Interaction of the convective energy cycle and large-scale dynamics

Journal of the Atmospheric Sciences

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--Manuscript Draft--

Manuscript Number:	
Full Title:	Interaction of the Convective Energy Cycle and Large-Scale Dynamics
Article Type:	Article
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Interaction of the Convective Energy Cycle and Large-Scale Dynamics

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ABSTRACT: The importance of the convective life cycle in tropical large-scale dynamics has long been emphasized, but without explicit analysis. The present work provides it by coupling the convective energy cycle under the framework of Arakawa and Schubert's (1974) convection parameterization with a shallow-water analogue atmosphere. The square frequency of linear convectively–coupled waves is given by a squared sum of the dry gravity-wave and the convective energy-cycle frequencies, shortening the period of the convective cycle through the large-scale coupling. In a weakly nonlinear regime, the system follows an equation analogous to the Kortweg–de Vries equation, which exhibits a solitary–wave solution, with behavior reminiscent of observed tropical westerly–wind bursts.

DOC/CQE/energy-cycle/1D-LS/ms.tex, 7 April 2023

Significance Statement: The present work suggests that a nonlinear description of a large–scale tropical system with an explicit convective life cycle may provide a simple model of tropical westerly–wind bursts. At the same time, an important lesson to learn is that, if the focus of a study is on the global scale of the atmosphere, it is wise not to try to include a convective life cycle explicitly into the model. Such a configuration will simply be dominated by the short convective–scale variabilities, that one would wish to filter out.

1. Introduction

It is commonly accepted that tropical atmospheric dynamics is essentially described by the interactions between large-scale equatorial waves and small-scale convection: $cf.$, critical reviews in introductions of Yano and Tribbia (2017), Yano and Wedi (2021), and further references therein. A standard approach has been to introduce parameterized convection to the large–scale dynamics under a general framework of convective quasi-equilibrium ($cf.$, Yano and Plant 2012a), which assumes that small–scale convection is in equilibrium with the large-scale dynamics in a certain manner. This general conceptual framework can cover a wide range of formulations, including the original one by Arakawa and Schubert (1974), but also a more straightforward assumption of convective neutrality of the large scale, originally suggested by Betts (1986), observationally supported by Xu and Emanuel (1989), and applied to theoretical studies by Emanuel (1987) and Neelin *et al.* (1987). More classical approaches of wave–CISK (Hayashi 1970, Lindzen 1974) can also be included in this category in the present context. All of these approaches have in common that they *do not* introduce an explicit process characterized by a convective time scale.

At the same time, there has been a persistent feeling in the tropical community that a finite time scale for the life cycle of small-scale convection plays a critical role in the tropical large-scale dynamics. This feeling may be, for example, reflected upon through brief, albeit rather obscure discussions on the convective life cycle leading to his Eqs. (2.2) and (3.6) in Kuo (1974), the emphasis on mesoscale processes for convection parameterizations in the review by Houze and Betts (1981), and probably most succinctly summarized by an argument of activation-control by Mapes (1997).

The most straightforward way to include a convective time scale within a parameterization is to introduce it as a finite-time adjustment process towards an equilibrium. A parameterization by Betts (1986) follows this approach, although his main focus in the formulation is in defining an equilibrium profile. Neelin and Yu (1994) and Yu and Neelin (1994) introduced this finite-time convective adjustment in the context of large-scale dynamic studies. Similar approaches are adopted by $e.g.$, Frierson *et al.* (2004), Stechmann and Majda (2006), Bouchut *et al.* (2009), Lambaerts *et al.* (2011). However, these convective adjustment approaches are still short of introducing a life-cycle of convection: adjustment only describes a monotonic approach towards an equilibrium, without going through anything like a cycle. A simple model for the convective life cycle was introduced by Yano and Plant (2012b).

Yano and Plant (2012b) showed that a basic behavior of atmospheric deep convection, especially its tendency for following a cycle of discharge and recharge ($cf.$, Blade and Hartmann 1993), can be described by an energy cycle, as originally introduced by Arakawa and Schubert (1974) as their Eqs. (132) and (140), but by adding simple closures to this system ($cf.$, Eq. 2.5 below). A key simplification in the formulation of Yano and Plant (2012b) is to consider only a single, deep convection mode so that the integral kernel, defined by Eqs. (B36) and (B37) in Arakawa and Schubert (1974), reduces to a single scalar parameter.

The purpose of the present study is to couple this convective energy cycle system with a simple large-scale dynamics described by a shallow-water analogue, and to present its basic behavior. The most fascinating finding from this study is the existence of a solitary wave solution under weak nonlinearity, whose behavior is reminiscent of observed tropical westerly–wind bursts ($cf.$, Hartten 1996, Yano *et al.* 2004).

The convective energy-cycle system introduced by Yano and Plant (2012b) is reviewed in the next section. As a first step for investigating the coupled dynamics of this system, we adopt a simple horizontally one-dimensional shallow-water analogue for the large-scale dynamics, as introduced in Sec. 3. A complete formulation of the system is presented in Sec. 4 in a nondimensional form. The derived system is analyzed over Secs. 5–7 in three steps: steady solutions (Sec. 5), linear waves (Sec. 6), and a weakly nonlinear analysis (Sec. 7). The paper is concluded by Sec. 9 after further discussions in Sec. 8.

2. Convective Energy-Cycle System (Dimensional)

Following Yano and Plant (2012b), the convective energy-cycle system is given by:

$$\frac{dK}{dt} = AM_B - D, \tag{2.1a}$$
$$\frac{dA}{dt} = -\gamma M_B + F \tag{2.1b}$$

with the convective kinetic energy, K, and the cloud work function, A, as prognostic variables. These are defined by

$$K = \int_{z_B}^{z_T} \sigma \frac{\rho}{2} w_c^2 dz, \tag{2.2a}$$
$$A = \int_{z_B}^{z_T} \eta b dz. \tag{2.2b}$$

Here, notably, σ is the fractional area occupied by convection, η is a normalized vertical profile of convective mass flux, and M_B is the convective mass flux at the convection base. The other variables introduced in Eqs. (2.2a, b) are: ρ the air density, w_c the convective vertical velocity, z the vertical coordinate, and b the buoyancy.

Arakawa and Schubert (1974) assumed an entraining plume profile in defining the cloud work function, A. In this case, the profile, η, is normalized by the value at the convective base. However, Yano *et al.* (2005) show that the concept of the cloud work function can be applied to any vertical convective profile, η, as a measure of the potential energy convertibility (PEC), as seen on the first term in the right-hand of Eq. (2.1a). Note further that if we set $\eta = 1$, the cloud work function (PEC) reduces to a form of convective available potential energy (CAPE). It fully reduces to CAPE if the buoyancy, b, is defined as that of a lifting parcel. However, the definition of the buoyancy is kept open in Eq. (2.2b): for example, it could be taken as the buoyancy as defined in explicit convection simulations, averaged over the convective area.

We assume that the convective damping, D, is expressed by a Rayleigh damping:

$$D = \frac{K}{\tau_D} \tag{2.2c}$$

with the damping time scale, $\tau_D \sim 10^3$ sec. γ measures the efficiency with which convection consumes the cloud work function (PEC), A, with time, corresponding to the kernel, $\mathcal{K}$, introduced by Arakawa and Schubert (1974), but reducing it to a scalar by only considering a single convective mode here.

The large-scale forcing, F, was taken to be a prescribed constant in Yano and Plant (2012b) in order to consider the convection dynamics in a stand alone manner. For the present purpose of considering a coupling of this energy-cycle system with the large-scale dynamics, the large-scale forcing must evolve following the evolution of the large-scale state. Thus, we define it by

$$F \simeq \int_{z_B}^{z_T} \frac{g\eta}{\bar{T}} \left(\bar{w} \frac{\partial \bar{\theta}}{\partial z} - Q_R \right) dz, \tag{2.2d}$$

where g is the acceleration due to gravity, $\bar{T}$ the large-scale temperature, $\bar{w}$ the large-scale velocity, $\bar{\theta}$ the large-scale potential temperature, and Q_R the radiative heating rate. It is important to note that we neglect a contribution of boundary-layer processes to the large-scale forcing in the definition (2.2d). This simplification is consistent with that which Arakawa and Schubert (1974) adopted in their quasi-equilibrium diagnosis, as well as the observationally–proposed approximation of parcel–environment quasi–equilibrium (Zhang 2002, 2003, Donner and Phillips 2003).

Finally, the vertical integrals in Eqs. (2.2a, b, d) are, in principle, performed from the convection base, z_B, to its top, z_T. However, for the sake of simplifying the coupling with the large-scale dynamics, we re-set them to be the surface, $z_B = 0$, and the top of the atmosphere, z_T. By adopting an equivalent vertical coordinate in the large-scale dynamics ($cf.$, Sec. 3), z_T, can easily be re-interpreted as the top of the troposphere.

For achieving the simplest possible coupling, we still assume that the radiative heating rate, Q_R, is prescribed, but modify the first term in the definition (2.2d) above, by following the evolution of the large-scale vertical velocity, $\bar{w}$. We assume a normalized vertical profile of the vertical velocity, $\bar{w}$, to be W so that

$$\bar{w} = \tilde{w}(x,t)W(z). \tag{2.3a}$$

Here, $\tilde{w}(x,t)$ designates the horizontal dependence of the large-scale vertical velocity, and x is the only horizontal coordinate. Throughout the paper, vertical profiles are designated by upper–case letters, and keep in mind that all of the vertical profiles are defined to be nondimensional, and also normalized to $O(1)$. Furthermore, the tilde sign is added to distinguish the horizontal components until the end of Sec. 3.

As a result, the large-scale forcing may be re-written as:

$$F = \mu \tilde{w} + F_R, \tag{2.3b}$$

where

$$\begin{aligned}\mu &= \int_{z_B}^{z_T} \frac{g\eta}{\bar{T}} W \frac{d\bar{\theta}}{dz} dz \\ &\sim \frac{gH}{T_0}\frac{d\bar{\theta}}{dz} \sim \frac{10\,\mathrm{m/s^2} \times 30\,\mathrm{K}}{300\,\mathrm{K}} \\ &\sim 1\,\mathrm{m/s^2}\end{aligned} \tag{2.4a}$$

measures the efficiency with which large-scale ascent generates the cloud work function (PEC), A. The second term in Eq. (2.3b),

$$F_R = -\int_{z_B}^{z_T} \frac{g\eta}{\bar{T}} Q_R dz, \tag{2.4b}$$

measures the rate at which the cloud work function (PEC) is generated by radiative cooling.

Finally, for closing the system, as in Yano and Plant (2012b), we assume a relation

$$K = \beta M_B, \tag{2.5}$$

where β is a constant estimated to be $\beta \sim 10^4$ m^2/s.

3. Large-Scale System

As a first step in constructing a large-scale system to be coupled with the convective energy cycle system introduced in the last section, we consider the large-scale heat equation in Sec. 3.a, because it is the key equation to achieve a coupling of the two scales. The formulation is completed more formally by introducing the normal mode decomposition of the linear primitive equation system in Sec. 3.b. The presentation is rather backwards, because the first subsection has to quote some of the results to be obtained in the following subsection. Nevertheless, we present in this order for the sake of making the physical motivations clear before a more complete mathematical formulation is provided. The system is assumed linear throughout this section. Nonlinear advection terms will be considered later in Sec. 4.e.

a. Large-Scale Heat Equation

A major feedback of convection to the large-scale state is found in the heat equation, which may be written as

$$\frac{\partial \theta}{\partial t} + w\frac{d\bar{\theta}}{dz} = Q_c + Q_R. \tag{3.1}$$

Here, Q_c is the convective heating rate, approximately given by:

$$Q_c = \sigma w_c \frac{d\bar{\theta}}{dz} \tag{3.2}$$

neglecting the effect of detrainment for simplicity ($cf.$, Yano and Plant, 2020). Recall that Q_R is the radiative heating.

Because the convective dynamics is described in terms of a single vertical mode, it is appropriate to reduce the large-scale dynamics similarly. For this reason, we have already assumed only a single vertical mode for the large-scale dynamics by writing the vertical velocity in the form of Eq. (2.3a) in Sec. 2, and equivalently, the potential temperature is represented by:

$$\theta = \tilde{\theta}(x,t)\Theta(z). \tag{3.3}$$

Here, Θ is a nondimensional, normalized vertical profile and $\tilde{\theta}$ describes the horizontal dependence. We also set

$$\sigma w_c = \frac{\eta}{\rho_0} M_B = \eta \tilde{w}_c,$$

where ρ_0 is the surface density.

As a standard procedure for projecting an equation onto a given vertical mode, we multiply Eq. (3.1) by Θ, and integrate it vertically. As a result, we obtain

$$\frac{\partial \tilde{\theta}}{\partial t} + \frac{\theta^*}{z_T}\tilde{w} = \hat{\eta}\tilde{w}_c - \hat{Q}_R^* w_R, \tag{3.4}$$

where

$$\frac{\theta^*}{z_T} = \left\langle W\Theta\frac{d\bar{\theta}}{dz}\right\rangle = \frac{\theta_0 h_E}{z_T^2}, \tag{3.5a}$$

$$\hat{\eta} = \left\langle \eta\Theta\frac{d\bar{\theta}}{dz}\right\rangle, \tag{3.5b}$$

$$\hat{Q}_R^* = -\langle \Theta Q_R \rangle. \tag{3.5c}$$

Here, we define the angled brackets as an integral operator

$$\langle * \rangle = \frac{1}{z_T}\int_0^{z_T} * dz,$$

setting $z_B = 0$ in previous vertical integrals, as already discussed. We have also assumed that Θ is normalized by

$$\langle \Theta^2 \rangle = 1.$$

We further introduce θ^* as a characteristic scale for θ. An alternative representation is also given in Eq. (3.5a) in terms of a reference value of potential temperature, θ_0 and an equivalent depth, h_E: this form will prove convenient later.

It will be shown in next subsection that the vertical–wind profile, W, is related to the potential–temperature profile, Θ, by:

$$W = \frac{\theta^*}{z_T}\left(\frac{d\bar{\theta}}{dz}\right)^{-1}\Theta. \tag{3.5d}$$

from Eq. (3.11b) to be derived below.

Additionally, the nondimensional radiative vertical velocity, w_R, has been introduced in Eq. (3.4), in order to represent a possible horizontal distribution of radiation. This study assumes the radiation to be horizontally homogeneous and thus we will simply set it to unity in the following, but explicitly re-introduce it whenever important to indicate the role of radiation in a given equation.

With the final goal of reducing the system to a shallow-water analogue in mind, it is convenient to replace the potential temperature, $\tilde{\theta}$, in the heat equation (3.4) by the height field, $\tilde{h}$. These two variables are linked together through hydrostatic balance, as will be obtained in Eq. (3.12b) below:

$$\tilde{h} = -\frac{h_E}{\theta^*}\tilde{\theta} = -\frac{z_T}{\theta_0}\tilde{\theta}. \tag{3.6}$$

As a result, the heat equation reduces to:

$$\frac{\partial \tilde{h}}{\partial t} - \hat{S}(\tilde{w} - \alpha \tilde{w}_c) = \hat{Q}_R. \tag{3.7}$$

Here, the introduced nondimensional parameters are estimated as:

$$\hat{S} = \frac{h_E}{z_T} \sim 10^{-2}, \tag{3.8a}$$

$$\alpha = \frac{z_T}{\theta^*}\hat{\eta} = \frac{z_T^2}{\theta_0 h_E}\hat{\eta} \sim 1, \tag{3.8b}$$

$$\hat{Q}_R = \frac{h_E}{\theta^*}\hat{Q}_R^* = \frac{z_T}{\theta_0}\hat{Q}_R^*. \tag{3.8c}$$

Recall that $\hat{\eta}$ has already been defined by Eq. (3.5b). The orders of magnitude estimates in (3.8a, b) are based on $h_E \sim 10^2$ m, $z_T \sim 10$ km, $\theta_0 \simeq 300$ K, and $\theta^* \sim 3$ K.

b. Normal–Mode Decomposition of the Linear Primitive Equation System

A thermodynamic formulation for a shallow-water analogue atmosphere has been introduced in the last subsection, in which the large–scale heat equation reduces to a height equation for shallow water. To complete the construction of a shallow–water analogue of the tropical atmosphere large-scale dynamics, we now consider a full, linear primitive equation system to see how the vertical profiles of the variables may be defined consistently. These profiles are usually called normal modes ($cf.$, Kasahara and Puri 1981).

We consider a linear horizontally one-dimensional system with the Boussinesq approximation:

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x}, \tag{3.9a}$$

$$\frac{\partial \phi}{\partial z} = g\frac{\theta}{\theta_0}, \tag{3.9b}$$

$$\frac{\partial \theta}{\partial t} + w\frac{d\bar{\theta}}{dz} = Q, \tag{3.9c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{3.9d}$$

Here, θ_0 is a constant reference potential temperature, u is the horizontal velocity, and ϕ is the geopotential. The total diabatic heating has been set to $Q = Q_c + Q_R$ as in the last subsection.

To apply the above system to a realistic atmosphere, the system is best re-interpreted as a consequence of transforming the pressure coordinate, p, into an equivalent geometrical coordinate, z, by the relation $dp = -\rho_0 g dz$ with ρ_0 a reference density, but with a minor modification to the hydrostatic balance (3.9b) of multiplying by an additional factor, $\rho_0\theta_0/\rho\bar{\theta}$ on the right-hand side. Keep in mind that all of the vertical integrals considered in the convective energy cycle formulation must also be re-interpreted accordingly.

We introduce a separation of variables by Eqs. (2.3a) and (3.3), as well as:

$$u = \Phi\tilde{u},\ \phi = \Phi\tilde{\phi},\ Q = \Theta\tilde{Q}. \tag{3.10a, b}$$

By substituting Eqs. (2.3a), (3.3), and (3.10a, b) into Eqs. (3.9a, b, c, d), we find that the vertical profiles must mutually satisfy the relations:

$$z_T \frac{d\Phi}{dz} = -\Theta, \tag{3.11a}$$

$$\Theta = \frac{z_T}{\theta^*}\frac{d\bar{\theta}}{dz}W, \tag{3.11b}$$

$$\Phi = z_T \frac{dW}{dz}. \tag{3.11c}$$

The two scales, z_T and θ^*, have been introduced so that all the vertical profiles consistently remain nondimensional, and also of the order unity.

By further substituting (3.11a, c) into (3.11b), we find:

$$\left[\frac{d^2}{dz^2} + \frac{1}{z_T}\left(\frac{1}{\theta^*}\frac{d\bar{\theta}}{dz}\right)\right]W = 0.$$

Here, $z_T\theta^*$ constitutes an eigenvalue in this equation. A more commonly accepted form is obtained by re–writing the above into:

$$\left[\frac{d^2}{dz^2} + \frac{1}{h_E}\left(\frac{1}{\theta_0}\frac{d\bar{\theta}}{dz}\right)\right]W = 0$$

with the equivalent depth,

$$h_E = \frac{\theta^*}{\theta_0}z_T,$$

constituting the standard engivenvalue of this problem (cf. Eq. 3.5a). It can be seen that the equivalent depth is the scaled–down version of the vertical scale by the relative fluctuation of the buoyancy with respect to the reference state.

Consequently, the equations for the horizontal components are given by:

$$\frac{\partial \tilde{u}}{\partial t} = -\frac{\partial \tilde{\phi}}{\partial x}, \tag{3.12a}$$

$$\tilde{\phi} = -\frac{gh}{\theta_0}\tilde{\theta} = -\frac{gh_E}{\theta^*}\tilde{\theta}, \tag{3.12b}$$

$$\frac{\partial \tilde{\theta}}{\partial t} + \frac{\theta^*}{z_T}\tilde{w} = \tilde{Q}, \tag{3.12c}$$

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\tilde{w}}{z_T} = 0. \tag{3.12d}$$

By further setting, $\tilde{\phi} = g\tilde{h}$, re-writing Eq. (3.12c) in terms of $\tilde{h}$, we recover Eq. (3.7) already introduced. By eliminating the vertical velocity with the help of the mass continuity (3.12d), we find that the governing equation set for the horizontal components constitute an analogue of the shallow–water system with the equivalent depth, h_E playing the role of the depth.

4. Nondimensionalization

For ease of further analyses, we now nondimensionalize the system derived over Secs. 2–3.

a. Convective Energy-Cycle System

To nondimensionalize the convective energy cycle, we first note that the equilibrium state is given at the convective scale by:

$$A = A_0 \equiv \beta/\tau_D \sim 10\,\mathrm{J/kg}, \tag{4.1a}$$

$$M_B = M_0 \equiv F_R/\gamma \sim 10^{-2}\,\mathrm{kg/m^2}/s, \tag{4.1b}$$

where F_R is the radiative contribution to convective forcing. Estimates are based on the values of $\beta \sim 10^4$ m^2/s, $\tau_D \sim 10^3$ sec, $F_R \sim 10^{-2}$ m^2/s^3, $\gamma \sim 1$ m^4/s^{2}kg by following Yano and Plant (2012b). Setting, for now, the large-scale equilibrium to be simply quiescent, $\tilde{w} = \tilde{h} = 0$, we find that the convection-base mass flux is further constrained to satisfy

$$M_B = \frac{\rho_0 \hat{Q}_R}{\alpha \hat{S}} \tag{4.1c}$$

from Eq. (3.7). Recall that $\hat{Q}_R$ is a measure of the radiative cooling rate, as defined by Eq. (3.8c). Obviously, this value must also agree with F_R/γ given by Eq. (4.1b).

We nondimensionalize the large-scale vertical velocity by:

$$\tilde{w} = w_0 \tilde{w}_*.$$

where the subscript $*$ suggests a nondimensionalized horizontal dependence, and w_0 is the scale of the vertical velocity. Keep in mind that the subscript $*$ will be tentative, and it will be removed as soon as the nondimensionalization is accomplished.

The appropriatetime scale, τ_c, and vertical-velocity scale, w_0 for nondimensionalization are given by

$$\tau_c = (\beta/F_R)^{1/2} \sim 10^3 \,\text{sec}, \tag{4.2a}$$

$$w_0 = F_R/\mu \sim 10^{-2} \text{m/s}. \tag{4.2b}$$

The convective-scale variables are nondimensionalized into k_c and a by setting

$$M_B = M_0 k_c, \tag{4.3a}$$

$$A = \frac{\tau_D}{\tau_c} A_0 a, \tag{4.3b}$$

such that the resulting nondimensionalized equations are:

$$\frac{\partial k_c}{\partial t'} = a k_c - \frac{k_c}{\tau_D^*}, \tag{4.4a}$$

$$\frac{\partial a}{\partial t'} = -k_c + w + w_R, \tag{4.4b}$$

where the dependent variables are defined by:

- $k_c = w_c$: convective kinetic energy (or the convective mass flux)

- a : the cloud work function (which may conceptually be interpreted as a convective potential energy) .

$\tau_D^* = \tau_D/\tau_c$ is a nondimensional damping time scale, and $w_R(=1)$ is a normalized radiative vertical velocity. In Eqs. (4.4a, b) the subscript $*$ indicating nondimensional variables has already been removed.

As required, we use the following notations in an interchangeable manner

$$k_c = w_c \tag{4.5}$$

depending on the context. Note further that a prime sign is added to the nondimensional time, t', because a different nondimensionalization of time will be introduced for the large-scale dynamics in the next subsection.

b. Large-Scale System

We nondimensionalize the large-scale system by introducing the scales u_0, h_0, τ_L, and L, marking the nondimensional variables with the subscript $*$ for now, thus, *e.g.*,

$$\frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x_*}.$$

By substituting into Eqs. (3.12a, b, c), we find that convenient nondimensionalization scales are:

$$h_0 = h_E,\ u_0 = c_g,\ \tau_L = L/c_g, \tag{4.6a, b, c}$$

where $c_g = (gh_E)^{1/2}$ is the gravity-wave speed, and the characteristic horizontal scale, L, is left to be determined. We set $L = 3\times10^3$ km provisionally, for the purpose of some numerical estimates.

After removing the tilde signs, and removing the subscripts $*$ from nondimensional variables, the resulting nondimensional set of equations are:

$$\frac{\partial u}{\partial t} = -\frac{\partial h}{\partial x}, \tag{4.7a}$$

$$\frac{\partial h}{\partial t} + \frac{\partial u}{\partial x} = -Q, \tag{4.7b}$$

$$w = -\hat{r}_L \frac{\partial u}{\partial x}. \tag{4.7c}$$

Here,

$$Q = \hat{\alpha}(w_c - w_R) = \hat{\alpha} w_c - \hat{Q}_R, \tag{4.8a}$$

$$\hat{r}_L = \frac{c_g z_T}{w_0 L} \sim 10, \tag{4.8b}$$

$$\hat{\alpha} = \alpha / \hat{r}_L, \tag{4.8c}$$

and $\hat{r}_L$ may be considered an effective aspect ratio of the system. Alternatively, it can be interpreted as a ratio of two characteristic horizontal scales:

$$\hat{r}_L = L_D / L,$$

where

$$L_D = \frac{c_g z_T}{w_0} \sim 3 \times 10^4 \text{km}.$$

Also keep in mind that the total depth of the shallow water is: $h_T = 1 + h$.

Recall from Eq. (3.7) that α controls the relative contributions of large-scale and convective-scale velocities to the stratification. The parameter, $\hat{\alpha}$ introduced by Eq. (4.8c) thus measures the efficiency of convection in modifying the stratification of the atmosphere, while $1 - \alpha$ may be considered a nondimensional measure of the effective stratification (or gross moist stability: Neelin and Held 1987). In particular, when $\alpha = 1$, the convective atmosphere is effectively neutrally stratified. Here, w_0 is a characteristic scale of the large–scale vertical velocity and, by nondimensionalization, the radiatively–driven vertical velocity is $w_R = 1$.

c. Two Time Scales

To couple together the two systems for convection and the large scale, we need to take care of the two different time scales adopted for the systems in nondimensionalization, τ_c (Eq. 4.2a) and $\tau_L = L/c_g$ (Eq. 4.6c). The ratio of the two is

$$\hat{r}_c = \tau_c/\tau_L \sim 10^{-2}. \tag{4.9c}$$

We will henceforth use τ_L for both systems for consistency. As a result, Eqs. (4.4a, b) are expressed as:

$$\hat{r}_c \frac{\partial k_c}{\partial t} = a k_c - \frac{k_c}{\tau_D^*}, \tag{4.9a}$$

$$\hat{r}_c \frac{\partial a}{\partial t} = -k_c + w + w_R. \tag{4.9b}$$

Note that for a large-scale horizontal scale of $L \simeq 30$ km, $\hat{r}_c \simeq 1$, and the two time scales match.

d. Coupling Problem

Through the considerations over the last subsections, we have arrived at a complete nondimensional set of equations given by (4.7a, b, c) and (4.9a, b). However, there remains one more issue to be addressed: the large–scale height, h, which is also related to the potential temperature, θ by Eq. (3.6), is effectively equivalent to the convective–scale cloud work function (PEC), a, because by neglecting contributions from the boundary layer, the buoyancy integral that defines a is determined exclusively by contributions of the environmental potential temperature, also neglecting the virtual effect for the present purpose. Thus, a is nothing other than an alternative measure of the tropospheric potential temperature, in addition to h. Here, strictly speaking, we can still distinguish them by taking different vertical profiles in the definitions. However, retaining two measures of the potential temperature in a single–layer shallow–water analogue model would be rather redundant. Thus, we now reduce them to a single equation by establishing the equivalence of the two.

This is accomplished in the following manner, by introducing two additional constraints. By comparing between the right–hand side of Eq. (4.9b) and the definition (4.8a), we find that

$$\hat{r}_c\hat{\alpha}\frac{\partial a}{\partial t} - \hat{\alpha}w = -Q, \tag{4.10a}$$

also recalling that $k_c = w_c$. For comparison, the height equation (4.7b) is re-written with the help of Eq. (4.7c) as:

$$\frac{\partial h}{\partial t} - \frac{w}{\hat{r}_L} = -Q. \tag{4.10b}$$

These two expressions suggest that the two variables become equivalent by setting:

$$h = \hat{r}_c\hat{\alpha}a. \tag{4.11a}$$

Furthermore, for consistency of the large-scale vertical advection term (2nd on the left-hand side) in both equations (4.10a, b), a further constraint is required to establish the equivalence:

$$\hat{\alpha} = 1/\hat{r}_L. \tag{4.11b}$$

By further referring to the definition of $\hat{\alpha}$ in Eq. (4.8c), this condition simply reduces to

$$\alpha = 1. \tag{4.11c}$$

Recall from Sec. 4.b that the parameter α measures the efficiency of convection in modifying the stratification of the atmosphere.

The equivalence between CAPE (PEC) and the height in the shallow–water analogue atmosphere has been pointed out by Mapes (1998). We just establish this connection in a more formal manner. As a result, there is no longer a need to consider the time evolution of PEC, a, separately.

Consequently Eq. (4.9a) describes the convective–scale process, alongside the equation set (4.7a, b, c) for the large scale. With the help of Eq. (4.11a), the PEC can be eliminated from Eq. (4.9a) which becomes:

$$\hat{\epsilon}\frac{\partial k_c}{\partial t} = \hat{\alpha}hk_c - \frac{k_c}{\tilde{\tau}_D}, \tag{4.12a}$$

where

$$\tilde{\tau}_D = \tau_D^*/\hat{r}_c\hat{\alpha}^2 = \tau_D/\hat{r}_c\hat{\alpha}^2\tau_c \sim 10^4, \tag{4.12b}$$

$$\hat{\epsilon} = \hat{r}_c^2\hat{\alpha}^2 \sim 10^{-6}. \tag{4.12c}$$

Large and small values for these two parameters suggest shorter time scales involved with convection compared to those of the large scale.

e. Full System with Nonlinearity

It remains to add nonlinearity to the linear version of the large–scale system derived so far, Eqs. (4.7a, b, c). This final step turns out be rather involved, and the details are presented in the Appendix. Therein, we examine the physical consistency of the included nonlinear terms with the energy cycle of the system. Based on those examinations, we adopt the final large–scale equation set to be:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial h}{\partial x}, \tag{4.13a}$$

$$\frac{\partial h}{\partial t} + \frac{\partial u}{\partial x} = -Q, \tag{4.13b}$$

$$w = -\hat{r}_L\frac{\partial u}{\partial x}. \tag{4.13c}$$

Thus, the nonlinear advection term has been added only to the momentum equation (4.13a), but not to the continuity (heat) equation (4.13b).

In summary, the full nonlinear system consists of Eqs. (4.13a, b, c) and (4.12a).

5. Steady Solutions

We first examine the steady solutions. This serves two purposes: i) to define a basic state of the system, as a first step for performing perturbation analyses; and ii) to seek for the possibility of a solution with a steady circulation, as an idealized analogue of the Hadley–Walker circulation.

The steady heat budget of the system is obtained by substituting Eqs. (4.13c) and (4.8a) into Eq. (4.13b):

$$\bar{w} - \alpha\bar{w}_c + \hat{r}_L\hat{Q}_R = 0$$

or

$$\bar{w} - \alpha \bar{k}_c + \alpha w_R = 0 \tag{5.1}$$

Here, the overbars are added to denote a steady state. Also keep in mind that we retain two notations with $k_c = w_c$.

The equilibrium state of convection is obtained from (4.12a) as:

$$\bar{k}_c = \bar{w}_c = 0 \quad \text{or} \quad \bar{h} = 1/\hat{\alpha}\tilde{\tau}_D \sim 10^{-3}. \tag{5.2a, b}$$

In the following, we take the second choice (5.2b), which is only a matter of adding a constant height on perturbations. The first choice (5.2a) is less interesting with no possibility of convection in the basic state.

From the heat balance (5.1), we see that $\bar{w}$ and $\bar{w}_c$ can be chosen freely so long they are consistent with the dynamics. To seek a more specific solution, we set:

$$\bar{u} = \bar{u}_0 \sin kx \tag{5.3a}$$

with $\bar{u}_0$ a constant. Its substitution into the continuity equation (4.13c) leads to:

$$\bar{w} = -\bar{w}_0 \cos kx \tag{5.3b}$$

with $\bar{w}_0 = \hat{r}_L k u_0$. Furthermore, from Eq. (5.1),

$$\bar{w}_c = w_R - \frac{\bar{w}_0}{\alpha} \cos kx. \tag{5.3c}$$

To maintain the convective vertical velocity to be always positive definite, *i.e.*, $\bar{w}_c \geq 0$, we require $w_R \geq \bar{w}_0/\alpha$. If we further assume the minimum convective velocity to be zero, we obtain $\bar{w}_0 = \alpha w_R$.

Finally, the steady nonlinear momentum equation,

$$\frac{\partial}{\partial x}\frac{\bar{u}^2}{2} = -\frac{\partial \bar{h}}{\partial x},$$

must be satisfied. However, here we face a problem: by the convective equilibrium condition, we have already set $\bar{h}$ to be constant by Eq. (5.2b), and thus the right–hand side vanishes from the above, and there is no term to balance with the nonlinear advection on the left–hand side. We circumvent this difficulty by noting that the nonlinear advection term arising from a baroclinic circulation, actually projects onto a barotropic mode, and thus the height perturbation required to balance the right–hand side is also of a barotropic mode:

$$\frac{\partial}{\partial x}\frac{\bar{u}^2}{2} = -\frac{\partial \bar{h}_b}{\partial x},$$

with the subscript b standing for the barotropic mode, but also suggesting that this mode arises directly from the surface–boundary effect, $e.g.$, the SST distribution, partially reminiscent of the idea of Lindzen and Nigam (1987). The barotropic height field which balances with the nonlinear term is given by:

$$h_b = \frac{u_0^2}{4}(\cos 2kx - 1).$$

The short analysis of this section outlines very crudely how a consistent theory for steady tropical circulations can be developed in the context of a shallow-water analogue formulations: for further analyses we rewfer to $e.g.$, Gill (1980), Lindzen and Nigam (1987), Neelin and Held (1987), Yano (2023).

6. Linear Analysis

For performing perturbation analyses in the following two sections, we assume a homogeneous basic state with no large–scale circulation, $i.e.$, $\bar{u} = \bar{w} = 0$. The basic–state height is defined by Eq. (5.2b), and from Eq. (5.1), $\bar{w}_c = w_R = 1$, also recalling $\alpha = 1$ ($cf.$, Eq. 4.11c).

The resulting set of linear perturbation equations is:

$$\frac{\partial u'}{\partial t} = -\frac{\partial h'}{\partial x} \tag{6.1a}$$

$$\frac{\partial h'}{\partial t} + \frac{\partial u'}{\partial x} + \hat{\alpha} w'_c = 0 \tag{6.1b}$$

$$\hat{\epsilon}\frac{\partial w'_c}{\partial t} = \hat{\alpha} h' \tag{6.1c}$$

with the prime sign, $'$, denoting perturbation variables, and $w_c' = k_c'$.

We further assume a solution of the form, $\sim e^{i(kx+\omega t)}$. Then, the linear frequency is given by:

$$\omega^2 = k^2 + \hat{\alpha}^2/\hat{\epsilon}$$

or

$$\omega^2 = k^2 + \frac{1}{\hat{r}_c^2}. \tag{6.2}$$

Note that only a neutral wave solution is available, and the standard gravity–wave solution is recovered by setting $\hat{r}_c \to \infty$. Since $\hat{r}_c = \tau_c/\tau_L$ this limit corresponds to setting the convective time scale much longer than that of the large scale. Rather unintuitively, the presence of finite convective time scale ($i.e.$, τ_c finite) increases the frequency of the mode to be larger than that of the dry gravity wave: by further decreasing τ_c, the waves propagate faster. Note that in absence of a large–scale circulation, the system reduces to a linear version of the convective discharge–recharge system ($cf.$, Yano and Plant 2012b);

$$\frac{\partial h'}{\partial t} + \hat{\alpha} w_c' = 0,$$
$$\hat{\epsilon}\frac{\partial w_c'}{\partial t} = \hat{\alpha} h'.$$

This leads to an oscillating solution with $\omega = \hat{\alpha}/\hat{\epsilon}^{1/2} = 1/\hat{r}_c = \tau_L/\tau_c$. Effectively, the dispersion (6.2) is comprised of the square sum of the dry and convective frequencies.

7. Weakly Nonlinear Analysis

As an extension to the analysis of the last section, we now take into account a weak nonlinearity. For the purpose of developing a weakly–nonlinear formulation in a formal manner, we introduce an explicit perturbation parameter, which we choose to be $\hat{\epsilon}$, bearing in mind the numerical estimate of (4.12c). We also focus on the situation in which the system satisfies the free–ride balance

$$\frac{\partial u'}{\partial x} + \hat{\alpha} w_c' = 0 \tag{7.1}$$

($cf.$, Fraedrich and McBride 1989) to the leading order of Eq. (4.13b). This state, alternatively called the weak-temperature gradient approximation (Sobel *et al.* 2001), can also be considered to be a quasi–equilibrium closure under the given shallow–water formulation.

To obtain (7.1) to the leading order, the variables must be be re–scaled. It is found that appropriate re–scalings are:

$$h = \bar{h} + \hat{\epsilon}^3 h', \tag{7.2a}$$

$$w_c = \bar{w}_c + \hat{\epsilon} w'_c, \tag{7.2b}$$

$$u = \hat{\epsilon}^{3/2} u', \tag{7.2c}$$

and

$$\partial/\partial t = \hat{\epsilon}\partial/\partial\tau \tag{7.2d}$$

$$\partial/\partial x = \hat{\epsilon}^{-1/2}\partial/\partial\xi \tag{7.2e}$$

Thus, a longer time and shorter horizontal scales are introduced compared to the original nondimensionalization scales. Recall that $\bar{h}$ is defined by Eq. (5.2b).

After substituting these re–scalings into the full set of equations, we obtain to the leading order of Eqs. (4.12a) and (4.13a):

$$\frac{\partial u'}{\partial \tau} + u' \frac{\partial u'}{\partial \xi} = -\frac{\partial h'}{\partial \xi}, \tag{7.3a}$$

$$\frac{\partial w'_c}{\partial \tau} = \hat{\alpha} h'. \tag{7.3b}$$

From Eqs. (7.1) and Eqs. (7.3b), we find:

$$w'_c = -\frac{1}{\hat{\alpha}} \frac{\partial u'}{\partial \xi}, \tag{7.4a}$$

$$h' = \frac{1}{\hat{\alpha}} \frac{\partial w'_c}{\partial \tau}. \tag{7.4b}$$

Substituting those expressions into Eq. (7.3a), we obtain a single equation for u':

$$\frac{\partial u'}{\partial \tau} + u' \frac{\partial u'}{\partial \xi} - \hat{\alpha}^{-2} \frac{\partial^3 u'}{\partial \xi^2 \partial \tau} = 0. \tag{7.5}$$

Let us examine the linearized equation briefly:

$$\frac{\partial}{\partial \tau}\left(1 - \hat{\alpha}^{-2}\frac{\partial}{\partial \xi^2}\right) u' = 0,$$

which has the dispersion relation:

$$\omega(k^2 + \hat{\alpha}^2) = 0. \tag{7.6}$$

Thus, possible solutions are $\omega = 0$ and $k^2 = -\hat{\alpha}^2$. Keep in mind that the horizontal wavenumber, k, is defined in terms of the re–scaled horizontal scale. Thus, only evanescent waves are available in the linear limit with the frequency left undetermined. As argued in *e.g.*, Yano and Flierl (1994), and Yano and Tribbia (2017), linear evanescent waves can be consistent solutions only if nonlinearity becomes important at a certain part of the system.

To solve the nonlinear equation (7.5), it is worthwhile to note that it has a similar form to the Kortewig–de Vries equation (*cf.*, Secs. 13.11 and 13.12 of Whitham 1974, Part 2, Epilogue of Lighthill 1978):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

The latter is known to have a soliton solution:

$$u = 12k^2 \mathrm{sech}^2[k(x - x_0 - 4k^2 t)]$$

Here, recall that $\mathrm{sech}\, x = \cosh^{-1} x$, and k and x_0 are arbitrary constants, which adjust the solution form. Thus, we anticipate that a solution with a similar form may also be available with Eq. (7.5). To seek this possibility, we set

$$u' = u_0 \mathrm{sech}^2[k(\xi - \xi_0) - \omega\tau]$$

with u_0, k, ω the parameters to be determined. Its substitution into Eq. (7.5) yields:

$$u_0 = 6\omega/\hat{\alpha}, \tag{7.7a}$$

$$k = \hat{\alpha}/2, \tag{7.7b}$$

while ω remains an arbitrary constant. The final solutions are:

$$u' = \frac{6\omega}{\hat{\alpha}}\operatorname{sech}^2\varphi, \tag{7.8a}$$

$$w'_c = \frac{6\omega}{\hat{\alpha}}\operatorname{sech}^3\varphi\sinh\varphi, \tag{7.8b}$$

$$h' = \frac{6\omega^2}{\hat{\alpha}^2}(-3\operatorname{sech}^4\varphi + 2\operatorname{sech}^2\varphi) \tag{7.8c}$$

with

$$\varphi = \frac{\hat{\alpha}}{2}(\xi - \xi_0) - \omega\tau. \tag{7.9}$$

Note that the wavenumber, k, of the solitary–wave solution is controlled by $\hat{\alpha}$, which is proportional to the ratio of the two horizontal scales, *i.e.*, $\hat{\alpha} = \alpha/\hat{r}_L = \alpha L/L_D$. Also recall the stretching factor, $\hat{\epsilon}^{-1/2}$, applied to the horizontal coordinate. Thus, a characteristic horizontal scale of this solitary wave is inferred by writing:

$$\hat{\alpha}\xi = \hat{\alpha}\hat{\epsilon}^{-1/2}x$$

From Eq. (4.12b), $\hat{\epsilon} = \hat{r}_c^2\hat{\alpha}^2$, so that

$$\hat{\alpha}\xi = \frac{x}{\hat{r}_c} = x\frac{\tau_L}{\tau_c} = \frac{Lx}{c_g\tau_c},$$

also recalling the definitions (4.9c) and (4.6c). Bearing in mind that Lx is the dimensional length of the system, a characteristic wavelength of the solitary wave solution is identified as: $c_g\tau_c \sim 50$ km. Thus, this wave is typically localized to the mesoscale.

Also note that the velocity and the height, respectively, are scaled by the factors, $\omega/\hat{\alpha}$ and $\omega^2/\hat{\alpha}^2$. Thus, the wave amplitude increases with its frequency, ω, and in a more acerbated manner for the height than the velocities. More significantly, the westerly and easterly–wind bursts propagate eastwards and westwards, respectively. In particular, the overall behavior of the

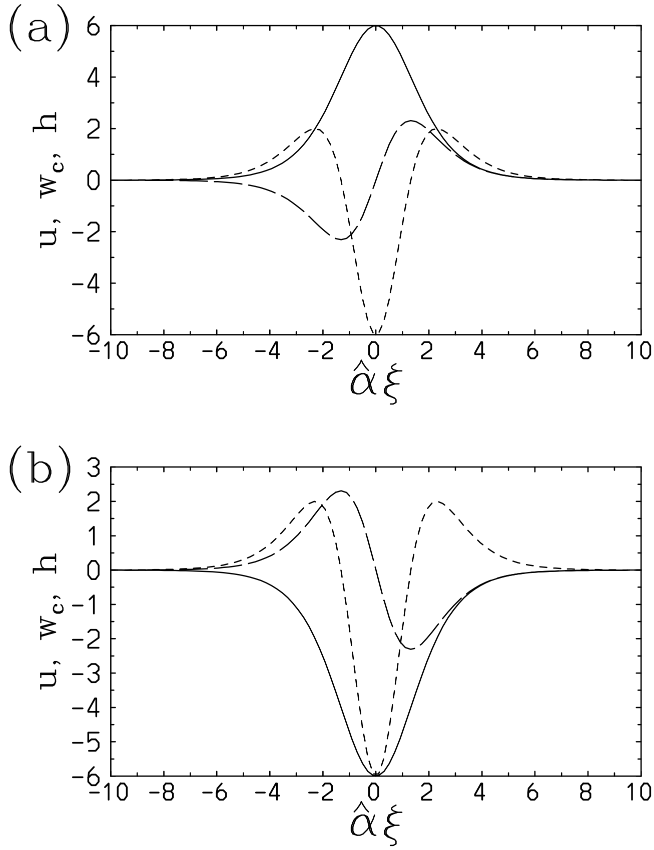

FIG. 1. Examples of the solitary–wave solutions (7.8a, b, c) with $\hat{\alpha} = 1$: (a) eastward propagating with $\omega = 1$, and (b) westward propagating with $\omega = -1$: the horizontal coordinate is $\hat{\alpha}\xi = \hat{\alpha}\hat{\epsilon}^{-1/2}x$ with the unit scale of about 50 km.

eastward–propagating solution is consistent with that of observed tropical westerly–wind bursts (*e.g.*, Hartten 1996, Yano *et al.* 2004).

Examples of the solutions with (a) $\omega = 1$ and (b) $\omega = -1$ are shown in Fig. 1. Here, curves are for the zonal wind, u' (solid), convection anomaly, w'_c (long dash), and the height, h' (short dash) with the horizontal coordinate given by $\hat{\alpha}\xi = \hat{\alpha}\hat{\epsilon}^{-1/2}x$.

8. Further Discussions

Atmospheric precipitating convection goes through a distinguished life cycle from a genesis to decay, and thus it is natural to expect that the convective life cycle may play an important role in its coupling to large-scale dynamics, especially over the tropics. From this perspective, the basic assumption of convective quasi-equilibrium adopted in convection parameterizations is unsatisfactory, because this approximation totally neglects life cycles associated with parameterized convection.

The present work shows what happens when a life cycle of convection is explicitly taken into account as a *part* of the large-scale dynamics. A qualitative consequence, even without performing any calculations, can even be intuitively expected: the short periodicity of convective life cycles dominate aspects of the coupled dynamics. This expected tendency is more explicitly demonstrated by a linear analysis, which shows that the squared frequency of a linear wave is obtained by a squared sum of the characteristic frequency of the convective life cycle and a dry gravity-wave frequency, under an analysis assuming no Coriolis force.

The convective life cycle used in the present study is based on the convective energy cycle originally introduced by Arakawa and Schubert (1974), in seeking a basis for a closure of their mass-flux parameterization. The energy cycle is closed by following Yano and Plant (2012b). The large-scale dynamics adopted is a shallow-water analogue.

The high-frequency characteristic of convectively-coupled waves obtained with explicit convective life cycles is in marked contrast to the typical characteristic under standard formulations with a convective quasi-equilibrium assumption. In the latter case, convection is found to slow down the dry large-scale waves by decreasing the effective stratification of the atmosphere. This behavior arises because any explicit periodicities associated with convection are effectively eliminated by averaging them out through the convective quasi-equilibrium assumption. The approach of the present paper explicitly retains such a high convective-scale frequency, and thus this frequency is added to a full spectrum of the whole system. An explicit emergence of the convective-scale high frequencies into the large-scale dynamics is obviously an unfavorable feature if the focus of an analysis is on the long timescale phenomena.

A more attractive feature emerges when the system is scaled down to a mesoscale regime, also introducing a weak nonlinearity. This re-scaling is performed in such a manner that the free-ride balance (Fraedrich and McBride 1989: see also Sobel *et al.* 2001) is obtained to the leading order. The analysis leads to a nonlinear equation analogous to the Kortweg-de Vries equation, and like the latter, it contains a solitary-wave solution. The obtained mesoscale solution is reminiscent of tropical westerly–wind bursts.

Although an analysis with the rotation effect is still to be performed, it is evident that the eastward–propagating solitary gravity wave solution obtained can be re-interpreted as a Kelvin wave in the presence of rotation so long as we can assume that the equatorial deformation radius is much larger than the longitudinal wavelength. Nevertheless, a full analysis of this system with the rotation effect will be worthwhile to explore rich possibilities of nonlinear interactions between convective life cycles and the equatorial waves. This investigation may also be considered a natural extension of dry solitary equatorial waves as investigated by Boyd (1980, 1983, 1984, 1985).

9. Conclusions

The most important lesson to learn from the present study is that if the focus is solely on the global scale of the atmosphere, then one should not try to include a convective life cycle explicitly into a model, how attractive this approach might appear to be at first sight.

On the other hand, for those who wish to investigate tropical atmospheric dynamics in its full spectrum, the convective energy-cycle system coupled with large-scale dynamics provides an attractive option to pursue. Although only a preliminary investigation has been performed, an identified solitary-wave solution, reminiscent of tropical westerly–wind bursts, already suggests a rich behavior of this system under full nonlinearity. However, we should also keep in mind that convection is still parameterized, using a mass-flux-based formulation.

Data Availability

No data is used in the present study.

Appendix: Energy Cycle Analysis

The purpose of this Appendix is identify the physically most consistent form of nonlinearity for the shallow–water analogue system from the point of view of the energy–cycle of the system. The most straightforward way to add nonlinearity to the linear large–scale system (4.7a, b, c) would be in the identical form as that which appears in the actual shallow-water system:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial h}{\partial x}, \tag{A.1a}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}u(1+h) = -Q, \tag{A.1b}$$

$$w = -\hat{r}_L \frac{\partial u}{\partial x}. \tag{A.1c}$$

Here, we are going to show that this form leads to a physically unacceptable interpretation from the point of view of the energy cycle. We show further that the problem arises with the postulated nonlinear contribution to Eq. (A.1b) but that the nonlinear advection term in Eq. (A.1a) may be retained.

a. Kinetic Energy

To derive the kinetic–energy budget, we first re-write the momentum equation (A.1a) in a flux form by multiplying it by $h_T = 1 + h$, and adding by Eq. (A.1b) multiplied by u:

$$\frac{\partial u h_T}{\partial t} + \frac{\partial}{\partial x}u^2 h_T = -\frac{\partial}{\partial x} y h_T^2 2 - uQ. \tag{A.2}$$

Multiplying Eq. (A.1a) by uh_T and Eq. (A.2) by u, we obtain the budget:

$$\frac{\partial}{\partial t}\frac{h_T}{2}u^2 + \frac{\partial}{\partial x}\frac{h_T u^3}{2} = -u\frac{\partial}{\partial x}\frac{h_T^2}{2} - \frac{u^2}{2}Q. \tag{A.3}$$

Here is the first key point to note: from a physical consideration, we expect that the large–scale kinetic energy would *not directly* be modified by a convective process or by diabatic heating. Thus, Eq. (A.3) is not physically consistent by containing a source term due to diabatic heating.

We can trace this physical inconsistency to the fact that the kinetic energy is defined by $h_T u^2/2$ above. Although this is a physically consistent definition of kinetic energy in the original shallow–water system, that is no longer the case for this shallow–water analogue atmosphere. This conclusion stems from the fact that in the shallow–water analogue atmosphere, the height is better interpreted as a representation of the potential-temperature anomaly rather than a representation of a fluid depth, as in the original definition of the shallow–water system.

Based on this consideration, we conclude that the kinetic energy is better defined as $u^2/2$. With this definition, the kinetic–energy budget is obtained by multiplying Eq. (A.1a) by u:

$$\frac{\partial}{\partial t}\frac{u^2}{2}+\frac{\partial}{\partial x}\frac{u^3}{3}=-u\frac{\partial h}{\partial x}. \tag{A.4}$$

Here, the form of the divergence term is rather unfortunate, and a minor negative consequence from the redefinition.

b. Potential Energy

A similar consideration also applies when defining the potential energy of this shallow–water analogue system. As already suggested above, the total depth, h_T, of the system does not have much physical significance: it is better to take the height perturbation, h, as a measure of the potential temperature perturbation, θ, under the relation (3.12b). Thus, it also follows that the potential energy is better defined by $h^2/2$ rather than $h_T^2/2$. Its budget is obtained by multiplying Eq. (A.1b) by h, so that:

$$\frac{\partial}{\partial t}\frac{h^2}{2}+h\frac{\partial}{\partial x}uh_T=-hQ. \tag{A.5}$$

We may note above that the advection term does not turn into a flux form as expected.

c. Total Energy Budget

Finally, by taking sum of Eqs. (A.4) and (A.5), we obtain the conservation law of the total energy as:

$$\frac{\partial}{\partial t}\left(\frac{u^2+h^2}{2}\right)+\frac{\partial}{\partial x}\frac{u^3}{3}+h\frac{\partial}{\partial x}uh_T+u\frac{\partial h}{\partial x}=-hQ.$$

To express the last two terms on the left–hand side closer to a flux form, recall that $h_T=1+h$, thus

$$h\frac{\partial}{\partial x}uh_T+u\frac{\partial h}{\partial x}=\frac{\partial}{\partial x}uh+h\frac{\partial}{\partial x}uh.$$

We can recognize that the remaining non–flux term on the left–hand side arises from the nonlinear term in the height equation (A.1b). This result suggests that it is unphysical to add a nonlinear advection term to the height (heat) equation under the present shallow–water analogue formulation. Thus, the choice of the form (4.13b) follows. After this modification, the total–energy conservation law reduces to:

$$\frac{\partial}{\partial t}\left(\frac{u^2+h^2}{2}\right)+\frac{\partial}{\partial x}\left(\frac{u^3}{3}+uh\right)=-hQ. \tag{A.6}$$

d. Coupling with Convection

The final step is to add the convective kinetic energy to the energy budget (A.6) just obtained. Towards this goal, note first that the term hQ on the right–hand side of the potential energy budget (A.5) can be re–written with the help of Eq. (4.8a) as:

$$hQ = \hat{\alpha} h k_c - h\hat{Q}_R. \tag{A.7}$$

Hence, convective kinetic energy is generated (i.e., $\hat{\alpha} h k_c > 0$ on the right hand side of Eq. 4.12a) by consuming the potential energy (i.e., $hQ > 0$ through the same process: the right–hand side of Eq. A.5). By substituting the expression (A.7) into the right–hand side of Eq. (A.6), we obtain:

$$\frac{\partial}{\partial t}\left(\frac{u^2+h^2}{2}\right)+\frac{\partial}{\partial x}\left(\frac{u^3}{3}+uh\right)=-\hat{\alpha} h k_c + h\hat{Q}_R. \tag{A.8}$$

Taking the sum of Eqs. (A.8) and (4.12a), the total–energy budget including the contribution of the convective scale is:

$$\frac{\partial}{\partial t}\left(\frac{u^2+h^2}{2}+\hat{\epsilon} k_c\right)+\frac{\partial}{\partial x}\left(\frac{u^3}{3}+uh\right)=h\hat{Q}_R-\frac{k_c}{\tilde{\tau}_D}. \tag{A.9}$$

Thus, as a whole the radiation, $\hat{Q}_R$, is the only ultimate source of the energy to the system, and the only sink is the dissipative loss, $k_c/\tilde{\tau}_D$, of convective kinetic energy. Note that the large–scale dynamics has been assumed to be dissipationless for simplicity.

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