

# *Multilateral exchange rates: a multivariate regression framework*

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## Multilateral exchange rates: A multivariate regression framework

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### ABSTRACT

Currencies must be priced in terms of a *numéraire* when they are included in a regression model. The numéraire can be either a single-currency numéraire or a multicurrency numéraire: a weighted basket of numéraire currencies. Pricing currencies in terms of a multicurrency numéraire results in a system of multilateral exchange rates. A no-arbitrage condition enforces the movements in the system of multilateral exchange rates associated with the numéraire currencies to be a singular system, where the covariance matrix is singular and its ordinary inverse does not exist. Singular systems pose a methodological challenge in a multivariate regression model. This paper provides a solution to overcome this methodological challenge by imposing implicit restrictions on both the explanatory variables and the regression coefficients. In addition, the generalized least squares estimator is modified by replacing the ordinary inverse with the generalized inverse. The proposed solution provides a consistent multivariate regression model to explain the observed heterogeneity in the relative currency market.

### 1. Introduction

Currencies are typically priced in terms of a *numéraire*. The numéraire can be either a single-currency numéraire that is a numéraire currency or a multicurrency numéraire that is a basket of numéraire currencies. If a multicurrency numéraire is chosen as the numéraire, each currency in the system of currencies is represented by a multilateral exchange rate. However, a no-arbitrage condition enforces the system of multilateral exchange rates associated with the numéraire currencies to be a singular system, which poses a methodological challenge in a multivariate regression model. The motivation for this paper is to overcome this methodological challenge.

A numéraire must be chosen when currencies are included in both univariate and multivariate regression models. For example, all currencies are priced in terms of a common numéraire for a univariate Frankel-Wei regression model, which is used to measure the relationship between currencies (Frankel & Wei, 1994). The choice for the common numéraire is between a single-currency numéraire or a multicurrency numéraire. If a single-currency numéraire is chosen, the currencies are represented by bilateral exchange rates. As previously mentioned, if a multicurrency numéraire is chosen, the currencies are represented by multilateral exchange rates.

Bilateral exchange rates tend to be used more than multilateral exchange rates (Haynes & Stone, 1994). However, multilateral exchange rates have recently been shown to be superior to bilateral exchange rates in the univariate Frankel-Wei regression framework, as bilateral exchange rates result in a biased estimator of the regression coefficients (Kunkler, 2021). Two of the most popular multicurrency numéraires are the International Monetary Fund's (IMF) Special Drawing Right (SDR) that was created in 1969 and the trade-weighted basket of numéraire currencies used for the US dollar index that was created in 1973. Another multicurrency numéraire that results in numéraire-invariant multilateral exchange rates is an equally-weighted basket of numéraire currencies

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(Hovanov et al., 2004; Kunkler & MacDonald, 2015).

A multicurrency numéraire is a weighted basket of numéraire currencies. The numéraire currencies themselves can be priced in terms of the multicurrency numéraire, even though the numéraire currencies are members of the multicurrency numéraire. For example, the US dollar can be priced in terms of the Special Drawing Right (SDR), even though the US dollar is a numéraire currency of the SDR. A no-arbitrage condition enforces the log returns of the system of multilateral exchange rates associated with the numéraire currencies to be a singular system, where the covariance matrix is singular and its ordinary inverse does not exist. Singular systems pose a methodological challenge when it comes to multivariate regression models. For example, when the disturbance covariance matrix from the multivariate regression model is singular, the ordinary matrix inverse does not exist. As a consequence, the generalized least squares (GLS) estimator of the regression coefficients also does not exist.

Much of the work in this area has been stimulated by the work on allocation models (see Haupt & Oberhofer, 2006, and the references within). There are two main solutions, namely, the deletion of one of the equations (Barten, 1969), or modifying the generalized least squares estimator of the regression coefficients (Theil, 1971). The former solution leads to the question of whether the deleted equation impacts the estimated parameters (see Bewley, 1986; Fiebig, 2001; Haupt & Oberhofer, 2006). The latter solution involves modifying the generalized least squares (GLS) estimator of the regression coefficients by replacing the ordinary inverse of the disturbance covariance matrix with the Moore-Penrose generalized inverse of the disturbance covariance (see Kreijger & Neudecker, 1977; Theil, 1971).

The solution that modifies the generalized least squares estimator of the regression coefficient (Theil, 1971) requires a singular disturbance covariance matrix. However, using any set of explanatory variables does not necessarily result in a singular disturbance covariance matrix. There are necessary and sufficient conditions for the existence of the generalized-inverse estimator of the regression coefficients (Dhrymes & Schwarz, 1987a). As a consequence, implicit restrictions are required on the explanatory variables and the regression coefficients to enforce the disturbance covariance matrix to be singular. If the implicit restrictions are satisfied, the generalized-inverse estimator of the regression coefficients will always exist (Dhrymes & Schwarz, 1987b).

This paper contributes to the literature by highlighting the methodological challenge of estimating regression coefficients when the numéraire currencies of a multicurrency numéraire are the dependent variables in a multivariate regression model. A solution is proposed based on Theil (1971) that modifies the generalized least squares (GLS) estimator of the regression coefficients by replacing the ordinary inverse with the generalized inverse. However, certain conditions need to be satisfied for the disturbance covariance matrix from the multivariate regression model to be singular. As a result, the solution imposes implicit restrictions on the explanatory variables and the regression coefficients to guarantee a singular disturbance covariance matrix. The multivariate regression framework can be classified as a model-derived singularity (Haupt & Oberhofer, 2002). The solution provides a consistent multivariate regression model to explain the observed heterogeneity in the relative currency market.

Seven developed market currencies are considered, namely, the US dollar, the Japanese yen, the Eurozone euro, the Swiss franc, the British pound, the Canadian dollar, and the Australian dollar. The proposed multivariate regression framework does not pre-specify which explanatory variables should be used in a multivariate regression model: similar to the Arbitrage Pricing Theory (see Ross, 1976). In addition, choosing explanatory variables for a multivariate regression model is a non-trivial task. For example, it is well-known that there is an exchange-rate disconnect puzzle between exchange rates and fundamentals (Engel & West, 2005; Meese & Rogoff, 1983a, 1983b). Thus, choosing the best set of explanatory variables is beyond the scope of this paper and is left for future research.

A focal point of this paper is to illustrate the implicit restrictions and to provide an application of the proposed multivariate regression framework. Currencies have been shown to move heterogeneously with global equity markets (Campbell et al., 2010). As a consequence, a group of local equity market indexes is used as explanatory variables, where there is one local equity market index associated with each currency. The results section confirms the previously reported heterogeneity in the currency market by showing that all currencies cannot move with, or against, the movements in a global equity market index. Furthermore, all currencies move in a common (panel-like) fashion with, or against, to the movements in a group of relative (idiosyncratic) equity market indexes.

The rest of this paper is organized as follows. Section 2 provides the material and methods. Section 3 describes the data sample and presents the results, and Section 4 concludes.

## 2. Material and methods

### 2.1. Bilateral exchange rates

A bilateral exchange rate is the price of one currency in terms of another currency: a single-currency numéraire. In log terms,<sup>1</sup> for a system of  $N_C$  currencies, let  $p_{B,t}^{i/j}$  represent the  $i$ th/ $j$ th bilateral exchange rate at time  $t$  for the  $i$ th currency in terms of the  $j$ th currency, where  $i, j = 1, \dots, N_C$ ; and  $t = 0, \dots, T$ . There is a no-arbitrage condition between the bilateral exchange rates for any three currencies given by:

$$p_{B,t}^{i/j} = p_{B,t}^{i/k} - p_{B,t}^{j/k} \quad (1)$$

<sup>1</sup> Bilateral exchange rates are modelled in log terms so the model does not suffer from the well-known Siegel Paradox (Siegel, 1972).

where  $i, j, k = 1, \dots, N_C; t = 0, \dots, T; p_{B,t}^{i/j}$  is the  $i$ th/ $j$ th bilateral exchange rate;  $p_{B,t}^{i/k}$  is the  $i$ th/ $k$ th bilateral exchange rate; and  $p_{B,t}^{j/k}$  is the  $j$ th/ $k$ th bilateral exchange rate (see Chacholiades, 1971). The no-arbitrage condition in (1) can be rearranged to give:

$$p_{B,t}^{i/j} + p_{B,t}^{k/i} + p_{B,t}^{j/k} = 0 \tag{2}$$

where  $i, j, k = 1, \dots, N_C; t = 0, \dots, T; p_{B,t}^{i/j}$  is the  $i$ th/ $j$ th bilateral exchange rate;  $p_{B,t}^{k/i}$  is the  $k$ th/ $i$ th bilateral exchange rate with  $p_{B,t}^{k/i} = -p_{B,t}^{i/k}$ ; and  $p_{B,t}^{j/k}$  is the  $j$ th/ $k$ th bilateral exchange rate. The no-arbitrage condition in (2) also applies to the log returns of the bilateral exchange rates for any three currencies given by:

$$\Delta p_{B,t}^{i/j} + \Delta p_{B,t}^{k/i} + \Delta p_{B,t}^{j/k} = 0 \tag{3}$$

where  $i, j, k = 1, \dots, N_C; t = 1, \dots, T; \Delta p_{B,t}^{i/j} = p_{B,t}^{i/j} - p_{B,t-1}^{i/j}$  is the log return of the  $i$ th/ $j$ th bilateral exchange rate;  $\Delta p_{B,t}^{k/i} = p_{B,t}^{k/i} - p_{B,t-1}^{k/i}$  is the log return of the  $k$ th/ $i$ th bilateral exchange rate;  $\Delta p_{B,t}^{j/k} = p_{B,t}^{j/k} - p_{B,t-1}^{j/k}$  is the log return of the  $j$ th/ $k$ th exchange rate; and  $\Delta$  is the first difference operator.

### 2.2. Multilateral exchange rates

A multilateral exchange rate is the price of one currency in terms of a weighted basket of numéraire currencies: a multicurrency numéraire. In log terms, for a system of  $N_C$  currencies, each multilateral exchange rate can be written as:

$$p_{M,t}^i = \sum_{j=1}^{N_C} w^j p_{B,t}^{i/j} \tag{4}$$

where  $i = 1, \dots, N_C; t = 0, \dots, T; p_{M,t}^i$  is the  $i$ th multilateral exchange rate for the  $i$ th currency;  $w^j$  is the numéraire weight associated with the  $j$ th numéraire currency; and  $p_{B,t}^{i/j}$  is the  $i$ th/ $j$ th bilateral exchange rate, with  $p_{B,t}^{i/i} = 0$ . It should be noted that the  $i$ th currency is also a numéraire currency of the multicurrency numéraire. For example, the US dollar is a numéraire currency of the Special Drawing Right (SDR) from the International Monetary Fund (IMF). The movements in the US dollar can be priced in terms of the SDR by:

$$\Delta p_{M,t}^{USD} = \sum_{j=1}^5 w^j \Delta p_{B,t}^{USD/j} = 0.4338 \Delta p_{B,t}^{USD/USD} + 0.2931 \Delta p_{B,t}^{USD/EUR} + 0.1228 \Delta p_{B,t}^{USD/CNY} + 0.0759 \Delta p_{B,t}^{USD/JPY} + 0.0744 \Delta p_{B,t}^{USD/GBP} \tag{5}$$

where  $t = 1, \dots, T; \Delta p_{M,t}^{USD}$  is the log return of the US dollar multilateral exchange rate in terms of the SDR; and  $\Delta p_{B,t}^{USD/j}$  is the log return of the  $USD/j$ th bilateral exchange rate with  $j = 1, \dots, 5$  and  $\Delta p_{B,t}^{USD/USD} = 0$ . The numéraire weights of the SDR are associated with the review in 2022, where the weights for the five numéraire currencies are: 0.4338 for the US dollar (USD), 0.2931 for the Eurozone euro (EUR), 0.1228 for the Chinese renminbi (CNY), 0.0759 for the Japanese yen (JPY), and 0.0744 for the British pound (GBP). Note that the numéraire weights of the SDR sum to one.

It is assumed throughout the rest of this paper that multilateral exchange rates are priced relative to an equally-weighted basket of  $N_C$  numéraire currencies so that  $w^j = 1/N_C$  for  $j = 1, \dots, N_C$ . In this case, each multilateral exchange rate in (4) can be rewritten as:

$$p_{M,t}^i = \frac{1}{N_C} \sum_{j=1}^{N_C} p_{B,t}^{i/j} \tag{6}$$

where  $i = 1, \dots, N_C; t = 0, \dots, T; p_{M,t}^i$  is the  $i$ th multilateral exchange rate for the  $i$ th currency; and  $p_{B,t}^{i/j}$  is the  $i$ th/ $j$ th bilateral exchange rate with  $j = 1, \dots, N_C$ .

### 2.3. Decomposing bilateral exchange rates

Bilateral exchange rates can be decomposed into the difference between two of the multilateral exchange rates from (6) by:

$$p_{B,t}^{i/j} = p_{M,t}^i - p_{M,t}^j \tag{7}$$

where  $i, j = 1, \dots, N_C; t = 0, \dots, T; p_{B,t}^{i/j}$  is the  $i$ th/ $j$ th bilateral exchange rate;  $p_{M,t}^i$  is the  $i$ th multilateral exchange rate; and  $p_{M,t}^j$  is the  $j$ th multilateral exchange rate (Kunkler & MacDonald, 2015). The decomposition in (7) also applies to the log returns of the bilateral exchange rates by:

$$\Delta p_{B,t}^{i/j} = \Delta p_{M,t}^i - \Delta p_{M,t}^j \tag{8}$$

where  $i, j = 1, \dots, N_C; t = 1, \dots, T; \Delta p_{B,t}^{i/j} = p_{B,t}^{i/j} - p_{B,t-1}^{i/j}$  is the log return of the  $i$ th/ $j$ th bilateral exchange rate;  $\Delta p_{M,t}^i = p_{M,t}^i - p_{M,t-1}^i$  is the log return of the  $i$ th multilateral exchange rate; and  $\Delta p_{M,t}^j = p_{M,t}^j - p_{M,t-1}^j$  is the log return of the  $j$ th multilateral exchange rate.

2.4. A multivariate regression model

In general, a multivariate regression model for the log returns of the  $N_C$  multilateral exchange rates with  $N_p$  explanatory variables can be written as:

$$\Delta p_M^i = \sum_{p=1}^{N_p} x_M^{p,i} \beta_M^{p,i} + \epsilon_M^i \tag{9}$$

where  $i = 1, \dots, N_C$ ;  $\Delta p_M^i$  is a  $T \times 1$  vector of log returns of the  $i$ th multilateral exchange rate;  $x_M^{p,i}$  is a  $T \times 1$  vector for the  $p$ th explanatory variable for the  $i$ th multilateral exchange rate;  $\beta_M^{p,i}$  is the  $i$ th regression coefficient for the  $p$ th explanatory variable for the  $i$ th multilateral exchange rate; and  $\epsilon_M^i$  is the  $i$ th disturbance term.

2.5. Singular system of equations

This section shows that the log returns for the system of  $N_C$  multilateral exchange rates for the  $N_C$  numéraire currencies is a singular system. In matrix notation, there is a no-arbitrage condition for the  $N_C$  multilateral exchange rates in (6) such that:

$$\sum_{i=1}^{N_C} p_M^i = \mathbf{0} \tag{10}$$

where  $p_M^i$  is a  $(T+1) \times 1$  vector of the  $i$ th multilateral exchange rate; and  $\mathbf{0}$  is a  $(T+1) \times 1$  vector of zeros (Kunkler & MacDonald, 2015). The no-arbitrage condition in (10) also applies to the log returns of the multilateral exchange rates by:

$$\sum_{i=1}^{N_C} \Delta p_M^i = \mathbf{0} \tag{11}$$

where  $\Delta p_M^i$  is a  $T \times 1$  vector of the log returns of the  $i$ th multilateral exchange rate;  $\Delta p_{M,t}^i = p_{M,t}^i - p_{M,t-1}^i$  is the log return of the  $i$ th multilateral exchange rate at time  $t$  with  $i = 1, \dots, N_C$ ; and  $\mathbf{0}$  is a  $T \times 1$  vector of zeros.

The no-arbitrage condition for the log returns of the system of  $N_C$  multilateral exchange rates in (11) creates a linear dependency, which can be seen by rearranging (11) to give:

$$\Delta p_M^i = - \sum_{j=1}^{N_C} I(i \neq j) \Delta p_M^j \tag{12}$$

where  $i = 1, \dots, N_C$ ;  $\Delta p_M^i$  is a  $T \times 1$  vector of the log returns of the  $i$ th multilateral exchange rate;  $\Delta p_M^j$  is a  $T \times 1$  vector of the log returns of the  $j$ th multilateral exchange rate; and  $I(i \neq j)$  is an indicator variable that is one when  $i \neq j$  and zero otherwise.

The linear dependency between the log returns of the system of  $N_C$  multilateral exchange rates results in a singular covariance matrix of the log returns of the multilateral exchange rates, where the ordinary matrix inverse does not exist. Let the covariance matrix be represented by:

$$\Sigma_{\Delta p} = \text{cov}(\Delta p_M^1, \dots, \Delta p_M^{N_C}) \tag{13}$$

where  $\Sigma_{\Delta p}$  is an  $N_C \times N_C$  covariance matrix for the log returns of the system of  $N_C$  multilateral exchange rates; and  $\sigma_{\Delta p}^{ij} = \text{cov}(\Delta p_M^i, \Delta p_M^j)$  is the covariance between  $\Delta p_M^i$  and  $\Delta p_M^j$ , and represents the  $i$ th row and  $j$ th column of  $\Sigma_{\Delta p}$ , with  $i, j = 1, \dots, N_C$ .

In summary, the log returns of the system of  $N_C$  multilateral exchange rates for the  $N_C$  numéraire currencies is a singular system. Thus, the covariance matrix of the log returns for the system of  $N_C$  multilateral exchange rates is singular and the ordinary inverse does not exist. Singular systems pose a challenge when it comes to multivariate regression analysis. For example, when the disturbance covariance matrix is also singular, the ordinary matrix inverse does not exist and consequently the generalized least squares (GLS) estimator of the regression coefficients will not exist. Theil (1971) suggested that the generalized inverse of the disturbance covariance matrix replace the ordinary inverse. Under these circumstances, there are necessary and sufficient conditions on the existence of the generalized-inverse estimator of the regression coefficients (Dhrymes & Schwarz, 1987a). In addition, if the adding up conditions are imposed the generalized-inverse estimator of the regression coefficients will always exist (Dhrymes & Schwarz, 1987b).

2.6. Implicit restrictions

2.6.1. Introduction

Using any set of explanatory variables does not necessarily result in a singular disturbance covariance matrix. Consequently, implicit restrictions are imposed on the explanatory variables and the regression coefficients, so the disturbance covariance matrix from the multivariate regression is singular. If the implicit restrictions are satisfied, the generalized-inverse estimator of the regression coefficients will always exist (Dhrymes & Schwarz, 1987b).

The covariance matrix of the system of  $N_C$  disturbance terms in (9) is given by:

$$\Sigma_{\epsilon} = \text{cov}(\epsilon_M^1, \dots, \epsilon_M^{N_C}) \tag{14}$$

where  $\Sigma_{\epsilon}$  is an  $N_C \times N_C$  disturbance covariance matrix; and  $\sigma_{\epsilon}^{ij} = \text{cov}(\epsilon_M^i, \epsilon_M^j)$  is the covariance between  $\epsilon_M^i$  and  $\epsilon_M^j$ , and represents

the  $i$ th row and  $j$ th column of  $\Sigma_\epsilon$ , with  $i, j = 1, \dots, N_C$ . One way for the disturbance covariance matrix to be singular is to impose implicit restrictions on the explanatory variables and the regression coefficients so that:

$$\sum_{i=1}^{N_C} \epsilon_M^i = \mathbf{0} \tag{15}$$

where  $\epsilon_M^i$  is a  $T \times 1$  vector of the  $i$ th disturbance term with  $i = 1, \dots, N_C$ ; and  $\mathbf{0}$  is a  $T \times 1$  vector of zeros.

Thus, implicit restrictions can be imposed on the explanatory variables and the regression coefficients so that the disturbance covariance matrix will be singular. Subsequently, the generalized least squares (GLS) estimator of the regression coefficients can be modified by replacing the ordinary inverse with the generalized-inverse (Theil, 1971). If the implicit restrictions are satisfied, the disturbance covariance matrix will be singular. In addition, the generalized-inverse generalized least squares (GI-GLS) estimator of the regression coefficients will always exist.

Different implicit restrictions are required for different explanatory variables. It is assumed that explanatory variables belong to one of three main types, namely, absolute explanatory variables, a group of relative explanatory variables, or a group of hybrid explanatory variables. Each of the three main types of explanatory variables, together with the implicit restrictions, are discussed in the subsections below.

### 2.6.2. Absolute explanatory variables

An absolute explanatory variable is a common explanatory variable for all currencies. Each currency has a separate regression coefficient associated with the absolute explanatory variable. For example, the movements in a global equity market index is an absolute explanatory variable, as there is only one global equity market index and all currencies have their own exposure to the movements in the global equity market index. It is assumed that there are  $N_A$  absolute explanatory variables in a multivariate regression model.

The implicit regression-coefficient restriction for an absolute explanatory variable is that the sum of the regression coefficients across the system of  $N_C$  currencies is equal to zero:

$$\sum_{i=1}^{N_C} \beta_A^{a,i} = 0 \tag{16}$$

where  $a = 1, \dots, N_A$ ; and  $\beta_A^{a,i}$  is the  $i$ th regression coefficient associated for the  $a$ th absolute explanatory variable with  $i = 1, \dots, N_C$ .

In matrix form, a multivariate regression model for an absolute explanatory variable ( $N_A = 1$ ) and the associated regression coefficients can be written by stacking the system of  $N_C$  equations to give:

$$\Delta p_M = \begin{bmatrix} \Delta p_M^1 \\ \vdots \\ \Delta p_M^{N_C} \end{bmatrix} = \begin{bmatrix} x_A^a & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & x_A^a \end{bmatrix} \begin{bmatrix} \beta_A^{a,1} \\ \vdots \\ \beta_A^{a,N_C} \end{bmatrix} + \begin{bmatrix} \epsilon_M^1 \\ \vdots \\ \epsilon_M^{N_C} \end{bmatrix} = X_A^a \beta_A^a + \epsilon_M \tag{17}$$

where  $a = 1$ ;  $\Delta p_M = (\Delta p_M^1, \dots, \Delta p_M^{N_C})$  is a  $(N_C * T) \times 1$  stacked vector of the log returns of the system of  $N_C$  multilateral exchange rates;  $x_A^a$  is a  $T \times 1$  vector of the  $a$ th absolute explanatory variable;  $\beta_A^{a,i}$  is the  $i$ th regression coefficient associated for the  $a$ th absolute explanatory variable with  $i = 1, \dots, N_C$ ;  $X_A^a = \text{diag}(x_A^a, \dots, x_A^a)$  is a  $(N_C * T) \times N_C$  diagonal matrix for  $a$ th absolute explanatory variable;  $\beta_A^a = (\beta_A^{a,1}, \dots, \beta_A^{a,N_C})$  is a  $N_C \times 1$  vector the regression coefficients associated for the  $a$ th absolute explanatory variable; and  $\epsilon_M = (\epsilon_M^1, \dots, \epsilon_M^{N_C})$  is a  $(N_C * T) \times 1$  stacked vector of disturbance terms.

If the implicit restriction for the absolute explanatory variable in (16) is satisfied, the cross-sectional sum of the disturbance terms is given by:

$$\sum_{i=1}^{N_C} \epsilon_M^i = \sum_{i=1}^{N_C} \Delta p_M^i - \sum_{i=1}^{N_C} x_A^a \beta_A^{a,i} = \sum_{i=1}^{N_C} \Delta p_M^i - x_A^a \sum_{i=1}^{N_C} \beta_A^{a,i} = \mathbf{0} \tag{18}$$

where  $\epsilon_M^i$  is the  $i$ th disturbance term in (17);  $\sum_{i=1}^{N_C} \Delta p_M^i = \mathbf{0}$  from (11); and  $\sum_{i=1}^{N_C} \beta_A^{a,i} = 0$  from (16). In this situation, the disturbance covariance matrix  $\Sigma_\epsilon$  is singular.

### 2.6.3. Relative explanatory variables

A relative explanatory variable is an idiosyncratic explanatory variable for a specific currency. The relative explanatory variable belongs to a group of  $N_C$  relative explanatory variables, where there is one for each currency. For example, the group of lagged log returns of the  $N_C$  multilateral exchange rates is a group of relative explanatory variables. It is assumed that there are  $N_R$  groups of relative explanatory variables in a multivariate regression model, where each group contains  $N_C$  relative explanatory variables: one for each currency.

The implicit regression-coefficient restriction for each group of relative explanatory variables is that all currencies share a common (pooled) regression coefficient, represented by  $\beta_R^r$  for the  $r$ th group of relative explanatory variables, where  $r = 1, \dots, N_R$ . In addition, the implicit explanatory-variable restriction for each group of relative explanatory variables is that the cross-sectional sum, at each time  $t$ , is equal to zero:

$$\sum_{i=1}^{N_C} x_{R,t}^{r,i} = 0 \tag{19}$$

where  $r = 1, \dots, N_R, t = 1, \dots, T$ ; and  $x_{R,t}^{r,i}$  is the  $i$ th relative explanatory variable in the  $r$ th group of relative explanatory variables.

In matrix form, a multivariate regression model for a group of relative explanatory variables ( $N_R = 1$ ) and the associated common regression coefficient can be written by stacking the system of  $N_C$  equations to give:

$$\Delta p_M = \begin{bmatrix} \Delta p_M^1 \\ \vdots \\ \Delta p_M^{N_C} \end{bmatrix} = \begin{bmatrix} x_R^{r,1} \\ \vdots \\ x_R^{r,N_C} \end{bmatrix} \beta_R^r + \begin{bmatrix} \epsilon_M^1 \\ \vdots \\ \epsilon_M^{N_C} \end{bmatrix} = X_R^r \beta_R^r + \epsilon_M \tag{20}$$

where  $r = 1$ ;  $\Delta p_M = (\Delta p_M^1; \dots; \Delta p_M^{N_C})$  is a  $(N_C * T) \times 1$  stacked vector of the log returns of the system of  $N_C$  multilateral exchange rates;  $x_R^{r,i}$  is a  $T \times 1$  vector of the  $i$ th relative explanatory variable for the  $r$ th group of relative explanatory variables with  $i = 1, \dots, N_C$ ;  $\beta_R^r$  is a common regression coefficient associated for the  $r$ th group of relative explanatory variables;  $X_R^r = (x_R^{r,1}; \dots; x_R^{r,N_C})$  is a  $(N_C * T) \times 1$  stacked vector of the  $r$ th group of relative explanatory variables; and  $\epsilon_M = (\epsilon_M^1; \dots; \epsilon_M^{N_C})$  is a  $(N_C * T) \times 1$  stacked vector of disturbance terms.

If the implicit restrictions are satisfied for a group of relative explanatory variables, the cross-sectional sum of the disturbance terms is given by:

$$\sum_{i=1}^{N_C} \epsilon_M^i = \sum_{i=1}^{N_C} \Delta p_M^i - \sum_{i=1}^{N_C} x_R^{r,i} \beta_R^r = \sum_{i=1}^{N_C} \Delta p_M^i - \beta_R^r \sum_{i=1}^{N_C} x_R^{r,i} = 0 \tag{21}$$

where  $\epsilon_M^i$  is the  $i$ th disturbance term in (20);  $\sum_{i=1}^{N_C} \Delta p_M^i = 0$  from (11); and  $\sum_{i=1}^{N_C} x_R^{r,i} = 0$  from (19). In this situation, the disturbance covariance matrix  $\Sigma_e$  is singular.

#### 2.6.4. Hybrid explanatory variables

Hybrid explanatory variables are similar to relative explanatory variables. For example, a hybrid explanatory variable is a currency-specific explanatory variable and belongs to a group of hybrid explanatory variables. For example, the movements in local equity market indexes are a group of hybrid explanatory variables, where there is one local equity market index associated with each currency. It is assumed that there are  $N_H$  groups of hybrid explanatory variables in the multivariate regression model, where each group contains  $N_H$  hybrid explanatory variables: one for each currency.

The difference between a group of hybrid explanatory variables and a group of relative explanatory variables is that the cross-sectional sum for a group of hybrid explanatory variables, at each time  $t$ , does not equal to zero:

$$\sum_{i=1}^{N_C} x_{H,t}^{h,i} \neq 0 \tag{22}$$

where  $h = 1, \dots, N_H; t = 1, \dots, T$ ; and  $x_{H,t}^{h,i}$  is the  $i$ th hybrid explanatory variable in the  $h$ th group of hybrid explanatory variables with  $i = 1, \dots, N_C$ .

The implicit explanatory-variable restriction decomposes each group of  $N_C$  hybrid explanatory variables into an absolute explanatory variable and a group of  $N_C$  relative explanatory variables by:

$$x_{H,t}^{h,i} = x_{A,t}^h + x_{R,t}^{h,i} \tag{23}$$

where  $h = 1, \dots, N_H; i = 1, \dots, N_C; t = 1, \dots, T$ ;  $x_{H,t}^{h,i}$  is the  $i$ th hybrid explanatory variable in the  $h$ th group of hybrid explanatory variables;  $x_{A,t}^h$  is the absolute explanatory variable for the  $h$ th group; and  $x_{R,t}^{h,i}$  is the  $i$ th relative explanatory variable for the  $h$ th group.

The absolute explanatory variable in (23) can be created by calculating the cross-sectional average of the group of hybrid explanatory variables to give:

$$x_{A,t}^h = \frac{1}{N_C} \sum_{i=1}^{N_C} x_{H,t}^{h,i} \tag{24}$$

where  $h = 1, \dots, N_H; t = 1, \dots, T$ ;  $x_{A,t}^h$  is the absolute explanatory variable for the  $h$ th group; and  $x_{H,t}^{h,i}$  is the  $i$ th hybrid explanatory variable for the  $h$ th group. Subsequently, the relative explanatory variables in (23) can be calculated by subtracting the absolute explanatory variable for the  $h$ th group in (24) from the  $i$ th hybrid explanatory variable for the  $h$ th group to give:

$$x_{R,t}^{h,i} = x_{H,t}^{h,i} - x_{A,t}^h \tag{25}$$

where  $h = 1, \dots, N_H; t = 1, \dots, T$ ;  $x_{R,t}^{h,i}$  is the  $i$ th relative explanatory variable for the  $h$ th group;  $x_{H,t}^{h,i}$  is the  $i$ th hybrid explanatory variable for the  $h$ th group; and  $x_{A,t}^h$  is the absolute explanatory variable for the  $h$ th group in (24).

The cross-sectional sum of the group of relative explanatory variables in (25) is:



$$\sum_{i=1}^{N_C} x_{R,t}^{h,i} = \sum_{i=1}^{N_C} (x_{H,t}^{h,i} - x_{A,t}^h) = \sum_{i=1}^{N_C} x_{H,t}^{h,i} - N_C \left( \frac{1}{N_C} \sum_{i=1}^{N_C} x_{H,t}^{h,i} \right) = 0 \tag{26}$$

where  $h = 1, \dots, N_H; t = 1, \dots, T; x_{R,t}^{h,i} = x_{H,t}^{h,i} - x_{A,t}^h$  is the  $i$ th relative explanatory variable for the  $h$ th group in (25);  $x_{H,t}^{h,i}$  is the  $i$ th hybrid explanatory variable for the  $h$ th group; and  $x_{A,t}^h = \frac{1}{N_C} \sum_{i=1}^{N_C} x_{H,t}^{h,i}$  is the absolute explanatory variable for the  $h$ th group in (24). Thus, the group of relative explanatory variables for the  $h$ th group in (25) satisfies the implicit explanatory-variable restriction in (19), where the cross-sectional sum of the group of relative explanatory variables is equal to zero.

In matrix form, a multivariate regression model for the decomposition of a group of  $N_C$  hybrid explanatory variables ( $N_H = 1$ ) into an absolute explanatory variable and a group of  $N_C$  relative explanatory variables can be written by stacking the system of  $N_C$  equations to give:

$$\Delta p_M = \begin{bmatrix} \Delta p_M^1 \\ \vdots \\ \Delta p_M^{N_C} \end{bmatrix} = \begin{bmatrix} x_A^h & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & x_A^h \end{bmatrix} \begin{bmatrix} \beta_A^{h,1} \\ \vdots \\ \beta_A^{h,N_C} \end{bmatrix} + \begin{bmatrix} x_R^{h,1} \\ \vdots \\ x_R^{h,N_C} \end{bmatrix} \beta_R^h + \begin{bmatrix} \epsilon_M^1 \\ \vdots \\ \epsilon_M^{N_C} \end{bmatrix} = X_A^h \beta_A^h + X_R^h \beta_R^h + \epsilon_M \tag{27}$$

where  $h = 1; \Delta p_M = (\Delta p_M^1; \dots; \Delta p_M^{N_C})$  is a  $(N_C * T) \times 1$  stacked vector of the log returns of the system of  $N_C$  multilateral exchange rates;  $x_A^h$  is a  $T \times 1$  vector of the absolute explanatory variable for the  $h$ th group;  $X_A^h = \text{diag}(x_A^h, \dots, x_A^h)$  is a  $(N_C * T) \times N_C$  diagonal matrix for the absolute explanatory variable for the  $h$ th group;  $\beta_A^h = (\beta_A^{h,1}; \dots; \beta_A^{h,N_C})$  is a  $N_C \times 1$  vector the regression coefficients for the  $h$ th group;  $\beta_A^{h,i}$  is the  $i$ th regression coefficient for the  $h$ th group with  $i = 1, \dots, N_C; x_R^{h,i}$  is a  $T \times 1$  vector of the  $i$ th relative explanatory variable for the  $h$ th group;  $\beta_R^h$  is the regression coefficient associated for the  $h$ th group;  $X_R^h = (x_R^{h,1}; \dots; x_R^{h,N_C})$  is a  $(N_C * T) \times 1$  stacked vector of the  $h$ th group of relative explanatory variables; and  $\epsilon_M = (\epsilon_M^1; \dots; \epsilon_M^{N_C})$  is a  $(N_C * T) \times 1$  stacked vector of disturbance terms.

If the implicit restrictions are satisfied for the group of hybrid explanatory variables, the cross-sectional sum of the disturbance terms is given by:

$$\sum_{i=1}^{N_C} \epsilon_M^i = \sum_{i=1}^{N_C} \Delta p_M^i - \sum_{i=1}^{N_C} (x_A^h \beta_A^{h,i} + x_R^h \beta_R^h) = \sum_{i=1}^{N_C} \Delta p_M^i - x_A^h \sum_{i=1}^{N_C} \beta_A^{h,i} + \beta_R^h \sum_{i=1}^{N_C} x_R^h = \mathbf{0} \tag{28}$$

where  $\epsilon_M^i$  is the  $i$ th disturbance term in (27);  $\sum_{i=1}^{N_C} \Delta p_M^i = \mathbf{0}$  from (11);  $\sum_{i=1}^{N_C} \beta_A^{h,i} = 0$  from (16); and  $\sum_{i=1}^{N_C} x_R^h = \mathbf{0}$  from (19). In this situation, the disturbance covariance matrix  $\Sigma_\epsilon$  is singular.

2.6.5. Summary

In general, explanatory variables can be classified into one of three main types, namely, absolute explanatory variables, a group of relative explanatory variables, or a group of hybrid explanatory variables. However, a group of hybrid explanatory variables can be decomposed into an absolute explanatory variable and a group of relative explanatory variables. Thus, each explanatory variable in a multivariate regression model is either an absolute explanatory variable or a relative explanatory variable. As a result, there are only two core types of explanatory variables, namely, absolute explanatory variables and relative explanatory variables.

2.7. A multivariate regression model with implicit restrictions

The form of a multivariate regression model for the log returns of each multilateral exchange rate with  $N_A$  absolute explanatory variables (including one for the intercept) and  $N_R$  groups of relative explanatory variables can be written as:

$$\Delta p_M^i = \sum_{a=1}^{N_A} x_A^a \beta_A^{a,i} + \sum_{r=1}^{N_R} x_R^{r,i} \beta_R^r + \epsilon_M^i \tag{29}$$

where  $i = 1, \dots, N_C; \Delta p_M^i$  is a  $T \times 1$  vector of log returns of the  $i$ th multilateral exchange rate;  $x_A^a$  is a  $T \times 1$  vector for the  $a$ th absolute explanatory variable;  $\beta_A^{a,i}$  is the  $i$ th regression coefficient for the  $a$ th absolute explanatory variable;  $x_R^{r,i}$  is a  $T \times 1$  vector for the  $i$ th explanatory variable for the  $r$ th group of relative explanatory variables;  $\beta_R^r$  is the common regression coefficient for the  $r$ th group of relative explanatory variables; and  $\epsilon_M^i$  is the  $i$ th disturbance term. It should be noted that the multivariate regression model in (29) also includes the hybrid explanatory variables, subsequent to the decomposition in (23).

In matrix form, the multivariate regression model in (29) can be written by stacking the system of  $N_C$  equations to give:

$$\Delta p_M = \begin{bmatrix} \Delta p_M^1 \\ \vdots \\ \Delta p_M^{N_C} \end{bmatrix} = [X_A^1 \quad \dots \quad X_A^{N_A}] \begin{bmatrix} \beta_A^1 \\ \vdots \\ \beta_A^{N_A} \end{bmatrix} + [X_R^1 \quad \dots \quad X_R^{N_R}] \begin{bmatrix} \beta_R^1 \\ \vdots \\ \beta_R^{N_R} \end{bmatrix} + \begin{bmatrix} \epsilon_M^1 \\ \vdots \\ \epsilon_M^{N_C} \end{bmatrix} = X_A \beta_A + X_R \beta_R + \epsilon_M \tag{30}$$

where  $\Delta p_M = (\Delta p_M^1; \dots; \Delta p_M^{N_C})$  is a  $(N_C * T) \times 1$  stacked vector of the log returns of the system of  $N_C$  multilateral exchange rates;  $X_A^a =$

$\text{diag}(\mathbf{x}_A^a, \dots, \mathbf{x}_A^a)$  is a  $(N_C * T) \times N_C$  diagonal matrix for the  $a$ th absolute explanatory variables in (17) with  $a = 1, \dots, N_A$ ;  $\beta_A^a$  is a  $N_C \times 1$  vector regression coefficients for the  $a$ th absolute explanatory variables in (17);  $\mathbf{x}_R^r$  is a  $(N_C * T) \times 1$  stacked vector for the  $r$ th group of relative explanatory variables in (20) with  $r = 1, \dots, N_R$ ;  $\beta_R^r$  is the common (panel-like) regression coefficient for the  $r$ th group of relative explanatory variables in (20); and  $\boldsymbol{\varepsilon}_M = (\boldsymbol{\varepsilon}_M^1, \dots, \boldsymbol{\varepsilon}_M^{N_C})$  is a  $(N_C * T) \times 1$  stacked vector of disturbance terms.

If the implicit restrictions are satisfied, the cross-sectional sum of the disturbance terms is given by:

$$\sum_{i=1}^{N_C} \boldsymbol{\varepsilon}_M^i = \sum_{i=1}^{N_C} \Delta \mathbf{p}_M^i - \sum_{i=1}^{N_C} \left( \sum_{a=1}^{N_A} \mathbf{x}_A^a \beta_A^{a,i} + \sum_{r=1}^{N_R} \mathbf{x}_R^r \beta_R^{r,i} \right) = \sum_{i=1}^{N_C} \Delta \mathbf{p}_M^i - \sum_{a=1}^{N_A} \mathbf{x}_A^a \sum_{i=1}^{N_C} \beta_A^{a,i} + \sum_{r=1}^{N_R} \beta_R^r \sum_{i=1}^{N_C} \mathbf{x}_R^{r,i} = \mathbf{0} \quad (31)$$

where  $\boldsymbol{\varepsilon}_M^i$  is the  $i$ th disturbance term in (30);  $\sum_{i=1}^{N_C} \Delta \mathbf{p}_M^i = \mathbf{0}$  from (11);  $\sum_{i=1}^{N_C} \beta_A^{a,i} = 0$  from (16); and  $\sum_{i=1}^{N_C} \mathbf{x}_R^{r,i} = \mathbf{0}$  from (19). In this situation, the disturbance covariance matrix  $\Sigma_\varepsilon$  is singular.

The full covariance matrix of the disturbance terms is given by:

$$\Omega_\varepsilon = \Sigma_\varepsilon \otimes \mathbf{I}_T \quad (32)$$

where  $\Omega_\varepsilon$  is a  $(N_C * T) \times (N_C * T)$  full covariance matrix;  $\Sigma_\varepsilon$  is an  $N_C \times N_C$  disturbance covariance matrix in (14); and  $\otimes$  is the Kronecker product. The generalized least squares (GLS) estimator of the regression coefficients of the multivariate regression in (30) is given by:

$$E(\boldsymbol{\beta}) = (\mathbf{X}' \Omega_\varepsilon^{-1} \mathbf{X})^{-1} (\mathbf{X}' \Omega_\varepsilon^{-1} \mathbf{y}). \quad (33)$$

where  $\Omega_\varepsilon$  is a  $(N_C * T) \times (N_C * T)$  full covariance matrix of the disturbance terms in (32);  $\mathbf{X} = [\mathbf{X}_A \quad \mathbf{X}_R]$  is a  $(N_C * T) \times (N_A * N_C + N_R)$  matrix of the combined explanatory variables, with  $N_A$  absolute explanatory variables and  $N_R$  groups of relative explanatory variables; and  $\boldsymbol{\beta} = (\beta_A; \beta_R)$  is a  $(N_A * N_C + N_R) \times 1$  vector of the combined regression coefficients.

However, the full disturbance covariance matrix in (32) is singular from (31) so the ordinary inverse of the disturbance covariance matrix  $\Omega_\varepsilon^{-1}$  does not exist. Consequently, the generalized least squares (GLS) estimator of the regression coefficients in (33) also does not exist. In contrast, when the disturbance covariance matrix is singular, the generalized inverse of the disturbance covariance matrix  $\Omega_\varepsilon^{-g}$  does exist. The generalized-inverse generalized least squares (GI-GLS) estimator of the regression coefficients can be written as:

$$E_g(\boldsymbol{\beta}) = (\mathbf{X}' \Omega_\varepsilon^{-g} \mathbf{X})^{-1} (\mathbf{X}' \Omega_\varepsilon^{-g} \mathbf{y}). \quad (34)$$

where  $\Omega_\varepsilon^{-g}$  is the generalized inverse of the full covariance matrix of the disturbance terms in (32); and all other terms are the same as described in (33). Thus, if the implicit restrictions are satisfied, the generalized-inverse generalized least squares (GI-GLS) estimator of the regression coefficients in (34) will always exist.

### 3. Results

#### 3.1. Data

The data consists of a system of seven ( $N_C = 7$ ) currencies, namely, the US dollar (USD), the Eurozone euro (EUR), the Japanese yen (JPY), the Swiss franc (CHF), the British pound (GBP), the Canadian dollar (CAD) and the Australian dollar (AUD). A system of six bilateral exchange rates against the US dollar are sourced from Bloomberg with 49 years of monthly data from Bloomberg from 1st January 1973 to 31st December 2021.

A group of seven local equity market indexes are considered as explanatory variables. The system of seven equity market indexes together with the associated three letter currency code is: United States (USD), Europe (EUR), Japan (JPY), Switzerland (CHF), United Kingdom (GBP), Canada (CAD) and Australia (AUD). The equity market indices are the MSCI total return indices. The German market total return index was used as a proxy for the equity market index associated with the euro.

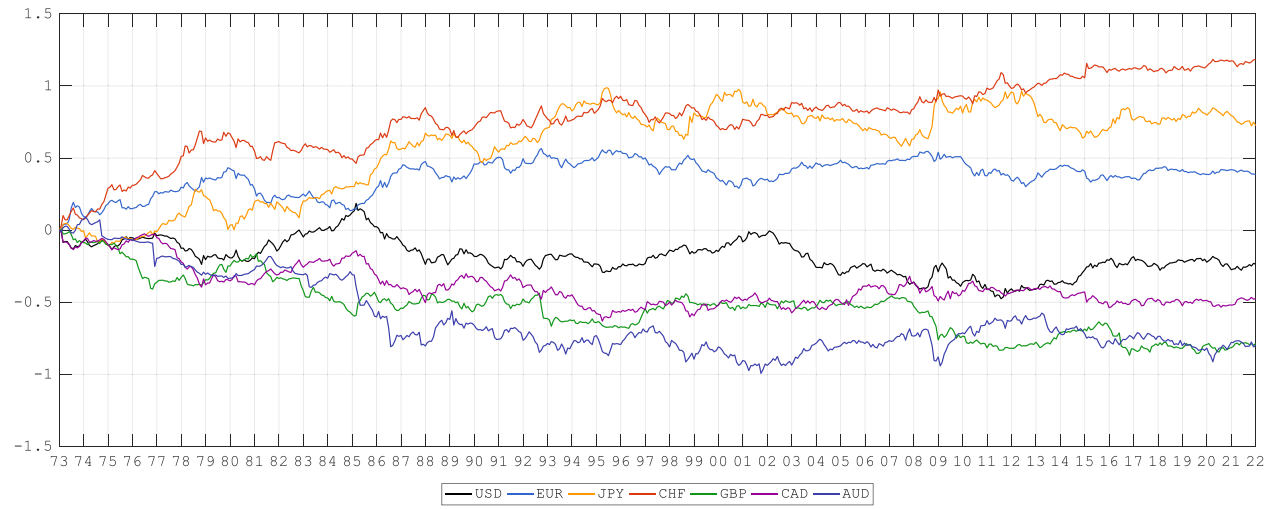
#### 3.2. Multivariate regression model

The form of a multivariate regression model for the log returns of each of the seven ( $N_C = 7$ ) multilateral exchange rates with two ( $N_A = 2$ ) absolute explanatory variables (including one for the intercept) and one ( $N_R = 1$ ) group of relative explanatory variables can be written as:

$$\Delta \mathbf{p}_M^i = \mathbf{x}_A^I \alpha_A^{I,i} + \mathbf{x}_A^E \beta_A^{E,i} + \mathbf{x}_R^E \beta_R^{E,i} + \boldsymbol{\varepsilon}_M^i \quad (35)$$

where  $i = 1, \dots, N_C$ ;  $\Delta \mathbf{p}_M^i$  is a  $T \times 1$  vector of log returns of the  $i$ th multilateral exchange rate;  $\mathbf{x}_A^I$  is a  $T \times 1$  vector of ones;  $\alpha_A^{I,i}$  is the  $i$ th intercept coefficient;  $\mathbf{x}_A^E$  is a  $T \times 1$  vector of the log returns of the global equity market index;  $\beta_A^{E,i}$  is the  $i$ th regression coefficient for the log returns of the global equity market index;  $\mathbf{x}_R^E$  is a  $T \times 1$  vector of the log returns of the  $i$ th relative equity market index;  $\beta_R^{E,i}$  is the common regression coefficient for the log returns of the group of relative equity market indexes; and  $\boldsymbol{\varepsilon}_M^i$  is the  $i$ th disturbance term.

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**Fig. 1.** Cumulative log returns for the multilateral exchange rates. *Notes:* Figure displays the cumulative log returns of the seven multilateral exchange rates.

**Table 1**  
Descriptive statistics for the movements in the multilateral exchange rates.

	Average	Std. Dev.	Skew	Kurt	ADF	KPSS	Jarque-Bera
USD	-0.48 %	6.34 %	-0.003	4.03**	-23.6**	0.046	25.9**
EUR	0.79 %	6.07 %	0.049	4.27**	-23.1**	0.043	40.1**
JPY	1.48 %	8.49 %	0.784**	6.12**	-23.5**	0.041	299.5**
CHF	2.41 %	7.17 %	0.352**	4.81**	-24.5**	0.091	92.9**
GBP	-1.60 %	6.55 %	-0.694**	6.56**	-22.6**	0.063	358.0**
CAD	-0.97 %	6.41 %	-0.252*	4.00**	-24.8**	0.034	30.8**
AUD	-1.63 %	8.58 %	-1.301**	8.38**	-23.6**	0.035	876.8**

Notes: This Table reports the summary statistics for the log returns of each multilateral exchange rate. The summary statistics consist of: annualised average return (Average); the annualised standard deviation (Std. Dev.); skewness (Skew), kurtosis (Kurt), Augmented Dickey-Fuller unit root test (ADF) displaying the test statistic, KPSS test displaying the test statistic; and the Jarque-Bera normality test. Significance levels for skewness, kurtosis, ADF, KPSS, and Jarque-Bera are denoted by \* for 5 %, and \*\* for 1 %.

**Table 2**  
Correlation matrix.

	USD	EUR	JPY	CHF	GBP	CAD	AUD
USD	1.000	-0.521**	-0.053	-0.496**	-0.165**	0.439**	-0.105*
EUR	-0.521**	1.000	-0.174**	0.626**	0.035	-0.471**	-0.348**
JPY	-0.053	-0.174**	1.000	0.012	-0.263**	-0.376**	-0.356**
CHF	-0.496**	0.626**	0.012	1.000	-0.062	-0.605**	-0.423**
GBP	-0.165**	0.035	-0.263**	-0.062	1.000	-0.172**	-0.226**
CAD	0.439**	-0.471**	-0.376**	-0.605**	-0.172**	1.000	0.271**
AUD	-0.105*	-0.348**	-0.356**	-0.423**	-0.226**	0.271**	1.000

Notes: This Table reports the correlation matrix for the log returns of the multilateral exchange rates. Significance levels for the correlations are denoted by \* for 5 %, and \*\* for 1 %.

### 3.3. Dependent variables

The dependent variables consist of the log returns of the seven ( $N_C = 7$ ) multilateral exchange rates. The log returns of the  $N_C$  multilateral exchange rates are priced in terms of an equally-weighted basket of all  $N_C$  numéraire currencies in (6). Note that all  $N_C$  currencies are numéraire currencies of the multicurrency numéraire. For example, the US dollar multilateral exchange rate is calculated using (6) to give:

$$\Delta p_{M,t}^{USD} = \frac{1}{7} \sum_{j=1}^7 \Delta p_{B,t}^{USD/j} = \frac{1}{7} \Delta p_{B,t}^{USD/USD} + \frac{1}{7} \Delta p_{B,t}^{USD/EUR} + \frac{1}{7} \Delta p_{B,t}^{USD/JPY} + \frac{1}{7} \Delta p_{B,t}^{USD/CHF} + \frac{1}{7} \Delta p_{B,t}^{USD/GBP} + \frac{1}{7} \Delta p_{B,t}^{USD/CAD} + \frac{1}{7} \Delta p_{B,t}^{USD/AUD} \quad (36)$$

where  $t = 1, \dots, T$ ;  $\Delta p_{M,t}^{USD}$  is the log return of the US dollar multilateral exchange rate; and  $\Delta p_{B,t}^{USD/j}$  is the log return of the USD/jth bilateral exchange rate with  $\Delta p_{B,t}^{USD/USD} = 0$  and  $j = 1, \dots, N_C$ .

Fig. 1 displays the cumulative log returns of each multilateral exchange rate. Table 1 reports the descriptive statistics for the log returns of the system of seven multilateral exchange rates. The Swiss franc (CHF) has the highest average annualised return of 2.41%, and the Australian dollar has the lowest average annualised return of -1.63%, which is closely followed by the British pound (GBP) with an average annualised return of -1.60%. The highest annualised volatilities of 8.58% and 8.49% are associated with the Australian dollar (AUD) and the Japanese yen (JPY), respectively. In contrast, the lowest annualised volatility of 6.07 % is associated with the Eurozone euro (EUR).

Significant positive skewness values of 0.784 and 0.352 are associated with the safe-haven currencies of the Japanese yen (JPY) and the Swiss franc (CHF), respectively. In contrast, significant negative skewness values of -1.301, -0.694, and -0.252 are associated with the Australian dollar (AUD), the British pound (GBP), and the Canadian dollar (CAD), respectively. The movements in all multilateral exchange rates experience significant ADF statistics (Dickey & Fuller, 1979) and insignificant KPSS statistics (Kwiatkowski et al., 1992), which provide significant evidence of stationarity. The movements in all of the multilateral exchange rates also experience significant kurtosis and significant Jarque-Bera statistics (Jarque & Bera, 1987), which indicate non-normal log returns.

Table 2 reports the correlation matrix for the log returns of the multilateral exchange rates. There are three highly significant positive observed correlations. The first positive observed correlation is 0.626 between the Eurozone euro (EUR) and the Swiss franc (CHF), which is driven by the geographically close proximity of the Eurozone and Switzerland. The second positive observed correlation is 0.439 between the US dollar (USD) and the Canadian dollar (CAD), which is driven by the geographically close proximity of the US and Canada. The third positive observed correlation is 0.271 between the Canadian dollar (CAD) and the Australian dollar (AUD), which is driven by both currencies being associated with commodity exporting countries: commodity currencies (Chen & Rogoff, 2003).

**Table 3**  
Descriptive statistics for the explanatory variables.

Panel A: Hybrid explanatory variables							
	Average	Std. Dev.	Skew	Kurt	ADF	KPSS	Jarque-Bera
USD	10.27%	15.40 %	-0.666**	5.53**	-22.7**	0.113	200.1**
EUR	7.83 %	19.62 %	-0.852**	6.01**	-22.5**	0.043	294.1**
JPY	4.89 %	18.07 %	-0.393**	4.51**	-22.1**	0.133	71.1**
CHF	7.94 %	15.76 %	-0.775**	6.50**	-21.3**	0.122	360.0**
GBP	9.86 %	18.33 %	0.262**	12.67**	-22.0**	0.057	2301.5**
CAD	8.88 %	16.20 %	-0.853**	6.62**	-22.3**	0.038	392.9**
AUD	10.10 %	18.88 %	-1.967**	20.61**	-23.1**	0.053	7989.0**
<b>AVG</b>	<b>8.54 %</b>	<b>17.47 %</b>	<b>-0.749</b>	<b>8.92</b>			
Panel B: Absolute explanatory variable							
	Average	Std. Dev.	Skew	Kurt	ADF	KPSS	Jarque-Bera
<b>GBL</b>	8.54 %	13.65 %	-1.312**	9.39**	-20.7**	0.083	1169.6**
Panel C: Relative explanatory variables							
	Average	Std. Dev.	Skew	Kurt	ADF	KPSS	Jarque-Bera
USD	1.73 %	7.90 %	0.469**	4.23**	-27.5**	0.069	58.5**
EUR	-0.71 %	12.13 %	-0.435**	5.47**	-24.9**	0.027	168.2**
JPY	-3.65 %	13.81 %	0.057	4.43**	-24.9**	0.144	50.6**
CHF	-0.60 %	9.11 %	0.001	3.84**	-23.7**	0.114	17.3**
GBP	1.32 %	10.94 %	1.105**	13.44**	-23.5**	0.016	2792.3**
CAD	0.34 %	9.56 %	-0.087	3.54**	-24.9**	0.061	8.0*
AUD	1.56 %	12.43 %	-0.313**	7.83**	-27.1**	0.028	581.8**
<b>AVG</b>	<b>0.00 %</b>	<b>10.84 %</b>	<b>0.114</b>	<b>6.11</b>			

Notes: This Table reports the summary statistics for the explanatory variables, with Panel A reporting the group of hybrid explanatory variables (one for each currency), Panel B reporting the absolute explanatory variable and Panel C reporting the group of relative explanatory variables (one for each currency). The summary statistics consist of: annualised average return (Average); the annualised standard deviation (Std. Dev.); skewness (Skew), kurtosis (Kurt), Augmented Dickey-Fuller unit root test (ADF) displaying the test statistic, KPSS test displaying the test statistic; and the Jarque-Bera normality test. Significance levels for skewness, kurtosis, ADF, KPSS, and Jarque-Bera are denoted by \* for 5 %, and \*\* for 1 %.

### 3.4. Explanatory variables

The explanatory variables consist of the movements in a group of local equity market indexes. There is one local equity market index associated with each currency. However, the cross-sectional sum of the movements in the group of local equity market indexes does not equal to zero. Thus, the movements in a group of local equity market indexes are classified as a group of hybrid explanatory variables. The decomposition of a group of hybrid explanatory variables in (23) is used to decompose the log return of the  $i$ th local equity market indexes at time  $t$  to give:

$$x_{H,t}^{E,i} = x_{A,t}^E + x_{R,t}^{E,i} \quad (37)$$

where  $i = 1, \dots, N_C$ ;  $t = 1, \dots, T$ ;  $x_{H,t}^{E,i}$  is the log returns of the  $i$ th equity market associated with the  $i$ th currency;  $x_{A,t}^E = \frac{1}{N_C} \sum_{i=1}^{N_C} x_{H,t}^{E,i}$  is the log return of the global equity market index (absolute explanatory variable) using (24); and  $x_{R,t}^{E,i} = x_{H,t}^{E,i} - x_{A,t}^E$  is the log return of the  $i$ th relative equity market index (relative explanatory variable) using (25).

Table 3 reports the descriptive statistics for the group of hybrid explanatory variables (Panel A), together with the absolute explanatory variable (Panel B) and the group of relative explanatory variables (Panel C). The absolute explanatory variable (Panel B) represents the movements in a global equity market index (GBL) and calculated by the cross-sectional average of the log returns for the group of local equity market indexes. The group of relative explanatory variables (one for each currency) are calculated from the differences between the movements in the local equity market indexes (Panel A) and the movements in the global equity market index (Panel B).

The movements in the global equity market index (absolute explanatory variable) have an annualised return of 8.54 %, and an annualised volatility of 13.65 %. In contrast, the group of the movements in the relative equity market index (relative explanatory variables) have an average annualised return of exactly 0.00 % by construction, and an average annualised volatility of 10.84 %. The movements in the relative US equity market index (USD) outperform the movements in the global equity market the most by 1.73 % annually. In contrast, movements in the relative Japanese equity market index (JPY) underperform the movements in the global equity market the most by - 3.65 % annually.

All of the explanatory variables experience significant ADF statistics (Dickey & Fuller, 1979) and insignificant KPSS statistics (Kwiatkowski et al., 1992), which provide significant evidence of stationarity. All of the explanatory variables also experience significant kurtosis and significant Jarque-Bera statistics (Jarque & Bera, 1987), which indicate non-normal log returns.

**Table 4**  
Multivariate regression results.

$i$	$\alpha_A^{I,i}$	$\beta_A^{E,i}$	$\beta_R^E$
USD	-0.0004	-0.0610**	-0.0247**
EUR	0.0007	-0.0383**	-0.0247**
JPY	0.0012*	-0.1375**	-0.0247**
CHF	0.0020**	-0.1174**	-0.0247**
GBP	-0.0013**	0.0159	-0.0247**
CAD	-0.0008*	0.1193**	-0.0247**
AUD	-0.0014*	0.2191**	-0.0247**
AVG	0.0000	0.0000	

Notes: This Table reports the observed regression coefficients from the multivariate regression model in (35). Significance levels for the regression coefficients are denoted by \* for 5 %, and \*\* for 1 %.

### 3.5. Multivariate regression output

Table 4 reports the observed regression coefficients from the multivariate regression model in (35) for the entire 49-year data sample. The regression coefficients are estimated using the generalized-inverse generalized least squares (GI-GLS) estimator in (34). The first data column of Table 4 reports the intercept coefficients ( $\alpha_A^{I,i}$ ) for the movements in the seven multilateral exchange rates. The Swiss franc (CHF) and the Japanese yen (JPY) have significant positive intercept coefficients of 0.0020 and 0.0012, respectively. In contrast, the Australian dollar (AUD), the British pound (GBP), and the Canadian dollar (CAD) have significant negative intercept coefficients of  $-0.0014$ ,  $-0.0013$  and  $-0.0008$ , respectively.

The second data column of Table 4 reports regression coefficients ( $\beta_A^{E,i}$ ) associated with the movements in the global equity market index (absolute explanatory variable). The Australian dollar (AUD) and the Canadian dollar (CAD) move significantly with the global equity market index, with observed regression coefficients of 0.2191 and 0.1193, respectively. In contrast, the Japanese yen (JPY), the Swiss franc (CHF), the US dollar (USD), and the Eurozone euro (EUR) move significantly against the global equity market index, with observed regression coefficients of  $-0.1375$ ,  $-0.1174$ ,  $-0.0610$ , and  $-0.0383$ , respectively.

The third data column of Table 4 reports the common regression coefficient ( $\beta_R^E$ ) associated with the movements in the group of relative equity market indexes (a group of relative explanatory variables). The common regression coefficient is the same for all currencies and is simply repeated for each currency. Currencies move significantly against the relative equity market indexes with a value of  $-0.0247$ . This shows that the currencies are shock-absorbers, so that when the local equity market index out (under) performs relative to the global equity market index, the associated currency under (out) performs.

## 4. Conclusion

A multicurrency numéraire is a weighted basket of numéraire currencies. A no-arbitrage condition enforces the movements in a system of multilateral exchange rates associated with the numéraire currencies of the multicurrency numéraire to be a singular system. Singular systems pose a methodological challenge in a multivariate regression model. This paper proposed a solution to overcome the methodological challenge that consisted of two parts. The first part imposed implicit restrictions on the explanatory variables and the regression coefficients to guarantee that the disturbance covariance matrix was singular. The second part replaced the ordinary inverse with the generalized inverse to modify the generalized least squares (GLS) estimator of the regression coefficients (see Theil, 1971).

The solution provided a consistent multivariate regression framework to explain the observed heterogeneity in the relative currency market. For example, all currencies cannot move with, or against, absolute explanatory variables. In this situation, the heterogeneity appears in the observed regression coefficients. In addition, all currencies move in a common (panel-like) fashion with, or against, a group of relative explanatory variables. In this situation, the heterogeneity appears in the currency-specific explanatory variables.

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