ker(G)) - \dim(\mathrm{Coker}(F))\\ &+ \dim(\mathrm{Coker}(GF)) - \dim(\mathrm{Ran}(i))\\ =&\dim(\ker(F)) - \dim(\ker(GF)) + \dim(\ker(G)) - \dim(\mathrm{Coker}(F))\\ &+ \dim(\mathrm{Coker}(GF)) - \dim(\mathrm{Coker}(G))\end{aligned}$$

which, by rearrangement, gives us

$$\begin{aligned}
0 =&\dim(\ker(F)) - \dim(\mathrm{Coker}(F)) + \dim(\ker(G)) - \dim(\mathrm{Coker}(G)) \\
&+ \dim(\mathrm{Coker}(GF)) - \dim(\ker(GF)) \\
=&\mathrm{ind}(F) + \mathrm{ind}(G) - \mathrm{ind}(GF).
\end{aligned}$$

□

Coburn's theorem, given below, is a useful demonstration of how our previously stated results, as well as a number of common results from functional analysis can be applied in studying Fredholm and essential spectral properties of an operator. The proof given here follows very closely that given in [21], Thm 3.15.

Theorem 3.19 (Coburn's theorem)**.** *Given a rational function $a : \mathbb{T} \to \mathbb{C}$, where $\mathbb{T}$ is the unit circle, then for $p > 1$, the corresponding Toeplitz operator*

$$\begin{aligned}
T_a : \ell^p(\mathbb{N}) &\to \ell^p(\mathbb{N}) \\
(T_a x)_j &= \sum_{k=1}^{\infty} a_{j-k} x_k, (j \in \mathbb{N})
\end{aligned}$$

is Fredholm if and only if $0 \notin a(\mathbb{T})$. Furthermore, we would have

$$\begin{aligned}
ind(T_a) &= -\omega(a; 0) \\
\sigma_{ess}(T_a) &= a(\mathbb{T}).
\end{aligned}$$

Proof. We consider the sequence $(a_n)_{n\in\mathbb{N}} \in c_{00}(\mathbb{Z})$, where

$$c_{00}(\mathbb{N}) := \{x \in \ell^\infty(\mathbb{N}) : \exists n_0 \in \mathbb{N}, \forall n \geq n_0 : x_n = 0\}$$

is taken to be the sequence which defines a, that is

$$a(x) := \sum_{n=\infty}^{\infty} a_n x^n.$$

We must have a is rational (the finiteness of this series a consequence of our definition of $(a_n)_{n\in\mathbb{N}}$). Taking the *backward shift* function to be

$$S : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$$

$$(Sx)_j := x_{j+1}, (j \in \mathbb{N})$$

and the forward shift function

$$\begin{aligned} T : \ell^p(\mathbb{N}) &\to \ell^p(\mathbb{N}) \\ (Tx)_j &:= x_{j-1}, (j \in \mathbb{N}) \\ (Tx)_j &:= 0, (j = 1). \end{aligned}$$

We can re-write the Toeplitz operator T_a, as a combination of backwards and forward shifts. Take $l, m \in \mathbb{Z}$ such that $a_{-l} \neq 0$, $a_m \neq 0$ and $a_n = 0$ for $n < -l$ and $n > m$.

$$\begin{aligned} \sum_{k=1}^{\infty} a_{j-k} x_k &= \sum_{k=1}^{j+l} a_{j-k} x_k \\ &= \sum_{k=-l}^{j-1} a_k x_{j-k} \\ &= \sum_{k=1}^{l} a_{-k} (S^k x)_j + a_0 x_j + \sum_{k=1}^{m} a_k (T^k x)_j, \end{aligned}$$

We prove below that $\sigma(S) = \sigma(T) = \overline{\mathbb{D}}$ where $\mathbb{D}$ is the closed unit disk, and that $\sigma_{ess}(S) = \sigma_{ess}(T) = \mathbb{T}$.

Observing that for $x = (x_1, x_2, x_3, ...), y = (y_1, y_2, y_3, ...)$

$$\langle Tx, y \rangle = \langle (0, x_1, x_2, x_3, ...), (y_1, y_2, y_3, ...) \rangle = \langle (x_1, x_2, x_3, ...), (y_2, y_3, y_4, ...) \rangle = \langle x, Sy \rangle,$$

we clearly have $S = T^*$. We also must have that 0 must be an eigenvalue of S. We can justify this as a result of the valid eigenvector $(k, 0, 0, 0, ...)$ for $k \in \mathbb{N}$. To determine the remaining point spectrum of S we consider the solutions of

$$\begin{aligned} ((S - \lambda I)x)_j &= x_{j+1} - \lambda x_j \text{ for } j \in \mathbb{N} \\ &= 0. \end{aligned}$$

Given this equation, we cannot have that $x_1 = 0$ as this would imply by iteration that all other coordinates are 0, and an eigenvector must be non-zero. Normalizing x such that $x_1 = 1$, the above equation then implies that $x_{j+1} = \lambda^j$. In order for x to be an element of $\ell^p(\mathbb{N})$ we must have

$$\|x\|^p = \sum_{j=0}^{\infty} |\lambda|^{np}$$

which we have converges if and only if $|\lambda| < 1$, implying that $\sigma_p(S) = \text{Int}(\mathbb{D})$. Additionally, we can determine $\|\sigma(S)\| = \|S\| = 1$. Given the closure of the spectrum and the spectral radius, we must have $\sigma(S) = \mathbb{D}$. We also note here that $\sigma(T) = \sigma(S^*) = \overline{\sigma(S)} = \mathbb{D}$ by the symmetry of $\mathbb{D} \subset \mathbb{C}$.

As we have shown above, for $|\lambda| < 1$, $(\text{Ker}(S - \lambda I)) = (1, \lambda, \lambda^2, ...)$ and so $\dim(\text{Ker}(S - \lambda I)) = 1$. Furthermore $\dim(\ell^p/(S - \lambda I)) = \dim(\text{Ker}(S^* - \lambda I)) = \dim(\text{Ker}(T - \lambda I))$. Additionally, for $|\lambda| \leq 1$,

$$(Tx - \lambda x)_j = x_{j-1} - \lambda x_j$$

Where x_0 =0. Thus if $(Tx - \lambda x) = 0$, then we must have $x_1 = 0$ implying, by iteration that $x_j = 0$ for all $j \in \mathbb{N}$. That is, $\text{Ker}(T - \lambda I) = \{0\}$ and so we must have $\dim(\text{Ker}(T - \lambda I)) = 0$. So for $\|\lambda\| < 1$, $S - \lambda I$ and $T - \lambda I$ must be Fredholm with indexes 1 and -1 respectively. Furthermore, we have for $|\lambda| < 1$

$$\|(T - \lambda)x\| \geq |\|Tx\| - \|\lambda x\|| = (\|T\| - |\lambda|)\|x\|,$$

and so $\text{Ran}(T - \lambda I)$ is closed. Thus we must have $\sigma_{ap}(T) \subset \mathbb{T}$. Taking $|\lambda| = 1$, we define the sequence, $(x^n)_{n \in \mathbb{N}} \subset X$, by

$$x^n := \frac{1}{n^{1/p}}(-\frac{1}{\lambda}, ..., -\frac{1}{\lambda^n}, 0, 0, ...).$$

giving us, firstly that

$$\begin{aligned}
\|x^n\|_{\ell^p} &= \frac{1}{n^{1/p}}(|\frac{1}{\lambda^p}| + ... + |\frac{1}{\lambda^n p}|)^{1/p} \\
&= \frac{1}{n^{1/p}} n^{1/p} \\
&= 1
\end{aligned}$$

and secondly

$$\begin{aligned}
\|(T - \lambda)x^n\| &= \frac{1}{n^{1/p}} \|(1, 0, ..., 0, -\frac{1}{\lambda^n}, 0, 0, ...)\| \\
&= \frac{2^{1/p}}{n^{1/p}}.
\end{aligned}$$

So $\|(T - \lambda)x^n\|$ tends to 0 as n tends to ∞. Thus we must also have $\mathbb{T} \subset \sigma_{ap}(T)$ and so $\sigma_{ap}(T) = \mathbb{T}$.

For $|\lambda| = 1$ we have $(T - \lambda I)x = 0$ implies $x = 0$ so $\mathrm{Ran}(T - \lambda I)$ is not closed, and hence $T - \lambda I$ cannot be Fredholm. Thus we must have $\sigma_{ess}(S) = \sigma_{ess}(T) = \mathbb{T}$.

It is possible to show that S is a regularizer of T, since $ST = I$ and $TS = (I - K)$ where

$$K : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N}),$$
$$(x_1, x_2, x_3, ...) \to (x_1, 0, 0, 0, ...).$$

Thus, T is Fredholm, and invertible in the Calkin algebra, from which we can deduce via the spectral mapping theorem of polynomials (which we have is valid for Banach algebras, see [5],Thm 19.9), that

$$\sigma_{ess}(T_a) = a(\mathbb{T}).$$

Given T_a is Fredholm, and using the property that $S^l T^l = S^{l-k} T^{l-k} = I$, we may write

$$\begin{aligned}\sum_{k=1}^{l} a_{-k} S^k + a_0 I + \sum_{k=1}^{m} a_k (T^k &= S^l \left(\sum_{k=1}^{l} a_{-k} T^{l-k} + a_0 T^l + \sum_{k=1}^{m} a_k T^{k+l} \right) \\ &= S^l \sum_{k=0}^{m+l} a_{k-l} T^k\end{aligned}$$

We may expand the polynomial $\sum_{k=0}^{m+l} a_{k-l} x^k = x^l a(x)$ in terms of its roots, λ_k,

$$\sum_{k=0}^{m+l} a_{k-l} x^k = a_m \prod_{k=0}^{m+l} (x - \lambda_k)$$

and so

$$T_a = a_m S^l \prod_{k=0}^{m+l} (T - \lambda_k).$$

Given S and T are both Fredholm, we can apply the Index Formula and separate out the index of T_a thus,

$$\mathrm{ind}(T_a) = l\mathrm{ind}(S) + \sum_{k=0}^{m+l} \mathrm{ind}(T - \lambda_k).$$

As we saw above in our analyisis of S and T $\mathrm{ind}(S) = 1$ and $\mathrm{ind}(T - \lambda_k I) = -1$ for

$\lambda_k \in \mathbb{D}$ or $\mathrm{ind}(T - \lambda_k I) = 0$ for $\lambda_k \in \mathbb{C} \setminus \overline{\mathbb{D}}$. Thus

$$\mathrm{ind}(T_a) = l - \text{No.of roots } \lambda_k \in \mathbb{D},$$

however within the unit circle the number of roots $\lambda_k \in \mathbb{D}$ must equal the winding number $\omega(\sum_{k=0}^{m+l} a_{k-l}x^k; 0)$ and given the winding number of a product of paths about the origin number splits into the sum of the winding numbers of said paths, we get, since $a(x) = x^l \sum_{k=0}^{m+l} a_{k-l}x^k$ that $\mathrm{ind}(T_a) = -\omega(a(x); 0)$. □

4 The Mellin transform and its properties

The purpose of this chapter is to provide the reader with a sufficient understanding of the relevant theory of of the Mellin transform, in particular, with regards to those properties upon which we go on to build our own results. As described in our introductory chapter, the Mellin transform is essential in determining a suitable unitarily equivalent operator matrix, in particular with regards to the properties of it"s convolution.

Additionally, we will here expound on the results relevant to Mellin transformed operators, which will enable us to compare the spectral theory of our operator matrix on the case of infinitely large pairs ofwedges with their localizations.

We begin with the following working decription of the Mellin transform and its core properties. The *Mellin transform* $\mathcal{M} : L^2(\mathbb{R}_+, \frac{ds}{s}) \longrightarrow L^2(\mathbb{R})$ of a function f is a unitary integral transform given by

$$\mathcal{M}f(\xi) = \int_{\mathbb{R}_+} s^{i\xi} f(s) \frac{ds}{s}$$

with *convolution* given by

$$j(t) \star f(t) := \int_{\mathbb{R}_+} j\left(\frac{t}{s}\right) f(s) \frac{ds}{s}.$$

In addition, the Mellin transform has the property that, for $f \in L^1(\frac{dt}{t})$ and $g \in L^2(\frac{dt}{t})$

$$\mathcal{M}(f \star g) = \mathcal{M}f\mathcal{M}g.$$

From the above property, it should be apparent that, if it is possible to characterize an integral operator with kernel k as a Mellin convolution, then, under the Mellin transform we then must have that their exists a unitarily equivalent multiplication operator, this form in particular being advantageous in the determining of our operators spectrum.

Indeed this convolution property is visibly comparable to that of the Fourier transform, and indeed it can be shown how the two results are equivalent.

It can be demonstrated that we may re-write the Mellin transform, from its definition in terms of substitutions in other existing integral operators.

In particular, this is the case for the Fourier transform, that is, for $f \in L^1(\mathbb{R})$

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx.$$

Explicitly we have, by subbing in $e^x = s$

$$\begin{aligned}
\int_{\mathbb{R}} f(x) e^{-2\pi\xi(ix)} dx &= \int_{\mathbb{R}_+} f(\ln(s)) e^{-\ln(s) 2\pi\xi i} \frac{ds}{s}, \\
&= \int_{\mathbb{R}_+} f(\ln(s)) e^{\ln(s^{-2\pi\xi i})} \frac{ds}{s} \\
&= \int_{\mathbb{R}_+} f(\ln(s)) s^{-2\pi\xi i} \frac{ds}{s} \\
&= \int_{\mathbb{R}_+} f(\ln(s)) s^{\theta i} \frac{ds}{s} = \mathcal{M}g(\theta).
\end{aligned}$$

for $g(s) = f(\ln(s))$ and $\theta = -2\pi\xi$.

As a result of this relationship, we can draw the conclusion that those results which hold on the Fourier transform may be modified to apply to the Mellin transform. For example, we will make use of the following result.

Lemma 4.1 (Riemann-Lebesgue)**.** *For $f \in L^1(\mathbb{R})$ we have that its Fourier transform $\hat{f}(x)$ tends to* 0 *as $|x|$ tends to ∞.*

Proof. Given a function $f \in L^1(\mathbb{R})$, the density of the simple functions in $L^1(\mathbb{R})$, that is, functions of the form

$$g(x) = \sum_{n=1}^{N} c_n \mathcal{X}_{(a_n,b_n)}(x),$$

where $c_n \in \mathbb{R}$ and $\mathcal{X}_{(a_n,b_n)}$ is the characteristic function on the interval (a_n, b_n), gives us that there exists such a simple function g which is arbitrarily close to f in the L^1 norm.

We also observe that

$$\int_{\mathbb{R}} \mathcal{X}_{(a,b)} e^{2\pi i\xi x} dx = \int_a^b e^{2\pi i\xi x} dx$$
$$= \frac{e^{2\pi i\xi b} - e^{2\pi i\xi a}}{2\pi i\xi}$$

that is, we have an identity for the Fourier transform of the characteristic function $\mathcal{X}_{(a,b)}$, which we observe tends to 0 as $|\xi|$ tends to infinity. We can apply this accross all simple functions, giving us the same limit as we take n sufficiently large. Thus we can choose n sufficiently large that, for all $\epsilon > 0$,

$$\begin{aligned}\left|\int_{\mathbb{R}} f(x)e^{2\pi i\xi x}dx\right| &= \left|\int_{\mathbb{R}} (f(x) - g(x))e^{2\pi i\xi x}dx + \int_{\mathbb{R}} g(x)e^{inx}dx\right| \\ &\leq \left|\int_{\mathbb{R}} (f(x) - g(x))e^{2\pi i\xi x}dx\right| + \left|\int_{\mathbb{R}} g(x)e^{2\pi i\xi x}dx\right| \\ &\leq \int_{\mathbb{R}} |(f(x) - g(x))e^{2\pi i\xi x}|dx + \left|\int_{\mathbb{R}} g(x)e^{2\pi i\xi x}dx\right| \\ &\leq \int_{\mathbb{R}} |(f(x) - g(x))|dx + \left|\int_{\mathbb{R}} g(x)e^{2\pi i\xi x}dx\right| < 2\epsilon\end{aligned}$$

by Euler's formula and our previous inequalities. Taking ϵ arbitrarily small gives us the result. $\square$

By the previously described substitution, it then immediately follows from this result that for $f \in L^1(\mathbb{R}_+, \frac{ds}{s})$, as $|\xi|$ tends to ∞, we have $\mathcal{M}f(\xi)$ will tend to 0.

The following definitions are essential for the understanding of the result below, drawn from [33]. In particular this will provide the relevant backgound results that will enable us to characterize the spectrum of our operator matrix both in the initial case and its localizations, by unifying the two results in terms of their matrix determinant and potential Fredholm properties.

The paper being drawn on for these properties uses the following, more common definition of the Mellin transform,

$$\tilde{f}(z) = \int_{\mathbb{R}_+} s^z f(s) \frac{ds}{s}$$

for $z \in \mathbb{C}$. Note we are simply substituting $i\xi$ for z where, again, $\xi \in \mathbb{C}$ and so the two representations of the Mellin transform are equivalent. The following result [13] is fundamental in determining both the well-definedness of the Mellin transform and its inverse.

Theorem 4.2 (Mellin inversion formula)**.** *Take $f \in C^2((0,\infty))$ such that for some $c \in \mathbb{R}$ we have*

$$s^c f(s), s^{c+1} f'(s), s^{c+2} f''(s) \in L^1\left(\mathbb{R}_+, \frac{ds}{s}\right)$$

then the Mellin transform, as described above, exists for $\mathcal{R}(z) = c$, satisfying the growth estimate $\mathcal{M}f(c+it) = O((1+|t|)^{-2})$. Furthermore, for all $s \in (0,\infty)$ the inversion formula holds:

$$f(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s^{-z}\tilde{f}(z)dz$$

where the inverse integral transform takes place over the complex line

$$\Upsilon_c = \{z \in \mathbb{C} : \mathcal{R}(z) = c\}$$

Note that, given the real part of said line remains within an appropriate interval, the inversion formula is equally well defined when evaluated over any of the lines in $\mathbb{C}$ parallel to $\mathbb{R}$. By examination one can determine that for a suitable range of functions, this definition can be shifted so as to be understood in terms of our original definition of the Mellin transform.

With this in mind, we give the next few statements in terms of the latter definition, bearing in mind that we may then apply them to our own results, again, after said substitution of variables.

Definition 4.3. A Frechet space Θ is a topological vector space space wich is locally convex, possesses a *translation invariant* metric $d(\cdot,\cdot)$, i.e.

$$d(x,y) = d(x+z, y+z), \forall x, y, z \in \Theta,$$

which induces the aforementioned topology, and any such metric on Θ is also complete.

We observe here that we will be restricting the results and definitions given in [33] so as to ensure a clearer relevence to our own results.

Definition 4.4. The strip $\Upsilon_{0,1}$ is the subset of $\mathbb{C}$ given by

$$\Upsilon_{0,1} := \{z \in \mathbb{C} : \mathcal{R}(z) \in (0,1)\}.$$

Definition 4.5. The Frechet space $\Theta^m_{0,1}$ comprises those functions f which are holomorphic on the strip $\Upsilon_{0,1}$ such that for all l and every $[a,b] \subset (0,1)$

$$|f|_{l,[a,b]} = \sup_{\Upsilon_{a,b}} \left| (1+|z|)^{l-m} \frac{d^l}{dz^l} f(z) \right| < \infty.$$

Given $\mathcal{M}(k) \in \Theta^{-1}_{0,1}$, and the complex line

$$\Upsilon_\gamma := \{z \in \mathbb{C} : \mathcal{R}(z) = \gamma\} \quad (\gamma \in [0,1]),$$

we call the operator

$$Kf(t) := \int_0^1 k\left(\frac{t}{s}\right) f(s) \frac{ds}{s} = \frac{1}{2\pi i} \int_{\Upsilon_\gamma} t^{-z} \mathcal{M}(k)(z) \mathcal{M}(f)(z) dz$$

the *Hardy operator* with kernel $k(t)$.

It is apparent from this definition that a Hardy operator, as described above can be viewed as a localization of the Mellin convolution onto the interval $[0,1]$, or, equivalently the aforementioned Mellin inversion formula for $k(z)f(z)$.

Definition 4.6. Take $\mathcal{X}$ to be the space of operators A acting on $L^2([0,1])$ such that

$$A = cI + dH + \chi K_0 \chi + R\chi K_1 \chi R + \text{Comp}$$

where

- $c, d \in C([0,1])$;
- H is the finite Hilbert transform: $Hf(t) = p.v. \frac{1}{\pi} \int_0^1 \frac{1}{t-s} f(s) ds$;
- $\chi(t) \in \mathcal{C}_0^\infty([0,1/3))$, $\chi(t) = 1$ on $[0,1/4]$, a cut off function;
- K_0 and K_1 are Hardy operators with kernels k_0 and k_1 respectively, that is, $\mathcal{M}(k_0)$ and $\mathcal{M}(k_1)$ belong to $\Theta^{-1}_{0,1}$;
- For any $f \in L^2([0,1])$, $Rf(t) = f(1-t)$ is the reflection operator;
- Comp $\in K(L^2([0,1]))$, where $K(L^2([0,1]))$ is the class of compact operators on $L^2([0,1])$.

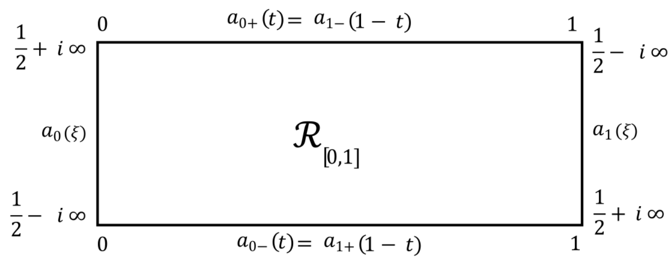
Furthermore, for any such operator A meeting the above definition, we may construct the following four functions from its components:

$$a_0(z) := c(0) + d(0)(-\cot \pi z) + \tilde{k}_0(z),$$

$$a_{0\pm}(t) := a_{1\mp}(1-t) = c(t) \pm id(t),$$
$$a_1(z) := c(1) + d(1)(\cot \pi z) + \tilde{k}_1(z),$$

using which, the definition below is formed.

Definition 4.7. Let $A \in \mathcal{X}$. The *principal symbol* of A as an operator on $L^2([0,1])$ denoted $\mathrm{Smbl}^{1/2}A$, is the quadruple of functions $a_0(\xi)$, $a_1(\xi)$ (on $(1/2 - i\infty, 1/2 + i\infty)$) and $a_{0-}(t) = a_{1+}(1-t)$, $a_{0+}(t) = a_{1-}(1-t)$ (on $(0,1)$), considered as a continuous function on the clockwise boundary $\mathcal{R}_{[0,1]}$ of the rectangle:



With these definitions in mind, we are finally given the result below. We note here that analogous to how the Fredholm Toeplitz operator has index given in terms of its winding number about the origin,in the below result we have that when our operator matrix A is Fredholm, its index is described in terms of the clockwise functions describing the boundary of $\mathcal{R}_{[0,1]}$. In this way, the result below might be considered a advanced version of Coburn's Theorem.

Theorem 4.8. *Let $A = (A_{i,j})_{i,j=1\ldots n}$ be an $n \times n$ matrix of operators in $\mathcal{X}$. Then A is Fredholm on $(L^2([0,1]))^n$ iff A is elliptic on $(L^2([0,1]))^n$, i.e., $Smbl^{1/2}A$ is nonsingular matrix on $\mathcal{R}_{[0,1]}$.*

If A is Fredholm on $(L^2([0,1]))^n$, its index is given by the change of argument in the determinant of $Smbl^{1/2}A$.

For the operator matrix A, note how the definition of $\mathrm{Symbl}^{1/2}A$ has been extended to matrices, entrywise.

We observe that for a given operator matrix with a suitable Mellin representation, or one that happens to be unitarily equivalent, this theorem gives us a direct connection between the determinant of such a matrix and the Fredholm property of its localizations, as well as any compact pertubations of such.

Indeed, by a careful selection of the component functions of A, we may use this result to determine the essential spectrum of our localised operator matrix.

5 The N.P. adjoint operator, and its spectrum for functions in L^2

We begin the following chapter by explicitly formulating an operator matrix based on the N.P. adjoint operator as it acts on those functions in the space $L^2(\partial\Omega)$ where the boundary of $\Omega \subset \mathbb{R}^2$ is formed of pairs of infinitely large 'wedges' in with coincident vertices. Applying the results on the Mellin transform discussed in the prior chapter, we generate a unitarily equivalent operator matrix with multiplication operator entries. This will greatly simplify the process of determining the spectrum of our operator matrix.

From there we will move to determine how we can localize any such results and compare these localizations to subsets of $\mathbb{R}^2$ bounded by bow-tie curves, before moving to determine the essential norm and further explicit spectral properties of this operator.

We find that the essential spectra of said localizations and these bow-tie curves coincide, and formulate the essential norm on each such set, before determining its value, particularly in the case where the angles determining our set are completely symmetric.

In the process we will show that on L^2 the essential norm exceeds expected values, particularly those found in similar studies, and so, based on these results we then go on to determine an exact formula for the spectrum of the operator, and examine the spectrum in specific scenarios, the outer limits of the spectrum in these cases appearing to match with this norm value.

We conclude this chapter determing the equality of the spectrum on infinite wedges and the essential spectrum on bow-tie curves, thereby allowing us to use the aforementioned results on the spectral radius.

5.1 Preliminaries and application of the Mellin transform

We begin by restating the form of the N.P. adjoint operator and outlining the initial parameterisation of the boundary for our 'wedges'. The adjoint of the Neumann-Poincaré operator on the boundary of some domain Ω is an integral operator given by,

$$K^*f(x) := \frac{2}{\omega_n}\int_{\delta\Omega}\frac{\langle x-y,\nu_x\rangle}{\pi|y-x|^n}f(y)d\sigma(y)$$

where x and y represent distinct points on $\partial\Omega$, ν_x is the outward oriented unit normal to $\partial\Omega$ at x and ω_n is the constant coefficient of the fundamental solution of the Laplacian in an n-dimensional space. Since we will be working in the case where $n = 2$, we then have $\omega_2 = 2\pi$.

When the Neumann-Poincaré adjoint operator is translated or rotated, the resultant operator is unitarily equivalent to the original. As such, we can consider the case where the boundary is composed of two wedges W_α, W_β of angles α and β, separated by some angle θ, the vertices of which coincide with the point $(0, 0)$ and where one of the edges of W_α rests on the positive x-axis, denoting the boundary formed by $\Gamma_{\alpha,\beta,\theta} = \partial W_\alpha \cup \partial W_\beta$.

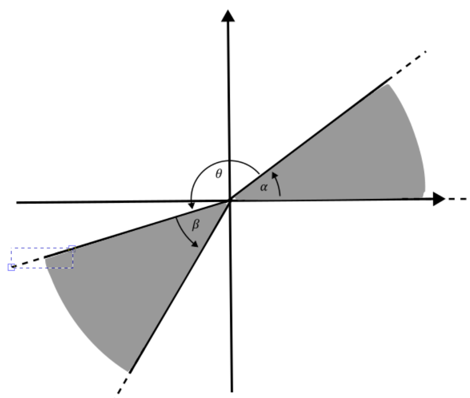


Figure 9: $\Gamma_{\alpha,\beta,\theta}$

Adopting the technique used in [36] we will parametrize each edge on $\mathbb{R}_+$ as,

$$\begin{aligned}
l_1(t) &= (t, 0),\\
l_2(t) &= (t\cos(\alpha), t\sin(\alpha)),\\
l_3(t) &= (t\cos(\alpha+\theta), t\sin(\alpha+\theta)),\\
l_4(t) &= (t\cos(\alpha+\theta+\beta), t\sin(\alpha+\theta+\beta))
\end{aligned}$$

and, using this parametrization, describe the Neumann-Poincaré adjoint operator on the weighted space $L^{2,a}(\Gamma_{\alpha,\beta,\theta}) = \bigoplus_{i=1}^{4} L^2(\mathbb{R}_+, t^a dt)$, $(a \in \mathbb{R})$, using the matrix,

$$K^a_{\Gamma_{\alpha,\beta,\theta}} = \begin{bmatrix} K^a_{1,1} & K^a_{1,2} & K^a_{1,3} & K^a_{1,4} \\ K^a_{2,1} & K^a_{2,2} & K^a_{2,3} & K^a_{2,4} \\ K^a_{3,1} & K^a_{3,2} & K^a_{3,3} & K^a_{3,4} \\ K^a_{4,1} & K^a_{4,2} & K^a_{4,3} & K^a_{4,4} \end{bmatrix}.$$

Here we have that each matrix component $K^a_{i,j}$ is an integral operator on $L^2(\mathbb{R}_+, t^a dt)$ determined by the kernel,

$$\widetilde{k_{i,j}} = \frac{\langle l_i - l_j, \nu_{l_i}\rangle}{\pi |l_j - l_i|^2},$$

of the N-P adjoint operator acting on the wedge formed by a pair of edges, where l_i and l_j represent arbitrary points on their respective parameterized edges and ν_{l_i} represents the unit normal to the edge l_i oriented to the exterior of $\Gamma_{\alpha,\beta,\theta}$.

We note that for each such a, whilst the domain each matrix entry acts on may differ, the kernel of each such integral operators remains the same. Our next step is to evaluate these matrix components.

Firstly, we have,

$$\begin{aligned}\widetilde{k_{i,i}} &= \frac{\langle l_i(x) - l_i(y), \nu_{l_i}\rangle}{\pi |l_i(y) - l_i(x)|^2} \\ &= 0\end{aligned}$$

as $l_i(x) - l_i(y)$ is orthogonal to ν_{l_i}, giving us,

$$K^a_{\Gamma_{\alpha,\beta,\theta}} = \begin{bmatrix} 0 & K^a_{12} & K^a_{13} & K^a_{14} \\ K^a_{21} & 0 & K^a_{23} & K^a_{24} \\ K^a_{31} & K^a_{32} & 0 & K^a_{34} \\ K^a_{41} & K^a_{42} & K^a_{43} & 0 \end{bmatrix}.$$

From the description of our graph, we observe that for any pair of edges (l_i, l_j), we only have that both the normal vectors oriented to the exterior of $\Gamma_{\alpha,\beta,\theta}$ are directed to the exterior of the wedge they form when considering the pairs (l_1, l_2), (l_1, l_4) and (l_3, l_4).

Indeed, for (l_1, l_3), we have that the normal vector on l_3 is oriented to the interior. Likewise for (l_2, l_4) and the normal vector on l_2, and in the case of the wedge formed by (l_2, l_3), the normal vectors on both edges are oriented inwards.

In evaluating our matrix entries, we will first consider the case of the wedge W_ω with normal vectors oriented to its exterior.

Using the parametrization of the two edges,

$$\begin{aligned} l_1(t) &= (t, 0) \\ l_2(t) &= (t\cos(\omega), t\sin(\omega)) \end{aligned}$$

on $\mathbb{R}_+$ where we have $\nu_{l_1} = (0, -1)$ and $\nu_{l_2} = (-\sin(\omega), \cos(\omega))$, evaluating $\widetilde{k_{1,2}}$ we get for $t, s \in \mathbb{R}_+$,

$$\begin{aligned}\widetilde{k_{1,2}} &= \frac{\langle (t,0)-(s\cos(\omega), s\sin(\omega)), (0,-1)\rangle}{\pi|(s\cos(\omega), s\sin(\omega))-(t,0)|^2} \\ &= \frac{s\sin(\omega)}{\pi(t^2+s^2-2st\cos(\omega))}\end{aligned}$$

and, by symmetry, we also get that

$$\begin{aligned}\widetilde{k_{2,1}} &= \frac{\langle (t\cos(\omega), t\sin(\omega))-(s,0), (-\sin(\omega), \cos(\omega))\rangle}{\pi|(s,0)-(t\cos(\omega), t\sin(\omega))|^2} \\ &= \frac{s\sin(\omega)}{\pi(t^2+s^2-2st\cos(\omega))}.\end{aligned}$$

Considering our previous description of $\Gamma_{\alpha,\beta,\theta}$, we can see the pairs of edges with suitably oriented normal are (l_1, l_2), (l_1, l_4) and (l_3, l_4) which gives us, for $s, t \in \mathbb{R}_+$

$$\begin{aligned}\widetilde{k_{1,2}} = \widetilde{k_{2,1}} &= \frac{s\sin(\alpha)}{\pi(t^2+s^2-2st\cos(\alpha))}, \\ \widetilde{k_{1,4}} = \widetilde{k_{4,1}} &= \frac{s\sin(\alpha+\theta+\beta)}{\pi(t^2+s^2-2st\cos(\alpha+\theta+\beta))}, \\ \widetilde{k_{3,4}} = \widetilde{k_{4,3}} &= \frac{s\sin(\omega)}{\pi(t^2+s^2-2st\cos(\beta))}.\end{aligned}$$

In the case of the remaining kernels, we must account for the orientation of the unit normal to our wedge with an appropriate change of sign. Since we are still able to use the invariance properties of the N.P. operator on each pair of edges with respect to rotation, we get that

$$\begin{aligned}\widetilde{k_{1,3}} &= \frac{\langle (t,0)-(s\cos(\alpha+\theta), s\sin(\alpha+\theta)), (0,-1)\rangle}{\pi|(s\cos(\alpha+\theta), s\sin(\alpha+\theta))-(t,0)|^2} \\ &= \frac{s\sin(\alpha+\theta)}{\pi(t^2+s^2-2st\cos(\alpha+\theta))} \\ &= -\frac{(-s\sin(\alpha+\theta))}{\pi(t^2+s^2-2st\cos(\alpha+\theta))} \\ &= -\frac{\langle (t\cos(\alpha+\theta), t\sin(\alpha+\theta))-(s,0), (\sin(\alpha+\theta), -\cos(\alpha+\theta))\rangle}{\pi|(s,0)-(t\cos(\alpha+\theta), t\sin(\alpha+\theta))|^2} \\ &= -\widetilde{k_{3,1}}, \\ \widetilde{k_{2,3}} &= \frac{\langle (t,0)-(s\cos(\theta), s\sin(\theta)), (0,1)\rangle}{\pi|(s\cos(\theta), s\sin(\theta))-(t,0)|^2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{-s\sin(\theta)}{\pi(t^2+s^2-2st\cos(\theta))}\\
&= \frac{\langle (t\cos(\theta), t\sin(\theta)) - (s,0), (\sin(\theta), -\cos(\theta))\rangle}{\pi|(s,0)-(t\cos(\theta),t\sin(\theta))|^2}\\
&= \widetilde{k_{3,2}},
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{k_{2,4}} &= \frac{\langle (t,0) - (s\cos(\theta+\beta), s\sin(\theta+\beta)), (0,1)\rangle}{\pi|(s\cos(\theta+\beta), s\sin(\theta+\beta)) - (t,0)|^2}\\
&= \frac{(-s\sin(\theta+\beta))}{\pi(t^2+s^2-2st\cos(\theta+\beta))}\\
&= -\frac{s\sin(\theta+\beta)}{\pi(t^2+s^2-2st\cos(\theta+\beta))}\\
&= -\frac{\langle (t\cos(\theta+\beta), t\sin(\theta+\beta)) - (s,0), (-\sin(\theta+\beta), \cos(\theta+\beta))\rangle}{\pi|(s,0)-(t\cos(\theta+\beta), t\sin(\theta+\beta))|^2}\\
&= -\widetilde{k_{4,2}}.
\end{aligned}
$$

So the full N.P. adjoint operator matrix is

$$
K^a_{\Gamma_{\alpha,\beta,\theta}} = \begin{bmatrix} 0 & K^a_\alpha & K^a_{\alpha+\theta} & K^a_{\alpha+\theta+\beta} \\ K^a_\alpha & 0 & -K^a_\theta & -K^a_{\theta+\beta} \\ -K^a_{\alpha+\theta} & -K^a_\theta & 0 & K^a_\beta \\ K^a_{\alpha+\theta+\beta} & K^a_{\theta+\beta} & K^a_\beta & 0 \end{bmatrix}
$$

where K^a_ω is the integral operator with kernel,

$$
\widetilde{k_\omega}(t,s) = \frac{s\sin(\omega)}{\pi(t^2+s^2-2st\cos(\omega))}.
$$

We have that

$$
\begin{aligned}
\widetilde{k_\omega}(\lambda t, \lambda s) &= \frac{\lambda s\sin(\omega)}{\pi((\lambda t)^2+(\lambda s)^2-2\lambda^2 st\cos(\omega))}\\
&= \frac{\lambda}{\lambda^2}\frac{s\sin(\omega)}{\pi(t^2+s^2-2st\cos(\omega))}\\
&= \frac{1}{\lambda}\frac{s\sin(\omega)}{\pi(t^2+s^2-2st\cos(\omega))}\\
&= \frac{1}{\lambda}\widetilde{k_\omega}(t,s),
\end{aligned}
$$

giving us

$$\widetilde{k_\omega}(t,s) = \frac{1}{s}\widetilde{k_\omega}\left(\frac{t}{s},1\right).$$

As such, we can rewrite the integral operator K_ω^a as

$$K_\omega^a f(t) = \int_{\mathbb{R}_+} \widetilde{k_\omega}\left(\frac{t}{s},1\right) f(s)\frac{ds}{s},$$

giving K_ω^a the form of a *Mellin convolution* on $\mathbb{R}_+$ with the Haar measure $\frac{ds}{s}$.

Our next step in determining the spectral properties of our operator matrix on $\Gamma_{\alpha,\beta,\theta}$ will be to find a unitarily equivalent operator that can also be described as a Mellin convolution, as applying the Mellin transform to such a convolution will result in a multiplication operator.

Again, following the approach of [36] we consider, for $\gamma \in \mathbb{R}$ the operator $V_\gamma f(t) = t^\gamma f(t)$. Indeed, since the N.P. adjoint operator acts on the weighted space $L^2(\Gamma_{\alpha,\beta,\theta}, t^a dt)$ we specifically consider the operator,

$$V_{\frac{a+1}{2}} : L^2(\mathbb{R}_+, t^a dt) \to L^2(\mathbb{R}_+, dt/t),$$

which we can prove to be unitary.

Lemma 5.1. *The operator* $V_{\frac{a+1}{2}} : L^2(\mathbb{R}_+, t^a dt) \to L^2(\mathbb{R}_+, dt/t)$ *is unitary*

Proof. The desired unitary property holds if and only if we have that

$$V_{\frac{a+1}{2}} : L^2(\mathbb{R}_+, t^a dt) \to L^2(\mathbb{R}_+, dt/t),$$

is both a surjective operator, and if the inner product is preserved under $V_{\frac{a+1}{2}}$. We begin by showing this surjective property. Taking $g \in L^2(\mathbb{R}_+, dt/t)$, we take some function f to be such that $f(x) := x^{(-a-1)/2} g(x)$. Hence,

$$\begin{aligned} V_{\frac{a+1}{2}} f(x) &= V_{\frac{a+1}{2}} x^{(-a-1)/2} g(x) \\ &= x^{(a+1)/2} x^{(-a-1)/2} g(x) \\ &= g(x). \end{aligned}$$

It then remains to show that $f \in L^2(\mathbb{R}_+, t^a dt)$, however

$$\int_{\mathbb{R}_+} |f(x)|^2 x^a dx = \int_{\mathbb{R}_+} |x^{(-a-1)/2} g(x)|^2 x^a dx$$

$$= \int_{\mathbb{R}_+} |x^{(-a-1)}||g(x)|^2 x^a dx$$
$$= \int_{\mathbb{R}_+} x^{-1}|g(x)|^2 dx \text{ given } x \in \mathbb{R}_+,$$
$$= \int_{\mathbb{R}_+} |g(x)|^2 \frac{dx}{x} < \infty.$$

Thus, we have surjectivity. For $f, g \in L^2(\mathbb{R}_+, t^a dt)$ we now consider the inner product acting on $V_{\frac{a+1}{2}} f(x)$ and $V_{\frac{a+1}{2}} g(x)$.

$$\begin{aligned}
\langle V_{\frac{a+1}{2}} f(x), V_{\frac{a+1}{2}} g(x) \rangle &= \int_{\mathbb{R}_+} V_{\frac{a+1}{2}} f(x) V_{\frac{a+1}{2}} g(x) \frac{dx}{x} \\
&= \int_{\mathbb{R}_+} x^{(a+1)/2} f(x) x^{(a+1)/2} g(x) \frac{dx}{x} \\
&= \int_{\mathbb{R}_+} x^{(a+1)} f(x) g(x) \frac{dx}{x} \\
&= \int_{\mathbb{R}_+} f(x) g(x) x^a dx \\
&= \langle f(x), g(x) \rangle.
\end{aligned}$$

Thus, the inner product is preserved. Hence the operator $V_{\frac{a+1}{2}}$, as defined above, is unitary. $\square$

Our integral operators, $K_\omega^a : L^2(\mathbb{R}_+, t^a dt) \to L^2(\mathbb{R}_+, t^a dt)$ are then unitarily equivalent to,

$$V_{\frac{a+1}{2}} K_\omega^a V_{-\frac{a+1}{2}} : L^2(\mathbb{R}_+, dt/t) \to L^2(\mathbb{R}_+, dt/t)$$
$$V_{\frac{a+1}{2}} K_\omega^a V_{-\frac{a+1}{2}} f(t) = \int_{\mathbb{R}_+} \left(\frac{t}{s}\right)^{\frac{a+1}{2}} \widetilde{k_\omega} \left(\frac{t}{s}, 1\right) f(s) \frac{ds}{s}.$$

This operator has the form of a Mellin convolution of f with the function $j_{\omega,a}(t) = t^{\frac{a+1}{2}} \widetilde{k_\omega}(t, 1)$. Examining the integral of $|j_{\omega,a}(t)|$, we see

$$\begin{aligned}
\int_{\mathbb{R}_+} |j_{\omega,a}(t)| \frac{dt}{t} &= \int_{\mathbb{R}_+} \left| t^{\frac{a+1}{2}} \widetilde{k_\omega}(t, 1) \right| \frac{dt}{t} \\
&= \int_{\mathbb{R}_+} \left| t^{\frac{a+1}{2}} \frac{\sin(\omega)}{\pi(t^2 + 1 - 2t\cos(\omega))} \right| \frac{dt}{t}
\end{aligned}$$

is asymptotically comparable to the integral

$$\int_1^\infty t^{-\frac{3-a}{2}} \frac{dt}{t} + \int_0^1 t^{\frac{a+1}{2}} \frac{dt}{t}$$

indicating that we have convergence only when $\frac{5-a}{2} > 1$ and $\frac{a-1}{2} > -1$, that is, taking $a \in (-1, 3)$ we have $j_{\omega,a} \in L^1(\frac{dt}{t})$. So for a in this range, using Young's inequality, which tells us for $k \in L^1(\frac{dt}{t})$ and $f \in L^2(\frac{dt}{t})$ that

$$||k \star f||_{L^2(\frac{dt}{t})} \leq ||k||_{L^1(\frac{dt}{t})} ||f||_{L^2(\frac{dt}{t})},$$

the operator given by the aforementioned Mellin convolution with $j_{\omega,a}(t)$ is bounded with respect to the norm in $L^2(\frac{dt}{t})$. Setting $j_{\omega,a} \in L^1(\frac{dt}{t})$ and $f \in L^2(\frac{dt}{t})$ we may take their Mellin convolution and apply the Mellin transform to them getting,

$$\mathcal{M}(j_{\omega,a} \star f) = \mathcal{M}(j_{\omega,a})\mathcal{M}(f),$$

a multiplication operator on $L^2(\mathbb{R})$. Note that as this transform is the result of composition of K^a_ω with unitary operators, this multiplication operator, which we shall denote $\widetilde{K}^a_\omega$, given by,

$$\begin{aligned} \widetilde{K}^a_\omega : L^2(\mathbb{R}) \to & L^2(\mathbb{R}) \\ f(\xi) \to & \mathcal{M}(j_{\omega,a}) f(\xi) \end{aligned}$$

is unitarily equivalent to K^a_ω.

Lemma 5.2. *Calculating the explicit form of the multiplication operator, we have that*

$$\mathcal{M}(j_{\omega,a}) = \frac{\sin(\left(\frac{a-1}{2} + i\xi\right)(\pi - \omega))}{\sin(\pi\left(\frac{a-1}{2} + i\xi\right))}.$$

Proof.

$$\mathcal{M}(j_{\omega,a}) = \frac{\sin\omega}{\pi} \int_{\mathbb{R}_+} \frac{s^{\frac{a-1}{2}+i\xi}}{s^2 - 2s\cos(\omega) + 1} ds.$$

Take $0 < r < R$ and set C_R and C_r to be the arcs in $\mathbb{C}$ defined by the circles of radius R and r respectively, centered at the origin. Set C_1 and C_2 to be the line segments in $\mathbb{C}$ that join the endpoints of C_R and C_r respectively, parameterized as $z = s \pm i\delta \in \mathbb{C}$ respectively for $s > 0$ and $\delta \in \mathbb{R}$ and where, if we fix the height $\delta < r$ then we take s

from the interval

$$\left[\sqrt{r^2-\delta^2}, \sqrt{R^2-\delta^2}\right).$$

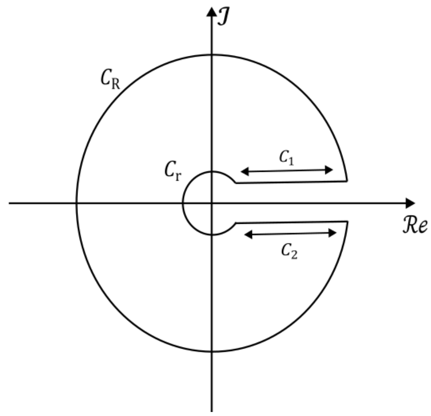


Figure 10: Integral Contour

For $s \in \mathbb{R}_+$, defining the integrand thus,

$$f(s) = \frac{s^{\frac{a-1}{2}+i\xi}}{s^2 - 2s\cos(\omega) + 1},$$

the residue theorem gives us:

$$\int_{C_R - C_r + C_1 - C_2} f(z)dz = 2\pi i \sum Res(f).$$

By applying the triangle and reverse triangle inequaity, we have

$$\left|\int_{C_R} \frac{z^{\frac{a-1}{2}+i\xi}}{z^2 - 2z\cos(\omega) + 1} dz\right| \leq \sup_{z \in C_R} \left|\frac{z^{\frac{a-1}{2}+i\xi}}{z^2 - 2z\cos(\omega) + 1}\right| 2\pi R$$
$$\left|\int_{C_r} \frac{z^{\frac{a-1}{2}+i\xi}}{z^2 - 2z\cos(\omega) + 1} dz\right| \leq \sup_{z \in C_r} \left|\frac{z^{\frac{a-1}{2}+i\xi}}{z^2 - 2z\cos(\omega) + 1}\right| 2\pi r$$

Analysing the supremum on C_R, we get

$$\sup_{z \in C_R} \left|\frac{z^{\frac{a-1}{2}+i\xi}}{z^2 - 2z\cos(\omega) + 1}\right| \leq \sup_{z \in C_R} \frac{|z^{\frac{a-1}{2}+i\xi}|}{|z^2 - 2z\cos(\omega) + 1|}$$
$$\sup_{z \in C_R} \leq \frac{|z^{\frac{a-1}{2}+i\xi}|}{||z|^2 - 2\cos(\omega)|z| - 1|}$$

$$
\begin{aligned}
&= \sup_{z\in C_R} \frac{|e^{(\frac{a-1}{2}+i\xi)\ln z}|}{||z|^2-2\cos(\omega)|z|-1|} \\
&= \sup_{z\in C_R} \frac{|e^{(\ln|z|+iarg(z))(\frac{a-1}{2}+i\xi)}|}{||z|^2-2\cos(\omega)|z|-1|} \\
&= \frac{e^{Re((\frac{a-1}{2}\ln R-\xi arg(z))+i(arg(z)\frac{a-1}{2}+i\xi\ln R))}}{|R^2-2\cos(\omega)R-1|} \\
&\leq \frac{R^{\frac{a-1}{2}}e^{2\pi|\xi|}}{|R^2-2\cos(\omega)R-1|},
\end{aligned}
$$

given ξ is fixed and $arg(z) \in [-2pi, 2pi]$.

By the same approach we find a similar upper bound on C_r. Taking the limit of the integral on C_R and using the fact that ξ is fixed and $arg(z) \in [-2pi, 2pi]$

$$
\begin{aligned}
\lim_{R\to\infty}\left|\int_{C_R}\frac{z^{\frac{a-1}{2}+i\xi}}{z^2-2z\cos(\omega)+1}dz\right| &\leq \lim_{R\to\infty} 2\pi R\frac{R^{\frac{a-1}{2}}e^{2\pi|\xi|}}{|R^2-2\cos(\omega)R-1|} \\
&= \lim_{R\to\infty} 2\pi\frac{R^{\frac{a+1}{2}}e^{2\pi|\xi|}}{|R^2-2\cos(\omega)R-1|} \\
&= 2\pi e^{2\pi|\xi|}\frac{\lim_{R\to\infty} R^{\frac{a+1}{2}-2}e^{2\pi|\xi|}}{\lim_{R\to\infty}|1-2\cos(\omega)R^{-1}-R^{-2}|}.
\end{aligned}
$$

As $a \in (-1, 3)$ we must have $\frac{a+1}{2} - 2 < 0$, and hence

$$
\lim_{R\to\infty}\left|\int_{C_R}\frac{z^{\frac{a-1}{2}+i\xi}}{z^2-2z\cos(\omega)+1}dz\right| \leq 2\pi\cdot e^{2\pi|\xi|}\cdot\frac{0}{1} = 0
$$

Similarly, for C_r, we have

$$
\begin{aligned}
\lim_{r\to 0}\left|\int_{C_r}\frac{z^{\frac{a-1}{2}+i\xi}}{z^2-2z\cos(\omega)+1}dz\right| &\leq \lim_{r\to 0} 2\pi r\frac{r^{\frac{a-1}{2}}e^{2\pi|\xi|}}{|r^2-2\cos(\omega)r-1|} \\
&= \lim_{r\to 0} 2\pi\frac{r^{\frac{a+1}{2}}e^{2\pi|\xi|}}{|r^2-2\cos(\omega)r-1|}
\end{aligned}
$$

Again, as $a \in (-1, 3)$, $\frac{a+1}{2} > 0$ so we have

$$
\lim_{r\to 0} 2\pi\frac{r^{\frac{a+1}{2}}e^{2\pi|\xi|}}{|r^2-2\cos(\omega)r-1|} = 0.
$$

Each point z on C_1 and C_2 also has circular representation given by $z = |z|e^{i\theta}$, for $\theta \in (0, 2\pi)$, with θ approaching 0 as C_1 approaches $\mathbb{R}_+$ and 2π as C_2 approaches $\mathbb{R}_+$ from the first and fourth quadrants respectively. With this in mind, and using our prior parameterization of C_1 and C_2

$$\lim_{\delta\to 0}\int_{\sqrt{r^2-\delta^2}}^{\sqrt{R^2-\delta^2}} f(s+i\delta)ds = \int_{[r,R]} f(s)ds$$
$$\lim_{\delta\to 0}\int_{\sqrt{r^2-\delta^2}}^{\sqrt{R^2-\delta^2}} f(s-i\delta)ds = \int_{[r,R]} f(s)e^{2\pi\left(\frac{a-1}{2}+i\xi\right)i}ds$$

Thus, taking the limits as r and R approach 0 and ∞ respectively

$$\lim_{r\to 0, R\to\infty}\int_{C_1} f(s)ds = \int_{\mathbb{R}_+} f(s)ds,$$
$$\lim_{r\to 0, R\to\infty}\int_{C_2} f(s)e^{2\pi\left(\frac{a-1}{2}+i\xi\right)i}ds = e^{2\pi\left(\frac{a-1}{2}+i\xi\right)i}\int_{\mathbb{R}_+} f(s)ds.$$

This implies

$$\begin{aligned}\left(1-e^{2\pi\left(\frac{a-1}{2}+i\xi\right)i}\right)\int_{\mathbb{R}_+} f(s)ds =& 2\pi i\sum Res(f)\\ =& 2\pi i\left(\frac{e^{\left(\frac{a-1}{2}+i\xi\right)\omega i}}{e^{i\omega}-e^{i(2\pi-\omega)}}+\frac{e^{\left(\frac{a-1}{2}+i\xi\right)(2\pi-\omega)i}}{e^{i(2\pi-\omega)}-e^{i\omega}}\right)\\ =& \pi\left(\frac{e^{\left(\frac{a-1}{2}+i\xi\right)\omega i}-e^{\left(\frac{a-1}{2}+i\xi\right)(2\pi-\omega)i}}{\sin(\omega)}\right).\end{aligned}$$

Expanding out the coefficient of the integral on the LHS,

$$\begin{aligned}1-e^{2\pi\left(\frac{a-1}{2}+i\xi\right)i} =& e^{\pi\left(\frac{a-1}{2}+i\xi\right)i}e^{-\pi\left(\frac{a-1}{2}+i\xi\right)i}-e^{2\pi\left(\frac{a-1}{2}+i\xi\right)i}\\ =& e^{\pi\left(\frac{a-1}{2}+i\xi\right)i}\left(e^{-\pi\left(\frac{a-1}{2}+i\xi\right)i}-e^{\pi\left(\frac{a-1}{2}+i\xi\right)i}\right)\\ =& -2ie^{\pi\left(\frac{a-1}{2}+i\xi\right)i}\sin(\pi\left(\frac{a-1}{2}+i\xi\right)).\end{aligned}$$

So, dividing through by $-2ie^{\pi\left(\frac{a-1}{2}+i\xi\right)i}\sin(\pi\left(\frac{a-1}{2}+i\xi\right)$ we get

$$\begin{aligned}
\int_{\mathbb{R}_+} f(s)ds &= \pi\left(\frac{e^{\left(\frac{a-1}{2}+i\xi\right)(\omega-\pi)i} - e^{\left(\frac{a-1}{2}+i\xi\right)(\pi-\omega)i}}{-2i\sin(\pi\left(\frac{a-1}{2}+i\xi\right))\sin(\omega)}\right) \\
&= 2i\pi\left(\frac{\sin(\left(\frac{a-1}{2}+i\xi\right)(\omega-\pi))}{-2i\sin(\pi\left(\frac{a-1}{2}+i\xi\right)\sin(\omega)}\right) \\
&= \pi\left(\frac{\sin(\left(\frac{a-1}{2}+i\xi\right)(\pi-\omega))}{\sin(\pi\left(\frac{a-1}{2}+i\xi\right)\sin(\omega)}\right),
\end{aligned}$$

a representation of the integral. Finally giving us,

$$\mathcal{M}(j_{\omega,a}) = \frac{\sin(\left(\frac{a-1}{2}+i\xi\right)(\pi-\omega))}{\sin(\pi\left(\frac{a-1}{2}+i\xi\right))}.$$

□

We can show $\mathcal{M}(j_{\omega,a})$ to be essentially bounded as we have,

$$\begin{aligned}
|\mathcal{M}(j_{\omega,a})(s)| &= \left|\int_{\mathbb{R}_+} t^{i\xi} j_{\omega,a}(t)\frac{dt}{t}\right| \\
&\leq \int_{\mathbb{R}_+} \left|t^{i\xi}\right| |j_{\omega,a}(t)| \frac{dt}{t} \\
&\leq \int_{\mathbb{R}_+} |j_{\omega,a}(t)| \frac{dt}{t}.
\end{aligned}$$

Thus $\mathcal{M}(j_{\omega,a}) \in L^\infty(\mathbb{R})$ follows from $j_{\omega,a} \in L^1(\frac{dt}{t})$.

By taking the matrix product $\widetilde{V}_{\frac{a+1}{2}} K^a_{\Gamma_{\alpha,\beta,\theta}} \widetilde{V}_{\frac{a+1}{2}}$ where

$$\widetilde{V}_\gamma = \begin{bmatrix} V_\gamma & 0 & 0 & 0 \\ 0 & V_\gamma & 0 & 0 \\ 0 & 0 & V_\gamma & 0 \\ 0 & 0 & 0 & V_\gamma \end{bmatrix}$$

and again, applying the Mellin convolution to each entry, we have a unitarily equivalent representation of the N.P. adjoint operator over $\Gamma_{\alpha,\beta,\theta}$ as

$$\widetilde{K}^a_{\Gamma_{\alpha,\beta,\theta}} = \begin{bmatrix} 0 & \widetilde{K}^a_{\alpha} & \widetilde{K}^a_{\alpha+\theta} & \widetilde{K}^a_{\alpha+\theta+\beta} \\ \widetilde{K}^a_{\alpha} & 0 & -\widetilde{K}^a_{\theta} & -\widetilde{K}^a_{\theta+\beta} \\ -\widetilde{K}^a_{\alpha+\theta} & -\widetilde{K}^a_{\theta} & 0 & \widetilde{K}^a_{\beta} \\ \widetilde{K}^a_{\alpha+\theta+\beta} & \widetilde{K}^a_{\theta+\beta} & \widetilde{K}^a_{\beta} & 0 \end{bmatrix}.$$

With this more manageable, equivalent form of our matrix operator in mind, we can begin re-examining the specific parameterized subset of $\mathbb{R}^2$ being acted on. While not unwieldy in terms of analysis, there is a practical motivation to determine how the spectral results on such 'infinite wedges' compare with those on subsets of those wedges local to their shared vertex, and further how such spaces compare to locally similar, closed bow-tie curves.

5.2 Localization and closed bow-tie curves

Again, defining the set $\Gamma_{\alpha,\beta,\theta}$ as the union of the boundary of wedges W_α and W_β, determined by the angles α and β respectively, whose vertices coincide with the origin and are separated by some angle θ, we will begin to consider localizations of our previous results regarding the N.P. adjoint operator. Specifically, we shall be comparing them to those bow tie curves in $\mathbb{R}^2$ which are identical to said localizations within a sufficiently small neighbourhood of the shared vertex.

In localizing the N.P. adjoint operator from the $\Gamma_{\alpha,\beta,\theta}$ case we begin by taking indicator function $\psi : \mathbb{R} \to [0,1]$ where

$$\psi(t) = \begin{cases} 1 & t \in [0,1/3), \\ 0 & t \in (1/3,\infty). \end{cases}$$

Combining this function, with the kernels of the integral operators that define the N.P. adjoint operator matrix on $L^2(\Gamma_{\alpha,\beta,\theta})$, with respect to our chosen variables $t,s \in [0,1]$ as below,

$$K^{loc}_{\omega} f(t) = \int_{[0,1]} \psi(t)\widetilde{k_\omega}(t,s)\psi(s)f(s)ds = \int_{[0,1]} \psi(t)\psi(s)\frac{s\sin(\omega)}{\pi(t^2+s^2-2st\cos(\omega))}f(s)ds,$$

we effectively restrict the domain of the functions each integral operator acts on to a set of finite curves. For $s,t \in [0,1/3)$, the closed curves our operators act on are identical to the parameterization of $\Gamma_{\alpha,\beta,\theta}$ restricted to this $[0,1/3)$ interval and 'cut off' as our variable reaches 1/3.

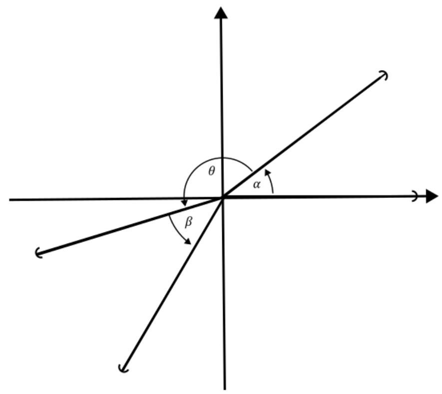


Figure 11: Localization of $\Gamma_{\alpha,\beta,\theta}$ boundary

Given this localization, we must define the bow-tie curves we compare them with accordingly. We will denote our bow-tie curve $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ and describe it as the union of the images of four C^2 class curves $\gamma_i : [0,1] \to \mathbb{R}^2$, $i \in \{1,2,3,4\}$, such that

- $\gamma_i(t) = \psi(t) l_i(t)$, for $t \in [0, 1/3)$ for all $i \in \{1,2,3,4\}$
- $\gamma_i(0) = \gamma_j(0) = 0$ for all $i, j \in \{1,2,3,4\}$
- $\gamma_1(1) = \gamma_2(1)$, $\gamma_3(1) = \gamma_4(1)$
- The complete boundary of $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ has parameterization of class C^2

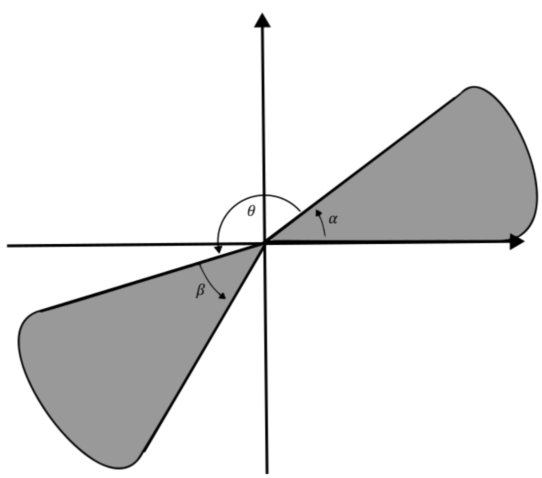


Figure 12: $\widetilde{\Gamma}_{\alpha,\beta,\theta}$

From the manner in which we have determined $\widetilde{\Gamma}_{\alpha,\beta,\theta}$, we then have that both the N.P. adjoint operator determined for $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ and our localization of the operator on $\Gamma_{\alpha,\beta,\theta}$ are acting on the space $L^2(\widetilde{\Gamma}_{\alpha,\beta,\theta}) = \bigoplus_{i=1}^{4} L^2([0,1])$.

Explicitly, we will determine our N.P. adjoint operator on $L^2(\widetilde{\Gamma}_{\alpha,\beta,\theta})$ using the four by four operator matrix

$$K_{\widetilde{\Gamma}_{\alpha,\beta,\theta}} := \begin{bmatrix} \overline{K}_{1,1} & \overline{K}_{1,2} & \overline{K}_{1,3} & \overline{K}_{1,4} \\ \overline{K}_{2,1} & \overline{K}_{2,2} & \overline{K}_{2,3} & \overline{K}_{2,4} \\ \overline{K}_{3,1} & \overline{K}_{3,2} & \overline{K}_{3,3} & \overline{K}_{3,4} \\ \overline{K}_{4,1} & \overline{K}_{4,2} & \overline{K}_{4,3} & \overline{K}_{4,4} \end{bmatrix},$$

where $\overline{K}_{i,j}$ represents the integral operator on $L^2([0,1])$ defined by the kernel,

$$\overline{k}_{i,j} := \frac{\langle \gamma_i - \gamma_j, \nu_{\gamma_i} \rangle}{\pi |\gamma_j - \gamma_i|^2},$$

again with ν_{γ_i} representing the outward oriented unit normal at γ_i.

We wish to prove that the operator defined by the difference of the kernels for the bow-tie curves and the localization is a Hilbert Schmidt integral operator or, more specifically, that

$$\int_{[0,1]} \int_{[0,1]} \left| \overline{k}_{i,j}(t,s) - \psi(t) \widetilde{k_\omega}(t,s) \psi(s) \right|^2 dtds < \infty.$$

where ω is the angle determining the integral operator in the (i,j) entry of the N.P. adjoint operator matrix on $\Gamma_{\alpha,\beta,\theta}$. Given the parameterisations of the sets we are working on are equal on $[0,1/3)$ and our localizaion becomes 0 for all $s,t \in (1/3,1]$ we can separate this inequality into

$$\begin{aligned} &\int_{[1/3,1]} \int_{[1/3,1]} \left| \overline{k}_{i,j}(t,s) \right|^2 dtds + \\ &\int_{[0,1/3]} \int_{[1/3,1]} \left| \overline{k}_{i,j}(t,s) - \psi(t) \widetilde{k_\omega}(t,s) \psi(s) \right|^2 dtds + \\ &\int_{[1/3,1]} \int_{[0,1/3]} \left| \overline{k}_{i,j}(t,s) - \psi(t) \widetilde{k_\omega}(t,s) \psi(s) \right|^2 dtds < \infty. \end{aligned}$$

Naturally, we have that this is the case if $\overline{k}_{i,j}(t,s)$ and $\widetilde{k_\omega}(t,s)$ are continuous on $(1/3,1] \times (1/3,1]$, $(1/3,1] \times [0,1/3]$ and $[0,1/3] \times (1/3,1]$.

We can see from the fact that both kernels are composed of continuous functions divided by some power of the difference of two points on $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ that for $i \neq j$ and $t \neq s$ these kernels are both well-defined and continuous on the aforementioned subsets of $\mathbb{R}^2$.

Indeed, it would appear that when $i = j$ and $t = s$ the integral operators defined by such kernels are singular. This however is not the case. Essentially, returning to our more general description of the N.P. adjoint operator, when we have that $x = y$, for $x, y \in \partial\Omega$ the kernel is not well defined.

Note given there only remains to examine the case $i = j$ and $t = s$, at this point the problem has become a matter of proving the inequality,

$$\int_{[1/3,1]} \int_{[1/3,1]} \left|\overline{k}_{i,i}(t,s)\right|^2 dtds < \infty$$

holds, since for $i = j$ our original kernel is always 0.

From Lemma 2.2 we get that for ∂D of class C^2 and for $x, y \in \partial D$

$$\frac{\langle x - y, \nu_x \rangle}{|y - x|^n} \leq \frac{L}{|y - x|^{n-2}}, \quad x \neq y,$$

that is, the N.P. adjoint operator kernel is bounded from above by some $L > 0$ on $\mathbb{R}^2$.

Given our parameterization of $\widetilde{\Gamma}_{\alpha,\beta,\theta}$, and specifically focused on the parameterisation on $(1/3, 1]$, our operators are acting on boundaries of class C^2.
Thus, using the above Lemma we have that on $(1/3, 1] \times (1/3, 1]$, $\overline{k}_{i,i}(t, s)$ is bounded from above by some $L > 0$, and so

$$\int_{[1/3,1]} \int_{[1/3,1]} \left|\overline{k}_{i,i}(t,s)\right|^2 dtds < \int_{[1/3,1]} \int_{[1/3,1]} |L|^2 dtds = L^2(1 - 1/3)^2 < \infty$$

holds on the set $\widetilde{\Gamma}_{\alpha,\beta,\theta}$. As such we must have that the difference of the integral operators for our localization and bow-tie curve is a Hilbert Schmidt integral operator and thus must also be compact. Given that the essential spectrum of an operator is invariant under compact perturbations, we then get,

Theorem 5.3. *For $K_{\widetilde{\Gamma}_{\alpha,\beta,\theta}}$ the N.P. adjoint operator acting on $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ and $K^{loc}_{\Gamma_{\alpha,\beta,\theta}}$ the localization of the N.P. adjoint operator as described above,*

$$\sigma_{ess}(K_{\widetilde{\Gamma}_{\alpha,\beta,\theta}}) = \sigma_{ess}(K^{loc}_{\Gamma_{\alpha,\beta,\theta}}).$$

With this in mind, we may begin determining the specific spectral properties of our operator with regard to these bow-tie curves $\widetilde{\Gamma}_{\alpha,\beta,\theta}$. Of particular interest is the potential essential spectral radius of our operator, and whether such a value is comparable to those found in related studies such as [6].

5.3 Determining the essential norm of the N.P. adjoint for functions in L^2

We now will consider determining essential norm results for the Matrix operator on the space $L^2(\widetilde{\Gamma}_{\alpha,\beta,\theta}) = \bigoplus_{i=1}^4 L^2([0,1],dt)$.

As we have already determined, the localization of our matrix operator to $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ is a compact perturbation of the localization of $\Gamma_{\alpha,\beta,\theta}$. As such, in determining an upper bound to the essential norm of our operator it is enough to look at our localization.

In the same manner described at the start of Section 5.2, we will use the cutoff function ψ to localize each operator element, acting on $\Gamma_{\alpha,\beta,\theta}$, giving us

$$K^{loc}_{\Gamma_{\alpha,\beta,\theta}} = \begin{bmatrix} 0 & K^{loc}_{\alpha} & K^{loc}_{\alpha+\theta} & K^{loc}_{\alpha+\theta+\beta} \\ K^{loc}_{\alpha} & 0 & -K^{loc}_{\theta} & -K^{loc}_{\theta+\beta} \\ -K^{loc}_{\alpha+\theta} & -K^{loc}_{\theta} & 0 & K^{loc}_{\beta} \\ K^{loc}_{\alpha+\theta+\beta} & K^{loc}_{\theta+\beta} & K^{loc}_{\beta} & 0 \end{bmatrix}$$

Considering this operator under the L^2 norm, we get, for $\phi(t) = [\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)]^T \in \bigoplus\limits_{i=1}^{4} L^2([0,1],dt)$,

$$\begin{aligned} ||K^{loc}_{\Gamma_{\alpha,\beta,\theta}}\phi(t)||_{L^2(\widetilde{\Gamma}_{\alpha,\beta,\theta})} =& ||K^{loc}_{\Gamma_{\alpha,\beta,\theta}}\phi(t)||_{\bigoplus\limits_{i=1}^{4} L^2([0,1])} \\ =& \left\| \begin{bmatrix} 0 & K^{loc}_{\alpha} & K^{loc}_{\alpha+\theta} & K^{loc}_{\alpha+\theta+\beta} \\ K^{loc}_{\alpha} & 0 & -K^{loc}_{\theta} & -K^{loc}_{\theta+\beta} \\ -K^{loc}_{\alpha+\theta} & -K^{loc}_{\theta} & 0 & K^{loc}_{\beta} \\ K^{loc}_{\alpha+\theta+\beta} & K^{loc}_{\theta+\beta} & K^{loc}_{\beta} & 0 \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \\ \phi_4(t) \end{bmatrix} \right\|_{\bigoplus\limits_{i=1}^{4} L^2([0,1])} \end{aligned}$$

Given each matrix element has constant sign, and the unitary equivalence of our matrix operator with the matrix with multiplication operator entries,

$$\left\|\begin{bmatrix} 0 & K_{\alpha}^{loc} & K_{\alpha+\theta}^{loc} & K_{\alpha+\theta+\beta}^{loc} \\ K_{\alpha}^{loc} & 0 & -K_{\theta}^{loc} & -K_{\theta+\beta}^{loc} \\ -K_{\alpha+\theta}^{loc} & -K_{\theta}^{loc} & 0 & K_{\beta}^{loc} \\ K_{\alpha+\theta+\beta}^{loc} & K_{\theta+\beta}^{loc} & K_{\beta}^{loc} & 0 \end{bmatrix}\begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \\ \phi_4(t) \end{bmatrix}\right\|_{\bigoplus_{i=1}^{4} L^2([0,1])}$$

$$\leq \left\|\begin{bmatrix} 0 & K_{\alpha} & K_{\alpha+\theta} & K_{\alpha+\theta+\beta} \\ K_{\alpha} & 0 & -K_{\theta} & -K_{\theta+\beta} \\ -K_{\alpha+\theta} & -K_{\theta} & 0 & K_{\beta} \\ K_{\alpha+\theta+\beta} & K_{\theta+\beta} & K_{\beta} & 0 \end{bmatrix}\begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \\ \phi_4(t) \end{bmatrix}\right\|_{\bigoplus_{i=1}^{4} L^2(\mathbb{R}_+)}$$

$$= \left\|\begin{bmatrix} 0 & \mathcal{M}(j_{\alpha})(\xi) & \mathcal{M}(j_{\alpha+\theta})(\xi) & \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) \\ \mathcal{M}(j_{\alpha})(\xi) & 0 & -\mathcal{M}(j_{\theta})(\xi) & -\mathcal{M}(j_{\theta+\beta})(\xi) \\ -\mathcal{M}(j_{\alpha+\theta})(\xi) & -\mathcal{M}(j_{\theta})(\xi) & 0 & \mathcal{M}(j_{\beta})(\xi) \\ \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) & \mathcal{M}(j_{\theta+\beta})(\xi) & \mathcal{M}(j_{\beta})(\xi) & 0 \end{bmatrix}\begin{bmatrix} \mathcal{M}(V^{\frac{1}{2}}\phi_1)(\xi) \\ \mathcal{M}(V^{\frac{1}{2}}\phi_2)(\xi) \\ \mathcal{M}(V^{\frac{1}{2}}\phi_3)(\xi) \\ \mathcal{M}(V^{\frac{1}{2}}\phi_4)(\xi) \end{bmatrix}\right\|_{\bigoplus_{i=1}^{4} L^2(\mathbb{R})}$$

where we extend each ϕ_i to $\mathbb{R}_+$ by setting $\phi_i(t) = 0$ for $[1, \infty)$, and then apply the Mellin transform to get a system of vector valued functions with coordinates based on $\mathbb{R}$.

Considering this,

$$\left\| \begin{bmatrix} 0 & \mathcal{M}(j_\alpha)(\xi) & \mathcal{M}(j_{\alpha+\theta})(\xi) & \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) \\ \mathcal{M}(j_\alpha)(\xi) & 0 & -\mathcal{M}(j_\theta)(\xi) & -\mathcal{M}(j_{\theta+\beta})(\xi) \\ -\mathcal{M}(j_{\alpha+\theta})(\xi) & -\mathcal{M}(j_\theta)(\xi) & 0 & \mathcal{M}(j_\beta)(\xi) \\ \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) & \mathcal{M}(j_{\theta+\beta})(\xi) & \mathcal{M}(j_\beta)(\xi) & 0 \end{bmatrix} \begin{bmatrix} \mathcal{M}(V^{\frac{1}{2}}\phi_1)(\xi) \\ \mathcal{M}(V^{\frac{1}{2}}\phi_2)(\xi) \\ \mathcal{M}(V^{\frac{1}{2}}\phi_3)(\xi) \\ \mathcal{M}(V^{\frac{1}{2}}\phi_4)(\xi) \end{bmatrix} \right\|^2_{\bigoplus_{i=1}^{4} L^2(\mathbb{R})}$$

$$= \int_0^\infty \left\| \begin{bmatrix} 0 & \mathcal{M}(j_\alpha)(\xi) & \mathcal{M}(j_{\alpha+\theta})(\xi) & \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) \\ \mathcal{M}(j_\alpha)(\xi) & 0 & -\mathcal{M}(j_\theta)(\xi) & -\mathcal{M}(j_{\theta+\beta})(\xi) \\ -\mathcal{M}(j_{\alpha+\theta})(\xi) & -\mathcal{M}(j_\theta)(\xi) & 0 & \mathcal{M}(j_\beta)(\xi) \\ \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) & \mathcal{M}(j_{\theta+\beta})(\xi) & \mathcal{M}(j_\beta)(\xi) & 0 \end{bmatrix} \begin{bmatrix} \mathcal{M}(V^{\frac{1}{2}}\phi_1)(\xi) \\ \mathcal{M}(V^{\frac{1}{2}}\phi_2)(\xi) \\ \mathcal{M}(V^{\frac{1}{2}}\phi_3)(\xi) \\ \mathcal{M}(V^{\frac{1}{2}}\phi_4)(\xi) \end{bmatrix} \right\|^2_{\ell^2(\mathbb{C}^4)} d\xi$$

$$\leq \int_{-\infty}^\infty \left\| \begin{bmatrix} 0 & \mathcal{M}(j_\alpha)(\xi) & \mathcal{M}(j_{\alpha+\theta})(\xi) & \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) \\ \mathcal{M}(j_\alpha)(\xi) & 0 & -\mathcal{M}(j_\theta)(\xi) & -\mathcal{M}(j_{\theta+\beta})(\xi) \\ -\mathcal{M}(j_{\alpha+\theta})(\xi) & -\mathcal{M}(j_\theta)(\xi) & 0 & \mathcal{M}(j_\beta)(\xi) \\ \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) & \mathcal{M}(j_{\theta+\beta})(\xi) & \mathcal{M}(j_\beta)(\xi) & 0 \end{bmatrix} \right\|^2_{B(\ell^2(\mathbb{C}^4))} \sum_{i=1}^{4} \left| \mathcal{M}(V^{\frac{1}{2}}\phi_i)(\xi) \right|^2 d\xi$$

$$\leq \sup_{\xi \in \mathbb{R}} \left\| \begin{bmatrix} 0 & \mathcal{M}(j_\alpha)(\xi) & \mathcal{M}(j_{\alpha+\theta})(\xi) & \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) \\ \mathcal{M}(j_\alpha)(\xi) & 0 & -\mathcal{M}(j_\theta)(\xi) & -\mathcal{M}(j_{\theta+\beta})(\xi) \\ -\mathcal{M}(j_{\alpha+\theta})(\xi) & -\mathcal{M}(j_\theta)(\xi) & 0 & \mathcal{M}(j_\beta)(\xi) \\ \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) & \mathcal{M}(j_{\theta+\beta})(\xi) & \mathcal{M}(j_\beta)(\xi) & 0 \end{bmatrix} \right\|^2_{B(\ell^2(\mathbb{C}^4))} \int_{-\infty}^\infty \sum_{i=1}^{4} \left| \mathcal{M}(V^{\frac{1}{2}}\phi_i)(\xi) \right|^2 d\xi$$

$$= N^2 \left\| \begin{bmatrix} \mathcal{M}(V^{\frac{1}{2}}\phi_1)(\xi) \\ \mathcal{M}(V^{\frac{1}{2}}\phi_2)(\xi) \\ \mathcal{M}(V^{\frac{1}{2}}\phi_3)(\xi) \\ \mathcal{M}(V^{\frac{1}{2}}\phi_4)(\xi) \end{bmatrix} \right\|^2_{\bigoplus_{i=1}^{4} L^2(\mathbb{R})},$$

where

$$N = \sup_{\xi \in \mathbb{R}} \left\| \begin{bmatrix} 0 & \mathcal{M}(j_\alpha)(\xi) & \mathcal{M}(j_{\alpha+\theta})(\xi) & \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) \\ \mathcal{M}(j_\alpha)(\xi) & 0 & -\mathcal{M}(j_\theta)(\xi) & -\mathcal{M}(j_{\theta+\beta})(\xi) \\ -\mathcal{M}(j_{\alpha+\theta})(\xi) & -\mathcal{M}(j_\theta)(\xi) & 0 & \mathcal{M}(j_\beta)(\xi) \\ \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) & \mathcal{M}(j_{\theta+\beta})(\xi) & \mathcal{M}(j_\beta)(\xi) & 0 \end{bmatrix} \right\|_{B(\ell^2(\mathbb{C}^4))}.$$

and finally, we have,

$$N^2 \left\| \begin{bmatrix} \mathcal{M}(V^{\frac{1}{2}}\phi_1)(\xi) \\ \mathcal{M}(V^{\frac{1}{2}}\phi_2)(\xi) \\ \mathcal{M}(V^{\frac{1}{2}}\phi_3)(\xi) \\ \mathcal{M}(V^{\frac{1}{2}}\phi_4)(\xi) \end{bmatrix} \right\|^2_{\bigoplus_{i=1}^{4} L^2(\mathbb{R})} = N^2 \left\| \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \\ \phi_4(t) \end{bmatrix} \right\|^2_{\bigoplus_{i=1}^{4} L^2(\mathbb{R}_+)} = N^2 \left\| \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \\ \phi_4(t) \end{bmatrix} \right\|^2_{\bigoplus_{i=1}^{4} L^2([0,1])}$$

This immediately gives us that the essential norm of the operator $\widetilde{K}$ is bounded on

$L(\widetilde{\Gamma}_{\alpha+\theta+\beta})$, by

$$\sup_{\xi\in\mathbb{R}} \left\| \begin{bmatrix} 0 & \mathcal{M}(j_\alpha)(\xi) & \mathcal{M}(j_{\alpha+\theta})(\xi) & \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) \\ \mathcal{M}(j_\alpha)(\xi) & 0 & -\mathcal{M}(j_\theta)(\xi) & -\mathcal{M}(j_{\theta+\beta})(\xi) \\ -\mathcal{M}(j_{\alpha+\theta})(\xi) & -\mathcal{M}(j_\theta)(\xi) & 0 & \mathcal{M}(j_\beta)(\xi) \\ \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) & \mathcal{M}(j_{\theta+\beta})(\xi) & \mathcal{M}(j_\beta)(\xi) & 0 \end{bmatrix} \right\|_{B(\ell^2(\mathbb{C}^4))}.$$

Below we have the following figures graphing the largest singular values of our matrix operator for different $\widetilde{\Gamma}_{\alpha+\theta+\beta}$, looking firstly at cases where the angles defining $\widetilde{\Gamma}_{\alpha+\theta+\beta}$ are completely symmetric, and at further asymmetric cases for $\widetilde{\Gamma}_{\alpha+\theta+\beta}$.

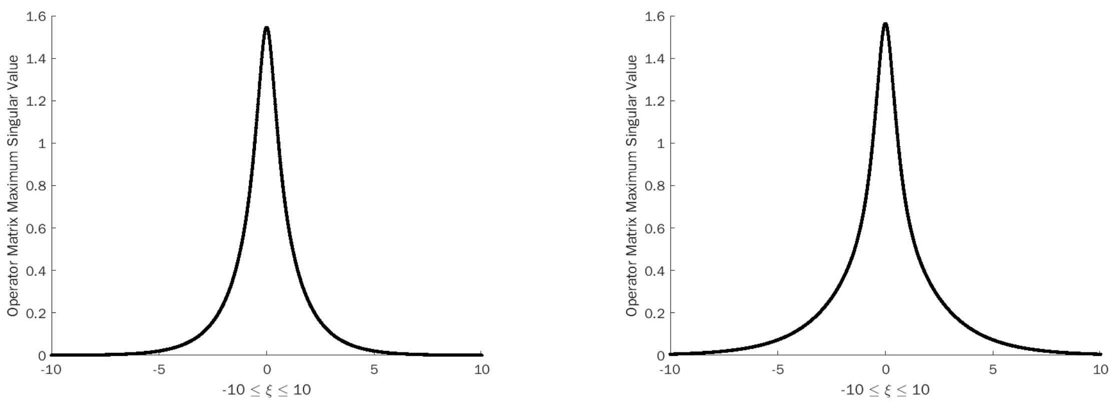


Figure 13: Max Singular values of matrix operator on $\widetilde{\Gamma}_{\frac{\pi}{3}\frac{3\pi}{4}\frac{2\pi}{3}}$ and $\widetilde{\Gamma}_{\frac{\pi}{6}\frac{\pi}{2}\frac{\pi}{4}}$

We observe how in each of these cases, there is a distinct downward trend from the apparent maximum at 0, and that in each case our maximum value exceeds 1. This however is unlike those results found in [6], where the essential norm was bounded above by 1.

Given each scenario depicted above indicates that the upper bound N is greater than 1, we are motivated to determine if this holds in all cases. In order to do so, we begin by determining a formula of our operator's maximum singular value when $\xi = 0$, before calculating its upper bound.

Calculating via Matlab (see Appendices), we get the following formula for the largest singular value $S(\alpha, \beta, \theta)$ at $\xi = 0$:

$$\sqrt{\begin{gathered}\frac{\sin(\frac{\alpha}{2}+\frac{\theta}{2}+\frac{\beta}{2}-\frac{\pi}{2})^2}{2}+\frac{\sin(\frac{\alpha}{2}+\frac{\theta}{2}-\frac{\pi}{2})^2}{2} \\ +\frac{\sin(\frac{\theta}{2}+\frac{\beta}{2}-\frac{\pi}{2})^2}{2}+\frac{\sin(\frac{\alpha}{2}-\frac{\pi}{2})^2}{2} \\ +\frac{\sin(\frac{\theta}{2}-\frac{\pi}{2})^2}{2}+\frac{\sin(\frac{\beta}{2}-\frac{\pi}{2})^2}{2}\end{gathered} + \tfrac{1}{2}\sqrt{\left(\begin{gathered}\cos(\frac{\alpha}{2})^2\cos(\frac{\theta}{2})^2 \\ +\cos(\frac{\theta}{2})^2\cos(\frac{\beta}{2})^2 \\ +\cos(\frac{\alpha}{2})^2+\cos(\frac{\theta}{2})^2 \\ +\cos(\frac{\beta}{2})^2+\sin(\frac{\alpha}{2})^2\sin(\frac{\theta}{2})^2 \\ +\sin(\frac{\theta}{2})^2\sin(\frac{\beta}{2})^2-2\cos(\frac{\alpha}{2})\cos(\frac{c}{2}) \\ +\cos(\frac{\alpha}{2})^2\sin(\frac{\theta}{2})^2\sin(\frac{\beta}{2})^2+\cos(\frac{\theta}{2})^2\sin(\frac{a}{2})^2\sin(\frac{\beta}{2})^2 \\ +\cos(\frac{\beta}{2})^2\sin(\frac{\beta}{2})^2\sin(\frac{\theta}{2})^2+2\cos(\frac{\theta}{2})^2\sin(\frac{\alpha}{2})\sin(\frac{\beta}{2}) \\ +2\sin(\frac{\alpha}{2})\sin(\frac{\theta}{2})^2\sin(\frac{\beta}{2})+\cos(\frac{\alpha}{2})^2\cos(\frac{\theta}{2})^2\cos(\frac{\beta}{2})^2 \\ -2\cos(\frac{\alpha}{2})\cos(\frac{\theta}{2})\sin(\frac{\alpha}{2})\sin(\frac{\theta}{2}) \\ -2\cos(\frac{\theta}{2})\cos(\frac{\beta}{2})\sin(\frac{\theta}{2})\sin(\frac{\beta}{2}) \\ -2\cos(\frac{\alpha}{2})\cos(\frac{\theta}{2})\cos(\frac{\beta}{2})^2\sin(\frac{\alpha}{2})\sin(\frac{\theta}{2}) \\ -2\cos(\frac{\alpha}{2})\cos(\frac{\theta}{2})^2\cos(\frac{\beta}{2})\sin(\frac{\alpha}{2})\sin(\frac{\beta}{2}) \\ -2\cos(\frac{\alpha}{2})^2\cos(\frac{\theta}{2})\cos(\frac{\beta}{2})\sin(\frac{\theta}{2})\sin(\frac{\beta}{2}) \\ +2\cos(\frac{\alpha}{2})\cos(\frac{\theta}{2})\sin(\frac{\alpha}{2})\sin(\frac{\theta}{2})\sin(\frac{\beta}{2})^2 \\ +2\cos(\frac{\alpha}{2})\cos(\frac{\beta}{2})\sin(\frac{\alpha}{2})\sin(\frac{\theta}{2})^2\sin(\frac{\beta}{2}) \\ +2\cos(\frac{\theta}{2})\cos(\frac{\beta}{2})\sin(\frac{\alpha}{2})^2\sin(\frac{\theta}{2})\sin(\frac{\beta}{2})\end{gathered}\right)\left(\begin{gathered}\cos(\frac{\alpha}{2})^2\cos(\frac{\theta}{2})^2+\cos(\frac{\theta}{2})^2\cos(\frac{\beta}{2})^2 \\ +\cos(\frac{\alpha}{2})^2+\cos(\frac{\theta}{2})^2+\cos(\frac{\beta}{2})^2 \\ +\sin(\frac{\alpha}{2})^2\sin(\frac{\theta}{2})^2 \\ +\sin(\frac{\theta}{2})^2\sin(\frac{\beta}{2})^2+2\cos(\frac{\alpha}{2})\cos(\frac{\beta}{2}) \\ +\cos(\frac{\alpha}{2})^2\sin(\frac{\theta}{2})^2\sin(\frac{\beta}{2})^2 \\ +\cos(\frac{\theta}{2})^2\sin(\frac{\alpha}{2})^2\sin(\frac{\beta}{2})^2 \\ +\cos(\frac{\beta}{2})^2\sin(\frac{\alpha}{2})^2\sin(\frac{\theta}{2})^2 \\ -2\cos(\frac{\theta}{2})^2\sin(\frac{\alpha}{2})\sin(\frac{\beta}{2}) \\ -2\sin(\frac{\alpha}{2})\sin(\frac{\theta}{2})^2\sin(\frac{\beta}{2}) \\ +\cos(\frac{\alpha}{2})^2\cos(\frac{\theta}{2})^2\cos(\frac{\beta}{2})^2 \\ -2\cos(\frac{\alpha}{2})\cos(\frac{\theta}{2})\sin(\frac{\alpha}{2})\sin(\frac{\theta}{2}) \\ -2\cos(\frac{\theta}{2})\cos(\frac{\beta}{2})\sin(\frac{\theta}{2})\sin(\frac{\beta}{2}) \\ -2\cos(\frac{\alpha}{2})\cos(\frac{\theta}{2})\cos(\frac{\beta}{2})^2\sin(\frac{\alpha}{2})\sin(\frac{\theta}{2}) \\ -2\cos(\frac{\alpha}{2})\cos(\frac{\theta}{2})^2\cos(\frac{\beta}{2})\sin(\frac{\alpha}{2})\sin(\frac{\beta}{2}) \\ -2\cos(\frac{\alpha}{2})^2\cos(\frac{\theta}{2})\cos(\frac{\beta}{2})\sin(\frac{\theta}{2})\sin(\frac{\beta}{2}) \\ +2\cos(\frac{\alpha}{2})\cos(\frac{\theta}{2})\sin(\frac{\alpha}{2})\sin(\frac{\theta}{2})\sin(\frac{\theta}{2})^2 \\ +2\cos(\frac{\alpha}{2})\cos(\frac{\beta}{2})\sin(\frac{\alpha}{2})\sin(\frac{\theta}{2})^2\sin(\frac{\beta}{2}) \\ +2\cos(\frac{\theta}{2})\cos(\frac{\beta}{2})\sin(\frac{\alpha}{2})^2\sin(\frac{\theta}{2})\sin(\frac{\beta}{2})\end{gathered}\right)}}$$

Given we have that each of our three variables is one of the angles determining $\widetilde{\Gamma}_{\alpha,\beta,\theta}$, each of which is dependent on the other, we will examine the specific scenario of when the angles determining $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ are completely symmetric.

Assuming that we have

$$\begin{aligned} \alpha + \theta &= \pi, \\ \alpha &= \beta, \end{aligned}$$

then by substitution we get

$$\begin{aligned}
S(\alpha, \pi - \alpha, \alpha)^2 =& \frac{\frac{\sin(\frac{\alpha}{2})^2}{2} + \frac{\cos(\frac{\alpha}{2})^2}{2} +}{\frac{\sin(\frac{\alpha}{2})^2}{2} + \frac{\cos(\frac{\alpha}{2})^2}{2}} + \frac{1}{2}\sqrt{\begin{pmatrix} 6\sin(\frac{\alpha}{2})^2\cos(\frac{\alpha}{2})^2 + 2\cos(\frac{\alpha}{2})^2 + \sin(\frac{\alpha}{2})^2 \\ -2\cos(\frac{\alpha}{2})^2 + 5\cos(\frac{\alpha}{2})^4\sin(\frac{\alpha}{2})^2 + \sin(\frac{\alpha}{2})^6 + 2\sin(\frac{\alpha}{2})^4 \\ -4\cos(\frac{\alpha}{2})^2\sin(\frac{\alpha}{2})^2 - 2\sin(\frac{\alpha}{2})^2\cos(\frac{\alpha}{2})^4 \\ -4\cos(\frac{\alpha}{2})^2\sin(\frac{\alpha}{2})^4 + 4\cos(\frac{\alpha}{2})^2\sin(\frac{\alpha}{2})^4 \end{pmatrix} \begin{pmatrix} 4\cos(\frac{\alpha}{2})^2\sin(\frac{\alpha}{2})^2 + \sin(\frac{\alpha}{2})^2 + 4\cos(\frac{\alpha}{2})^2 \\ +7\cos(\frac{\alpha}{2})^4\sin(\frac{\alpha}{2})^2 + \sin(\frac{\alpha}{2})^6 - 2\sin(\frac{\alpha}{2})^4 \\ -6\sin(\frac{\alpha}{2})^2\cos(\frac{\alpha}{2})^2 - 2\cos(\frac{\alpha}{2})^2\sin(\frac{\alpha}{2})^4 \\ -4\cos(\frac{\alpha}{2})^4\sin(\frac{\alpha}{2})^2 + 2\sin(\frac{\alpha}{2})^4\cos(\frac{\alpha}{2})^2 \end{pmatrix}} \\
=& 1 + \frac{1}{2}\sqrt{\begin{pmatrix} 2\sin(\frac{\alpha}{2})^2\cos(\frac{\alpha}{2})^2 + \sin(\frac{\alpha}{2})^2 \\ +3\cos(\frac{\alpha}{2})^4\sin(\frac{\alpha}{2})^2 + \sin(\frac{\alpha}{2})^6 + 2\sin(\frac{\alpha}{2})^4 \end{pmatrix} \begin{pmatrix} \sin(\frac{\alpha}{2})^2 + 4\cos(\frac{\alpha}{2})^2 + 3\cos(\frac{\alpha}{2})^4\sin(\frac{\alpha}{2})^2 \\ +\sin(\frac{\alpha}{2})^6 - 2\sin(\frac{\alpha}{2})^4 - 2\sin(\frac{\alpha}{2})^2\cos(\frac{\alpha}{2})^2 \end{pmatrix}} \\
=& 1 + \frac{1}{2}\sqrt{\left(3\sin(\frac{\alpha}{2})^2 + 3\cos(\frac{\alpha}{2})^4\sin(\frac{\alpha}{2})^2 + \sin(\frac{\alpha}{2})^6\right) \begin{pmatrix} 4\cos(\frac{\alpha}{2})^2 + 3\cos(\frac{\alpha}{2})^4\sin(\frac{\alpha}{2})^2 \\ +\sin(\frac{\alpha}{2})^6 - \sin(\frac{\alpha}{2})^2 \end{pmatrix}} \\
=& 1 + \frac{1}{2}\sqrt{\frac{1}{16}\sin^2(\alpha)\begin{pmatrix} 63 + 4\cos^2(\alpha) - \cos^2(2\alpha) \\ +32\cos(\alpha) - 2\cos(2\alpha) \end{pmatrix}}
\end{aligned}$$

We observe that on $(0, \pi)$

$$\sin^2(\alpha), \cos^2(\alpha) > 0,$$

and furthermore

$$\begin{aligned} |-\cos^2(2\alpha) + 32\cos(\alpha) - 2\cos(2\alpha)| &\leq |\cos^2(2\alpha)| + 32|\cos(\alpha)| + 2|\cos(2\alpha)| \\ &\leq 35. \end{aligned}$$

As such we must have

$$63 + 4\cos^2(\alpha) - \cos^2(2\alpha) + 32\cos(\alpha) - 2\cos(2\alpha) \geq 28 > 0.$$

Thus, the term under the interior root must be strictly positive, and so:

$$S(\alpha, \pi - \alpha, \alpha) > 1.$$

As such, we must have that when the angles defining $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ are completely symmetric, $N > 1$. Indeed, by a more careful analysis of the above argument we get the following result:

Theorem 5.4. *The essential norm of $K_{\tilde{\Gamma}_{\alpha,\beta,\theta}}$ on $L^2(\widetilde{\Gamma}_{\alpha+\theta+\beta})$ is given by*

$$\sup_{\xi\in\mathbb{R}} \left\| \begin{bmatrix} 0 & \mathcal{M}(j_\alpha)(\xi) & \mathcal{M}(j_{\alpha+\theta})(\xi) & \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) \\ \mathcal{M}(j_\alpha)(\xi) & 0 & -\mathcal{M}(j_\theta)(\xi) & -\mathcal{M}(j_{\theta+\beta})(\xi) \\ -\mathcal{M}(j_{\alpha+\theta})(\xi) & -\mathcal{M}(j_\theta)(\xi) & 0 & \mathcal{M}(j_\beta)(\xi) \\ \mathcal{M}(j_{\alpha+\theta+\beta})(\xi) & \mathcal{M}(j_{\theta+\beta})(\xi) & \mathcal{M}(j_\beta)(\xi) & 0 \end{bmatrix} \right\|_{B(\ell^2(\mathbb{C}^4))}.$$

and thus,

Corollary 5.5. *In the case where the angles defining $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ are completely symmetric, the essential norm of the N.P. adjoint operator on $L^2(\widetilde{\Gamma}_{\alpha,\beta,\theta})$ is strictly greater than* 1.

Again, given the unexpectedness of this result for the upper-bound, it is clear that a more complete analysis of the spectrum of this operator is necessary. To that end, we return to our study of the N.P. adjoint operator on the set determined by infinitely large wedges, with the intention of giving an explicit formula for the spectrum of our operator on such $\Gamma_{\alpha,\beta,\theta}$.

5.4 Formulating the spectrum of the N.P. Adjoint for functions in L^2

Given we have that $K^a_{\Gamma_{\alpha,\beta,\theta}}$ and $\widetilde{K}^a_{\Gamma_{\alpha,\beta,\theta}}$ are unitarily equivalent, their spectrum's must coincide. Determining the spectrum of one gives us the spectrum of the other.
In determining sufficient and necessary criteria for $\widetilde{K}^a_{\Gamma_{\alpha,\beta,\theta}} - \lambda I$ to be invertible, we make use of the below result, which follows directly from [31].

Proposition 5.1. *Let $\mathcal{B} = (B_{i,j})_{n\times n} \in \oplus_1^n L^2(\mathbb{R})$ describe an $n \times n$ operator matrix where the operators $B_{i,j}$ are bounded and pairwise commutative on $L^2(\mathbb{R})$. Then we have that $\mathcal{B}$ is invertible if and only if the formal determinant operator* $\det \mathcal{B}$ *is invertible on $L^2(\mathbb{R})$.*

Applying this result to the case of a 4×4 matrix, and given the commutativity of multiplication operators, we have that the operator $\widetilde{K}^a_{\Gamma_{\alpha,\beta,\theta}} - \lambda I$ is invertible if and only if we have that its formal determinant $\det((\widetilde{K}^a_{\Gamma_{\alpha,\beta,\theta}} - \lambda I))$ is an invertible operator also.

We can represent the formal determinant as a multiplication operator determined by the constant λ and the products of our $\mathcal{M}(j_{\omega,a})$ functions. Calculating $\det((\widetilde{K}^a_{\Gamma_{\alpha,\beta,\theta}} - \lambda I))$ gives us the following multiplication operator for $f \in L^2(\mathbb{R})$,

$$\det((\widetilde{K}^a_{\Gamma_{\alpha,\beta,\theta}} - \lambda I))f = (\lambda^4 + S\lambda^2 - T)f$$

where the functions S and T are given by

$$\begin{aligned} S(\xi) := &-\mathcal{M}(j_{\alpha,a})(\xi)^2 + \mathcal{M}(j_{\alpha+\theta,a})(\xi)^2 \\ &-\mathcal{M}(j_{\alpha+\theta+\beta,a})(\xi)^2 - \mathcal{M}(j_{\theta,a})(\xi)^2 \\ &+\mathcal{M}(j_{\theta+\beta,a})(\xi)^2 - \mathcal{M}(j_{\beta,a})(\xi)^2 \end{aligned}$$

$$\begin{aligned} T(\xi) = &\mathcal{M}(j_{\theta+\beta,a})(\xi)^2 \mathcal{M}(j_{\alpha+\theta,a})(\xi)^2 \\ &+\mathcal{M}(j_{\alpha+\theta+\beta,a})(\xi)^2 \mathcal{M}(j_{\theta,a})(\xi)^2 \\ &+\mathcal{M}(j_{\alpha,a})(\xi)^2 \mathcal{M}(j_{\beta,a})(\xi)^2 \\ &-2\mathcal{M}(j_{\theta+\beta,a})(\xi)\mathcal{M}(j_{\alpha+\theta,a})(\xi) \\ &\quad\times\mathcal{M}(j_{\alpha+\theta+\beta,a})(\xi) \\ &-2\mathcal{M}(j_{\alpha,a})(\xi)\mathcal{M}(j_{\theta+\beta,a})(\xi) \\ &\quad\times\mathcal{M}(j_{\alpha+\theta,a})(\xi)\mathcal{M}(j_{\beta,a})(\xi) \\ &+2\mathcal{M}(j_{\alpha,a})(\xi)\mathcal{M}(j_{\alpha+\theta+\beta,a})(\xi) \\ &\quad\times\mathcal{M}(j_{\theta,a})(\xi)\mathcal{M}(j_{\beta,a})(\xi). \end{aligned}$$

for $\xi \in \mathbb{R}$. Note that from this representation we can see that $(\lambda^4 + S\lambda^2 - T) \in L^\infty(\mathbb{R})$, since the same holds for each $\mathcal{M}(j_{\omega,a})$. Furthermore, from [4](Chp 2, p.44, Thm 2.1.4) we have the following result:

Lemma 5.6. *For $f \in L^2(\mathbb{R})$ and $m \in L^\infty(\mathbb{R})$ define the multiplication operator M by*

$$Mf(x) = m(x)f(x),$$

then we have that the spectrum of M is equal to the essential range of m, that is, the set of all $\lambda \in \mathbb{C}$ such that the set $E \subset \mathbb{R}$ defined by

$$E := \{x \in \mathbb{R} \mid |m(x) - \lambda| < \epsilon\},$$

has nonzero measure for all $\epsilon > 0$

Proof. For λ not in the essential range of m, we have that the set E is a null set, and that for $x \in \mathbb{R} \cap E^c$, $|m(x) - \lambda| \geq \epsilon$. Thus, defining the function $r_\lambda(x) := (\lambda - m(x))^{-1}$, we have that for such x,

$$\begin{aligned} |r_\lambda(x)| &= \left|(\lambda - m(x))^{-1}\right| \\ &= |\lambda - m(x)|^{-1} \\ &\leq \frac{1}{\epsilon}. \end{aligned}$$

That is, we have r_λ bounded almost everywhere or, more specifically, $r_\lambda \in L^\infty(\mathbb{R})$. Furthermore, for such x we see by inspection how $r_\lambda \circ (m(x) - \lambda) = (m(x) - \lambda) \circ r_\lambda = Id$. So the operator $(m(x) - \lambda)$ has a bounded a.e. inverse on $L^2(\mathbb{R})$ and thus λ does not belong to the spectrum of M.

If λ lies in the essential range of m then, taking $\epsilon = \frac{1}{n}$ the sets

$$S_n := \{x : |\lambda - m(x)| < \frac{1}{n}\}$$

have non zero measures for all n. If the measure of S_n is infinite then we replace S_n with a subset of non-zero, finite measure. Taking ϕ_n to be the characteristic function of the set S_n, we have $0 \neq \phi_n \in L^2(\mathbb{R})$ and

$$\|(\lambda - M)\phi_n\|_{L^2} \leq \frac{1}{n} \|\phi_n\|_{L^2}.$$

Since this holds for all $n \in \mathbb{N}$, and given $\frac{1}{n} \to 0$ as $n \to \infty$, we must have $\|(\lambda - M)\phi_n\|_{L^2} = 0$, hence there can exist no $p > 0$ for the operator such that $p\|\phi_n\|_{L^2} < \|(\lambda - M)\phi_n\|_{L^2}$. However, by [32] we know that being bounded from below is a necessary condition for $(\lambda - M)$ to be a bounded and invertible operator, and so $\lambda \in \sigma(M)$. $\square$

So given we can represent $\det((\widetilde{K}^a_{\Gamma_{\alpha,\beta,\theta}} - \lambda I))$ as a multiplication operator defined by a bounded function, the above result gives us that we can determine that λ is an element of the spectrum of the $K^a_{\Gamma_{\alpha,\beta,\theta}}$ if and only if it belongs to the essential range of the symbol of $\det((\widetilde{K}^a_{\Gamma_{\alpha,\beta,\theta}} - \lambda I))$.
Given the function defining the $\det((\widetilde{K}^a_{\Gamma_{\alpha,\beta,\theta}} - \lambda I))$ operator has the form of a polynomial in terms of λ and coefficients defined by the product of $\mathcal{M}(j_{\omega,a})$ functions, we can then re-write our multiplier function in factored form, as the product of functions $(\lambda - m(\xi))$, for $\xi \in \mathbb{R}$. Using the pairwise commutativity of these functions, carried over from $\mathcal{M}(j_{\omega,a})$, we then immediately have from our previous result that the spectrum of $\det((\widetilde{K}^a_{\Gamma_{\alpha,\beta,\theta}} - \lambda I))$ is given by those $m(\xi) \in \mathbb{C}$ that form the roots of said polynomial at $\xi \in \mathbb{R}$. Further, given we are taking the essential range of our $\det((\widetilde{K}^a_{\Gamma_{\alpha,\beta,\theta}} - \lambda I))$ operator, we must also include the limits of these roots as $|\xi|$ tends to ∞, using the *Riemman-Lebesgue Lemma*

Given we can define the Mellin transform in terms of the Fourier transform and $j_{\omega,a} \in L^1(\frac{dt}{t})$, we then similarly have that $\mathcal{M}(j_{\omega,a})(\xi)$ tends to 0 as $|\xi|$ tends to ∞. Finally by combining this result with our decomposition of $\det((\widetilde{K}^a_{\Gamma_{\alpha,\beta,\theta}} - \lambda I))$ we obtain,

$$\sigma(K^a_{\Gamma_{\alpha,\beta,\theta}}) = \left\{ m(\xi) = \pm\sqrt{\frac{-S(\xi) \pm \sqrt{S(\xi)^2 - 4T(\xi)}}{2}} : \xi \in \mathbb{R} \right\} \cup \{0\},$$

for S and T complex-valued functions, defined as above.

Having determined in Theorem 5.3 that for localizations of our matrix operator on $\Gamma_{\alpha,\beta,\theta}$ we may identify the essential spectrum of said localized operators with their compact perturbations, and now having determined the form of the spectrum of our matrix operator on the whole of $\Gamma_{\alpha,\beta,\theta}$ our next step is to determine to what degree the spectra of these localizations and by extension those of said compact perturbations coincides with the spectrum as determined above on the infinite double wedge.

To this end we make use of the results from [33], the details of which can be seen in Section 4, which we shall modify here for the scenario expressed in our research.

We begin by determining if we can apply these results to our operator. Given some interval $(a, b) \subset [0, 1]$ we recall the definition of a strip:

$$\Upsilon_{a,b} := \{z \in \mathbb{C} : \mathrm{Re}(z) \in (a, b)\}.$$

We fix $r := \min\{1 - b, a\}$, the radius of a disc centered at ξ on $\Upsilon_{\alpha,\beta}$, the boundary of which is given by the curve γ. We have that, by the Cauchy differentiation formula and the holomorphic nature of $\sin(z)$,

$$\begin{aligned}
\sup_{\Upsilon_{a,b}}\left|(1+|\xi|)^{\ell-1}\frac{\partial^\ell}{\partial\xi^\ell}\frac{\sin((\pi-\alpha)\xi)}{\sin(\pi\xi)}\right| =& \sup_{\Upsilon_{a,b}}\left|(1+|\xi|)^{\ell-1}\frac{\ell!}{2\pi i}\oint_\gamma\frac{\frac{\sin((\pi-\alpha)z)}{\sin(\pi z)}}{(z-\xi)^{\ell+1}}dz\right| \\
=&\frac{\ell!}{2\pi}\sup_{\Upsilon_{a,b}}\left|(1+|\xi|)^{\ell-1}\right|\left|\oint_\gamma\frac{\frac{\sin((\pi-\alpha)z)}{\sin(\pi z)}}{(z-\xi)^{\ell+1}}dz\right| \\
\leq&\frac{\ell!}{2\pi}\sup_{\Upsilon_{a,b}}\left|(1+|\xi|)^{\ell-1}\right|\oint_\gamma\left|\frac{\frac{\sin((\pi-\alpha)z)}{\sin(\pi z)}}{(z-\xi)^{\ell+1}}\right|dz \\
=&\frac{\ell!}{2\pi}\sup_{\Upsilon_{a,b}}\left|(1+|\xi|)^{\ell-1}\right|\oint_\gamma\frac{\left|\frac{\sin((\pi-\alpha)z)}{\sin(\pi z)}\right|}{|(z-\xi)^{\ell+1}|}dz
\end{aligned}$$

Here we observe the following:

$$\begin{aligned}
\frac{\sin((\pi-\alpha)z)}{\sin(\pi z)} &= \frac{e^{i(\pi-\alpha)z}-e^{-i(\pi-\alpha)z}}{e^{i\pi z}-e^{-i\pi z}} \\
&= \frac{e^{-i(\pi-\alpha)z}\big(e^{2i(\pi-\alpha)z}-1\big)}{e^{-i\pi z}\big(e^{2i\pi z)}-1\big)} \\
&= \frac{e^{i\alpha z}\big(e^{2i(\pi-\alpha)z}-1\big)}{\big(e^{2i\pi z}-1\big)}
\end{aligned}$$

where, as $\mathrm{Im}(z)$ tends to ∞ we have

$$\frac{(e^{2i(\pi-\alpha)z}-1)}{(e^{2i\pi z}-1)}\to\frac{0-1}{0-1}=1$$

and

$$e^{i\alpha z}\to 0.$$

Similarly we have,

$$\begin{aligned}
\frac{\sin((\pi-\alpha)z)}{\sin(\pi z)} &= \frac{e^{i(\pi-\alpha)z}-e^{-i(\pi-\alpha)z}}{e^{i\pi z}-e^{-i\pi z}} \\
&= \frac{e^{i(\pi-\alpha)z}\big(1-e^{-2i(\pi-\alpha)z}\big)}{e^{i\pi z}\big(1-e^{-2i\pi z}\big)}
\end{aligned}$$

$$= \frac{e^{-i\alpha z}(1 - e^{-2i(\pi-\alpha)z})}{(1 - e^{-2i\pi z})}$$

where, as $\mathrm{Im}(z)$ tends to $-\infty$ we have

$$\frac{(1 - e^{-2i(\pi-\alpha)z})}{(1 - e^{-2i\pi z})} \to \frac{1-0}{1-0} = 1$$

and

$$e^{-i\alpha z} \to 0.$$

So we have finiteness at the limits of $\mathrm{Im}(z)$.

In combination with the finite boundedness of $\mathrm{Re}(z)$, we have for some $C > 0$ that

$$\left| \frac{(e^{|2(\pi-\alpha)z|} - 1)}{(e^{|2\pi z|} - 1)} \right| \leq C$$

and thus, factoring out $\mathrm{Re}(z)$, we get

$$\begin{aligned} \left| \frac{\sin((\pi-\alpha)z)}{\sin(\pi z)} \right| &\leq C \frac{e^{|\mathrm{Im}(z)||\pi-\alpha|}}{e^{|\mathrm{Im}(z)||\pi|}} \\ &= Ce^{(|\pi-\alpha|-|\pi|)|\mathrm{Im}(z)|} \end{aligned}$$

and, given we must have the angle $\alpha \in (0, 2\pi)$, this implies our exponential must have negative sign. Hence,

$$\frac{\ell!}{2\pi} \sup_{\Upsilon_{a,b}} \left| (1+|\xi|)^{\ell-1} \right| \oint_\gamma \frac{\left| \frac{\sin((\pi-\alpha)z)}{\sin(\pi z)} \right|}{|(z-\xi)^{\ell+1}|} dz \leq \frac{\ell!}{2\pi} \sup_{\Upsilon_{a,b}} \left| (1+|\xi|)^{\ell-1} \right| \frac{Ce^{-\alpha(|\xi|-r)}}{r^{\ell+1}}.$$

Given the exponential decrease as $|\mathrm{Im}(z)|$ tends to ∞ must supercede the increase in the polynomial, we therefore must have

$$\sup_{\Upsilon_{a,b}} \left| (1+|\xi|)^{\ell-1} \frac{\partial^\ell}{\partial \xi^\ell} \frac{\sin((\pi-\alpha)\xi)}{\sin(\pi\xi)} \right| < \infty$$

from which it follows by the Definition 4.5 that

$$\frac{\sin((\pi-\alpha)\xi)}{\sin(\pi\xi)} \in \Theta^{-1}_{0,1}.$$

We recall how we determined in Lemma 5.2 that, taking

$$j_{\alpha,1}(s) = \frac{\sin(\omega)}{\pi(s^2+1-2s\cos(\omega))}$$

to be the kernel of the Mellin convolution representation of the entries in our operator matrix, we then must have that under the Mellin transform these entries have the form

$$\mathcal{M}(j_{\alpha,1}) = \frac{\sin(i\xi\,(\pi-\alpha))}{\sin(i\pi\xi)}.$$

Thus we must have that

$$\int_0^1 \frac{\sin(\omega)}{\pi(s^2+1-2s\cos(\omega))} f(s)\frac{ds}{s}$$

is a Hardy operator acting on f.

Substituting $i\xi$ for ξ, this gives us the Mellin transform of the N.P. adjoint operator kernel acting on the boundary of a wedge determined by angle α, and so we can conclude that the multiplication operators that define the elements of our operator matrix are also Hardy operators, meaning that we can apply the previously discussed results from [33] to our operator matrix.

We can now write the elements of our operator matrix $K^{loc}_{\Gamma_{\alpha,\beta,\theta}} - \lambda I$ in the manner described in Definition 4.6, as operators in $\mathcal{X}$. For $i,j \in \{1,2,3,4\}$ we have $K_0 := \widetilde{K}^{i,j}_{\Gamma_{\alpha,\beta,\theta}}$, $K_1 := 0$, $d(t) := 0$ and

$$c(t) := \begin{cases} -\lambda & i=j \\ 0 & i \neq j. \end{cases}$$

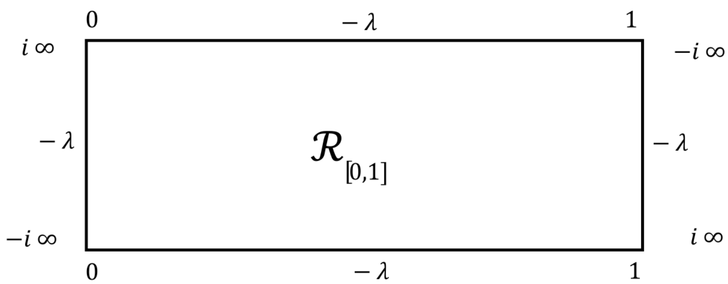
explicitly, we have:

$$\begin{aligned} A &:= (\chi \widetilde{K}_{\Gamma_{\alpha,\beta,\theta}} \chi - \lambda) \\ &= K^{loc}_{\Gamma_{\alpha,\beta,\theta}} - \lambda I + \text{compact} \end{aligned}$$

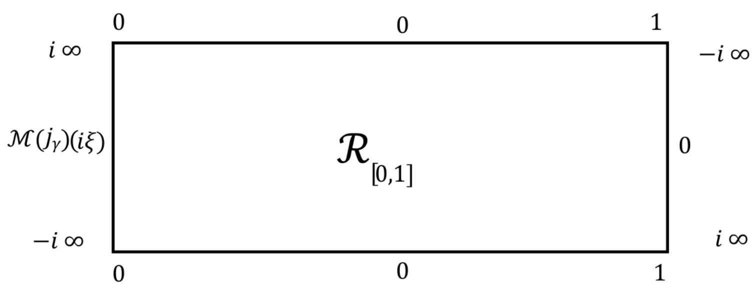
where our diagonal elements are all $-\lambda$ and the remaining elemnts are given by our wedge convolution operator $j_{\gamma,1}(t)$, where γ represents their defining angle.

As such, identifying the essential spectrum of our operator is now a matter of determining when we have that the operator matrix $A := (A^{i,j})_{i,j\in\{1,2,3,4\}}$ is Fredholm. As determined in [33] we have that this holds only when $\mathrm{Smbl}^{1/2}A$ is nonsingular on $\mathcal{R}_{[0,1]}$. We may determine the elements of $\mathrm{Smbl}^{1/2}A$ indiviually as $\mathrm{Smbl}^{1/2}A^{(i,j)}$ for $(i,j) \in$

$\{1,2,3,4\}^2$. For $i = j$ we then have $\mathrm{Smbl}^{1/2} A^{(i,i)}$ given by



and for $i \neq j$,



where γ represents the angle defining $A^{(i,j)}$. Note how in both cases, we are evaluating on the line $i\mathbb{R}$ rather than $\frac{1}{2} + i\mathbb{R}$ as in Section 4.7. The shift of line was already accounted for in Section 5.1 by our choice of weighting ($a = 0$), see Lemma 5.1.

It remains to determine when we have $\det(\mathrm{Smbl}^{1/2} A) \neq 0$, as, by Theorem 4.8, A is only Fredholm when $\mathrm{Smbl}^{1/2} A$ is nonsingular. Because of the manner in which we have defined our operator matrix, this immediately reduces to determining

$$\begin{aligned}\det(\mathrm{Smbl}^{1/2} A) &= \det\left(\begin{bmatrix} -\lambda & \widetilde{K}_\alpha & \widetilde{K}_{\alpha+\theta} & \widetilde{K}_{\alpha+\theta+\beta} \\ \widetilde{K}_\alpha & -\lambda & -\widetilde{K}_\theta & -\widetilde{K}_{\theta+\beta} \\ -\widetilde{K}_{\alpha+\theta} & -\widetilde{K}_\theta & \lambda & \widetilde{K}_\beta \\ \widetilde{K}_{\alpha+\theta+\beta} & \widetilde{K}_{\theta+\beta} & \widetilde{K}_\beta & -\lambda \end{bmatrix}\right) \\ &= \det(\widetilde{K}_{\Gamma_{\alpha,\beta,\theta}} - \lambda I)\end{aligned}$$

where $\widetilde{K}_{\alpha+\theta+\beta} = \widetilde{K}^0_{\alpha+\theta+\beta}$ (see the equation for $\widetilde{K}^a_{\alpha+\theta+\beta}$ from the previous section). Thus we have the following result,

Theorem 5.7. *For $K^{loc}_{\Gamma_{\alpha,\beta,\theta}}$ the localization of the N.P. adjoint operator matrix on $\Gamma_{\alpha,\beta,\theta}$.*

Then we have that the operator and its compact perturbations have essential spectrum given by

$$\sigma_{ess}(K^{loc}_{\Gamma_{\alpha,\beta,\theta}}) = \sigma_{ess}(K^{loc}_{\Gamma_{\alpha,\beta,\theta}} + Comp) = \sigma(K_{\Gamma_{\alpha,\beta,\theta}}),$$

the formula for which is given by,

$$\sigma(K_{\Gamma_{\alpha,\beta,\theta}}) = \left\{ m(\xi) = \pm\sqrt{\frac{-S(\xi) \pm \sqrt{S(\xi)^2 - 4T(\xi)}}{2}} : \xi \in \mathbb{R} \right\} \cup \{0\},$$

Where, S and T are given by

$$\begin{aligned} S(\xi) = &-\mathcal{M}(j_\alpha)(\xi)^2 + \mathcal{M}(j_{\alpha+\theta})(\xi)^2 \\ &-\mathcal{M}(j_{\alpha+\theta+\beta})(\xi)^2 - \mathcal{M}(j_\theta)(\xi)^2 \\ &+\mathcal{M}(j_{\theta+\beta})(\xi)^2 - \mathcal{M}(j_\beta)(\xi)^2 \end{aligned}$$

$$\begin{aligned} T(\xi) = &\mathcal{M}(j_{\theta+\beta})(\xi)^2\mathcal{M}(j_{\alpha+\theta})(\xi)^2 \\ &+\mathcal{M}(j_{\alpha+\theta+\beta})(\xi)^2\mathcal{M}(j_\theta)(\xi)^2 \\ &+\mathcal{M}(j_\alpha)(\xi)^2\mathcal{M}(j_\beta)(\xi)^2 \\ &-2\mathcal{M}(j_{\theta+\beta})(\xi)\mathcal{M}(j_{\alpha+\theta})(\xi) \\ &\quad\times\mathcal{M}(j_{\alpha+\theta+\beta})(\xi) \\ &-2\mathcal{M}(j_\alpha)(\xi)\mathcal{M}(j_{\theta+\beta})(\xi) \\ &\quad\times\mathcal{M}(j_{\alpha+\theta})(\xi)\mathcal{M}(j_\beta)(\xi) \\ &+2\mathcal{M}(j_\alpha)(\xi)\mathcal{M}(j_{\alpha+\theta+\beta})(\xi) \\ &\quad\times\mathcal{M}(j_\theta)(\xi)\mathcal{M}(j_\beta)(\xi). \end{aligned}$$

and for a given angle γ we have,

$$\mathcal{M}(j_\gamma)(\xi) = \frac{\sin((\pi-\gamma)(i\xi - 1/2))}{\sin(\pi(i\xi - 1/2))}$$

.

6 The spectral properties of the N.P. operator on the space of continuous functions

We shall now discuss the application of Kress's techniques (see Section 2.4) as used on continuously parameterized curvilinear polygons to the scenario of bow-tie curves. Specifically, we will be using those same techniques to examine the possibility of determining an upper bound for the essential spectrum of a modified form of the N.P. operator, when acting on bow-tie curves, parameterized as in Section 5.2.

As is the case in Kress, we have that our parameterization is composed of a number of class C^2 closed arcs that either meet at one of the two internal corners or are such that within a sufficiently small neighbourhood of their meeting point the section of boundary contained therein is also of class C^2.

Furthermore, following our previous parameterization, we also have that within sufficiently small neighbourhoods of our corners each arc is a straight-line segment, again, matching the localization of our previous work on the boundaries of infinitely large wedges.

In this manner we meet the required space/boundary conditions in order to apply Kress's techniques to our bow-tie curve.

A key difference between our set and the curvilinear polygons used by Kress is the occurrence of our corners. Specifically, while we do have that two distinct internal corners exist, their vertices coincide meaning that we a have a single point of discontinuity along the boundary.

Taking this into account, we must formulate an operator matrix on $C(\widetilde{\Gamma}_{\alpha,\beta,\theta})$ analogous to those used in our prior resuts, with entries based on Kress's modification to the N.P. operator for each pair of curves in our parameterization. Here each entry will be treated as a single corner, as in the scenario described by Kress.

Given we will be determining a matrix operator, we must ascertain a suitatably analogous space on which our operator can act. For $\psi \in C(\widetilde{\Gamma}_{\alpha,\beta,\theta})$ we have

$$\begin{aligned} \phi &: [0,1] \to \mathbb{R}^4, \\ x &\to [\psi(\gamma_1(x)), \psi(\gamma_2(x)), \psi(\gamma_3(x)), \psi(\gamma_4(x))]^T . \end{aligned}$$

By the properties defined on each γ_i, we can identify an isomorphism with a vector subspace of $C[0,1]^4$ that our operator matrix acts upon:

$$C(\widetilde{\Gamma}_{\alpha,\beta,\theta}) \simeq \left\{ \phi = (\psi_1, \psi_2, \psi_3, \psi_4) \in C[0,1]^4 \middle| \begin{matrix} \psi_1(0) = \psi_2(0) = \psi_3(0) = \psi_4(0), \\ \psi_1(1) = \psi_2(1), \psi_3(1) = \psi_4(1) \end{matrix} \right\}.$$

Furthermore, we observe here that given the specifics of our parameterization of $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ we have that our shared vertex occurs precisely when $x = 0$.

With this in mind we define our matrix operator on $C(\widetilde{\Gamma}_{\alpha,\beta,\theta})$ by,

$$\tilde{K}\phi(x) := \begin{bmatrix} \tilde{K}_{1,1} & \tilde{K}_{1,2} & \tilde{K}_{1,3} & \tilde{K}_{1,4} \\ \tilde{K}_{2,1} & \tilde{K}_{2,2} & \tilde{K}_{2,3} & \tilde{K}_{2,4} \\ \tilde{K}_{3,1} & \tilde{K}_{3,2} & \tilde{K}_{3,3} & \tilde{K}_{3,4} \\ \tilde{K}_{4,1} & \tilde{K}_{4,2} & \tilde{K}_{4,3} & \tilde{K}_{4,4} \end{bmatrix} \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix} \tag{6.1}$$

where, for $x \in [0,1]$, we have

$$\tilde{K}_{i,j}\psi_i(x) = \begin{cases} K\psi_i(x), & x \neq 0, \\ K\psi_i(x) + (\frac{\epsilon}{\pi} - 1)\psi_i(0), & x = 0. \end{cases}$$

Here K is the N.P. operator given by

$$K(\psi_i(x)) := \int_{[0,1]} \frac{\partial\Phi(\gamma_i(x), \gamma_j(y))}{\partial\nu(\gamma_j(y))} \psi_i(y) ds(\gamma_j(y)), \quad x, y \in [0,1].$$

and, for $x = 0$, $\tilde{K}_{i,j}\psi_i(x)$ is determined by the sum of the interior and exterior limits given by Kress in his modification of the jump relations of the double layer potential for boundaries with corners (Section 2.4), where ϵ represents the angle between the line-segments on γ_i and γ_j. Similarly, we reintroduce Kress's localization K_n of the N.P. operator on $C(\widetilde{\Gamma}_{\alpha,\beta,\theta})$ as an operator matrix,

$$K_n(\phi(x)) := \begin{bmatrix} K_n^{1,1} & K_n^{1,2} & K_n^{1,3} & K_n^{1,4} \\ K_n^{2,1} & K_n^{2,2} & K_n^{2,3} & K_n^{2,4} \\ K_n^{3,1} & K_n^{3,2} & K_n^{3,3} & K_n^{3,4} \\ K_n^{4,1} & K_n^{4,2} & K_n^{4,3} & K_n^{4,4} \end{bmatrix} \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix}$$

where

$$K_n^{i,j}(\psi_i(x)) := \int_{[0,1]} h(n|\gamma_i(x) - \gamma_j(y)|) \frac{\partial\Phi(\gamma_i(x), \gamma_j(y))}{\partial\nu(\gamma_j(y))} \psi_i(y) ds(\gamma_j(y)), \quad x, y \in [0,1].$$

Here, again, we have that h is a continuous function on $[0, \infty)$, such that $h(x) = 0$ for $x \in [0, \frac{1}{2}]$ and $h(x) = 1$ for $x > 1$.
Finally we shall define the operator matrix $\widetilde{K}_n := \widetilde{K} - K_n$ to have entries of the form $\widetilde{K}_n^{i,j} := \widetilde{K}_{i,j} - K_n^{i,j}$.

We now wish to examine how the operator matrix $\widetilde{K}_n$ acts on $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ within neighbourhoods of the straight edges at each corner. In doing so, we may again determine an over-all upper bound to the essential norm of our operator matrix.

6.1 Determining the upper bound on $C(\widetilde{\Gamma}_{\alpha,\beta,\theta})$

Following Kress's method, we set $n \in \mathbb{N}$ large enough such that for $x \in (0, 1]$ sufficiently small, on each pair γ_i, γ_j the ball $B[\gamma_i(x), 1/n]$ intersects with γ_i and γ_j only on those parts of the curve that are segments of straight edges.

Considering our matrix entries $\widetilde{K}_n^{i,j}$ as they act on $(\gamma_i \cup \gamma_j) \cap B[\gamma_i(x), 1/n]$, we again observe that for $i = j$, the points $\gamma_i(x)$ and $\gamma_j(y)$ occur on the same segment of straight line for all $x, y \in [0, 1]$, and thus the normal vector is perpendicular to the edge at both points, giving us

$$\begin{aligned} \frac{\partial \Phi(\gamma_i(x), \gamma_j(y))}{\partial \nu(\gamma_j(y))} &= \frac{\langle \nu(\gamma_j(y)), (\gamma_i(x) - \gamma_j(y)) \rangle}{|\gamma_i(x) - \gamma_j(y)|^n} \\ &= 0. \end{aligned}$$

As such we may rewrite our operator matrix as

$$\widetilde{K}_n \phi(x) = \begin{bmatrix} 0 & \widetilde{K}_n^{1,2} & \widetilde{K}_n^{1,3} & \widetilde{K}_n^{1,4} \\ \widetilde{K}_n^{2,1} & 0 & \widetilde{K}_n^{2,3} & \widetilde{K}_n^{2,4} \\ \widetilde{K}_n^{3,1} & \widetilde{K}_n^{3,2} & 0 & \widetilde{K}_n^{3,4} \\ \widetilde{K}_n^{4,1} & \widetilde{K}_n^{4,2} & \widetilde{K}_n^{4,3} & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix}$$

where, for $i \neq j$ the second scenario holds for all $K_n^{i,j}$.

Furthermore, given our choice of n has $B(\gamma_i(x), 1/n)$ intersect with those parts of γ_i and γ_j which are segments of straight edges, we have for each $\widetilde{K}_n^{i,j}$ a scenario analogous to that discussed in Section 5.2 of localizations of $\Gamma_{\alpha,\beta,\theta}$. Thus we may rewrite $\widetilde{K}_n$ where each entry is given in terms of the angle of separation between the two edges being acted upon:

$$\widetilde{K}_n\phi(x) = \begin{bmatrix} 0 & \widetilde{K}_n^{\alpha} & \widetilde{K}_n^{\alpha+\theta} & \widetilde{K}_n^{\alpha+\theta+\beta} \\ \widetilde{K}_n^{\alpha} & 0 & -\widetilde{K}_n^{\theta} & -\widetilde{K}_n^{\theta+\beta} \\ -\widetilde{K}_n^{\alpha+\theta} & -\widetilde{K}_n^{\theta} & 0 & \widetilde{K}_n^{\beta} \\ \widetilde{K}_n^{\alpha+\theta+\beta} & \widetilde{K}_n^{\theta+\beta} & \widetilde{K}_n^{\beta} & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix}.$$

Applying the ∞-norm, for a given function ϕ we have, with Kress's results for a single 'corner' that

$$\begin{aligned}
\|(\widetilde{K}_n)\phi(x)\|_\infty \;&\leq\; \max\left\{\begin{matrix} |\widetilde{K}_n^{\alpha}\psi_2(x)| + |\widetilde{K}_n^{\alpha+\theta}\psi_3(x)| + |\widetilde{K}_n^{\alpha+\theta+\beta}\psi_4(x)|, \\ |\widetilde{K}_n^{\alpha}\psi_1(x)| + |\widetilde{K}_n^{\theta}\psi_3(x)| + |\widetilde{K}_n^{\theta+\beta}\psi_4(x)|, \\ |\widetilde{K}_n^{\alpha+\theta}\psi_1(x)| + |\widetilde{K}_n^{\theta}\psi_2(x)| + |\widetilde{K}_n^{\beta}\psi_4(x)|, \\ |\widetilde{K}_n^{\alpha+\theta+\beta}\psi_1(x)| + |\widetilde{K}_n^{\theta+\beta}\psi_2(x)| + |\widetilde{K}_n^{\beta}\psi_3(x)| \end{matrix}\right\} \\
&\leq\; \max\left\{\begin{matrix} \left|1-\frac{\alpha}{\pi}\right||\psi_2(x)| + \left|1-\frac{\alpha+\theta}{\pi}\right||\psi_3(x)| + \left|1-\frac{\alpha+\theta+\beta}{\pi}\right||\psi_4(x)|, \\ \left|1-\frac{\alpha}{\pi}\right||\psi_1(x)| + \left|1-\frac{\theta}{\pi}\right||\psi_3(x)| + \left|1-\frac{\theta+\beta}{\pi}\right||\psi_4(x)|, \\ \left|1-\frac{\alpha+\theta}{\pi}\right||\psi_1(x)| + \left|1-\frac{\theta}{\pi}\right||\psi_2(x)| + \left|1-\frac{\beta}{\pi}\right||\psi_4(x)|, \\ \left|1-\frac{\alpha+\theta+\beta}{\pi}\right||\psi_1(x)| + \left|1-\frac{\theta+\beta}{\pi}\right||\psi_2(x)| + \left|1-\frac{\beta}{\pi}\right||\psi_3(x)| \end{matrix}\right\} \\
&\leq\; \max\left\{\begin{matrix} \left|1-\frac{\alpha}{\pi}\right| + \left|1-\frac{\alpha+\theta}{\pi}\right| + \left|1-\frac{\alpha+\theta+\beta}{\pi}\right|, \\ \left|1-\frac{\alpha}{\pi}\right| + \left|1-\frac{\theta}{\pi}\right| + \left|1-\frac{\theta+\beta}{\pi}\right|, \\ \left|1-\frac{\alpha+\theta}{\pi}\right| + \left|1-\frac{\theta}{\pi}\right| + \left|1-\frac{\beta}{\pi}\right|, \\ \left|1-\frac{\alpha+\theta+\beta}{\pi}\right| + \left|1-\frac{\theta+\beta}{\pi}\right| + \left|1-\frac{\beta}{\pi}\right| \end{matrix}\right\} \max\left\{\begin{matrix} |\psi_1(x)|, \\ |\psi_2(x)|, \\ |\psi_3(x)|, \\ |\psi_4(x)| \end{matrix}\right\} \\
&=\; \max\left\{\begin{matrix} \left|1-\frac{\alpha}{\pi}\right| + \left|1-\frac{\alpha+\theta}{\pi}\right| + \left|1-\frac{\alpha+\theta+\beta}{\pi}\right|, \\ \left|1-\frac{\alpha}{\pi}\right| + \left|1-\frac{\theta}{\pi}\right| + \left|1-\frac{\theta+\beta}{\pi}\right|, \\ \left|1-\frac{\alpha+\theta}{\pi}\right| + \left|1-\frac{\theta}{\pi}\right| + \left|1-\frac{\beta}{\pi}\right|, \\ \left|1-\frac{\alpha+\theta+\beta}{\pi}\right| + \left|1-\frac{\theta+\beta}{\pi}\right| + \left|1-\frac{\beta}{\pi}\right| \end{matrix}\right\} \|\phi(x)\|_\infty
\end{aligned}$$

For brevity we shall hence denote this upper bound for $\widetilde{K}_n$ by M.

Given this holds for all $x_0 \in B[x, 1/n] \setminus \{0\}$ when x is sufficiently small and given the continuity established by Kress for each 'corner', we immediately have that this upper bound must also apply at the corner, that is, when $x = 0$. Thus, we have for our choice of n that

$$\|\widetilde{K}_n\|_\infty \leq M.$$

Furthermore, as each $K_n^{i,j}$ can be characterised as the limit of a sequence of finite-rank operators, we can equally determine the operator matrix K_n as the limit of a sequence of operator matrices, the entries of each term in the sequence corresponding to the same term in the sequences that tend to the entries of K_n.

The rank of each term in the sequence of operator-matrices being the sum of the ranks of each matrix entry in that term, each of which is finite, and there being only a finite number of entries gives us that each term in the operator-matrix sequence must be of finite rank. Hence, we also must have K_n is a compact operator.

Thus, we have that the essential norm of $\widetilde{K}$ is similarly bounded, that is

$$\|\widetilde{K}\|_{ess} \leq M.$$

It seems likelly that this bound is sharp, that is

$$\|\widetilde{K}\|_{ess} = M.$$

however we do not prove this here, instead we begin examining specifically when we have that said bound is less than or equal to one.

6.2 Subcases of the upper bound

Note that in each case we work with the assumption that none of our angles can be 0. In addition to this given we cannot have the sum of our angles exceed 2π, we observe that we cannot have that more than one angle, or a pair of angles exceeds π in magnitude.The remaining two angles or one angle respectively cannot exceed π either individually or as a sum.

We will examine each case individually, demonstrating how a contradiction arises in all but the final case.

For $\alpha + \theta + \beta \leq \pi$

We have, under our assumption, that the four components that comprise M can be simplified as follows:

$$\begin{aligned} 1 - \frac{\alpha}{\pi} + 1 - \frac{\alpha+\theta}{\pi} + 1 - \frac{\alpha+\theta+\beta}{\pi} &= 3 - \frac{3\alpha + 2\theta + \beta}{\pi}, \\ 1 - \frac{\alpha}{\pi} + 1 - \frac{\theta}{\pi} + 1 - \frac{\theta+\beta}{\pi} &= 3 - \frac{\alpha + 2\theta + \beta}{\pi}, \\ 1 - \frac{\alpha+\theta}{\pi} + 1 - \frac{\theta}{\pi} + 1 - \frac{\beta}{\pi} &= 3 - \frac{\alpha + 2\theta + \beta}{\pi}, \\ 1 - \frac{\alpha+\theta+\beta}{\pi} + 1 - \frac{\theta+\beta}{\pi} + 1 - \frac{\beta}{\pi} &= 3 - \frac{\alpha + 2\theta + 3\beta}{\pi}. \end{aligned}$$

Furthermore, our assumption necessitates that each of these components is non-negative. Given we wish to consider the case when each such component is bounded from above by 1, we then consider

$$\begin{aligned} 3 - \frac{3\alpha + 2\theta + \beta}{\pi} &\in [0, 1], \\ 3 - \frac{\alpha + 2\theta + \beta}{\pi} &\in [0, 1], \\ 3 - \frac{\alpha + 2\theta + 3\beta}{\pi} &\in [0, 1], \end{aligned}$$

which, by rearrangement, gives us

$$\begin{aligned} 3\alpha + 2\theta + \beta &\in [2\pi, 3\pi], \\ \alpha + 2\theta + \beta &\in [2\pi, 3\pi], \\ \alpha + 2\theta + 3\beta &\in [2\pi, 3\pi]. \end{aligned}$$

However, by our assumption $\alpha + \theta + \beta \leq \pi$, and so

$$\alpha + 2\theta + \beta \in [2\pi, 3\pi] \Rightarrow \theta \geq \pi \Rightarrow \alpha + \theta + \beta > \pi.$$

This is a contradiction; thus we cannot have an overall upper bound of 1 for $\alpha + \theta + \beta \leq \pi$.

For $\alpha > \pi$

We have, under our assumption, that the four components that comprise M can be simplified as follows:

$$\begin{aligned}
\frac{\alpha}{\pi} - 1 + \frac{\alpha+\theta}{\pi} - 1 + \frac{\alpha+\theta+\beta}{\pi} - 1 &= \frac{3\alpha+2\theta+\beta}{\pi} - 3 \\
\frac{\alpha}{\pi} - 1 + 1 - \frac{\theta}{\pi} + 1 - \frac{\theta+\beta}{\pi} &= 1 - \frac{2\theta+\beta-\alpha}{\pi} \\
\frac{\alpha+\theta}{\pi} - 1 + 1 - \frac{\theta}{\pi} + 1 - \frac{\beta}{\pi} &= 1 - \frac{\beta-\alpha}{\pi} \\
\frac{\alpha+\theta+\beta}{\pi} - 1 + 1 - \frac{\theta+\beta}{\pi} + 1 - \frac{\beta}{\pi} &= 1 - \frac{\beta-\alpha}{\pi}
\end{aligned}$$

Furthermore, our assumption necessitates that each of these components is non-negative. Given we wish to consider the case when each such component is bounded from above by 1, we then consider

$$\begin{aligned}
\frac{3\alpha+2\theta+\beta}{\pi} - 3 &\in [0,1], \\
1 - \frac{2\theta+\beta-\alpha}{\pi} &\in [0,1], \\
1 - \frac{\beta-\alpha}{\pi} &\in [0,1],
\end{aligned}$$

which, by rearrangement, gives us

$$\begin{aligned}
3\alpha+2\theta+\beta &\in [3\pi, 4\pi], \\
\alpha+2\theta+\beta &\in [0, \pi], \\
\beta-\alpha &\in [0, \pi].
\end{aligned}$$

However, by our assumption $\alpha > \pi$, and so

$$\beta - \alpha \in [0,\pi] \Rightarrow \beta > \alpha \Rightarrow \beta > \pi \Rightarrow \alpha + \beta + \theta > 2\pi.$$

This is a contradiction; thus we cannot have an overall upper bound of 1 for $\alpha > \pi$.

For $\beta > \pi$

We have, under our assumption, that the four components that comprise M can be simplified as follows:

$$\begin{aligned}1 - \frac{\alpha}{\pi} + 1 - \frac{\alpha+\theta}{\pi} + \frac{\alpha+\theta+\beta}{\pi} - 1 &= 1 - \frac{\alpha-\beta}{\pi}\\ 1 - \frac{\alpha}{\pi} + 1 - \frac{\theta}{\pi} + \frac{\theta+\beta}{\pi} - 1 &= 1 - \frac{\alpha-\beta}{\pi}\\ 1 - \frac{\alpha+\theta}{\pi} + 1 - \frac{\theta}{\pi} + \frac{\beta}{\pi} - 1 &= 1 - \frac{\alpha+2\theta-\beta}{\pi}\\ \frac{\alpha+\theta+\beta}{\pi} - 1 + \frac{\theta+\beta}{\pi} - 1 + \frac{\beta}{\pi} - 1 &= \frac{\alpha+2\theta+3\beta}{\pi} - 3\end{aligned}$$

Furthermore, our assumption necessitates that each of these components is non-negative. Given we wish to consider the case when each such component is bounded from above by 1, we then consider

$$\begin{aligned}1 - \frac{\alpha-\beta}{\pi} &\in [0,1],\\ 1 - \frac{\alpha+2\theta-\beta}{\pi} &\in [0,1],\\ \frac{\alpha+2\theta+3\beta}{\pi} - 3 &\in [0,1],\end{aligned}$$

which, by rearrangement, gives us

$$\begin{aligned}\alpha - \beta &\in [0,\pi],\\ \alpha + 2\theta + \beta &\in [0,\pi],\\ \alpha + 2\theta + 3\beta &\in [3\pi, 4\pi].\end{aligned}$$

However, by our assumption $\beta > \pi$, and so

$$\alpha - \beta \in [0,\pi] \Rightarrow \alpha > \beta \Rightarrow \alpha > \pi \Rightarrow \alpha + \beta + \theta > 2\pi.$$

This is a contradiction; thus we cannot have an overall upper bound of 1 for $\beta > \pi$.

For $\theta > \pi$

We have, under our assumption, that the four components that comprise M can be simplified as follows:

$$
\begin{aligned}
1 - \frac{\alpha}{\pi} + \frac{\alpha+\theta}{\pi} - 1 + \frac{\alpha+\theta+\beta}{\pi} - 1 &= -1 + \frac{\alpha+2\theta+\beta}{\pi} \\
1 - \frac{\alpha}{\pi} + \frac{\theta}{\pi} - 1 + \frac{\theta+\beta}{\pi} - 1 &= -1 + \frac{\beta+2\theta-\alpha}{\pi} \\
\frac{\alpha+\theta}{\pi} - 1 + \frac{\theta}{\pi} - 1 + 1 - \frac{\beta}{\pi} &= -1 + \frac{\alpha+2\theta-\beta}{\pi} \\
\frac{\alpha+\theta+\beta}{\pi} - 1 + \frac{\theta+\beta}{\pi} - 1 + 1 - \frac{\beta}{\pi} &= -1 + \frac{\alpha+2\theta+\beta}{\pi}
\end{aligned}
$$

Furthermore, our assumption necessitates that each of these components is non-negative. Given we wish to consider the case when each such component is bounded from above by 1, we then consider

$$
\begin{aligned}
-1 + \frac{\alpha+2\theta+\beta}{\pi} &\in [0,1], \\
-1 + \frac{\beta+2\theta-\alpha}{\pi} &\in [0,1],
\end{aligned}
$$

which, by rearrangement, gives us

$$
\begin{aligned}
\alpha + 2\theta + \beta &\in [\pi, 2\pi], \\
\beta + 2\theta - \alpha &\in [\pi, 2\pi].
\end{aligned}
$$

However, by our assumption $\theta > \pi$, and so

$$2\theta > 2\pi \Rightarrow \theta \notin [\pi, 2\pi].$$

This is a contradiction; thus we cannot have an overall upper bound of 1 for $\theta > \pi$.

For $\alpha + \theta > \pi$, $\alpha \leq \pi$, $\theta \leq \pi$, $\alpha + \beta \leq \pi$, $\theta + \beta \leq \pi$

We have, under our assumption, that the four components that comprise M can be simplified as follows:

$$\begin{aligned}
1 - \frac{\alpha}{\pi} + \frac{\alpha+\theta}{\pi} - 1 + \frac{\alpha+\theta+\beta}{\pi} - 1 &= -1 + \frac{\alpha+2\theta+\beta}{\pi} \\
1 - \frac{\alpha}{\pi} + 1 - \frac{\theta}{\pi} + 1 - \frac{\theta+\beta}{\pi} &= 3 - \frac{\alpha+2\theta+\beta}{\pi} \\
\frac{\alpha+\theta}{\pi} - 1 + 1 - \frac{\theta}{\pi} + 1 - \frac{\beta}{\pi} &= 1 + \frac{\alpha-\beta}{\pi} \\
\frac{\alpha+\theta+\beta}{\pi} - 1 + 1 - \frac{\theta+\beta}{\pi} + 1 - \frac{\beta}{\pi} &= 1 + \frac{\alpha-\beta}{\pi}
\end{aligned}$$

Furthermore, our assumption necessitates that each of these components is non-negative. Given we wish to consider the case when each such component is bounded from above by 1, we then consider

$$\begin{aligned}
-1 + \frac{\alpha+2\theta+\beta}{\pi} &\in [0,1], \\
3 - \frac{\alpha+2\theta+\beta}{\pi} &\in [0,1], \\
1 + \frac{\alpha-\beta}{\pi} &\in [0,1],
\end{aligned}$$

which, by rearrangement, gives us

$$\begin{aligned}
\alpha + 2\theta + \beta &\in [\pi, 2\pi], \\
\alpha + 2\theta + \beta &\in [2\pi, 3\pi], \Rightarrow \\
\beta - \alpha &\in [0, \pi].
\end{aligned}
\quad
\begin{aligned}
\alpha + 2\theta + \beta &= 2\pi, \\
\beta - \alpha &\in [0, \pi].
\end{aligned}$$

By our assumption $\alpha + \theta > \pi$, and so

$$\alpha + 2\theta + \beta = 2\pi \Rightarrow \beta + \theta < \pi \Rightarrow \alpha > \beta \Rightarrow \alpha = \beta.$$

However,

$$\alpha = \beta \Rightarrow \alpha + \theta = \theta + \beta > \pi \Rightarrow \alpha + 2\theta + \beta > 2\pi.$$

This is a contradiction.

For $\theta+\beta>\pi$, $\theta\le\pi$, $\beta\le\pi$, $\alpha+\beta\le\pi$, $\alpha+\theta\le\pi$

We have, under our assumption, that the four components that comprise M can be simplified as follows:

$$\begin{aligned}
1-\frac{\alpha}{\pi}+1-\frac{\alpha+\theta}{\pi}+\frac{\alpha+\theta+\beta}{\pi}-1&=1-\frac{\alpha-\beta}{\pi}\\
1-\frac{\alpha}{\pi}+1-\frac{\theta}{\pi}+\frac{\theta+\beta}{\pi}-1&=1-\frac{\alpha-\beta}{\pi}\\
1-\frac{\alpha+\theta}{\pi}+1-\frac{\theta}{\pi}+1-\frac{\beta}{\pi}&=3-\frac{\alpha+2\theta+\beta}{\pi}\\
\frac{\alpha+\theta+\beta}{\pi}-1+\frac{\theta+\beta}{\pi}-1+1-\frac{\beta}{\pi}&=-1+\frac{\alpha+2\theta+\beta}{\pi}
\end{aligned}$$

Furthermore, our assumption necessitates that each of these components is non-negative. Given we wish to consider the case when each such component is bounded from above by 1, we then consider

$$\begin{aligned}
1-\frac{\alpha-\beta}{\pi}&\in[0,1],\\
3-\frac{\alpha+2\theta+\beta}{\pi}&\in[0,1],\\
-1+\frac{\alpha+2\theta+\beta}{\pi}&\in[0,1],
\end{aligned}$$

which, by rearrangement, gives us

$$\begin{aligned}
&\alpha-\beta\in[0,\pi] & &\\
&\alpha+2\theta+\beta\in[2\pi,3\pi], \Rightarrow & &\begin{aligned}\alpha-\beta&\in[0,\pi],\\ \alpha+2\theta+\beta&=2\pi.\end{aligned}\\
&\alpha+2\theta+\beta\in[\pi,2\pi], & &
\end{aligned}$$

By our assumption $\theta+\beta>\pi$, and so

$$\alpha+2\theta+\beta=2\pi\Rightarrow\alpha+\theta<\pi\Rightarrow\beta>\alpha\Rightarrow\alpha=\beta.$$

However,

$$\alpha=\beta\Rightarrow\alpha+\theta=\theta+\beta>\pi\Rightarrow\alpha+2\theta+\beta>2\pi.$$

This is a contradiction.

For $\alpha+\beta>\pi$, $\alpha+\theta>\pi$, $\theta+\beta>\pi$ where $\alpha\leq\pi$, $\beta\leq\pi$, $\theta\leq\pi$

We have, under our assumption, that the four components that comprise M can be simplified as follows:

$$\begin{aligned}1-\frac{\alpha}{\pi}+\frac{\alpha+\theta}{\pi}-1+\frac{\alpha+\theta+\beta}{\pi}-1&=-1+\frac{\alpha+2\theta+\beta}{\pi}\\ 1-\frac{\alpha}{\pi}+1-\frac{\theta}{\pi}+\frac{\theta+\beta}{\pi}-1&=1-\frac{\beta-\alpha}{\pi}\\ \frac{\alpha+\theta}{\pi}-1+1-\frac{\theta}{\pi}+1-\frac{\beta}{\pi}&=1-\frac{\alpha-\beta}{\pi}\\ \frac{\alpha+\theta+\beta}{\pi}-1+\frac{\theta+\beta}{\pi}-1+1-\frac{\beta}{\pi}&=-1+\frac{\alpha+2\theta+\beta}{\pi}\end{aligned}$$

Furthermore, our assumption necessitates that each of these components is non-negative. Given we wish to consider the case when each such component is bounded from above by 1, we then consider

$$\begin{aligned}-1+\frac{\alpha+2\theta+\beta}{\pi}&\in[0,1],\\ 1-\frac{\beta-\alpha}{\pi}&\in[0,1],\\ 1-\frac{\alpha-\beta}{\pi}&\in[0,1],\end{aligned}$$

which, by rearrangement, gives us

$$\begin{aligned}\alpha+2\theta+\beta&\in[\pi,2\pi]\\ \alpha-\beta&\in[0,\pi]\\ \beta-\alpha&\in[0,\pi]\end{aligned}\quad\Rightarrow\quad\begin{aligned}\alpha&=\beta\\ \alpha+2\theta+\beta&\in[\pi,2\pi]\end{aligned}$$

However, by assumption $\alpha+\theta>\pi$ and $\theta+\beta>\pi$, so

$$\alpha+\theta+\theta+\beta>2\pi.$$

This is a contradiction.

For $\alpha+\beta>\pi$, $\alpha+\theta>\pi$, where $\alpha\leq\pi$, $\beta\leq\pi$, $\theta\leq\pi$ $\theta+\beta\leq\pi$

We have, under our assumption, that the four components that comprise M can be simplified as follows:

$$\begin{aligned}
1-\frac{\alpha}{\pi}+\frac{\alpha+\theta}{\pi}-1+\frac{\alpha+\theta+\beta}{\pi}-1 &= -1+\frac{\alpha+2\theta+\beta}{\pi}\\
1-\frac{\alpha}{\pi}+1-\frac{\theta}{\pi}+1-\frac{\theta+\beta}{\pi} &= 3-\frac{\alpha+2\theta+\beta}{\pi}\\
\frac{\alpha+\theta}{\pi}-1+1-\frac{\theta}{\pi}+1-\frac{\beta}{\pi} &= 1-\frac{\alpha-\beta}{\pi}\\
\frac{\alpha+\theta+\beta}{\pi}-1+1-\frac{\theta+\beta}{\pi}+1-\frac{\beta}{\pi} &= 1+\frac{\alpha-\beta}{\pi}
\end{aligned}$$

Furthermore, our assumption necessitates that each of these components is non-negative. Given we wish to consider the case when each such component is bounded from above by 1, we then consider

$$\begin{aligned}
-1+\frac{\alpha+2\theta+\beta}{\pi} &\in[0,1],\\
3-\frac{\alpha+2\theta+\beta}{\pi} &\in[0,1],\\
1-\frac{\alpha-\beta}{\pi} &\in[0,1]
\end{aligned}$$

which, by rearrangement, gives us

$$\begin{array}{c}\alpha+2\theta+\beta\in[\pi,2\pi]\\ \alpha+2\theta+\beta\in[2\pi,3\pi]\\ \beta-\alpha\in[0,\pi]\end{array}\Rightarrow\begin{array}{c}\alpha+2\theta+\beta=2\pi,\\ \beta-\alpha\in[0,\pi].\end{array}$$

However, by assumption $\theta+\beta\leq\pi$, and

$$\beta-\alpha+2\pi=\beta-\alpha+(\alpha+2\theta+\beta)=2\beta+2\theta\Rightarrow 2\beta+2\theta\in[2\pi,3\pi]\Rightarrow\theta+\beta\in[\pi,\frac{3\pi}{2}]\Rightarrow\theta+\beta=\pi$$

But, again, by our assumption that $\alpha+\theta>\pi$, this gives us

$$\alpha+2\theta+\beta>2\pi.$$

This is a contradiction.

For $\alpha+\beta>\pi$, $\theta+\beta>\pi$,where $\alpha\leq\pi$, $\beta\leq\pi$, $\theta\leq\pi$, $\alpha+\theta\leq\pi$

We have, under our assumption, that the four components that comprise M can be simplified as follows:

$$\begin{aligned}
1-\frac{\alpha}{\pi}+1-\frac{\alpha+\theta}{\pi}+\frac{\alpha+\theta+\beta}{\pi}-1 &= 1-\frac{\alpha-\beta}{\pi}\\
1-\frac{\alpha}{\pi}+1-\frac{\theta}{\pi}+\frac{\theta+\beta}{\pi}-1 &= 1-\frac{\alpha-\beta}{\pi}\\
1-\frac{\alpha+\theta}{\pi}+1-\frac{\theta}{\pi}+1-\frac{\beta}{\pi} &= 3-\frac{\alpha+2\theta+\beta}{\pi}\\
\frac{\alpha+\theta+\beta}{\pi}-1+\frac{\theta+\beta}{\pi}-1+1-\frac{\beta}{\pi} &= -1+\frac{\alpha+2\theta+\beta}{\pi}
\end{aligned}$$

Furthermore, our assumption necessitates that each of these components is non-negative. Given we wish to consider the case when each such component is bounded from above by 1, we then consider

$$\begin{aligned}
1-\frac{\alpha-\beta}{\pi} &\in [0,1],\\
3-\frac{\alpha+2\theta+\beta}{\pi} &\in [0,1],\\
-1+\frac{\alpha+2\theta+\beta}{\pi} &\in [0,1],
\end{aligned}$$

which, by rearrangement, gives us

$$\begin{aligned}
\alpha-\beta &\in [0,\pi]\\
\alpha+2\theta+\beta &\in [2\pi,3\pi] \Rightarrow \quad \begin{aligned}\alpha-\beta &\in [0,\pi]\\ \alpha+2\theta+\beta &= 2\pi,\end{aligned}\\
\alpha+2\theta+\beta &\in [\pi,2\pi]
\end{aligned}$$

However, by assumption $\alpha+\theta\leq\pi$, and

$$\alpha-\beta+2\pi=\alpha-\beta+(\alpha+2\theta+\beta)=2\alpha+2\theta \Rightarrow 2\alpha+2\theta\in[2\pi,3\pi] \Rightarrow \alpha+\theta\in[\pi,\frac{3\pi}{2}] \Rightarrow \theta+\beta=\pi$$

But, again, by our assumption that $\theta+\beta>\pi$, this gives us

$$\alpha+2\theta+\beta>2\pi.$$

This is a contradiction.

For $\alpha + \theta > \pi$, $\theta + \beta > \pi$, where $\alpha \leq \pi$, $\beta \leq \pi$, $\theta \leq \pi$, $\alpha + \beta \leq \pi$

We have, under our assumption, that the four components that comprise M can be simplified as follows:

$$\begin{aligned}
1 - \frac{\alpha}{\pi} + 1 - \frac{\alpha + \theta}{\pi} + \frac{\alpha + \theta + \beta}{\pi} - 1 &= -1 + \frac{\alpha + 2\theta + \beta}{\pi} \\
1 - \frac{\alpha}{\pi} + 1 - \frac{\theta}{\pi} + \frac{\theta + \beta}{\pi} - 1 &= 1 - \frac{\alpha - \beta}{\pi} \\
\frac{\alpha + \theta}{\pi} - 1 + 1 - \frac{\theta}{\pi} + 1 - \frac{\beta}{\pi} &= 1 - \frac{\beta - \alpha}{\pi} \\
\frac{\alpha + \theta + \beta}{\pi} - 1 + \frac{\theta + \beta}{\pi} - 1 + 1 - \frac{\beta}{\pi} &= -1 + \frac{\alpha + 2\theta + \beta}{\pi}
\end{aligned}$$

Furthermore, our assumption necessitates that each of these components is non-negative. Given we wish to consider the case when each such component is bounded from above by 1, we then consider

$$\begin{aligned}
-1 + \frac{\alpha + 2\theta + \beta}{\pi} &\in [0, 1], \\
1 - \frac{\alpha - \beta}{\pi} &\in [0, 1], \\
1 - \frac{\beta - \alpha}{\pi} &\in [0, 1],
\end{aligned}$$

which, by rearrangement, gives us

$$\begin{aligned}
\alpha + 2\theta + \beta &\in [\pi, 2\pi] \\
\beta - \alpha &\in [0, \pi] \\
\alpha - \beta &\in [0, \pi]
\end{aligned} \quad \Rightarrow \quad \begin{aligned}
\alpha + 2\theta + \beta &= [\pi, 2\pi], \\
\alpha &= \beta.
\end{aligned}$$

However, by our assumption $\alpha + \theta > \pi$ and $\theta + \beta > \pi$, and so

$$\alpha = \beta \Rightarrow \alpha + 2\theta + \beta = 2\alpha + 2\theta = 2\theta + 2\beta \in [\pi, 2\pi] \Rightarrow \alpha + \theta = \theta + \beta \in [\frac{\pi}{2}, \pi].$$

This is a contradiction.

For $\alpha+\beta > \pi$, where $\alpha \leq \pi$, $\beta \leq \pi$, $\alpha+\theta \leq \pi$, $\theta+\beta \leq \pi$ Or For $\alpha+\theta+\beta > \pi$, where $\alpha \leq \pi$, $\beta \leq \pi$, $\alpha+\theta \leq \pi$,$\alpha+\beta \leq \pi$, $\alpha+\theta \leq \pi$, $\theta+\beta \leq \pi$

We have, under our assumption, that the four components that comprise M can be simplified as follows:

$$\begin{aligned}1-\frac{\alpha}{\pi}+1-\frac{\alpha+\theta}{\pi}+\frac{\alpha+\theta+\beta}{\pi}-1&=1+\frac{\beta-\alpha}{\pi}\\ 1-\frac{\alpha}{\pi}+1-\frac{\theta}{\pi}+1-\frac{\theta+\beta}{\pi}&=3-\frac{\alpha+2\theta+\beta}{\pi}\\ 1-\frac{\alpha+\theta}{\pi}+1-\frac{\theta}{\pi}+1-\frac{\beta}{\pi}&=3-\frac{\alpha+2\theta+\beta}{\pi}\\ \frac{\alpha+\theta+\beta}{\pi}-1+1-\frac{\theta+\beta}{\pi}+1-\frac{\beta}{\pi}&=1+\frac{\alpha-\beta}{\pi}\end{aligned}$$

Furthermore, our assumption necessitates that each of these components is non-negative. Given we wish to consider the case when each such component is bounded from above by 1, we then consider

$$\begin{aligned}1+\frac{\beta-\alpha}{\pi}&\in[0,1],\\ 3-\frac{\alpha+2\theta+\beta}{\pi}&\in[0,1],\\ 1+\frac{\alpha-\beta}{\pi}&\in[0,1],\end{aligned}$$

which, by rearrangement, gives us

$$\begin{matrix}\alpha-\beta\in[0,\pi]\\ \alpha+2\theta+\beta\in[2\pi,3\pi]\\ \beta-\alpha\in[0,\pi]\end{matrix}\Rightarrow\begin{matrix}\alpha=\beta\\ \alpha+2\theta+\beta\in[2\pi,3\pi]\end{matrix}\Rightarrow\begin{matrix}\alpha=\beta\\ \alpha+\theta,\in[\pi,\frac{3\pi}{2}]\end{matrix}$$

As part of our assumption $\alpha+\theta\leq\pi$. As such

$$\begin{aligned}\alpha+\theta\in[\pi,\frac{3\pi}{2}]&\Rightarrow\alpha+\theta=\pi\\ &\Rightarrow 2\alpha+2\theta=\alpha+\beta+2\theta=2\pi,\end{aligned}$$

that is, we must have that our angles are completely symmetric.

We now can investigate the only case does not lead to a contradiction. Here we will examine each identity in M under the assumption that the angles forming $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ are completely symmetric and deduce that each component of M is bounded by 1. For the components in M we have:

$$\begin{aligned}
\left|1-\frac{\alpha}{\pi}\right|+\left|1-\frac{\alpha+\theta}{\pi}\right|+\left|1-\frac{\alpha+\theta+\beta}{\pi}\right| &= \left|1-\frac{\alpha}{\pi}\right|+\left|1-\frac{\pi}{\pi}\right|+\left|1-\frac{\pi+\alpha}{\pi}\right| \\
&= \left|1-\frac{\alpha}{\pi}\right|+\left|-\frac{\alpha}{\pi}\right| \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\left|1-\frac{\alpha}{\pi}\right|+\left|1-\frac{\theta}{\pi}\right|+\left|1-\frac{\theta+\beta}{\pi}\right| &= \left|1-\frac{\alpha}{\pi}\right|+\left|1-\frac{\pi-\alpha}{\pi}\right|+\left|1-\frac{\pi}{\pi}\right| \\
&= \left|1-\frac{\alpha}{\pi}\right|+\left|\frac{\alpha}{\pi}\right| \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\left|1-\frac{\alpha+\theta}{\pi}\right|+\left|1-\frac{\theta}{\pi}\right|+\left|1-\frac{\beta}{\pi}\right| &= \left|1-\frac{\pi}{\pi}\right|+\left|1-\frac{\pi-\beta}{\pi}\right|+\left|1-\frac{\beta}{\pi}\right| \\
&= \left|1-\frac{\beta}{\pi}\right|+\left|\frac{\beta}{\pi}\right| \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\left|1-\frac{\alpha+\theta+\beta}{\pi}\right|+\left|1-\frac{\theta+\beta}{\pi}\right|+\left|1-\frac{\beta}{\pi}\right| &= \left|1-\frac{\pi+\beta}{\pi}\right|+\left|1-\frac{\pi}{\pi}\right|+\left|1-\frac{\beta}{\pi}\right| \\
&= \left|1-\frac{\beta}{\pi}\right|+\left|\frac{-\beta}{\pi}\right| \\
&= 1
\end{aligned}$$

Thus we have each component bounded from above by 1.

Theorem 6.1. *Let $\widetilde{K} : C(\widetilde{\Gamma}_{\alpha,\beta,\theta}) \to C[0,1]^4$ be the matrix operator as defined in 6.1.*

Then we have that there exists an upper bound $\|\widetilde{K}\|_{ess} \leq M$, where

$$M = \max \left\{ \begin{array}{l} \left|1 - \frac{\alpha}{\pi}\right| + \left|1 - \frac{\alpha+\theta}{\pi}\right| + \left|1 - \frac{\alpha+\theta+\beta}{\pi}\right|, \\ \left|1 - \frac{\alpha}{\pi}\right| + \left|1 - \frac{\theta}{\pi}\right| + \left|1 - \frac{\theta+\beta}{\pi}\right|, \\ \left|1 - \frac{\alpha+\theta}{\pi}\right| + \left|1 - \frac{\theta}{\pi}\right| + \left|1 - \frac{\beta}{\pi}\right|, \\ \left|1 - \frac{\alpha+\theta+\beta}{\pi}\right| + \left|1 - \frac{\theta+\beta}{\pi}\right| + \left|1 - \frac{\beta}{\pi}\right| \end{array} \right\}$$

and furthermore we have that $M \leq 1$ if and only if

$$\alpha = \beta$$
$$\alpha + \theta = \pi,$$

that is, the angles that determine $\Gamma_{\alpha,\beta,\theta}$ are completely symmetric.

We also wish to determine sufficient and necessary conditions for which we have that the image of $\widetilde{K}$ acting on $\phi \in C(\widetilde{\Gamma}_{\alpha,\beta,\theta})$ is in $C(\widetilde{\Gamma}_{\alpha,\beta,\theta})$. We observe here that

$$\widetilde{K} = \widetilde{K}_n + K_n,$$

The case for the continuity of $K_n\phi$ follows immediately as it must have a continuous kernel on $\widetilde{\Gamma}_{\alpha,\beta,\theta}$, thus to determine the continuity of $\widetilde{K}$ it is only necessary to determine the continuity of $\widetilde{K}_n\phi$.

We observe that $\widetilde{K}_n\phi$ has continuous kernel at the endpoints $\gamma_1(1) = \gamma_2(1)$ and $\gamma_3(1) = \gamma_4(1)$ respectively and thus must also be continuous at these points.

It remains to determine when we have continuity for $x = 0$, that is, we must find when the value of each component function of the resultant vector-function is equal at 0. If we can establish this then we have sufficient conditions for $\widetilde{K}_n\phi \in C(\widetilde{\Gamma}_{\alpha,\beta,\theta})$.

Using the result for single corners described by Kress (Section 2.4) for each entry, we again take n sufficiently large that the intersection of $B[x, 1/n]$ with any two γ_i, γ_j is a straight-line segment. With this in mind and given each ψ_i must be equal at 0, setting $x = 0$, we get

$$\widetilde{K}_n\phi(0) = \begin{bmatrix} \left(1-\frac{\alpha}{\pi}\right)\psi_2(0) + \left(1-\frac{\alpha+\theta}{\pi}\right)\psi_3(0) + \left(1-\frac{\alpha+\theta+\beta}{\pi}\right)\psi_4(0) \\ \left(1-\frac{\alpha}{\pi}\right)\psi_1(0) - \left(1-\frac{\theta}{\pi}\right)\psi_3(0) - \left(1-\frac{\theta+\beta}{\pi}\right)\psi_4(0) \\ -\left(1-\frac{\alpha+\theta}{\pi}\right)\psi_1(0) - \left(1-\frac{\theta}{\pi}\right)\psi_2(0) + \left(1-\frac{\beta}{\pi}\right)\psi_4(0) \\ \left(1-\frac{\alpha+\theta+\beta}{\pi}\right)\psi_1(0) + \left(1-\frac{\theta+\beta}{\pi}\right)\psi_2(0) + \left(1-\frac{\beta}{\pi}\right) \end{bmatrix}$$
$$= \begin{bmatrix} \left(1-\frac{\alpha}{\pi}\right)\psi_i(0) + \left(1-\frac{\alpha+\theta}{\pi}\right)\psi_i(0) + \left(1-\frac{\alpha+\theta+\beta}{\pi}\right)\psi_i(0) \\ \left(1-\frac{\alpha}{\pi}\right)\psi_i(0) - \left(1-\frac{\theta}{\pi}\right)\psi_i(0) - \left(1-\frac{\theta+\beta}{\pi}\right)\psi_i(0) \\ -\left(1-\frac{\alpha+\theta}{\pi}\right)\psi_i(0) - \left(1-\frac{\theta}{\pi}\right)\psi_i(0) + \left(1-\frac{\beta}{\pi}\right)\psi_i(0) \\ \left(1-\frac{\alpha+\theta+\beta}{\pi}\right)\psi_i(0) + \left(1-\frac{\theta+\beta}{\pi}\right)\psi_i(0) + \left(1-\frac{\beta}{\pi}\right)\psi_i(0) \end{bmatrix}$$
$$= \begin{bmatrix} \left(3-\frac{3\alpha+2\theta+\beta}{\pi}\right)\psi_i(0) \\ \left(-1+\frac{-\alpha+\beta+2\theta}{\pi}\right)\psi_i(0) \\ \left(-1+\frac{\alpha+2\theta-\beta}{\pi}\right)\psi_i(0) \\ \left(3-\frac{\alpha+2\theta+3\beta}{\pi}\right)\psi_i(0) \end{bmatrix}$$

If we first assume continuity, then we have that

$$\begin{aligned}\left(3-\frac{3\alpha+2\theta+\beta}{\pi}\right)\psi_i(0) &= \left(-1+\frac{-\alpha+\beta+2\theta}{\pi}\right)\psi_i(0) \\ &= \left(-1+\frac{\alpha+2\theta-\beta}{\pi}\right)\psi_i(0) = \left(3-\frac{\alpha+2\theta+3\beta}{\pi}\right)\psi_i(0),\end{aligned}$$

but,

$$\left(-1+\frac{-\alpha+\beta+2\theta}{\pi}\right)\psi_i(0) = \left(-1+\frac{\alpha+2\theta-\beta}{\pi}\right)\psi_i(0) \Rightarrow \beta-\alpha = \alpha-\beta \Rightarrow \alpha = \beta.$$

So we have that continuity implies symmetry. Given our prior result, we will examine the case where $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ is symmetric.

Assuming we have

$$\begin{aligned}\left(3-\frac{3\alpha+2\theta+\beta}{\pi}\right)\psi_i(0) &= \left(-1+\frac{-\alpha+\beta+2\theta}{\pi}\right)\psi_i(0) \\ &= \left(-1+\frac{\alpha+2\theta-\beta}{\pi}\right)\psi_i(0) = \left(3-\frac{\alpha+2\theta+3\beta}{\pi}\right)\psi_i(0),\end{aligned}$$

then if we assume $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ to be symmetric, i.e. $\alpha = \beta$, we can reduce this down to

$$\left(3 - \frac{4\alpha + 2\theta}{\pi}\right) = \left(-1 + \frac{2\theta}{\pi}\right)$$

from which rearrangement we can deduce that

$$\alpha + \theta = \pi.$$

And conversely, if we assume complete symmetry, i.e. $\alpha = \beta$ and $\alpha + \theta = \pi$, then we get

$$\left(3 - \frac{3\alpha + 2\theta + \beta}{\pi}\right) = \left(3 - \frac{\alpha + 2\theta + 3\beta}{\pi}\right) = \left(3 - \frac{4\alpha + 2\theta}{\pi}\right)$$
$$\left(-1 + \frac{-\alpha + \beta + 2\theta}{\pi}\right) = \left(-1 + \frac{\alpha + 2\theta - \beta}{\pi}\right) = \left(-1 + \frac{2\theta}{\pi}\right)$$

and

$$\begin{aligned} 3 - \frac{4\alpha + 2\theta}{\pi} &= 3 - \frac{2\alpha + 2\pi}{\pi} \\ &= 1 - \frac{2\alpha}{\pi} \\ &= -1 + \frac{2\pi - 2\alpha}{\pi} \\ &= -1 + \frac{2\theta}{\pi}. \end{aligned}$$

Thus, complete symmetry implies

$$\begin{aligned} \left(3 - \frac{3\alpha + 2\theta + \beta}{\pi}\right) \psi_i(0) &= \left(-1 + \frac{-\alpha + \beta + 2\theta}{\pi}\right) \psi_i(0) \\ &= \left(-1 + \frac{\alpha + 2\theta - \beta}{\pi}\right) \psi_i(0) = \left(3 - \frac{\alpha + 2\theta + 3\beta}{\pi}\right) \psi_i(0), \end{aligned}$$

That is, $\widetilde{K}_n$ is continous if and only if the angles that define $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ are completely symmetric. This, combined with the continuity of K_n then gives us the following result:

Theorem 6.2. *$\widetilde{K}$ is an operator on $C(\widetilde{\Gamma}_{\alpha,\beta,\theta})$ if and only if the angles that define $\widetilde{\Gamma}_{\alpha,\beta,\theta}$ are completely symmetric, that is $\alpha = \beta$ and $\alpha + 2\theta + \beta = 2\pi$.*

These two results (Theorems 6.1 and 6.2) together are consistent with those found in [6], specifically in that we have established sufficient and neccesary conditions for continuity and for the essential spectrum of $\widetilde{K}$ to be bounded from above by 1.

7 Appendices

7.1 Matlab codes

Code for $\mathcal{M}(j_{\alpha,a})(\xi)$:

```
function[F1] = F1(x,a,d)
     %Let x represent the variable '\xi'
    %Let a represent the interior angle '\alpha'
    %Let d represent the space weighting denoted 'a'
    F1=sin((pi-a).*(i.*x +(d-1)./2))./sin(pi.*(i.*x +(d-1)./2));
end
```

Code for $\mathcal{M}(j_{\alpha,\theta,a})(\xi)$:

```
function[F2] = F2(x,a,b,d)
     %Let x represent the variable '\xi'
    %Let a represent the interior angle '\alpha'
    %Let b represent the exterior angle '\theta'
    %Let d represent the space weighting denoted 'a'
    F2=sin((pi-a-b).*(i.*x +(d-1)./2))./sin(pi.*(i.*x +(d-1)./2));
end
```

Code for $\mathcal{M}(j_{\alpha,\theta,\beta,a})(\xi)$:

```
function[F3] = F3(x,a,b,c,d)
     %Let x represent the variable '\xi'
    %Let a represent the interior angle '\alpha'
    %Let b represent the exterior angle '\theta'
    %Let c represent the interior angle '\beta'
    %Let d represent the space weighting denoted 'a'
    F3=sin((pi-a-b-c).*(i.*x +(d-1)./2))./sin(pi.*(i.*x +(d-1)./2));
end
```

Code for $\mathcal{M}(j_{\theta,a})(\xi)$:

```
function[F4] = F4(x,b,d)
     %Let x represent the variable '\xi'
```

```
    %Let b represent the exterior angle '\theta'
    %Let d represent the space weighting denoted 'a'
    F4=sin((pi-b).*(i.*x +(d-1)./2))./sin(pi.*(i.*x +(d-1)
       ./2));
end
```

Code for $\mathcal{M}(j_{\theta\beta,a})(\xi)$:

```
function[F5] = F5(x,b,c,d)
     %Let x represent the variable '\xi'

    %Let b represent the exterior angle '\theta'
    %Let c represent the interior angle '\beta'
    %Let d represent the space weighting denoted 'a'
    F5=sin((pi-b-c).*(i.*x +(d-1)./2))./sin(pi.*(i.*x +(d-1)
       ./2));
end
```

Code for $\mathcal{M}(j_{\beta,a})(\xi)$:

```
function[F6] = F6(x,c,d)
     %Let x represent the variable '\xi'
    %Let c represent the interior angle '\beta'
    %Let d represent the space weighting denoted 'a'
    F6=sin((pi-c).*(i.*x +(d-1)./2))./sin(pi.*(i.*x +(d-1)
       ./2));
end
```

Code for $S(\xi)$:

```
function[K1] = K1(x,a,b,c,d)
     %Let x represent the variable '\xi'
    %Let a represent the interior angle '\alpha'
    %Let b represent the exterior angle '\theta'
    %Let c represent the interior angle '\beta'
    %Let d represent the space weighting denoted 'a'
    K1=-F1(x,a,d).^2 + F2(x,a,b,d).^2 - F3(x,a,b,c,d).^2 -
       F4(x,b,d).^2 + F5(x,b,c,d).^2 - F6(x,c,d).^2;
end
```

Code for $T(\xi)$:

```
function[K2] = K2(x,a,b,c,d)
    %Let x represent the variable '\xi'
    %Let a represent the interior angle '\alpha'
    %Let b represent the exterior angle '\theta'
    %Let c represent the interior angle '\beta'
    %Let d represent the space weighting denoted 'a'
    K2=-(-(F5(x,b,c,d).*F2(x,a,b,d)).^2 - (F3(x,a,b,c,d).*F4
       (x,b,d)).^2 +2.*F5(x,b,c,d).*F2(x,a,b,d).*F3(x,a,b,c,
       d).*F4(x,b,d) -(F1(x,a,d).*F6(x,c,d)).^2 +2.*F1(x,a,d
       ).*F5(x,b,c,d).*F2(x,a,b,d).*F6(x,c,d) - 2.*F1(x,a,d)
       .*F3(x,a,b,c,d).*F4(x,b,d).*F6(x,c,d));
end
```

Code for Operator Matrix:

```
function[A]=A(x,a,b,c,d)
     %Let x represent the variable '\xi'
    %Let a represent the interior angle '\alpha'
    %Let b represent the exterior angle '\theta'
    %Let c represent the interior angle '\beta'
    %Let d represent the space weighting denoted 'a'
    A=[ 0 F1(x,a,d) F2(x,a,b,d) F3(x,a,b,c,d); F1(x,a,d) 0 -
       F4(x,b,d) -F5(x,b,c,d); -F2(x,a,b,d) -F4(x,b,d) 0 F6(
       x,c,d); F3(x,a,b,c,d) F5(x,b,c,d) F6(x,c,d) 0];
end
```

Code for graphing singular values on $L^2(\Gamma_{\alpha,\beta,\theta})$:

```
for x=-10:.01:10; %'\xi'
a='\alpha';
b='\theta';
c='\beta';
d='weight-a';
hold on
B=A(x,a,b,c,d);
S=svds(B,1,'largest');
plot(x,S,'k.')
```

```
end
xlabel('-10 \leq \xi \leq 10')
ylabel('Operator Matrix Maximum Singular Value')
```

Code for returning the singular values of the operator matrix in symbolic form:

```

x=0;
d=0;
syms a b c real
A=A(x,a,b,c,d);
svd(A)

ans =

(sin(a/2 + b/2 + c/2 - pi/2)^2/2 + sin(a/2 + b/2 - pi/2)^2/2
  + sin(b/2 + c/2 - pi/2)^2/2 + sin(a/2 - pi/2)^2/2 + sin(
 b/2 - pi/2)^2/2 + sin(c/2 - pi/2)^2/2 - ((cos(a/2)^2*cos(
 b/2)^2 + cos(b/2)^2*cos(c/2)^2 + cos(a/2)^2 + cos(b/2)^2
 + cos(c/2)^2 + sin(a/2)^2*sin(b/2)^2 + sin(b/2)^2*sin(c
 /2)^2 - 2*cos(a/2)*cos(c/2) + cos(a/2)^2*sin(b/2)^2*sin(c
 /2)^2 + cos(b/2)^2*sin(a/2)^2*sin(c/2)^2 + cos(c/2)^2*sin
 (a/2)^2*sin(b/2)^2 + 2*cos(b/2)^2*sin(a/2)*sin(c/2) + 2*
 sin(a/2)*sin(b/2)^2*sin(c/2) + cos(a/2)^2*cos(b/2)^2*cos(
 c/2)^2 - 2*cos(a/2)*cos(b/2)*sin(a/2)*sin(b/2) - 2*cos(b
 /2)*cos(c/2)*sin(b/2)*sin(c/2) - 2*cos(a/2)*cos(b/2)*cos(
 c/2)^2*sin(a/2)*sin(b/2) - 2*cos(a/2)*cos(b/2)^2*cos(c/2)
 *sin(a/2)*sin(c/2) - 2*cos(a/2)^2*cos(b/2)*cos(c/2)*sin(b
 /2)*sin(c/2) + 2*cos(a/2)*cos(b/2)*sin(a/2)*sin(b/2)*sin(
 c/2)^2 + 2*cos(a/2)*cos(c/2)*sin(a/2)*sin(b/2)^2*sin(c/2)
  + 2*cos(b/2)*cos(c/2)*sin(a/2)^2*sin(b/2)*sin(c/2))*(cos
 (a/2)^2*cos(b/2)^2 + cos(b/2)^2*cos(c/2)^2 + cos(a/2)^2 +
  cos(b/2)^2 + cos(c/2)^2 + sin(a/2)^2*sin(b/2)^2 + sin(b
 /2)^2*sin(c/2)^2 + 2*cos(a/2)*cos(c/2) + cos(a/2)^2*sin(b
 /2)^2*sin(c/2)^2 + cos(b/2)^2*sin(a/2)^2*sin(c/2)^2 + cos
 (c/2)^2*sin(a/2)^2*sin(b/2)^2 - 2*cos(b/2)^2*sin(a/2)*sin
 (c/2) - 2*sin(a/2)*sin(b/2)^2*sin(c/2) + cos(a/2)^2*cos(b
```

```
/2)^2*cos(c/2)^2 - 2*cos(a/2)*cos(b/2)*sin(a/2)*sin(b/2)
- 2*cos(b/2)*cos(c/2)*sin(b/2)*sin(c/2) - 2*cos(a/2)*cos(
b/2)*cos(c/2)^2*sin(a/2)*sin(b/2) - 2*cos(a/2)*cos(b/2)
^2*cos(c/2)*sin(a/2)*sin(c/2) - 2*cos(a/2)^2*cos(b/2)*cos
(c/2)*sin(b/2)*sin(c/2) + 2*cos(a/2)*cos(b/2)*sin(a/2)*
sin(b/2)*sin(c/2)^2 + 2*cos(a/2)*cos(c/2)*sin(a/2)*sin(b
/2)^2*sin(c/2) + 2*cos(b/2)*cos(c/2)*sin(a/2)^2*sin(b/2)*
sin(c/2)))^(1/2)/2)^(1/2)
(sin(a/2 + b/2 + c/2 - pi/2)^2/2 + sin(a/2 + b/2 - pi/2)^2/2
 + sin(b/2 + c/2 - pi/2)^2/2 + sin(a/2 - pi/2)^2/2 + sin(
b/2 - pi/2)^2/2 + sin(c/2 - pi/2)^2/2 - ((cos(a/2)^2*cos(
b/2)^2 + cos(b/2)^2*cos(c/2)^2 + cos(a/2)^2 + cos(b/2)^2
+ cos(c/2)^2 + sin(a/2)^2*sin(b/2)^2 + sin(b/2)^2*sin(c
/2)^2 - 2*cos(a/2)*cos(c/2) + cos(a/2)^2*sin(b/2)^2*sin(c
/2)^2 + cos(b/2)^2*sin(a/2)^2*sin(c/2)^2 + cos(c/2)^2*sin
(a/2)^2*sin(b/2)^2 + 2*cos(b/2)^2*sin(a/2)*sin(c/2) + 2*
sin(a/2)*sin(b/2)^2*sin(c/2) + cos(a/2)^2*cos(b/2)^2*cos(
c/2)^2 - 2*cos(a/2)*cos(b/2)*sin(a/2)*sin(b/2) - 2*cos(b
/2)*cos(c/2)*sin(b/2)*sin(c/2) - 2*cos(a/2)*cos(b/2)*cos(
c/2)^2*sin(a/2)*sin(b/2) - 2*cos(a/2)*cos(b/2)^2*cos(c/2)
*sin(a/2)*sin(c/2) - 2*cos(a/2)^2*cos(b/2)*cos(c/2)*sin(b
/2)*sin(c/2) + 2*cos(a/2)*cos(b/2)*sin(a/2)*sin(b/2)*sin(
c/2)^2 + 2*cos(a/2)*cos(c/2)*sin(a/2)*sin(b/2)^2*sin(c/2)
 + 2*cos(b/2)*cos(c/2)*sin(a/2)^2*sin(b/2)*sin(c/2))*(cos
(a/2)^2*cos(b/2)^2 + cos(b/2)^2*cos(c/2)^2 + cos(a/2)^2 +
 cos(b/2)^2 + cos(c/2)^2 + sin(a/2)^2*sin(b/2)^2 + sin(b
/2)^2*sin(c/2)^2 + 2*cos(a/2)*cos(c/2) + cos(a/2)^2*sin(b
/2)^2*sin(c/2)^2 + cos(b/2)^2*sin(a/2)^2*sin(c/2)^2 + cos
(c/2)^2*sin(a/2)^2*sin(b/2)^2 - 2*cos(b/2)^2*sin(a/2)*sin
(c/2) - 2*sin(a/2)*sin(b/2)^2*sin(c/2) + cos(a/2)^2*cos(b
/2)^2*cos(c/2)^2 - 2*cos(a/2)*cos(b/2)*sin(a/2)*sin(b/2)
- 2*cos(b/2)*cos(c/2)*sin(b/2)*sin(c/2) - 2*cos(a/2)*cos(
b/2)*cos(c/2)^2*sin(a/2)*sin(b/2) - 2*cos(a/2)*cos(b/2)
^2*cos(c/2)*sin(a/2)*sin(c/2) - 2*cos(a/2)^2*cos(b/2)*cos
(c/2)*sin(b/2)*sin(c/2) + 2*cos(a/2)*cos(b/2)*sin(a/2)*
sin(b/2)*sin(c/2)^2 + 2*cos(a/2)*cos(c/2)*sin(a/2)*sin(b
```

```
/2)^2*sin(c/2) + 2*cos(b/2)*cos(c/2)*sin(a/2)^2*sin(b/2)*
sin(c/2)))^(1/2)/2)^(1/2)
(sin(a/2 + b/2 + c/2 - pi/2)^2/2 + sin(a/2 + b/2 - pi/2)^2/2
 + sin(b/2 + c/2 - pi/2)^2/2 + sin(a/2 - pi/2)^2/2 + sin(
b/2 - pi/2)^2/2 + sin(c/2 - pi/2)^2/2 + ((cos(a/2)^2*cos(
b/2)^2 + cos(b/2)^2*cos(c/2)^2 + cos(a/2)^2 + cos(b/2)^2
+ cos(c/2)^2 + sin(a/2)^2*sin(b/2)^2 + sin(b/2)^2*sin(c
/2)^2 - 2*cos(a/2)*cos(c/2) + cos(a/2)^2*sin(b/2)^2*sin(c
/2)^2 + cos(b/2)^2*sin(a/2)^2*sin(c/2)^2 + cos(c/2)^2*sin
(a/2)^2*sin(b/2)^2 + 2*cos(b/2)^2*sin(a/2)*sin(c/2) + 2*
sin(a/2)*sin(b/2)^2*sin(c/2) + cos(a/2)^2*cos(b/2)^2*cos(
c/2)^2 - 2*cos(a/2)*cos(b/2)*sin(a/2)*sin(b/2) - 2*cos(b
/2)*cos(c/2)*sin(b/2)*sin(c/2) - 2*cos(a/2)*cos(b/2)*cos(
c/2)^2*sin(a/2)*sin(b/2) - 2*cos(a/2)*cos(b/2)^2*cos(c/2)
*sin(a/2)*sin(c/2) - 2*cos(a/2)^2*cos(b/2)*cos(c/2)*sin(b
/2)*sin(c/2) + 2*cos(a/2)*cos(b/2)*sin(a/2)*sin(b/2)*sin(
c/2)^2 + 2*cos(a/2)*cos(c/2)*sin(a/2)*sin(b/2)^2*sin(c/2)
 + 2*cos(b/2)*cos(c/2)*sin(a/2)^2*sin(b/2)*sin(c/2))*(cos
(a/2)^2*cos(b/2)^2 + cos(b/2)^2*cos(c/2)^2 + cos(a/2)^2 +
 cos(b/2)^2 + cos(c/2)^2 + sin(a/2)^2*sin(b/2)^2 + sin(b
/2)^2*sin(c/2)^2 + 2*cos(a/2)*cos(c/2) + cos(a/2)^2*sin(b
/2)^2*sin(c/2)^2 + cos(b/2)^2*sin(a/2)^2*sin(c/2)^2 + cos
(c/2)^2*sin(a/2)^2*sin(b/2)^2 - 2*cos(b/2)^2*sin(a/2)*sin
(c/2) - 2*sin(a/2)*sin(b/2)^2*sin(c/2) + cos(a/2)^2*cos(b
/2)^2*cos(c/2)^2 - 2*cos(a/2)*cos(b/2)*sin(a/2)*sin(b/2)
- 2*cos(b/2)*cos(c/2)*sin(b/2)*sin(c/2) - 2*cos(a/2)*cos(
b/2)*cos(c/2)^2*sin(a/2)*sin(b/2) - 2*cos(a/2)*cos(b/2)
^2*cos(c/2)*sin(a/2)*sin(c/2) - 2*cos(a/2)^2*cos(b/2)*cos
(c/2)*sin(b/2)*sin(c/2) + 2*cos(a/2)*cos(b/2)*sin(a/2)*
sin(b/2)*sin(c/2)^2 + 2*cos(a/2)*cos(c/2)*sin(a/2)*sin(b
/2)^2*sin(c/2) + 2*cos(b/2)*cos(c/2)*sin(a/2)^2*sin(b/2)*
sin(c/2)))^(1/2)/2)^(1/2)
(sin(a/2 + b/2 + c/2 - pi/2)^2/2 + sin(a/2 + b/2 - pi/2)^2/2
 + sin(b/2 + c/2 - pi/2)^2/2 + sin(a/2 - pi/2)^2/2 + sin(
b/2 - pi/2)^2/2 + sin(c/2 - pi/2)^2/2 + ((cos(a/2)^2*cos(
b/2)^2 + cos(b/2)^2*cos(c/2)^2 + cos(a/2)^2 + cos(b/2)^2
```

```
+ cos(c/2)^2 + sin(a/2)^2*sin(b/2)^2 + sin(b/2)^2*sin(c
/2)^2 - 2*cos(a/2)*cos(c/2) + cos(a/2)^2*sin(b/2)^2*sin(c
/2)^2 + cos(b/2)^2*sin(a/2)^2*sin(c/2)^2 + cos(c/2)^2*sin
(a/2)^2*sin(b/2)^2 + 2*cos(b/2)^2*sin(a/2)*sin(c/2) + 2*
sin(a/2)*sin(b/2)^2*sin(c/2) + cos(a/2)^2*cos(b/2)^2*cos(
c/2)^2 - 2*cos(a/2)*cos(b/2)*sin(a/2)*sin(b/2) - 2*cos(b
/2)*cos(c/2)*sin(b/2)*sin(c/2) - 2*cos(a/2)*cos(b/2)*cos(
c/2)^2*sin(a/2)*sin(b/2) - 2*cos(a/2)*cos(b/2)^2*cos(c/2)
*sin(a/2)*sin(c/2) - 2*cos(a/2)^2*cos(b/2)*cos(c/2)*sin(b
/2)*sin(c/2) + 2*cos(a/2)*cos(b/2)*sin(a/2)*sin(b/2)*sin(
c/2)^2 + 2*cos(a/2)*cos(c/2)*sin(a/2)*sin(b/2)^2*sin(c/2)
 + 2*cos(b/2)*cos(c/2)*sin(a/2)^2*sin(b/2)*sin(c/2))*(cos
(a/2)^2*cos(b/2)^2 + cos(b/2)^2*cos(c/2)^2 + cos(a/2)^2 +
 cos(b/2)^2 + cos(c/2)^2 + sin(a/2)^2*sin(b/2)^2 + sin(b
/2)^2*sin(c/2)^2 + 2*cos(a/2)*cos(c/2) + cos(a/2)^2*sin(b
/2)^2*sin(c/2)^2 + cos(b/2)^2*sin(a/2)^2*sin(c/2)^2 + cos
(c/2)^2*sin(a/2)^2*sin(b/2)^2 - 2*cos(b/2)^2*sin(a/2)*sin
(c/2) - 2*sin(a/2)*sin(b/2)^2*sin(c/2) + cos(a/2)^2*cos(b
/2)^2*cos(c/2)^2 - 2*cos(a/2)*cos(b/2)*sin(a/2)*sin(b/2)
- 2*cos(b/2)*cos(c/2)*sin(b/2)*sin(c/2) - 2*cos(a/2)*cos(
b/2)*cos(c/2)^2*sin(a/2)*sin(b/2) - 2*cos(a/2)*cos(b/2)
^2*cos(c/2)*sin(a/2)*sin(c/2) - 2*cos(a/2)^2*cos(b/2)*cos
(c/2)*sin(b/2)*sin(c/2) + 2*cos(a/2)*cos(b/2)*sin(a/2)*
sin(b/2)*sin(c/2)^2 + 2*cos(a/2)*cos(c/2)*sin(a/2)*sin(b
/2)^2*sin(c/2) + 2*cos(b/2)*cos(c/2)*sin(a/2)^2*sin(b/2)*
sin(c/2)))^(1/2)/2)^(1/2)
```

Code for graphing the spectrum of the operator matrix on $L^2(\Gamma_{\alpha,\beta,\theta})$:

```
x=-10:.000001:10;%'\xi'
a='\alpha';
b='\theta';
c='\beta';
d='weight-a';
y1=sqrt((-K1(x,a,b,c,d)+sqrt((K1(x,a,b,c,d).^2 - 4.*K2(x,a,b
,c,d)))./2);
y2=-sqrt((-K1(x,a,b,c,d)+sqrt((K1(x,a,b,c,d).^2 - 4.*K2(x,a,
```

```
   b,c,d)))) ./2);
y3=sqrt((-K1(x,a,b,c,d)-sqrt((K1(x,a,b,c,d).^2 - 4.*K2(x,a,b
   ,c,d)))) ./2);
y4=-sqrt((-K1(x,a,b,c,d)-sqrt((K1(x,a,b,c,d).^2 - 4.*K2(x,a,
   b,c,d)))) ./2);
hold on
plot(real(y1),imag(y1),'k')
plot(real(y2),imag(y2),'k')
plot(real(y3),imag(y3),'k')
plot(real(y4),imag(y4),'k')
xlabel('R')
ylabel('Im')
```

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