# *Synthetic money: addressing the budget-constraint issue*

Article

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## Abstract

A *synthetic currency* attempts to mimic a *target currency* with an optimal portfolio of *other* currencies, without having a position in the target currency. The original construction methodology of a synthetic currency imposed a *budget constraint* such that the portfolio weights sum to a non-zero value, $x$ say. However, a synthetic currency is a portfolio of invariant currency indexes, rather than a portfolio of currency positions. In this paper, we show that invariant currency indexes are tradable multilateral exchange rates. Consequently, the sum-to-$x$ budget constraint unintentionally creates a non-zero position in the target currency. We address this *budget-constraint issue* by replacing the sum-to-$x$ budget constraint with a sum-to-zero budget constraint, which correctly enforces a zero position in the target currency. Once the budget-constraint issue is addressed, investors are faced with the fact that synthetic money is unable to mimic significant currency-specific movements in target currencies.



**Keywords:** Foreign exchange rates, Synthetic Money.

**JEL Codes:** F31.

## 1 Introduction

The concept of *synthetic money* builds on the idea of creating a *synthetic currency* to mimic a *target currency*, where *synthetic money* is the plural form of *synthetic currency*. More specifically, a synthetic currency is an optimal portfolio of *other* currencies, without having a position in the target currency (Hovanov et al., 2007). For example, a synthetic US dollar would be an optimal portfolio of other currencies that mimics the behaviour of the US dollar, without holding a US dollar position. A synthetic currency can be used in any situation where there is a restriction on the ownership of a target currency, where the restrictions could be based on political, cultural, social, or other reasons (Hovanov et al., 2007). A synthetic currency provides investors with the ability to replicate the behaviour of a target currency, without having a position in the target currency.

For the concept of synthetic money to be adhered to, a synthetic currency must not have a position in the target currency. For example, a synthetic US dollar must not have a US dollar position. The original construction methodology of a synthetic currency used a portfolio optimisation technique subject to a budget constraint such that the portfolio weights sum to a non-zero value (Hovanov et al., 2007). However, the optimal portfolio of other currencies is a portfolio of *invariant currency indexes*, rather than a portfolio of currency positions. Furthermore, the target currency is also an invariant currency index. In summary, a synthetic currency is an optimal portfolio of *other* invariant currency indexes that attempts to mimic a *target* invariant currency index, without having a position in the target currency.

For a collection of currencies, an invariant currency index is a normalised bilateral exchange rate that is independent of the choice of *numéraire* (Hovanov et al., 2004). Like invariant currency indexes, *relative currency rates* are also independent of the choice of numéraire for a collection of currencies. However, a relative currency rate is priced relative to an equally-weighted collection of currencies (Kunkler and MacDonald, 2015). As a result, relative currency rates are tradable multilateral exchange rates, which provide transparency on the underlying currency positions.

In this paper, we uncover the underlying currency positions of an invariant currency index and ultimately the underlying currency positions of a synthetic currency. We first show that invariant currency indexes are equivalent to relative currency rates. Consequently, invariant currency indexes are tradable multilateral exchange rates, which are priced relative to an equally-weighted collection of currencies. We use the equivalence between invariant currency indexes and relative currency rates to show that there is a *budget-constraint issue* with the original construction methodology for a synthetic currency. The optimal portfolio is estimated by maximising the return correlation between the synthetic currency and the target currency, subject to a budget constraint such that the portfolio weights sum a non-zero value, $x$ say. We show that this sum-to-$x$ budget constraint unintentionally creates a non-zero position in the target currency.

A non-zero position in the target currency invalidates the concept of a synthetic currency being a portfolio of "other" currencies. We address the budget-constraint issue by replacing the sum-to-$x$ budget constraint with a sum-to-zero budget constraint, which correctly enforces a zero position in the target currency. Sum-to-zero budget constraints are commonly used when constructing long-short portfolios (Jacobs et al., 1998).

For comparison purposes, we first consider the same collection of seven developed-market currencies used in Hovanov et al. (2007), which are some of the most liquid developed-market currencies. We then consider a collection of 28 currencies to provide a broader picture of synthetic money. The 28 currencies are some of the most liquid developed-market, and emerging-market, currencies. Once the budget-constraint issue is addressed, we find that the observed return correlations between each synthetic currency and the associated target currency for the collection of 28 currencies range between 0.403 and 0.945. Currencies have significant currency-specific (idiosyncratic) movements, which synthetic money is unable to mimic.

This paper continues by providing an overview of the related literature in Section 2. Section 3 shows that invariant currency indexes are equivalent to relative currency rates, as well as addressing the budget-constraint issue in the construction methodology of synthetic money. Section 4 reports the results using both data samples, and Section 5 concludes.

## 2 Related literature

Currencies are usually priced relative to a *numéraire*. More specifically, a bilateral exchange rate is the price of one currency priced relative to another currency (numéraire). For example, US dollar exchange rates are frequently used to model a collection of currencies, where the US dollar is the common numéraire for all exchange rates. In addition, multivariate statistical techniques typically classify the US dollar as a systematic factor when applied to a collection of US dollar exchange rates, as the US dollar is a common part of each US dollar exchange rate (Mahieu and Schotman, 1994; Lustig et al., 2011; Kunkler and MacDonald, 2016; Greenaway-McGrevy et al., 2018; Verdelhan, 2018). Furthermore, a US dollar systematic factor has been reported to have significant explanatory power for movements in US dollar exchange rates (Verdelhan, 2018).

The US dollar being an anchor currency is an alternative explanation for the US dollar being classified as a systematic factor. The most important anchor currency is the US dollar, which is followed by the Eurozone euro as a distant second (Ilzetzki et al., 2019). In this context, if many currencies are anchored to the US dollar (first-level anchors) or are anchored to US-dollar anchored currencies (second-level anchors), it is not surprising that researchers classify the US dollar as a systematic factor.

Empirical results for modelling a collection of currencies are dependent on the choice of numéraire. For example, researchers are required to choose a common numéraire of the underlying currencies in the Frankel-Wei regression framework, which measures the co-movements between currencies (Frankel and Wei, 1994). In addition, the choice of numéraire matters when measuring purchasing power parity (Papell and Theodoridis, 2001), as well as when testing the forward rate unbiasedness hypothesis (Razzak, 2002).

Researchers have proposed solutions to the *numéraire issue*. For example, purchasing power parity results are numéraire invariant when the contemporaneous cross-correlations are considered (O'Connell, 1998; Coakley and Fuertes, 2000). Another example is invariant currency indexes, which are normalised bilateral exchange rates that are independent of the choice of numéraire (Hovanov et al., 2004). Similarly, relative currency rates are also independent of the choice of numéraire and are priced relative to an equally-weighted collection of currencies, so are tradable multilateral exchange rates (Kunkler and MacDonald, 2015). Both invariant currency indexes and relative currency rates are based on a fixed predefined collection of currencies.

Applications of invariant currency indexes have been the creation of a stable aggregate currency and synthetic money. A stable aggregate currency represents a stable numéraire for international trade and finance, like the IMF's Special Drawing Rights (SDR) (Hovanov et al., 2004). Synthetic money attempts to mimic a target currency with an optimal portfolio of "other" currencies (Hovanov et al., 2007). Both a stable aggregate currency and synthetic money are portfolios of invariant currency indexes, rather than a portfolio of currency positions. When investors trade either a stable aggregate currency or synthetic money, they require clarification on the underlying currency positions.

Optimal portfolio weights are frequently estimated using mean-variance analysis, which maximises a portfolio's expected return for different levels of risk (Markowitz, 1959). In respect to a synthetic currency, the optimal portfolio is estimated by maximising the return correlation between the synthetic currency and the target currency subject to a constraint that the portfolio weights sum to a non-zero value, where the optimisation does not incorporate expected returns (Hovanov et al., 2007).

Synthetic money has similarities with the exchange rate arrangements literature, which attempts to classify a currency's regime (see Frankel and Xie, 2010; Ilzetzki et al., 2019). Some classification categories are: fixed, crawling peg, managed floating, freely falling, and freely floating (Ilzetzki et al., 2019). The Frankel-Wei regression framework is used to estimate a basket of "other" currencies to represent a *de facto* exchange rate regime for a target currency (Frankel and Xie, 2010). In addition, when a currency has a fixed regime (hard peg), it is expected that the target currency to be fully explained (100% r-squared) by one, or more, other currencies (Frankel, 2009). A de facto exchange rate is like a synthetic currency, as the de facto exchange rate can be thought of as mimicking a target currency.

## 3 Materials & methods

In general, we first consider a collection of $N_S$ bilateral exchange rates priced relative to a single numéraire ($\eta th$) currency: a single-currency numéraire. Consequently, there are $N_C$ currencies, which consists of $N_S$ non-numéraire currencies and the numéraire currency itself ($N_C = N_S + 1$). It is assumed that both foreign exchange rates and currency positions are expressed in nominal terms when written in uppercase letters and in log terms when written in lowercase letters. In addition, this section uses a collection of seven currencies, namely, the Australian dollar (AUD), the British pound (GBP), the Canadian dollar (CAD), the Eurozone euro (EUR), the Japanese yen (JPY), the Swiss franc (CHF), and the US dollar (USD).

Two currency positions result from trading a bilateral exchange rate, a positive position in one currency and a negative position in another currency. In nominal terms at time $t$, the $i$th/$\eta$th exchange rate can be written as:

$$P_{S,t}^{i/\eta} = Q_{C,t}^{\eta}/Q_{C,t}^{i} \tag{1}$$

where $i,\eta = 0,\dots,N_S$; $t = 0,\dots,T$; $P_{S,t}^{i/\eta}$ is the $i$th/$\eta$th exchange rate; $Q_{C,t}^{\eta}$ is the quantity of the $\eta$th currency; and $Q_{C,t}^{i}$ is the quantity of the $i$th currency. All cross-rates are possible so that $P_{S,t}^{i/j} = P_{S,t}^{i/\eta}/P_{S,t}^{j/\eta}$ and $P_{S,t}^{i/i} = 1$. By taking natural logarithms of (1), the $i$th/$\eta$th exchange rate at time $t$ can be written in log terms as:

$$p_{S,t}^{i/\eta} = q_{C,t}^{\eta} - q_{C,t}^{i} \tag{2}$$

where $i, \eta = 0, \ldots, N_S$; $t = 0, \ldots, T$; $p_{S,t}^{i/\eta}$ is the $i$th/$\eta$th exchange rate; $q_{C,t}^{\eta}$ is the quantity of the $\eta$th currency; and $q_{C,t}^{i}$ is the quantity of the $i$th currency.

Table 1 reports the nominal currency positions in US dollars from buying USD 1M (one million) worth of the EUR/USD exchange rate. The result is a positive USD 1M position in the Eurozone euro (EUR) and a negative USD 1M position in the US dollar (USD). More generally, trading the $i$th/$\eta$th exchange rate results in two currency positions: a position in the $i$th currency and an opposite-signed position in the $\eta$th currency.

**Table 1:** EUR/USD exchange rate

| | **USD** | **EUR** | **JPY** | **AUD** | **CHF** | **GBP** | **CAD** |
|---|---|---|---|---|---|---|---|
| $Q_{C,t}^{USD}$ | -1M | | | | | | |
| $Q_{C,t}^{EUR}$ | | 1M | | | | | |
| $P_{S,t}^{EUR/USD} = Q_{C,t}^{USD}/Q_{C,t}^{EUR}$ | -1M | 1M | | | | | |

*Notes:* This table reports the currency positions in US dollars that result from buying USD 1M (one million) worth of EUR/USD.

### *3.1 Equivalence*

This section shows that invariant currency indexes are equivalent to relative currency rates. Invariant currency indexes are normalised bilateral exchange rates that are independent of the choice of numéraire (Hovanov et al., 2004). In contrast, relative currency rates are tradable multilateral exchange rates, which are priced relative to an equally-weighted collection of $N_C$ currencies (Kunkler and MacDonald, 2015). We first present relative currency rates and then present invariant currency indexes.

Relative currency rates are a unique solution to a system of $N_C$ equations, which consist of $N_S$ bilateral-exchange-rate-decomposition equations and one equilibrium-condition equation, where $N_C = N_S + 1$. In log terms at time $t$, the system of $N_C$ equations consist of a collection of $N_S$ bilateral exchange rates decompositions given by:

$$p_{S,t}^{i/\eta} = p_{C,t}^{i} - p_{C,t}^{\eta}, \tag{3}$$

and an equilibrium-condition equation:

$$\sum_{i=0}^{N_S} p_{C,t}^{i} = 0 \tag{4}$$

where $i = 0, \ldots, N_S$ with $i \neq \eta$; $t = 0, \ldots, T$; $p_{S,t}^{i/\eta}$ is the $i$th/$\eta$th exchange rate; $p_{C,t}^{i}$ is the $i$th relative currency rate and $p_{C,t}^{\eta}$ is the numéraire ($\eta$th) relative currency rate (see Kunkler and MacDonald, 2015).

At each point in time $t$, the system of equations in (3) and (4) contain $N_C$ equations with $N_C$ unknown relative currency rates. Consequently, there is a unique solution where the $i$th relative currency rate at time $t$ is given by:

$$p_{C,t}^{i} = \frac{1}{N_C} \sum_{j=0}^{N_S} p_{S,t}^{i/j} \tag{5}$$

where $i = 0, \ldots, N_S$; $t = 0, \ldots, T$; and $p_{S,t}^{i/j}$ is the $i$th/$j$th exchange rate. The extra equilibrium-condition equation in (4) is added so that a unique solution is possible (Kunkler and MacDonald, 2015).

The underlying currency positions (quantities) that result from trading the $i$th relative currency rate are found by substituting (2) into (5) to give:

$$p_{C,t}^{i} = \frac{1}{N_C} \sum_{j=0}^{N_S} (q_{C,t}^{j} - q_{C,t}^{i}) = q_{C,t}^{CU} - q_{C,t}^{i} \tag{6}$$

where $i = 0, \ldots, N_S$; $t = 0, \ldots, T$; and $q_{C,t}^{CU}$ is the quantity of the collection of $N_C$ currencies:

$$q_{C,t}^{CU} = \frac{1}{N_C} \sum_{j=0}^{N_S} q_{C,t}^{j} \tag{7}$$

The $i$th relative currency rate in (6) is priced relative to the collection of currencies in (7), which are weighted equally.

For example, Table 2 reports the nominal currency positions in US dollars for buying USD 7M (seven million) worth of the US dollar (USD) relative currency rate. The result is a positive USD 6M position in the US dollar (USD) and negative USD 1M positions in the other six currencies. More generally, trading the $i$th relative currency rate results in $N_C$ currency positions: a large position in the $i$th currency and smaller opposite-signed positions in all the other currencies.

**Table 2:** US dollar relative currency rate

| | **USD** | **EUR** | **JPY** | **AUD** | **CHF** | **GBP** | **CAD** |
|---|---|---|---|---|---|---|---|
| $Q_{C,t}^{USD}$ | 7M | | | | | | |
| $Q_{C,t}^{CU}$ | -1M | -1M | -1M | -1M | -1M | -1M | -1M |
| $P_{C,t}^{USD} = Q_{C,t}^{CU}/Q_{C,t}^{USD}$ | 6M | -1M | -1M | -1M | -1M | -1M | -1M |

*Notes:* This table reports the currency positions in US dollars that result from buying USD 7M (seven million) worth of the US dollar relative currency rate.

Invariant currency indexes are normalised bilateral exchange rates that are independent of the choice of numéraire (Hovanov et al., 2004). We consider a collection of $N_S$ bilateral exchange rates that are priced relative to the numéraire ($\eta$th) currency, so there are $N_C$ currencies with $N_C = N_S + 1$. In nominal terms at time $t$, we can write the $i$th invariant currency index as:

$$H_{C,t}^{i} = P_{S,t}^{i/\eta} / \prod_{j=0}^{N_S} \left(P_{S,t}^{j/\eta}\right)^{\frac{1}{N_C}} \tag{8}$$

where $i = 0, \dots, N_S$; $t = 0, \dots, T$; $H_{C,t}^{i}$ is the invariant currency index for the $i$th currency; $P_{S,t}^{i/\eta}$ is the $i$th/$\eta$th exchange rate, $P_{S,t}^{j/\eta}$ is the $j$th/$\eta$th exchange rate with $P_{S,t}^{j/j} = 1$; and $\prod_{j=0}^{N_S} \left(P_{S,t}^{j/\eta}\right)^{\frac{1}{N_C}}$ is the normalisation term. It should be noted that the choice of the numéraire ($\eta$th) currency in (8) does not affect the value of $H_{C,t}^{i}$.

We can write the $i$th invariant currency index at time $t$ in log terms by taking natural logarithms of (8) to give:

$$\begin{aligned} h_{C,t}^{i} &= p_{S,t}^{i/\eta} - \frac{1}{N_C} \sum_{j=0}^{N_S} p_{S,t}^{j/\eta} \\ &= p_{S,t}^{i/\eta} + \frac{1}{N_C} \sum_{j=0}^{N_S} p_{S,t}^{\eta/j} \end{aligned} \tag{9}$$

where $i = 0, \dots, N_S$; $t = 0, \dots, T$; $h_{C,t}^{i}$ is the invariant currency index for the $i$th currency; $p_{S,t}^{i/\eta}$ is the $i$th/$\eta$th exchange rate; $p_{S,t}^{j/\eta}$ is the $j$th/$\eta$th exchange rate; and $p_{S,t}^{\eta/j} = -p_{S,t}^{j/\eta}$ can easily be seen from (3). We can rewrite (9) by substituting $p_{S,t}^{i/\eta} = p_{C,t}^{i} - p_{C,t}^{\eta}$ from (3) and $p_{C,t}^{\eta} = \frac{1}{N_C} \sum_{j=0}^{N_S} p_{S,t}^{\eta/j}$ from (5) to give:

$$\begin{aligned} h_{C,t}^{i} &= (p_{C,t}^{i} - p_{C,t}^{\eta}) + p_{C,t}^{\eta} \\ &= p_{C,t}^{i} \end{aligned} \tag{10}$$

where $i = 0, \dots, N_S$; $t = 0, \dots, T$; $h_{C,t}^{i}$ is the $i$th invariant currency index; and $p_{C,t}^{i}$ is the $i$th relative currency rate.

The $i$th invariant currency index in (10) is equivalent to the $i$th relative currency rate in (5) for $i = 0, \dots, N_S$ and $t = 0, \dots, T$. The normalisation term in (8) changes the numéraire of each currency from a bilateral exchange rate to a multilateral exchange rate. Thus, the $i$th invariant currency index is priced relative to the collection of currencies in (7), which are weighted equally.

For example, Table 3 reports the nominal currency positions in US dollars for buying USD 7M (seven million) worth of the US dollar (USD) invariant currency index, which is the same as buying USD 7M (seven million) worth of the US dollar (USD) relative currency rate. The result is a positive USD 6M position in the US dollar (USD) and negative USD 1M positions in the other six currencies. More generally, trading the $i$th invariant currency index results in $N_C$ currency positions: a large position in the $i$th currency and smaller opposite-signed positions in the other currencies.

**Table 3:** US dollar invariant currency index

| | **USD** | **EUR** | **JPY** | **AUD** | **CHF** | **GBP** | **CAD** |
|---|---|---|---|---|---|---|---|
| $Q_{C,t}^{USD}$ | 7M | | | | | | |
| $Q_{C,t}^{CU}$ | -1M | -1M | -1M | -1M | -1M | -1M | -1M |
| $H_{C,t}^{USD} = P_{C,t}^{USD}$ | 6M | -1M | -1M | -1M | -1M | -1M | -1M |

*Notes:* This table reports the currency positions in US dollars that result from buying USD 7M (seven million) worth of the US dollar invariant currency index.

The equivalence between invariant currency indexes and relative currency rates provides transparency of the underlying currency positions for each invariant currency index. We assume throughout the rest of the paper that the terms invariant currency index and relative currency rate are interchangeable, with a preference on the relative currency rate term.

### *3.2 Synthetic Money*

A synthetic currency attempts to mimic a target currency with an optimal portfolio of other currencies, where the portfolio does not contain a position in the target currency (Hovanov et al., 2007). The optimal portfolio of other currencies is a portfolio of relative currency rates (invariant currency indexes), rather than a portfolio of currency positions. A synthetic currency is estimated by maximising the return correlation between the target currency and the portfolio of *other* currencies subject to a budget constraint such that the portfolio weights sum to a non-zero value, $x$ say: a sum-to-$x$ budget constraint. It should be noted that we will refer to the synthetic currency for the $i$th currency as the *synthetic $i$th currency*.

In log terms at time $t$, the synthetic $i$th currency is:

$$s_{C,t}^{i} = \sum_{j=0}^{N_S} I_{(j \neq i)} w_{C,t}^{i,j} p_{C,t}^{j} \tag{11}$$

subject to a sum-to-$x$ budget constraint such that:

$$\bar{w}_{C,t}^{i} = \sum_{j=0}^{N_S} I_{(j \neq i)} w_{C,t}^{i,j} = x \tag{12}$$

where $i = 0, \dots, N_S$; $t = 0, \dots, T$; $s_{C,t}^{i}$ is the synthetic $i$th currency that mimics the $i$th relative currency rate (target currency); $I_{(j \neq i)}$ is an indicator function that equals one when $j \neq i$ and zero when $j = i$; $w_{C,t}^{i,j}$ is the portfolio weight of the $j$th relative currency rate for the synthetic $i$th currency; $p_{C,t}^{j}$ is the $j$th relative currency rate; and $\overline{w}_{C,t}^{i} = \sum_{j=0}^{N_S} I_{(j \neq i)} w_{C,t}^{i,j}$ is the sum of the portfolio weights. The indicator function enforces a zero portfolio weight for the $i$th relative currency rate.

The portfolio weight for the $j$th relative currency rate is not the same as the currency position weight in the $j$th currency. In a portfolio context, the signs of the quantities must be flipped when writing the $j$th relative currency rate in terms of quantities, so that $p_{C,t}^{j} = q_{C,t}^{j} - q_{C,t}^{CU}$ (Kunkler, 2021). For example, a *positive weight* in the $j$th relative currency rate requires a positive position in the $i$th currency and a negative position in the equally-weighted collection of $N_C$ currencies in (7). Similarly, a *negative weight* in the $j$th relative currency rate requires a negative position in the $i$th currency and a positive position in the equally-weighted collection of $N_C$ currencies in (7). Consequently, the underlying currency positions that result from trading the synthetic $i$th currency are found by substituting $p_{C,t}^{j} = q_{C,t}^{j} - q_{C,t}^{CU}$ into (11) to give:

$$
\begin{aligned}
s_{C,t}^{i} &= \sum_{j=0}^{N_S} I_{(j \neq i)} w_{C,t}^{i,j} (q_{C,t}^{j} - q_{C,t}^{CU}) \\
&= \sum_{j=0}^{N_S} I_{(j \neq i)} w_{C,t}^{i,j} q_{C,t}^{j} - \overline{w}_{C,t}^{i} q_{C,t}^{CU} \\
&= \sum_{j=0}^{N_S} I_{(j \neq i)} w_{C,t}^{i,j} q_{C,t}^{j} - \frac{1}{N_C} \overline{w}_{C,t}^{i} \sum_{j=0}^{N_S} q_{C,t}^{j} \\
&= \sum_{j=0}^{N_S} (I_{(j \neq i)} w_{C,t}^{i,j} - \frac{1}{N_C} \overline{w}_{C,t}^{i})\, q_{C,t}^{j} \\
&= \sum_{j=0}^{N_S} \delta_{C,t}^{i,j}\, q_{C,t}^{j} \qquad (13)
\end{aligned}
$$

where $i = 0, \dots, N_S$; $t = 0, \dots, T$; $\overline{w}_{C,t}^{i} = \sum_{j=0}^{N_S} I_{(j \neq i)} w_{C,t}^{i,j}$ is the sum of the portfolio weights; $q_{C,t}^{CU}$ is the quantities (currency positions) associated with the collection of $N_C$ currencies in (7) with $q_{C,t}^{CU} = \frac{1}{N_C} \sum_{\eta=0}^{N_S} q_{C,t}^{\eta}$; and $\delta_{C,t}^{i,j}$ is:

$$\delta_{C,t}^{i,j} = I_{(j \neq i)} w_{C,t}^{i,j} - \frac{1}{N_C} \overline{w}_{C,t}^{i} \qquad (14)$$

where $\delta_{C,t}^{i,j}$ is the *currency position weight* in the $j$th currency for the synthetic $i$th currency.

We can calculate the currency position weight in the target ($i$th) currency for the synthetic $i$th currency by substituting $j = i$ into (14) to give:

$$\delta_{C,t}^{i,i} = -\frac{1}{N_C} \overline{w}_{C,t}^{i} \qquad (15)$$

where $i = 0, \ldots, N_S$; $t = 0, \ldots, T$; $I_{(j \neq i)} = 0$ when $j = i$; $\delta_{C,t}^{i,i}$ is the currency position weight in the target ($i$th) currency; $\overline{w}_{C,t}^{i}$ is the sum of the portfolio weights; and $N_C$ represent the number of currencies. Thus, the currency position weight in the target ($i$th) currency is non-zero with any sum-to-$x$ budget constraint. In this situation, $x = \overline{w}_{C,t}^{i}$ represents the sum-to-$x$ budget constraint, which results in the currency position weight in the target ($i$th) currency being $\delta_{C,t}^{i,i} = -\frac{1}{N_C} x$.

Thus, a sum-to-$x$ budget constraint unintentionally creates a non-zero position in the US dollar (target currency), which invalidates the concept of a synthetic US dollar being an optimal portfolio of "other" currencies. For example, Table 4 reports the synthetic US dollar subject to a sum-to-$x$ budget constraint for the collection of seven currencies. The result is a non-zero currency position weight in the US dollar (USD) of $\delta_{C,t}^{USD,USD} = -\frac{1}{7} x = -\frac{1}{7} \overline{w}_{C,t}^{USD}$.

**Table 4:** Synthetic US dollar with a sum-to-$x$ budget constraint

| Currency | $w_{C,t}^{USD,j}$ | $\delta_{C,t}^{USD,j}$ |
|---|---|---|
| **USD** | 0 | $-\frac{1}{7}\overline{w}_{C,t}^{USD}$ |
| **EUR** | $w_{C,t}^{USD,EUR}$ | $w_{C,t}^{USD,EUR} - \frac{1}{7}\overline{w}_{C,t}^{USD}$ |
| **JPY** | $w_{C,t}^{USD,JPY}$ | $w_{C,t}^{USD,JPY} - \frac{1}{7}\overline{w}_{C,t}^{USD}$ |
| **AUD** | $w_{C,t}^{USD,AUD}$ | $w_{C,t}^{USD,AUD} - \frac{1}{7}\overline{w}_{C,t}^{USD}$ |
| **CHF** | $w_{C,t}^{USD,CHF}$ | $w_{C,t}^{USD,CHF} - \frac{1}{7}\overline{w}_{C,t}^{USD}$ |
| **GBP** | $w_{C,t}^{USD,GBP}$ | $w_{C,t}^{USD,GBP} - \frac{1}{7}\overline{w}_{C,t}^{USD}$ |
| **CAD** | $w_{C,t}^{USD,CAD}$ | $w_{C,t}^{USD,CAD} - \frac{1}{7}\overline{w}_{C,t}^{USD}$ |

*Notes:* This table reports the currency positions and currency position weights for the synthetic US dollar subject to a sum-to-$x$ budget constraint.

If a synthetic currency is attempting to mimic the target currency with an optimal portfolio of other currencies, the portfolio should not have a position in the target currency. The only way to enforce a zero position in the target currency is to have a budget constraint where the portfolio weights sum to zero: a *sum-to-zero budget constraint*. If we replace the sum-to-$x$ budget constraint with a sum-to-zero budget constraint so that $x = \overline{w}_{C,t}^{i} = 0$, the currency position weight in the target ($i$th) currency for the synthetic $i$th currency becomes:

$$\delta_{C,t}^{i,i} = -\frac{1}{N_C} \overline{w}_{C,t}^{i} = 0 \quad (16)$$

where $i = 0, \dots, N_S$; $t = 0, \dots, T$; $\delta_{C,t}^{i,i}$ is the currency position weight in the target ($i$th) currency; $\bar{w}_{C,t}^{i}$ is the sum of the portfolio weights; and $N_C$ is the number of currencies. In this situation, the currency position weights in the other currencies can be rewritten subject to a sum-to-zero budget constraint by substituting $\bar{w}_{C,t}^{i} = 0$ into (14) to give:

$$\delta_{C,t}^{i,j} = I_{(j \neq i)} w_{C,t}^{i,j} - \frac{1}{N_C} \bar{w}_{C,t}^{i} = w_{C,t}^{i,j} \qquad (17)$$

where $j = 0, \dots, N_S$; $j \neq i$; $t = 0, \dots, T$; $\delta_{C,t}^{i,j}$ is the currency position weight in the $j$th currency; and $w_{C,t}^{i,j}$ is the portfolio weight of the $j$th relative currency rate for the synthetic $i$th currency. Thus, for the synthetic $i$th currency subject to sum-to-zero budget constraint, the currency position weight in the $j$th currency is equal to the portfolio weight for the $j$th relative currency rate.

For example, Table 5 reports the synthetic US dollar subject to a sum-to-zero budget constraint for the collection of seven currencies. The result is a zero position in the US dollar (USD). Thus, a sum-to-zero budget constraint correctly enforces a zero position in the target currency.

**Table 5:** Synthetic US dollar with a sum-to-zero budget constraint

| Currency | $w_{C,t}^{USD,j}$ | $\delta_{C,t}^{USD,j}$ |
|---|---|---|
| **USD** | 0 | 0 |
| **EUR** | $w_{C,t}^{USD,EUR}$ | $w_{C,t}^{USD,EUR}$ |
| **JPY** | $w_{C,t}^{USD,JPY}$ | $w_{C,t}^{USD,JPY}$ |
| **AUD** | $w_{C,t}^{USD,AUD}$ | $w_{C,t}^{USD,AUD}$ |
| **CHF** | $w_{C,t}^{USD,CHF}$ | $w_{C,t}^{USD,CHF}$ |
| **GBP** | $w_{C,t}^{USD,GBP}$ | $w_{C,t}^{USD,GBP}$ |
| **CAD** | $w_{C,t}^{USD,CAD}$ | $w_{C,t}^{USD,CAD}$ |

*Notes:* This table reports the currency positions and currency position weights for the synthetic US dollar subject to a sum-to-zero budget constraint.

In summary, a sum-to-$x$ budget constraint unintentionally creates a synthetic currency with a non-zero position in the target currency. This invalidates the concept of a synthetic currency as being a portfolio of “other” currencies. In contrast, a sum-to-zero budget constraint creates a synthetic currency with a zero position in the target currency. Thus, a zero position in the target currency is only possible with a sum-to-zero budget constraint.

## 4 Results

The two data samples used in this section are reported in Table 6. The first data sample consists of seven currencies ($N_C = 7$) and the second data sample consists of 28 currencies ($N_C = 28$), where the first data sample is a subset of the second data sample. The seven-currency data sample contains some of the most liquid developed-market currencies. In contrast, the 28-currency data sample represents both developed-market, and emerging-market, currencies. All data is sourced from Bloomberg using end-of-the-month Tokyo closing prices for the period of the 1st of January 2000 to the 31st of December 2020.

**Table 6:** Two data samples

| **Code** | **Currency** | **7 Currencies** | **28 Currencies** |
|---|---|---|---|
| USD | US dollar | Yes | Yes |
| EUR | Eurozone euro | Yes | Yes |
| JPY | Japanese yen | Yes | Yes |
| AUD | Australian dollar | Yes | Yes |
| CHF | Swiss franc | Yes | Yes |
| GBP | British pound | Yes | Yes |
| CAD | Canadian dollar | Yes | Yes |
| NZD | New Zealand dollar | | Yes |
| NOK | Norwegian krone | | Yes |
| SEK | Swedish krona | | Yes |
| CNY | Chinese renminbi | | Yes |
| INR | Indian rupee | | Yes |
| IDR | Indonesian rupiah | | Yes |
| KRW | South Korean won | | Yes |
| MYR | Malaysian ringgit | | Yes |
| PHP | Philippine peso | | Yes |
| SGD | Singaporean dollar | | Yes |
| THB | Thai baht | | Yes |
| CZK | Czech koruna | | Yes |
| HUF | Hungarian forint | | Yes |
| ILS | Israeli shekel | | Yes |
| PLN | Polish zloty | | Yes |
| RUB | Russian ruble | | Yes |
| TRY | Turkish lira | | Yes |
| ZAR | South African rand | | Yes |
| BRL | Brazilian real | | Yes |
| CLP | Chilean peso | | Yes |
| MXN | Mexican peso | | Yes |

*Notes:* This table reports the two data samples, where the first data sample consists of seven currencies and the second data sample consists of 28 currencies.

Table 7 reports a summary of the return statistics for the 28 relative currency rates in the second data sample. The Swiss franc (CHF) has the largest average return of 3.71%, and the Turkish lira (TRY) has the smallest average return of -11.54%. The Turkish lira (TRY) has the largest volatility of 15.55%, and the Singaporean dollar (SGD) has the smallest volatility of 3.66%.

The safe-haven currencies show significant positive skewness, with 1.104 for the Japanese yen (JPY), 1.052 for the Swiss franc (CHF), and 0.769 for the US dollar (USD). In contrast, movements in the Turkish lira (TRY) show significant negative skewness, with an observed skewness of -3.025. Furthermore, the Turkish lira (TRY) also has the largest kurtosis of 23.60. The significant kurtosis for all currencies, together with some significant skewness, indicates that movements in the relative currency rates are non-normal.

**Table 7:** Summary statistics

| Currency | Average | Volatility | Skewness | Kurtosis |
|---|---|---|---|---|
| **USD** | 0.92% | 7.05% | 0.769*** | 5.54*** |
| **EUR** | 1.84% | 5.88% | 0.309** | 3.45*** |
| **JPY** | 0.86% | 10.03% | 1.104*** | 7.63*** |
| **AUD** | 1.69% | 7.01% | -0.462*** | 4.58*** |
| **CHF** | 3.71% | 7.22% | 1.052*** | 5.86*** |
| **GBP** | 0.12% | 6.90% | -0.709*** | 6.04*** |
| **CAD** | 1.51% | 6.00% | -0.298* | 3.74*** |
| **NZD** | 2.47% | 8.46% | -0.151 | 3.86*** |
| **NOK** | 0.62% | 7.21% | -0.052 | 3.57*** |
| **SEK** | 1.10% | 6.65% | 0.175 | 3.55*** |
| **CNY** | 2.04% | 6.61% | 0.784*** | 6.09*** |
| **INR** | -1.54% | 6.07% | 0.007 | 3.79*** |
| **IDR** | -2.32% | 9.42% | 0.982*** | 11.59*** |
| **KRW** | 1.08% | 7.66% | 0.072 | 8.61*** |
| **MYR** | 0.65% | 5.36% | -0.014 | 5.57*** |
| **PHP** | 0.05% | 6.53% | -0.064 | 4.55*** |
| **SGD** | 2.00% | 3.66% | 0.682*** | 7.16*** |
| **THB** | 1.91% | 5.44% | 0.475*** | 4.58*** |
| **CZK** | 3.34% | 7.74% | 0.127 | 3.89*** |
| **HUF** | 0.13% | 9.03% | -1.149*** | 7.64*** |
| **ILS** | 2.16% | 6.67% | 0.097 | 3.86*** |
| **PLN** | 1.41% | 8.47% | -0.692*** | 4.06*** |
| **RUB** | -3.83% | 10.94% | -1.167*** | 10.21*** |
| **TRY** | -11.54% | 15.55% | -3.025*** | 23.60*** |
| **ZAR** | -3.21% | 12.95% | -0.578*** | 4.10*** |
| **BRL** | -4.11% | 13.75% | -0.819*** | 6.25*** |
| **CLP** | -0.49% | 8.97% | 0.063 | 3.75*** |
| **MXN** | -2.58% | 8.58% | -0.617*** | 5.12*** |

*Notes:* This table reports the average return, volatility, skewness, and kurtosis. Significance levels for skewness and kurtosis are denoted by * for 10%, ** for 5%, and *** for 1%.

A *synthetic currency* attempts to mimic a *target currency* with an optimal portfolio of *other* currencies. The currency positions that result from the optimal portfolio should *not* contain a position in the target currency. In this section, we estimate a synthetic currency for each of the seven currencies in the first data sample. We maximise the sample return correlation between a target currency and the other currencies. For comparison, we estimate a synthetic currency subject to both a sum-to-$x$ budget constraint and a sum-to-zero budget constraint.

### *4.1 Return correlation matrix*

Table 8 reports the observed return correlation matrix for the seven relative currency rates in the first data sample. The observed return correlations that are significantly positive represent currencies that share similar loadings on one, or more, systematic factors. For example, currencies of countries that are geographically closer are more exposed to systematic risk (Lustig and Richmond, 2020).

The Eurozone and Swiss markets are geographically close. The observed return correlation is a significant 0.429 between the Eurozone euro (EUR) and the Swiss franc (CHF). In contrast, the US and Canadian markets are also geographically close. However, the observed return correlation is an insignificant -0.026 between the US dollar (USD) and the Canadian dollar (CAD). Similarly, the British and Eurozone markets are geographically close, but the observed return correlation is an insignificant -0.029 between the British pound (GBP) and the Eurozone euro (EUR).

In addition, the Australian and Canadian markets are not geographically close, but they are both commodity-exporting countries (Chen and Rogoff, 2003). The observed return correlation is a significant 0.276 between the Australian dollar (AUD) and the Canadian dollar (CAD). Furthermore, the US and Japanese markets are not geographically close, but the US dollar (USD) and the Japanese yen (JPY) are classified as safe-haven currencies. The observed return correlation is a significant 0.276 between the US dollar (USD) and the Japanese yen (JPY).

There are significant positive observed return correlations between different pairs of relative currency rates. The positive observed return correlations represent markets that share similar loadings on one, or more, systematic factors, such as geographic proximity, safe-haven currencies, and commodity currencies.

**Table 8:** Return correlation matrix

| | USD | EUR | JPY | AUD | CHF | GBP | CAD |
|---|---|---|---|---|---|---|---|
| **USD** | 1.000 | -0.450*** | 0.276*** | -0.561*** | -0.306*** | 0.000 | -0.026 |
| **EUR** | -0.450*** | 1.000 | -0.374*** | 0.044 | 0.429*** | -0.029 | -0.306*** |
| **JPY** | 0.276*** | -0.374*** | 1.000 | -0.481*** | -0.030 | -0.376*** | -0.349*** |
| **AUD** | -0.561*** | 0.044 | -0.481*** | 1.000 | -0.197*** | -0.144** | 0.276*** |
| **CHF** | -0.306*** | 0.429*** | -0.030 | -0.197*** | 1.000 | -0.205*** | -0.530*** |
| **GBP** | 0.000 | -0.029 | -0.376*** | -0.144** | -0.205*** | 1.000 | -0.056 |
| **CAD** | -0.026 | -0.306*** | -0.349*** | 0.276*** | -0.530*** | -0.056 | 1.000 |

*Notes:* This table reports the observed return correlation matrix for the relative currency rates. Significance levels are denoted by * for 10%, ** for 5%, and *** for 1%.

### *4.2 Sum-to-$x$ budget constraint*

We first estimate a synthetic currency for each of the seven currencies in the first data sample subject to a sum-to-$x$ budget constraint. We use regression analysis with constraints to estimate each synthetic currency. If the value of $x$ in the sum-to-$x$ budget constraint is equal to $-N_S$ in (12), all the optimal portfolio weights will equal minus one ($w_{C,t}^{i,j} = -1$) for all $j \neq i$. In this situation, the synthetic $i$th currency becomes:

$$s_{C,t}^{i} = \sum_{j=0}^{N_S} I_{(j\neq i)} w_{C,t}^{i,j} p_{C,t}^{j} = -\sum_{j=0}^{N_S} I_{(j\neq i)} p_{C,t}^{j} = p_{C,t}^{i} \quad (18)$$

where $i = 0, \dots, N_S$; $t = 0, \dots, T$; $s_{C,t}^{i}$ is the synthetic $i$th currency; $I_{(j\neq i)}$ is an indicator function that equals one when $j \neq i$ and zero when $j = i$; and $p_{C,t}^{i}$ is the $i$th relative currency rate in terms of the other currency rates, i.e., $p_{C,t}^{i} = -\sum_{j=0}^{N_S} I_{(j\neq i)} p_{C,t}^{j}$, which can be found by rearranging the equilibrium-condition equation in (4).

Thus, the synthetic $i$th currency in (18) is equal to the $i$th relative currency rate. In this situation, the currency position weight for the $i$th currency (target currency when $j = i$) is:

$$\delta_{C,t}^{i,i} = -\frac{\bar{w}_{C,t}^{i}}{N_S+1} = -\frac{-N_S}{N_S+1} = \frac{6}{7} = 0.8571 \quad (19)$$

and the currency position weight in the $j$th currency (*other* currency when $j \neq i$) is:

$$\delta_{C,t}^{i,j} = I_{(j\neq i)} w_{C,t}^{i,j} - \frac{\bar{w}_{C,t}^{i}}{N_S+1} -= -1 - \frac{-N_S}{N_S+1} = -1 + \frac{6}{7} = -\frac{1}{7} = -0.1429 \quad (20)$$

where $i, j = 0, \dots, N_S$; $t = 0, \dots, T$; $I_{(j \neq i)}$ is an indicator function that equals one when $j \neq i$, and zero when $j = i$.

Table 9 reports both the optimal portfolio weights $w_{C,t}^{i,j}$, and the currency position weights $\delta_{C,t}^{i,j}$, associated with the observed synthetic $i$th currency subject to a sum-to-$x$ budget constraint with $x = -N_S$. Table 9 also reports the observed return correlations between each synthetic currency and the associated target currency. The equivalence between invariant currency indexes and relative currency rates provides transparency of the underlying currency positions for each invariant currency index, and ultimately the underlying currency positions for each synthetic currency. All the observed return correlations are exactly one between the synthetic $i$th currency and the target $i$th currency (relative currency rate). These results demonstrate that the sum-to-$x$ budget constraint unintentionally recreates the target currency exactly, and contains a significant position in the target currency. This violates the concept of a synthetic currency being an optimal portfolio of other currencies.

**Table 9:** Synthetic money using log returns

| | **USD** | **EUR** | **JPY** | **AUD** | **CHF** | **GBP** | **CAD** | **Corr** |
|---|---|---|---|---|---|---|---|---|
| $w_{C,t}^{USD,j}$ | | -1.0000 | -1.0000 | -1.0000 | -1.0000 | -1.0000 | -1.0000 | **1.000** |
| $\delta_{C,t}^{USD,j}$ | **0.8571** | -0.1429 | -0.1429 | -0.1429 | -0.1429 | -0.1429 | -0.1429 | |
| $w_{C,t}^{EUR,j}$ | -1.0000 | | -1.0000 | -1.0000 | -1.0000 | -1.0000 | -1.0000 | **1.000** |
| $\delta_{C,t}^{EUR,j}$ | -0.1429 | **0.8571** | -0.1429 | -0.1429 | -0.1429 | -0.1429 | -0.1429 | |
| $w_{C,t}^{JPY,j}$ | -1.0000 | -1.0000 | | -1.0000 | -1.0000 | -1.0000 | -1.0000 | **1.000** |
| $\delta_{C,t}^{JPY,j}$ | -0.1429 | -0.1429 | **0.8571** | -0.1429 | -0.1429 | -0.1429 | -0.1429 | |
| $w_{C,t}^{AUD,j}$ | -1.0000 | -1.0000 | -1.0000 | | -1.0000 | -1.0000 | -1.0000 | **1.000** |
| $\delta_{C,t}^{AUD,j}$ | -0.1429 | -0.1429 | -0.1429 | **0.8571** | -0.1429 | -0.1429 | -0.1429 | |
| $w_{C,t}^{CHF,j}$ | -1.0000 | -1.0000 | -1.0000 | -1.0000 | | -1.0000 | -1.0000 | **1.000** |
| $\delta_{C,t}^{CHF,j}$ | -0.1429 | -0.1429 | -0.1429 | -0.1429 | **0.8571** | -0.1429 | -0.1429 | |
| $w_{C,t}^{GBP,j}$ | -1.0000 | -1.0000 | -1.0000 | -1.0000 | -1.0000 | | -1.0000 | **1.000** |
| $\delta_{C,t}^{GBP,j}$ | -0.1429 | -0.1429 | -0.1429 | -0.1429 | -0.1429 | **0.8571** | -0.1429 | |
| $w_{C,t}^{CAD,j}$ | -1.0000 | -1.0000 | -1.0000 | -1.0000 | -1.0000 | -1.0000 | | **1.000** |
| $\delta_{C,t}^{CAD,j}$ | -0.1429 | -0.1429 | -0.1429 | -0.1429 | -0.1429 | -0.1429 | **0.8571** | |

*Notes:* This table reports portfolio weights $w_{C,t}^{i,j}$, and the currency position weights $\delta_{C,t}^{i,j}$ that results from estimating the synthetic $i$th currency subject to a sum-to-$x$ budget constraint using log returns. This table also reports the observed return correlations (Corr) between each synthetic currency and the associated target currency.

The results in Table 9 for the synthetic US dollar are comparable with the results in Hovanov et al. (2007), which reported an observed return correlation of 0.99 between the synthetic US dollar and the US dollar. However, Hovanov et al. (2007) used proportional returns, rather than log returns. Consequently, the estimation procedure suffered from the well-known Siegel Paradox (Siegel, 1972).

Exchange rates are usually expressed in natural logarithms to overcome the well-known Siegel Paradox (see Taylor and Sarno, 1998). Siegel's Paradox is based on Jensen's Inequality, which states that for all convex functions:

$$f(E(Y)) \leq E(f(Y)) \tag{21}$$

where $f$ is a convex function; $Y$ is a continuous random variable; and $E$ is the expectation operator. In nominal terms at time $t$, if we apply Jensen's Inequality to the $i\text{th}/\eta\text{th}$ exchange rate, the inverse of the expectation of the $i\text{th}/\eta\text{th}$ exchange rate is less than or equal to the expectation of the inverse of the $i\text{th}/\eta\text{th}$ exchange rate:

$$1/E(P_{S,t}^{i/\eta}) \leq E(1/P_{S,t}^{i/\eta}) \tag{22}$$

where $i, \eta = 0, \ldots, N_S$; $t = 0, \ldots, T$; $P_{S,t}^{i/\eta}$ is the $i\text{th}/\eta\text{th}$ exchange rate; and the convex function is $f(Y) = 1/Y$. However, in log terms at time $t$, there is no convex function for the $i\text{th}/\eta\text{th}$ exchange rate, and we get:

$$E(-p_{S,t}^{i/\eta}) = -E(p_{S,t}^{i/\eta}) \tag{23}$$

where $i, \eta = 0, \ldots, N_S$; $t = 0, \ldots, T$; and $p_{S,t}^{i/\eta}$ is the $i\text{th}/\eta\text{th}$ exchange rate (see Taylor and Sarno, 1998).

Table 10 reports both the optimal portfolio weights $w_{C,t}^{i,j}$, and the currency position weights $\delta_{C,t}^{i,j}$, associated with the estimation of the synthetic $i$th currency subject to a sum-to-$x$ budget constraint using proportional returns, rather than the log returns used in Table 9. Table 10 also reports the observed return correlations between each synthetic currency and the associated target currency. Using a sum-to-$x$ budget constraint, we know that the observed return correlation should be exactly one between the *synthetic currency* and the associated *target currency*. However, Siegel's Paradox affects the estimated values, with the observed return correlations ranging between 0.994 and 0.998, which are comparable to Table 9, although the differences are small.

**Table 10:** Synthetic money using proportional returns

| | **USD** | **EUR** | **JPY** | **AUD** | **CHF** | **GBP** | **CAD** | **Corr** |
|---|---|---|---|---|---|---|---|---|
| $w_{C,t}^{USD,j}$ | | -1.0248 | -0.9749 | -1.0098 | -0.9681 | -1.0117 | -1.0106 | **0.996** |
| $\delta_{C,t}^{USD,j}$ | **0.8571** | -0.1677 | -0.1178 | -0.1526 | -0.1110 | -0.1546 | -0.1535 | |
| $w_{C,t}^{EUR,j}$ | -0.9872 | | -0.9791 | -1.0112 | -0.9834 | -1.0213 | -1.0179 | **0.994** |
| $\delta_{C,t}^{EUR,j}$ | -0.1301 | **0.8571** | -0.1220 | -0.1540 | -0.1262 | -0.1641 | -0.1607 | |
| $w_{C,t}^{JPY,j}$ | -0.9718 | -1.0269 | | -1.0063 | -0.9593 | -1.0195 | -1.0162 | **0.998** |
| $\delta_{C,t}^{JPY,j}$ | -0.1146 | -0.1697 | **0.8571** | -0.1492 | -0.1022 | -0.1623 | -0.1591 | |
| $w_{C,t}^{AUD,j}$ | -0.9847 | -1.0277 | -0.9800 | | -0.9697 | -1.0168 | -1.0211 | **0.997** |
| $\delta_{C,t}^{AUD,j}$ | -0.1276 | -0.1705 | -0.1229 | **0.8571** | -0.1125 | -0.1596 | -0.1640 | |
| $w_{C,t}^{CHF,j}$ | -0.9853 | -1.0013 | -0.9735 | -1.0050 | | -1.0148 | -1.0201 | **0.995** |
| $\delta_{C,t}^{CHF,j}$ | -0.1282 | -0.1442 | -0.1163 | -0.1479 | **0.8571** | -0.1576 | -0.1630 | |
| $w_{C,t}^{GBP,j}$ | -0.9935 | -1.0312 | -0.9782 | -1.0083 | -0.9699 | | -1.0189 | **0.995** |
| $\delta_{C,t}^{GBP,j}$ | -0.1363 | -0.1740 | -0.1211 | -0.1511 | -0.1128 | **0.8571** | -0.1618 | |
| $w_{C,t}^{CAD,j}$ | -0.9938 | -1.0267 | -0.9798 | -1.0147 | -0.9666 | -1.0184 | | **0.996** |
| $\delta_{C,t}^{CAD,j}$ | -0.1367 | -0.1695 | -0.1227 | -0.1575 | -0.1094 | -0.1613 | **0.8571** | |

*Notes:* This table reports portfolio weights $w_{C,t}^{i,j}$, and the currency position weights $\delta_{C,t}^{i,j}$ that results from estimating the synthetic $i$th currency subject to a sum-to-$x$ budget constraint using proportional returns. This table also reports the observed return correlations (Corr) between each synthetic currency and the associated target currency.

### *4.3 Sum-to-zero budget constraint*

We now estimate a synthetic currency for each of the seven currencies in the first data sample subject to a sum-to-zero budget constraint. We again use regression analysis with constraints to estimate each synthetic currency. Table 11 reports both the optimal portfolio weights $w_{C,t}^{i,j}$, and the currency position weights $\delta_{C,t}^{i,j}$, associated with the estimation of the synthetic $i$th currency. The sum-to-zero budget constraint for each synthetic currency correctly enforces a zero position in the associated target currency. Table 11 also reports the observed return correlations between each synthetic currency and the associated target currency.

The largest observed return correlation is 0.632 between the synthetic Swiss franc (CHF) and the Swiss franc (CHF). The synthetic Swiss franc (CHF) has a large portfolio weight of 0.5110 in the Eurozone euro (EUR). The second-largest observed return correlation is 0.620 between the synthetic Eurozone euro (EUR) and the Eurozone euro (EUR). The synthetic Eurozone euro (EUR) has a large portfolio weight of 0.3567 in the Swiss franc

(CHF). The observed return correlation is a significant 0.429 between the Eurozone euro (EUR) and the Swiss franc (CHF) (see Table 8).

Furthermore, the observed return correlation is 0.603 between the synthetic US dollar (USD) and the US dollar (USD). The synthetic US dollar (USD) has large portfolio weights of 0.2233 in the Canadian dollar (CAD) and 0.1807 in the Japanese yen (JPY).

In summary, the observed return correlations range from 0.385 to 0.632 for the seven developed-market currencies in the first data sample. The observed return correlations between a synthetic currency and the target currency are largest for currencies that also have high return correlations with other currencies in the collection of currencies. However, there are significant currency-specific movements in a target currency that cannot be mimicked by a synthetic currency.

**Table 11:** Synthetic money with sum-to-zero budget constraint

| Weights | USD | EUR | JPY | AUD | CHF | GBP | CAD | Corr |
|---|---|---|---|---|---|---|---|---|
| $w_{C,t}^{USD,j}$ | | -0.1312 | 0.1807 | -0.3318 | -0.0843 | 0.1432 | 0.2233 | **0.603** |
| $\delta_{C,t}^{USD,j}$ | **0.0000** | -0.1312 | 0.1807 | -0.3318 | -0.0843 | 0.1432 | 0.2233 | |
| $w_{C,t}^{EUR,j}$ | -0.1354 | | -0.167 | 0.0003 | 0.3567 | 0.0295 | -0.0842 | **0.620** |
| $\delta_{C,t}^{EUR,j}$ | -0.1354 | **0.0000** | -0.167 | 0.0003 | 0.3567 | 0.0295 | -0.0842 | |
| $w_{C,t}^{JPY,j}$ | 0.4919 | -0.2165 | | -0.118 | 0.2654 | -0.2818 | -0.141 | **0.554** |
| $\delta_{C,t}^{JPY,j}$ | 0.4919 | -0.2165 | **0.0000** | -0.118 | 0.2654 | -0.2818 | -0.141 | |
| $w_{C,t}^{AUD,j}$ | -0.4222 | 0.1866 | -0.1241 | | -0.0518 | -0.0585 | 0.47 | **0.615** |
| $\delta_{C,t}^{AUD,j}$ | -0.4222 | 0.1866 | -0.1241 | **0.0000** | -0.0518 | -0.0585 | 0.47 | |
| $w_{C,t}^{CHF,j}$ | -0.0936 | 0.5110 | 0.0321 | -0.0911 | | -0.1066 | -0.2519 | **0.632** |
| $\delta_{C,t}^{CHF,j}$ | -0.0936 | 0.5110 | 0.0321 | -0.0911 | **0.0000** | -0.1066 | -0.2519 | |
| $w_{C,t}^{GBP,j}$ | 0.2048 | 0.1833 | -0.2289 | -0.0735 | -0.0904 | | 0.0047 | **0.385** |
| $\delta_{C,t}^{GBP,j}$ | 0.2048 | 0.1833 | -0.2289 | -0.0735 | -0.0904 | **0.0000** | 0.0047 | |
| $w_{C,t}^{CAD,j}$ | 0.2201 | -0.0523 | -0.1419 | 0.268 | -0.2714 | -0.0225 | | **0.561** |
| $\delta_{C,t}^{CAD,j}$ | 0.2201 | -0.0523 | -0.1419 | 0.268 | -0.2714 | -0.0225 | **0.0000** | |

*Notes:* This table reports portfolio weights $w_{C,t}^{i,j}$, and the currency position weights $\delta_{C,t}^{i,j}$ that result from estimating the synthetic $i$th currency subject to a sum-to-zero budget constraint. This table also reports the observed return correlations (Corr) between each synthetic currency and the associated target currency.

### 4.4 *Currency-specific movements*

Hovanov et al. (2007) reported an observed return correlation of 0.99 between the synthetic US dollar and the US dollar and subsequently suggested that purchasing power parity and interest rate parity were responsible for this high observed correlation. In this context, it is assumed that most of the movements in the target currency are attributed to systematic factors, such as a purchasing-power-parity factor and an interest-rate-parity factor. However, power purchasing power parity is more of a long-run model, rather than having high explanatory power (see MacDonald, 1993). Similarly, interest rate parity is not known to have high explanatory power (Fama, 1984).

Although it was shown in this paper that the high observed correlation was due to the sum-to-$x$ budget constraint unintentionally creating a non-zero position in the *target currency*, we now investigate the relationship between synthetic money and systematic factors. More specifically, we decompose currency movements into two parts: systematic movements and idiosyncratic movements (see Kunkler and MacDonald, 2016). Systematic movements arise from common factors that affect the movements of many currencies, and idiosyncratic movements arise from currency-specific factors that are specific to a currency.

We use 28 currencies in the second data sample to highlight the relationship between synthetic money and systematic factors. A principal components analysis is used to estimate six systematic factors, which explain approximately 60% of the total cross-sectional movements (variation) in the collection of currencies. Table 12 reports the observed r-squared values for each currency from a multiple regression model using the six systematic factors as explanatory variables, where the rows are sorted in descending order. An observed r-squared represents the percentage of movements in a currency attributed to the systematic factors. The largest observed r-squared is for the US dollar (USD), where 85.8% of the movements can be attributed to systematic factors. In contrast, the smallest observed r-squared is for the South African rand (ZAR), where only 34.1% of the movements can be attributed to systematic factors.

Table 12 also reports the observed return correlations between each synthetic currency and the associated target currency. The largest observed return correlation of 0.944 is between the synthetic US dollar and the US dollar (USD). In contrast, the smallest observed return correlation of 0.395 is between the synthetic Chilean peso and the Chilean peso (CLP).

Figure 1 displays the observed r-squared values against the observed return correlations for all 28 currencies. Synthetic currencies with a high (low) systematic risk have high (low) observed return correlations. However, synthetic money cannot mimic the significant currency-specific movements in the target currencies.

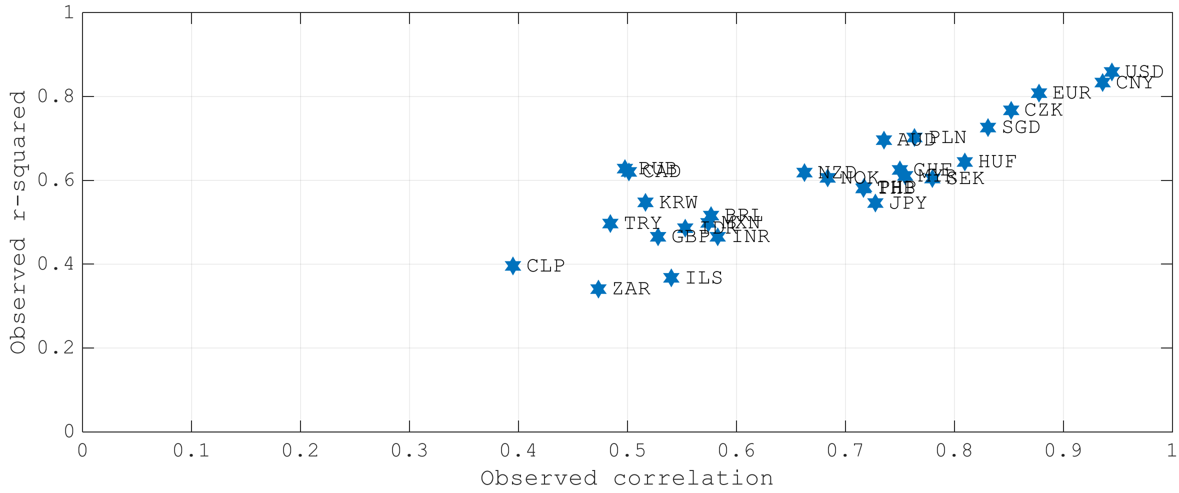


**Fig. 1.** Observed r-squared values versus observed return correlations

*Notes:* Figure 1 displays the observed r-squared values against the observed return correlations between the synthetic currency and the associated target currency, for all 28 currencies.

The US dollar is the most important anchor currency, with the Eurozone euro being a distant second (Ilzetzki et al., 2019). The last two columns of Table 12 report the observed return correlations between each relative currency rate and the US dollar (USD) relative currency rate, and the observed return correlations between each relative currency rate and the Eurozone euro (EUR) relative currency rate. This observed data provides evidence that some currencies may be anchored to the US dollar (USD) or the Eurozone euro (EUR).

For example, the results for the US dollar (USD) are interesting, where 85.8% of the movements in the US dollar (USD) relative currency rate can be attributed to systematic factors. In addition, the observed return correlation between the synthetic US dollar and the US dollar (USD) relative currency rate is 0.944. Recall that when a currency has a fixed regime, it is expected that the target currency to be fully explained (100% r-squared) by one, or more, other currencies (Frankel, 2009). The high numbers for the US dollar are, in part, driven by the currency regime of the Chinese renminbi, which was hard pegged to the US dollar until the 21st of July 2005, and a tight band with the US dollar subsequently. Consequently, the observed return correlation between the Chinese renminbi and the US dollar is 0.919 over the whole sample period.

The strong linkage between the Chinese renminbi and the US dollar helps to explain similar results of the Chinese renminbi (CNY). For example, 83.0% of the movements in the Chinese renminbi (CNY) relative currency rate can be attributed to systematic factors. In addition, the observed return correlation between the synthetic Chinese renminbi and the Chinese renminbi relative currency rate is 0.938.

Thus, if many currencies are anchored to the US dollar (first-level anchors) or are anchored to US-dollar anchored currencies (second-level anchors), it is hardly surprising that researchers classify the US dollar as a systematic factor. The results in this section provide strong evidence for the US dollar being classified as a systematic factor because it is an anchor currency.

**Table 12:** Observed r-squared values versus observed correlations

| Relative Currency Rate | Observed r-squared | Synthetic Currency Correlation | US dollar Correlation | Eurozone euro Correlation |
|---|---|---|---|---|
| **USD** | 0.858 | 0.944 | 1.000 | -0.140 |
| **CNY** | 0.833 | 0.936 | 0.919 | -0.169 |
| **EUR** | 0.808 | 0.878 | -0.140 | 1.000 |
| **CZK** | 0.767 | 0.852 | -0.339 | 0.719 |
| **SGD** | 0.726 | 0.831 | 0.643 | -0.117 |
| **PLN** | 0.702 | 0.763 | -0.473 | 0.335 |
| **AUD** | 0.696 | 0.735 | -0.576 | 0.002 |
| **HUF** | 0.644 | 0.809 | -0.480 | 0.547 |
| **RUB** | 0.628 | 0.498 | -0.166 | -0.205 |
| **CHF** | 0.625 | 0.750 | 0.030 | 0.599 |
| **CAD** | 0.621 | 0.501 | 0.082 | -0.168 |
| **NZD** | 0.618 | 0.662 | -0.425 | 0.082 |
| **MYR** | 0.610 | 0.755 | 0.484 | -0.300 |
| **NOK** | 0.605 | 0.683 | -0.343 | 0.390 |
| **SEK** | 0.604 | 0.780 | -0.383 | 0.593 |
| **PHP** | 0.583 | 0.717 | 0.586 | -0.177 |
| **THB** | 0.581 | 0.716 | 0.575 | -0.184 |
| **KRW** | 0.547 | 0.517 | -0.043 | -0.112 |
| **JPY** | 0.545 | 0.727 | 0.481 | 0.033 |
| **BRL** | 0.516 | 0.577 | -0.242 | -0.396 |
| **MXN** | 0.499 | 0.574 | -0.012 | -0.409 |
| **TRY** | 0.497 | 0.484 | -0.162 | -0.257 |
| **IDR** | 0.486 | 0.553 | 0.044 | -0.251 |
| **INR** | 0.466 | 0.583 | 0.404 | -0.267 |
| **GBP** | 0.466 | 0.528 | 0.217 | 0.216 |
| **CLP** | 0.395 | 0.395 | -0.025 | -0.226 |
| **ILS** | 0.367 | 0.540 | 0.377 | -0.054 |
| **ZAR** | 0.341 | 0.473 | -0.359 | -0.239 |

*Notes:* This table reports the observed r-squared value for each relative currency rate (Systematic r-squared), the observed return correlation between the synthetic currency and the associated target currency (Synthetic Currency Correlation), the observed return correlation between each relative currency rate and the US dollar relative currency rate (US dollar Correlation), and the observed return correlation between each relative currency rate and the Eurozone euro relative currency rate (Eurozone euro Correlation).

## 5 Conclusion

A *synthetic currency* attempts to mimic a *target currency* without having a position in the target currency. A synthetic currency is an optimal portfolio of invariant currency indexes and not a portfolio of currency positions. Invariant currency indexes are tradable multilateral exchange rates and are priced relative to an equally-weighted collection of currencies. The normalisation term of an invariant currency index changes the numéraire of the currency from a bilateral exchange rate to a multilateral exchange rate.

There is a *budget-constraint issue* with the original construction methodology of a synthetic currency, where a *sum-to-$x$ budget constraint* unintentionally creates a non-zero position in the target currency. In general, a non-zero position in the target currency invalidates the concept of synthetic money from being a portfolio of other currencies. The budget-constraint issue can be addressed by replacing the sum-to-$x$ budget constraint with a *sum-to-zero budget constraint*, which correctly enforces a zero position in the target currency.

Once the budget-constraint issue is addressed, investors are faced with the fact that synthetic money is unable to mimic the significant currency-specific (idiosyncratic) movements in the target currencies. For example, for a collection of 28 currencies, the observed return correlations between each synthetic currency and the associated target currency range from 0.403 to 0.945. The observed return correlations are largest for the anchor currencies of the US dollar and the Eurozone euro, which are driven by large exposures to the anchored currencies. In addition, currencies that anchor themselves to the anchor currencies also have high observed return correlations. Although a synthetic currency may not mimic the associated target currency exactly, it does provide transparency on the currencies that are linked, and possibly anchored to, the target currency.

## 6 Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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