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Currency attribution for bilateral exchange rates: A decomposition of the moments

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ABSTRACT

A bilateral exchange rate is the price of a *base currency* in terms of a *quote currency*. In this situation, how do international investors decide the contributions of both the base currency and the quote currency to the mean, variance, skewness, and kurtosis of the movements in each bilateral exchange rate? For a group of currencies, each bilateral exchange rate can be decomposed into a difference between two multilateral exchange rates: a *base-currency* multilateral exchange rate minus a *quote-currency* multilateral exchange rate. In this paper, the decomposition of bilateral exchange rates is used to decompose the moments of the movements in bilateral exchange rates. The result is a quantitative methodology to perform currency attribution, where the mean, variance, skewness, and kurtosis of the movements in each bilateral exchange rate can be attributed to the base currency and the quote currency.

1. Introduction

Currency attribution is a quantitative methodology to decompose the mean, variance, skewness, and kurtosis of the movements in a bilateral exchange rate into the base currency and the quote currency of the bilateral exchange rate. A bilateral exchange rate is the price of a *base currency* in terms of a *quote* (or *numéraire*) currency. For a group of currencies, each bilateral exchange rate can be decomposed into a difference between two multilateral exchange rates: one associated with the *base currency* and one associated with the *quote currency* (Kunkler & MacDonald, 2015). The motivation of this paper is to provide a quantitative methodology to perform currency attribution, which allows international investors a finer level of granularity to uncover the contributions of both the *base currency* and the *quote currency* to the mean, variance, skewness, and kurtosis of movements in bilateral exchange rates.

The *moments* of a statistical distribution describe the shape of that distribution. With respect to bilateral exchange rate movements, the *mean* is the first moment and measures the central location of the bilateral exchange rate movements. After the first moment, the subsequent moments are usually adjusted by the first moment and are called *central moments*. The *variance* is the second central moment and measures the dispersion of the bilateral exchange rate movements relative to the mean. After the second central moment, the subsequent central moments are usually normalised by the second moment and are called *normalised central moments*. *Skewness* is the third normalised central moment of bilateral exchange rate movements and measures the symmetry of the bilateral exchange rate movements relative to the mean. A number of papers have reported significant asymmetry in the movements of bilateral exchange rates (see Broll, 2016; Brunnermeier et al., 2008, pp. 313–347; Jurek, 2014; Patton, 2006). *Kurtosis* is the fourth normalised central moment of bilateral exchange rate movements and measures the tailedness, or peakedness, of the bilateral exchange rate movements relative to the mean. There have been a number papers that reported significant leptokurtic behaviour in the movements of bilateral exchange rates (see Bollerslev, 1987; León et al., 2005; Wali & Manzur, 2013).

The decomposition of a system of N bilateral exchange rates into $N+1$ currency factors was proposed by Mahieu and Schotman (1994). However, the system has infinitely many solutions, since there are fewer equations (N bilateral exchange rates) than unknown variables ($N+1$ currency factors): thus the system of equations is underdetermined. Mahieu and Schotman (1994) assumed structure on the factor loadings and used ordinary least squares on a pooled times series sample to estimate the currency factors. In contrast, Kunkler and MacDonald (2015) extended the system of equations used in Mahieu and Schotman (1994) by introducing an equilibrium condition for the $N+1$ currency factors, which added an extra equation to the system. The equilibrium condition changed the system from underdetermined to one where the number of equations (N bilateral exchange rates plus one equilibrium condition)

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was equal to the number of unknown variables ($N+1$ currency factors). In this situation, the system of equations has a *single unique solution* and the currency factors are tradeable equally-weighted multilateral exchange rates (see Kunkler & MacDonald, 2015). The multilateral exchange rates are independent of the choice of *numéraire* used for the system of N bilateral exchange rates and are equivalent to the *invariant currency indexes* of Hovanov et al. (2004).

This paper contributes to the literature by using the decomposition of bilateral exchange rates of Kunkler and MacDonald (2015) to further decompose the mean, variance, skewness, and kurtosis of the movements in bilateral exchange rates. This creates a quantitative methodology to perform currency attribution, which decomposes the mean, variance, skewness, and kurtosis of bilateral exchange rate movements into the base currency and the quote currency. It is shown that the mean can be decomposed into the difference between two terms associated with the two multilateral exchange rates: two mean terms. In addition, the variance can be decomposed into a weighted sum of three terms associated with the two multilateral exchange rates: two variance terms and one covariance term. Furthermore, the skewness can be decomposed into a weighted sum of four terms associated with the two multilateral exchange rates: two skewness terms and two co-skewness terms. Finally, the kurtosis can be decomposed into a weighted sum of five terms associated with the two multilateral exchange rates: two kurtosis terms and three co-kurtosis terms.

Currency attribution can be applied to the area of hedging exchange rate risk. When international investors buy foreign currencies, equities, or bonds, they indirectly hold positions in the foreign currency (Campbell et al., 2010). Thus, international investors are exposed to exchange rate risk and must decide whether or not to hedge the exchange rate risk. Risk-averse investors prefer positive skew to reduce exposure to extreme left-tail events and lower kurtosis to reduce exposure to extreme events generally (Kim et al., 2014). The standard approach to hedge bilateral exchange rate risk is by trading the associated bilateral exchange rate. However, the decomposition of bilateral exchange rates provides an alternative approach, where investors can hedge bilateral exchange rate risk by trading either or both of the two associated multilateral exchange rates: one for the *base currency* and one for the *quote currency*. In similar work, two optimal hedge ratios were estimated to adjust the currency exposures of international equity investors (Kunkler, 2021).

2. Material and methods

2.1. Sample moments

The sample moments are used to describe the shape of the unknown moments of a data sample. The kth sample moment of a data sample is given by:

$$m^k = E\left[x^k\right] = \frac{1}{N}\sum_{i=1}^{N} x_i^k \tag{1}$$

where $k=1,2,3,4,\ldots$; m^k is the kth sample moment; N is the sample size; and E is the expectation operator. For $k>1$, the kth central sample moment of a data sample is given by:

$$\overline{m}^k = E\left[(x-\mu)^k\right] = \frac{1}{N}\sum_{i=1}^{N} (x_i-\mu)^k \tag{2}$$

where $k=2,3,4,\ldots$; $\overline{m}^k$ is the kth central sample moment; $\mu = m^1$ is the first sample moment in Eq. (1); and N is the sample size. For $k>2$, the kth normalised central sample moment of a data sample is given by:

$$\widehat{m}^k = E\left[\frac{(x-\mu)^k}{\sigma^k}\right] = \frac{1}{N\sigma^k}\sum_{i=1}^{N} (x_i-\mu)^k \tag{3}$$

where $k=3,4,\ldots$; $\widehat{m}^k$ is the kth normalised central sample moment; $\sigma = \sqrt{\overline{m}^2}$ is the square root of the second central sample moment in Eq. (2); $\mu = m^1$ is the first sample moment in Eq. (1); and N is the sample size.

The central sample moments in Eq. (2) and the normalised central sample moments in Eq. (3) are biased, as they do not account for the degrees of freedom lost by estimating the moments used within the central sample moments or the normalised central sample moments. However, there are well known corrections that remove the bias. For example, the well-known *Bessel's correction* for the variance is:

$$\overline{m}^{2*} = \frac{N}{N-1}\overline{m}^2 \tag{4}$$

where $\overline{m}^{2*}$ is an unbiased estimator of the second central sample moment. For simplicity, the biased versions will be used throughout this paper.

2.2. Decomposition of bilateral exchange rates

A bilateral exchange rate is the price of a *base currency* in terms of a *quote* (or *numéraire*) *currency*. Bilateral exchange rates are typically modelled in log terms to overcome the Siegel Paradox (see Taylor & Sarno, 1998). Consequently, all exchange rates throughout this paper will be written in log terms. For a group of N currencies, let $p_{i/j}(t)$ represent the ith/jth bilateral exchange rate at time t, which is the ith (*base*) currency in terms of the jth (*quote*) currency, where $i,j=1,\ldots,N$ and $t=0,\ldots,T$. In addition, let $\Delta p_{i/j}(t)$ represent the log return of the ith/jth bilateral exchange rate at time t, where $i,j=1,\ldots,N$ and $t=1,\ldots,T$.

The movements in a multilateral exchange rate are calculated by an equally-weighted sum of the movements in a group of bilateral exchange rates given by:

$$\Delta p_i(t) = \frac{1}{N}\sum_{j=1}^{N} \Delta p_{i/j}(t) \tag{5}$$

where $i=1,\ldots,N$; $t=1,\ldots,T$; $\Delta p_i(t)$ is the log return of the ith multilateral exchange rate; and $\Delta p_{i/j}(t)$ is the log return of the ith/jth bilateral exchange rate (see Kunkler & MacDonald, 2015). The movements in a multilateral exchange rate in Eq. (5) are priced in terms of an equally-weighted basket of the N currencies. It should be noted that the multilateral exchange rates in Eq. (5) are independent of the choice of *numéraire* used for the group of N bilateral exchange rates (see Hovanov et al., 2004; Kunkler, 2022b; Kunkler & MacDonald, 2015).

The movements in each bilateral exchange rate can be decomposed into a difference between the movements in a *base-currency* multilateral exchange rate and the movements in a *quote-currency* multilateral exchange rate by:

$$\Delta p_{i/j}(t) = \Delta p_i(t) - \Delta p_j(t) \tag{6}$$

where $i,j=1,\ldots,N$; $t=1,\ldots,T$; $\Delta p_{i/j}(t)$ is the log return of the ith/jth bilateral exchange rate; $\Delta p_i(t)$ is the log return of the ith (*base-currency*) multilateral exchange rate; and $\Delta p_j(t)$ is the log return of the jth (*quote-currency*) multilateral exchange rate (see Kunkler & MacDonald, 2015).

2.3. Mean

The mean is the first sample moment of a data sample and measures the central location of the data sample. Using Eq. (1), the mean of the movements in a bilateral exchange rate is:

$$\mu_{i/j} = E\left[\Delta p_{i/j}\right] = \frac{1}{T}\sum_{t=1}^{T} \Delta p_{i/j}(t) \tag{7}$$

where $i,j=1,\ldots,N$; $\mu_{i/j}$ is the mean of the log returns of the ith/jth bilateral exchange rate; and $\Delta p_{i/j}(t)$ is the log return of the ith/jth bilateral exchange rate. Similarly, the mean of the movements in a multilateral exchange rate in is:

$$\mu_i = E[\Delta p_i] = \frac{1}{T}\sum_{t=1}^{T}\Delta p_i(t) \tag{8}$$

where $i = 1,\ldots,N$; μ_i is the mean of the log returns of the ith multilateral exchange rate; and $\Delta p_i(t)$ is the log return of the ith multilateral exchange rate.

The mean of the movements in a bilateral exchange rate can be decomposed by substituting Eq. (6) for $\Delta p_{i/j}(t)$ into Eq. (7) to give:

$$\begin{aligned}\mu_{i/j} &= \frac{1}{T}\sum_{t=1}^{T}\left(\Delta p_i(t) - \Delta p_j(t)\right)\\ &= \frac{1}{T}\sum_{t=1}^{T}\Delta p_i(t) - \frac{1}{T}\sum_{t=1}^{T}\Delta p_j(t)\\ &= \mu_i - \mu_j\end{aligned} \tag{9}$$

where $i,j = 1,\ldots,N$; $\mu_{i/j}$ is the mean of the log returns of the ith/ jth bilateral exchange rate; $\Delta p_i(t)$ is the log return of the ith (*base-currency*) multilateral exchange rate; and $\Delta p_j(t)$ is the log return of the jth (*quote-currency*) multilateral exchange rate; μ_i is the mean of the log returns of the ith (*base-currency*) multilateral exchange rate; and μ_j is the mean of the log returns of the jth (*quote-currency*) multilateral exchange rate.

2.4. Variance

The variance is the second central sample moment of a data sample and measures the dispersion of the data sample relative to the mean. The variance of the movements in a bilateral exchange rate is given by:

$$\sigma_{i/j}^2 = E\left[\left(\Delta p_{i/j} - \mu_{i/j}\right)^2\right] = \frac{1}{T}\sum_{t=1}^{T}\left(\Delta p_{i/j}(t) - \mu_{i/j}\right)^2 \tag{10}$$

where $i,j = 1,\ldots,N$; $\sigma_{i/j}^2$ is the variance of the log returns of the ith/ jth bilateral exchange rate; $\Delta p_{i/j}(t)$ is the log return of the ith/ jth bilateral exchange rate; and $\mu_{i/j}$ is the mean of the log returns of the ith/ jth bilateral exchange rate. The standard deviation is the square root of the variance:

$$\sigma_{i/j} = \sqrt{\sigma_{i/j}^2} \tag{11}$$

where $\sigma_{i/j}$ is the standard deviation of the log returns of the ith/ jth bilateral exchange rate; and $\sigma_{i/j}^2$ is the variance of the log returns of the ith/jth bilateral exchange rate.

The variance of the movements in a multilateral exchange rate is given by:

$$\sigma_i^2 = \mathrm{E}\left[(\Delta p_i - \mu_i)^2\right] = \frac{1}{T}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)^2 \tag{12}$$

where $i,j = 1,\ldots,N$; σ_i^2 is the variance of the log returns of the ith multilateral exchange rate; $\Delta p_i(t)$ is the log return of the ith multilateral exchange rate; and μ_i is the mean of the log returns of the ith multilateral exchange rate. The standard deviation is the square root of the variance:

$$\sigma_i = \sqrt{\sigma_i^2} \tag{13}$$

where σ_i is the standard deviation of the log returns of the ith multilateral exchange rate; and σ_i^2 is the variance of the log returns of the ith multilateral exchange rate.

The variance of the movements in a bilateral exchange rate can be decomposed by substituting Eq. (6) for $\Delta p_{i/j}(t)$ and Eq. (9) for $\mu_{i/j}$ into Eq. (10) to give:

$$\begin{aligned}\sigma_{i/j}^2 &= \frac{1}{T}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)^2 + \frac{1}{T}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)^2\\ &\quad -2\frac{1}{T}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)(\Delta p_i(t) - \mu_i)\\ &= \sigma_i^2 + \sigma_j^2 - 2\sigma_{i,j}\\ &= \sigma_i^2 + \sigma_j^2 - 2\sigma_i\sigma_j\rho_{i,j}\end{aligned} \tag{14}$$

where $i,j = 1,\ldots,N$; $\sigma_{i/j}^2$ is the variance of the log returns of the ith/jth bilateral exchange rate; σ_i^2 is the variance of the log returns of ith multilateral exchange rate; σ_j^2 is the variance of the log returns of jth multilateral exchange rate; $\sigma_{i,j} = \frac{1}{T}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)(\Delta p_i(t) - \mu_i)$ is the covariance between the log returns of ith multilateral exchange rate and the log returns of the jth multilateral exchange rate; and $\rho_{i,j} = \sigma_{i,j}/(\sigma_i\sigma_j)$ is the correlation between the log returns of ith multilateral exchange rate and the log returns of the jth multilateral exchange rate.

2.5. Skewness

Skewness is the third central normalised sample moment of a data sample and measures the symmetry of the data sample relative to the mean and normalised by the standard derivation to the power of three. The skewness in the movements of a bilateral exchange rate is given by:

$$s_{i/j} = \mathrm{E}\left[\frac{\left(\Delta p_{i/j} - \mu_{i/j}\right)^3}{\sigma_{i/j}^3}\right] = \frac{1}{T\sigma_{i/j}^3}\sum_{t=1}^{T}\left(\Delta p_{i/j}(t) - \mu_{i/j}\right)^3 \tag{15}$$

where $i,j = 1,\ldots,N$; $s_{i/j}$ is the skewness in the log returns of the ith/jth bilateral exchange rate; $\Delta p_{i/j}(t)$ is the log return of the ith/jth bilateral exchange rate; $\mu_{i/j}$ is the mean of the log returns of the ith/jth bilateral exchange rate; and $\sigma_{i/j}$ is the standard deviation of the log returns of the ith/jth bilateral exchange rate. The skewness in the movements of the multilateral exchange rate is given by:

$$s_i = E\left[\frac{(\Delta p_i - \mu_i)^3}{\sigma_i^3}\right] = \frac{1}{T\sigma_i^3}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)^3 \tag{16}$$

where $i,j = 1,\ldots,N$; s_i is the skewness in the log returns of the ith multilateral exchange rate; $\Delta p_i(t)$ is the log return of the ith multilateral exchange rate; μ_i is the mean of the log returns of the ith multilateral exchange rate; and σ_i is the standard deviation of the log returns of the ith multilateral exchange rate.

The co-skewness in the movements of three multilateral exchange rates measures the symmetry of the multilateral exchange rates relative to their average. The co-skewness in the movements of three multilateral exchange rates is given by:

$$\begin{aligned}cs_{i,j,k} &= \mathrm{E}\left[\frac{(\Delta p_i - \mu_i)}{\sigma_i}\frac{(\Delta p_j - \mu_j)}{\sigma_j}\frac{(\Delta p_k - \mu_k)}{\sigma_k}\right]\\ &= \frac{1}{T\sigma_i\sigma_j\sigma_k}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)(\Delta p_j(t) - \mu_j)(\Delta p_k(t) - \mu_k)\end{aligned} \tag{17}$$

where $i,j = 1,\ldots,N$; $cs_{i,j,k}$ is the co-skewness in the log returns of the ith, jth, and kth multilateral exchange rates; $\Delta p_i(t)$, $\Delta p_j(t)$, and $\Delta p_k(t)$ are the log returns of the ith, jth, and kth multilateral exchange rate, respectively; μ_i, μ_j, and μ_k are the means of the log returns of the ith, jth, and kth multilateral exchange rates, respectively; and σ_i, σ_j, and σ_k are the standard deviations of the log returns of the ith, jth, and kth multilateral exchange rates, respectively. Note that when all the multilateral exchange rates in Eq. (17) are the same, the co-skewness is equivalent to the skewness, with $cs_{i,i,i} = s_i$.

The skewness of the movements in a bilateral exchange rate can be decomposed by substituting Eq. (6) for $\Delta p_{i/j}(t)$ and Eq. (9) for $\mu_{i/j}$ into Eq. (15) to give:

$$s_{i/j} = \frac{1}{\sigma_{i/j}^3}\left(\frac{1}{T}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)^3 - 3\frac{1}{T}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)^2(\Delta p_j(t) - \mu_j)\right.$$
$$\left. + 3\frac{1}{T}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)(\Delta p_j(t) - \mu_j)^2 - \frac{1}{T}\sum_{t=1}^{T}(\Delta p_j(t) - \mu_j)^3\right)$$
$$= \frac{1}{\sigma_{i/j}^3}\left(\sigma_i^3 s_i - 3\sigma_i^2\sigma_j cs_{i,i,j} + 3\sigma_i\sigma_j^2 cs_{i,j,j} - \sigma_j^3 s_j\right)$$
$$= w_i s_i - w_{i,i,j}cs_{i,i,j} + w_{i,j,j}cs_{i,j,j} - w_j s_j \tag{18}$$

where $i,j = 1,\ldots,N$; $s_{i/j}$ is the skewness of the ith/jth bilateral exchange rate; $\sigma_{i/j}$ is the standard deviation of the ith/jth bilateral exchange rate; s_i is the skewness of the ith multilateral exchange rate; σ_i is the standard deviation of the ith multilateral exchange rate; s_j is the skewness of the jth multilateral exchange rate; σ_j is the standard deviation of the jth multilateral exchange rate; $cs_{i,i,j}$ is the co-skewness of the ith, ith, and jth multilateral exchange rates; $cs_{i,j,j}$ is the co-skewness of the ith, jth, and jth multilateral exchange rates; $w_i = \sigma_i^3/\sigma_{i/j}^3$ is the weight associated with s_i; $w_{i,i,j} = 3\sigma_i^2\sigma_j/\sigma_{i/j}^3$ is the weight associated with $cs_{i,i,j}$; $w_{i,j,j} = 3\sigma_i\sigma_j^2/\ \sigma_{i/j}^3$ is the weight associated with $cs_{i,j,j}$; and $w_j = \sigma_j^3/\sigma_{i/j}^3$ is the weight associated with s_j.

2.6. Kurtosis

Kurtosis is the fourth central normalised sample moment of a data sample and measures the extremity of the data sample relative to the mean and normalised by the standard derivation to the power of three. The kurtosis in the movements of a bilateral exchange rate is given by:

$$k_{i/j} = \mathrm{E}\left[\frac{\left(\Delta p_{i/j} - \mu_{i/j}\right)^4}{\sigma_{i/j}^4}\right] = \frac{1}{T\sigma_{i/j}^4}\sum_{t=1}^{T}\left(\Delta p_{i/j}(t) - \mu_{i/j}\right)^4 \tag{19}$$

where $i,j = 1,\ldots,N$; $k_{i/j}$ is the kurtosis in the log returns of the ith/ jth bilateral exchange rate; $\Delta p_{i/j}(t)$ is the log return of the ith/ jth bilateral exchange rate; $\mu_{i/j}$ is the mean of the log returns of the ith/ jth bilateral exchange rate; and $\sigma_{i/j}$ is the standard deviation of the log returns of the ith/jth bilateral exchange rate. The kurtosis in the movements of the multilateral exchange rate is given by:

$$k_i = \mathrm{E}\left[\frac{(\Delta \boldsymbol{p}_i - \mu_i)^4}{\sigma_i^4}\right] = \frac{1}{T\sigma_i^4}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)^4 \tag{20}$$

where $i, j = 1, \ldots, N$; k_i is the kurtosis in the log returns of the ith multilateral exchange rate; $\Delta p_i(t)$ is the log return of the ith multilateral exchange rate; μ_i is the mean of the log returns of the ith multilateral exchange rate; and σ_i is the standard deviation of the log returns of the ith multilateral exchange rate.

The co-kurtosis in the movements of three multilateral exchange rates measures the extremity of the multilateral exchange rates relative to their average. The co-kurtosis in the movements of four multilateral exchange rates is given by:

$$ck_{i,j,k,l} = \mathrm{E}\left[\frac{(\Delta p_i - \mu_i)}{\sigma_i}\frac{(\Delta p_j - \mu_j)}{\sigma_j}\frac{(\Delta p_k - \mu_k)}{\sigma_k}\frac{(\Delta p_l - \mu_l)}{\sigma_l}\right]$$
$$= \frac{1}{T\sigma_i\sigma_j\sigma_k\sigma_l}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)(\Delta p_j(t) - \mu_j)(\Delta p_k(t) - \mu_k)(\Delta p_l(t) - \mu_l) \tag{21}$$

where $i,j = 1,\ldots,N$; $ck_{i,j,k,l}$ is the co-kurtosis in the log returns of the ith, jth, kth, and lth multilateral exchange rates; $\Delta p_i(t)$, $\Delta p_j(t)$, $\Delta p_k(t)$, and $\Delta p_l(t)$ are the log returns of the ith, jth, kth, and lth multilateral exchange rate, respectively; μ_i, μ_j, μ_k, and μ_l are the means of the log returns of the ith, jth, kth, and lth multilateral exchange rates, respectively; and σ_i, σ_j, σ_k, and σ_l are the standard deviations of the log returns of the ith, jth, kth, and lth multilateral exchange rates, respectively. Note that when all the multilateral exchange rates in Eq. (21) are the same, the co-kurtosis is equivalent to the kurtosis, with $ck_{i,i,i,i} = k_i$.

The kurtosis of the movements in a bilateral exchange rate can be decomposed by substituting Eq. (6) for $\Delta p_{i/j}(t)$ and Eq. (9) for $\mu_{i/j}$ into Eq. (19) to give:

$$k_{i/j} = \frac{1}{\sigma_{i/j}^4}\left(\frac{1}{T}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)^4 - 4\frac{1}{T}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)^3(\Delta p_j(t) - \mu_j)\right.$$
$$+6\frac{1}{T}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)^2(\Delta p_j(t) - \mu_j)^2$$
$$\left. -4\frac{1}{T}\sum_{t=1}^{T}(\Delta p_i(t) - \mu_i)(\Delta p_j(t) - \mu_j)^3 + \frac{1}{T}\sum_{t=1}^{T}(\Delta p_j(t) - \mu_j)^4\right)$$
$$= \frac{1}{\sigma_{i/j}^4}\left(\sigma_i^4 k_i - 4\sigma_i^3\sigma_j ck_{i,i,i,j} + 6\sigma_i^2\sigma_j^2 ck_{i,i,j,j} - 4\sigma_i\sigma_j^3 ck_{i,j,j,j} + \sigma_j^4 k_j\right)$$
$$= w_i k_i - w_{i,i,i,j}ck_{i,i,i,j} + w_{i,i,j,j}ck_{i,i,j,j} - w_{i,j,j,j}ck_{i,j,j,j} + w_j k_j \tag{22}$$

where $i,j = 1,\ldots,N$; $k_{i/j}$ is the kurtosis of the ith/jth bilateral exchange rate; $\sigma_{i/j}$ is the standard deviation of the ith/jth bilateral exchange rate; k_i is the kurtosis of the ith multilateral exchange rate; σ_i is the standard deviation of the ith multilateral exchange rate; k_j is the kurtosis of the jth multilateral exchange rate; σ_j is the standard deviation of the jth multilateral exchange rate; $ck_{i,i,i,j}$ is the co-kurtosis of the ith, ith, ith, and jth multilateral exchange rates; $ck_{i,i,j,j}$ is the co-kurtosis of the ith, ith, jth, and jth multilateral exchange rates; $ck_{i,j,j,j}$ is the co-kurtosis of the ith, jth, jth, and jth multilateral exchange rates; $w_i = \sigma_i^4/\sigma_{i/j}^4$ is the weight associated with k_i; $w_{i,i,i,j} = 4\sigma_i^3\sigma_j/\sigma_{i/j}^4$ is the weight associated with $ck_{i,i,i,j}$; $w_{i,i,j,j} = 6\sigma_i^2\sigma_j^2/\sigma_{i/j}^4$ is the weight associated with $ck_{i,i,j,j}$; $w_{i,j,j,j} = 4\sigma_i\sigma_j^3/\sigma_{i/j}^4$ is the weight associated with $ck_{i,j,j,j}$; and $w_j = \sigma_j^4/\sigma_{i/j}^4$ is the weight associated with k_j.

2.7. Summary

Currency attribution can be performed on the moments of the movements of a bilateral exchange rate, such as the mean, variance, skewness, and kurtosis. The mean can be decomposed into the difference between the mean of the movements of the *base-currency* multilateral exchange rate and the mean of the movements of the *quote-currency* multilateral exchange rate.

In addition, the variance can be decomposed into the variance of the movements of the *base-currency* multilateral exchange rate plus the variance of the movements of *quote-currency* multilateral exchange rate minus twice the covariance between the movements of the *base-currency* multilateral exchange rate and the variance of the movements of *quote-currency* multilateral exchange rate.

Furthermore, the skewness can be decomposed into a weighted difference between the skewness of the movements of the *base-currency* multilateral exchange rate and the skewness of the movements of *quote-currency* multilateral exchange rate minus a weighted difference between the co-skewness of the movements of the *base-currency* multilateral exchange rate and the co-skewness of the movements of *quote-currency* multilateral exchange rate.

Finally, the kurtosis can be decomposed into a weighted sum between the kurtosis of the movements of the *base-currency* multilateral exchange rate and the kurtosis of the movements of *quote-currency* multilateral exchange rate, together with a weighted combination of three co-kurtosis terms between the movements of the *base-currency* multilateral exchange rate and the movements of *quote-currency* multilateral exchange rate.

The decomposition of the moments allow international investors a finer level of granularity to uncover the contributions of both the *base currency* and the *quote currency* to both the moments of the movements of a bilateral exchange rate.

3. Empirical analysis

3.1. Data sample

The data sample consists of monthly data from Bloomberg for a group of nine bilateral exchange rates against the US dollar, beginning on the January 1, 2000 and ending on the December 31, 2021. This results in a group of ten ($N = 10$) currencies, namely, the US dollar (USD), the Eurozone euro (EUR), the Japanese yen (JPY), the Australian dollar (AUD), the New Zealand dollar (NZD), the Swiss franc (CHF), the British pound (GBP), the Canadian dollar (CAD), the Norwegian krone (NOK), and the Swedish krona (SEK).

3.2. Multilateral exchange rates

Table 1 reports observed mean, standard deviation, skewness and kurtosis for the log returns of each multilateral exchange rate. The largest annualised mean is 2.19% for the Swiss franc (CHF), and the smallest annualised mean is −1.14% for the British pound (GBP). The highest annualised standard deviation is 9.40% for the Japanese yen (JPY), and the lowest annualised standard deviation is 4.54% for the Eurozone euro (EUR). Both the Japanese yen and the Swiss franc have highly significant positive observed skewness. In contrast, movements in the Australian dollar (AUD), the New Zealand dollar (NZD), the British pound (GBP), the Canadian dollar (CAD), and the Norwegian krone (NOK) have significant negative observed skewness. All currencies have highly significant positive observed kurtosis.

3.3. US dollar bilateral exchange rates

US dollar bilateral exchange rates are chosen to provide consistent examples of currency attribution for one set of bilateral exchange rates. However, it is straightforward to use bilateral exchange rates that are priced in terms of another *quote* (or *numéraire*) currency. Table 2 reports mean, standard deviation, skewness and kurtosis for the log returns of each US dollar bilateral exchange rate. The largest annualised mean is 2.53% for the Swiss franc (CHF), and the smallest annualised mean is −0.80% for the British pound (GBP). The highest annualised standard deviation is 12.98% for the New Zealand dollar (NZD), and the lowest annualised standard deviation is 8.64% for the British pound (GBP). There are no bilateral exchange rates with positive observed skewness. In contrast, movements in the Australian dollar (AUD), the New Zealand dollar (NZD), the British pound (GBP), the Canadian dollar (CAD), and the Norwegian krone (NOK) all have significant negative observed skewness. All US dollar bilateral exchange rates have highly significant positive observed kurtosis, with the Canadian dollar (CAD) having the largest positive observed kurtosis of 8.07. Thus, there are contrasting results for the US dollar bilateral exchange rates in Table 2 compared to the multilateral exchange rates in Table 1. These results will be compared further in the sections below.

Table 1
Multilateral exchange rates.

	Mean	Std. Deviation	Skewness	Kurtosis
USD	−0.33%	7.42%	0.251	4.38**
EUR	0.20%	4.54%	−0.006	4.15**
JPY	−0.86%	9.38%	1.210**	8.98**
GBP	−1.14%	6.13%	−1.084**	7.35**
AUD	0.15%	6.81%	−0.654**	4.81**
NZD	0.91%	7.84%	−0.322*	4.17**
CHF	2.19%	5.96%	1.482**	10.22**
CAD	0.27%	6.03%	−0.416**	3.78**
NOK	−0.77%	6.51%	−0.396**	3.86**
SEK	−0.62%	5.47%	−0.027	3.89**

Notes: Table 1 reports both the observed mean, observed standard deviation, observed skewness and observed kurtosis for the multilateral exchange rates. Significance levels for skewness and kurtosis are denoted by * for 5%, and ** for 1%.

Table 2
US dollar bilateral exchange rates.

	Mean	Std. Deviation	Skewness	Kurtosis
EUR	0.53%	9.63%	−0.267	4.55**
JPY	−0.53%	8.92%	−0.071	3.67**
GBP	−0.80%	8.62%	−0.349*	4.59**
AUD	0.48%	12.32%	−0.708**	6.01**
NZD	1.24%	12.96%	−0.378*	4.30**
CHF	2.53%	9.82%	0.148	4.74**
CAD	0.61%	8.77%	−0.921**	8.07**
NOK	−0.43%	11.66%	−0.344*	4.25**
SEK	−0.28%	11.30%	−0.115	3.97**

Notes: Table 2 reports both the observed mean, observed standard deviation, observed skewness and observed kurtosis for the US dollar bilateral exchange rates. Significance levels for skewness and kurtosis are denoted by * for 5%, and ** for 1%.

3.4. Mean

The mean is the first sample moment of a data sample and measures the central location of the data sample. Table 3 reports both the observed means for the log returns of the US dollar bilateral exchange rates and the observed means for the log returns of the multilateral exchange rates. Fig. 1 displays the observed means for both the US dollar bilateral exchange rates and the multilateral exchange rates. The observed means in the movements of the US dollar bilateral exchange rates are very intuitive when using the observed means in the movements of the multilateral exchange rates. The observed means in the movements of the US dollar bilateral exchange rates are simply the difference between the observed mean of the base-currency multilateral exchange rate minus the observed mean of the US dollar multilateral exchange rate. For example, the mean of the Australian dollar (AUD)/US dollar (USD) bilateral exchange rate is 0.48%. The decomposition of the mean in Eq. (9) can be used to understand this result by:

$$\begin{aligned}\mu_{AUD/USD} &= \mu_{AUD} - \mu_{USD} \\ &= (0.15\%) - (-0.33\%) \\ &= 0.48\%\end{aligned} \tag{23}$$

Table 3
Decomposition of the mean.

i	$\mu_{i/USD}$	μ_i	μ_{USD}
USD		−0.33%	−0.33%
EUR	0.53%	0.20%	−0.33%
JPY	−0.53%	−0.86%	−0.33%
GBP	−0.80%	−1.14%	−0.33%
AUD	0.48%	0.15%	−0.33%
NZD	1.24%	0.91%	−0.33%
CHF	2.53%	2.19%	−0.33%
CAD	0.61%	0.27%	−0.33%
NOK	−0.43%	−0.77%	−0.33%
SEK	−0.28%	−0.62%	−0.33%

Notes: Table 3 reports the decomposition of observed mean for the log returns of the US dollar bilateral exchange rates, together with the observed mean for the log returns of the multilateral exchange rates and the observed mean (repeated) for the log returns of the US dollar multilateral exchange rate.

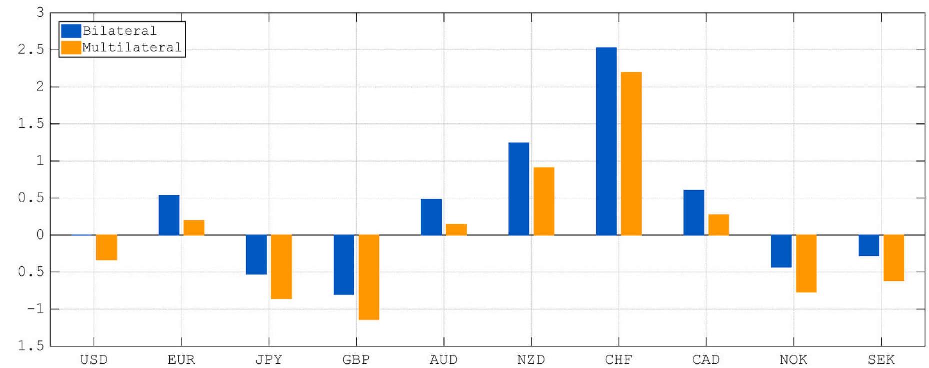

Fig. 1. Mean of both the US dollar bilateral, and the multilateral, exchange rates. *Notes:* Fig. 1 displays the observed mean in the movements of both the US dollar bilateral exchange rates and the multilateral exchange rates.

where $\mu_{AUD/USD}$ is the mean of the Australian dollar (AUD)/US dollar (USD) bilateral exchange; μ_{AUD} is the mean of the Australian dollar (AUD) multilateral exchange rate; and μ_{USD} is the mean of the US dollar (USD) multilateral exchange rate.

For US investors, buying a *base* (foreign) *currency* involves selling the US dollar. Historically, the negative annualised mean of − 0.33% for the US dollar multilateral exchange rate when subtracted from the observed mean of the base-currency multilateral exchange rate shifts the observed mean of the US dollar bilateral exchange rates up by an annualised 0.33%.

3.5. Variance

The variance is the second central sample moment of a data sample and measures the dispersion of the data sample relative to the mean. Table 4 reports both the observed standard deviations for the log returns of the US dollar bilateral exchange rates, the observed standard deviations for the log returns of the multilateral exchange rates, and the observed correlation between the log returns of each multilateral exchange rate and the log returns of the US dollar multilateral exchange rates. Fig. 2 displays the observed standard deviations for both the US dollar bilateral exchange rates and the multilateral exchange rates. The observed standard deviations in the movements of the bilateral exchange rates are less intuitive than the observed mean in the previous section. For example, the observed standard deviation for the movements of the Japanese yen/US dollar (JPY/USD) bilateral exchange rate is less than the observed standard deviation for the movements of the Japanese yen (JPY) multilateral exchange rate. The decomposition of the variance in Eq. (14) can be used to understand this result by:

Table 4
Decomposition of the standard deviations.

i	$\sigma_{i/USD}$	σ_i	σ_{USD}	$\rho_{i,USD}$
USD		7.42%	7.42%	1.000
EUR	9.63%	4.54%	7.42%	−0.256**
JPY	8.92%	9.38%	7.42%	0.456**
GBP	8.62%	6.13%	7.42%	0.202**
AUD	12.32%	6.81%	7.42%	−0.498**
NZD	12.96%	7.84%	7.42%	−0.442**
CHF	9.82%	5.96%	7.42%	−0.068
CAD	8.77%	6.03%	7.42%	0.161**
NOK	11.66%	6.51%	7.42%	−0.400**
SEK	11.30%	5.47%	7.42%	−0.527**

Notes: Table 4 reports the observed standard deviations for the log returns of the US dollar bilateral exchange rates, the observed standard deviations for the log returns of the multilateral exchange rates, the observed standard deviation (repeated) for the log returns of the US dollar multilateral exchange rate, and the observed correlation between the log returns of each multilateral exchange rate and the log returns of the US dollar multilateral exchange rates.

$$\begin{aligned}\sigma_{JPY/USD} &= \sqrt{\sigma^2_{JPY} + \sigma^2_{USD} - 2\sigma_{JPY}\sigma_{USD}\rho_{JPY,USD}} \\ &= \sqrt{9.38\%^2 + 7.42\%^2 - 2 \times 9.38\% \times 7.42\% \times 0.456} \\ &= \sqrt{0.88\% + 0.55\% - 0.63\%} \\ &= 8.92\%\end{aligned} \tag{24}$$

where $\sigma_{JPY/USD}$ is the standard deviation of the Japanese yen/US dollar (JPY/USD) bilateral exchange; σ_{JPY} is the standard deviation of the Japanese yen (JPY) multilateral exchange rate; σ_{USD} is the standard deviation of the US dollar (USD) multilateral exchange rate; and $\rho_{JPY,USD}$ is the correlation between the Japanese yen (JPY) multilateral exchange rate and the US dollar (USD) multilateral exchange rate. The positive correlation of 0.456 when subtracted from the sum of the two variance terms shifts the observed standard deviation of the Japanese yen/US dollar (JPY/USD) bilateral exchange downwards.

The opposite happens for the multilateral exchange rates that are negatively correlated with the US dollar (USD) multilateral exchange rate. For example, the observed correlation between the movements of the Australian dollar (AUD) multilateral exchange rate and the movements of the US dollar (USD) multilateral exchange rate is − 0.498. The decomposition of the variance in Eq. (14) can be used to decompose the standard deviation of the movements of the Australian dollar/US dollar (AUD/USD) bilateral exchange rate to give:

$$\begin{aligned}\sigma_{AUD/USD} &= \sqrt{\sigma^2_{AUD} + \sigma^2_{USD} - 2\sigma_{AUD}\sigma_{USD}\rho_{AUD,USD}} \\ &= \sqrt{6.81\%^2 + 7.42\%^2 - 2 \times 6.81\% \times 7.42\% \times (-0.498)} \\ &= \sqrt{0.46\% + 0.55\% + 0.50\%} \\ &= 12.32\%\end{aligned}$$

where $\sigma_{AUD/USD}$ is the standard deviation of the AUD/USD bilateral exchange; σ_{AUD} is the standard deviation of the Australian dollar (AUD) multilateral exchange rate; σ_{USD} is the standard deviation of the US dollar (USD) multilateral exchange rate; and $\rho_{AUD,USD}$ is the correlation between the Australian dollar (AUD) multilateral exchange rate and the US dollar (USD) multilateral exchange rate.

In summary, when two currencies are positively correlated the observed standard deviation of the associated bilateral exchange rate is shifted downwards. In contrast, when two currencies are negative correlated the observed standard deviation of the associated bilateral exchange rate is shifted upwards.

3.5.1. Correlation

The correlation between the log returns of *base-currency* multilateral exchange rate and the log returns of the *quote-currency* multilateral exchange rate plays an integral role in the magnitude of the variance of the associated bilateral exchange rate. Table 5 reports the correlation matrix for the log returns of the multilateral exchange rates. Fig. 3 displays the

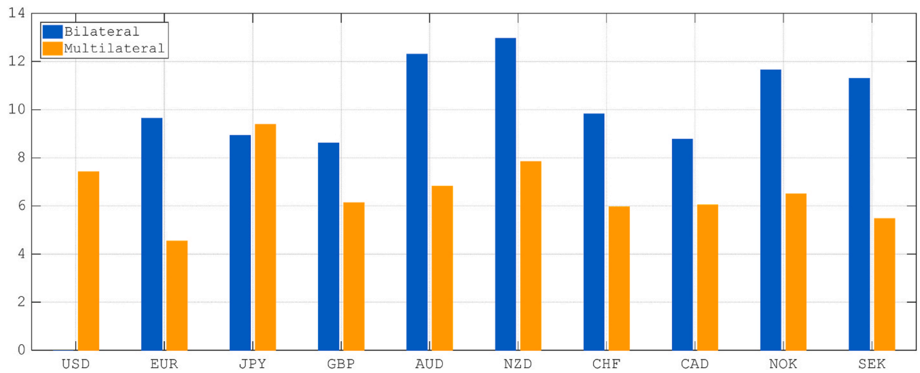

Fig. 2. Std. deviation of both the US dollar bilateral, and the multilateral, exchange rates. *Notes:* Fig. 2 displays the observed standard deviation in the movements of both the US dollar bilateral exchange rates and the multilateral exchange rates.

Table 5
Correlation matrix for the movements of the multilateral exchange rates.

	USD	EUR	JPY	GBP	AUD	NZD	CHF	CAD	NOK	SEK	AVG
USD	1.000	−0.256**	0.456**	0.202**	−0.498**	−0.442**	−0.068	0.161**	−0.400**	−0.527**	−0.152
EUR	−0.256**	1.000	−0.207**	−0.078	−0.232**	−0.174**	0.395**	−0.416**	0.128*	0.375**	−0.052
JPY	0.456**	−0.207**	1.000	−0.150*	−0.450**	−0.385**	0.148*	−0.153*	−0.395**	−0.403**	−0.171
GBP	0.202**	−0.078	−0.150*	1.000	−0.265**	−0.232**	−0.151*	0.003	−0.086	−0.145*	−0.100
AUD	−0.498**	−0.232**	−0.450**	−0.265**	1.000	0.525**	−0.368**	0.170**	0.074	0.065	−0.109
NZD	−0.442**	−0.174**	−0.385**	−0.232**	0.525**	1.000	−0.183**	−0.047	−0.159**	0.018	−0.120
CHF	−0.068	0.395**	0.148*	−0.151*	−0.368**	−0.183**	1.000	−0.497**	−0.129*	0.012	−0.093
CAD	0.161**	−0.416**	−0.153*	0.003	0.170**	−0.047	−0.497**	1.000	−0.051	−0.258**	−0.121
NOK	−0.400**	0.128*	−0.395**	−0.086	0.074	−0.159**	−0.129*	−0.051	1.000	0.356**	−0.074
SEK	−0.527**	0.375**	−0.403**	−0.145*	0.065	0.018	0.012	−0.258**	0.356**	1.000	−0.056
AVG	−0.152	−0.052	−0.171	−0.100	−0.109	−0.120	−0.093	−0.121	−0.074	−0.056	−0.105

Notes: Table 5 reports the observed correlation for the movements of the multilateral exchange rates. Significance levels for correlations are denoted by * for 5%, and ** for 1%.

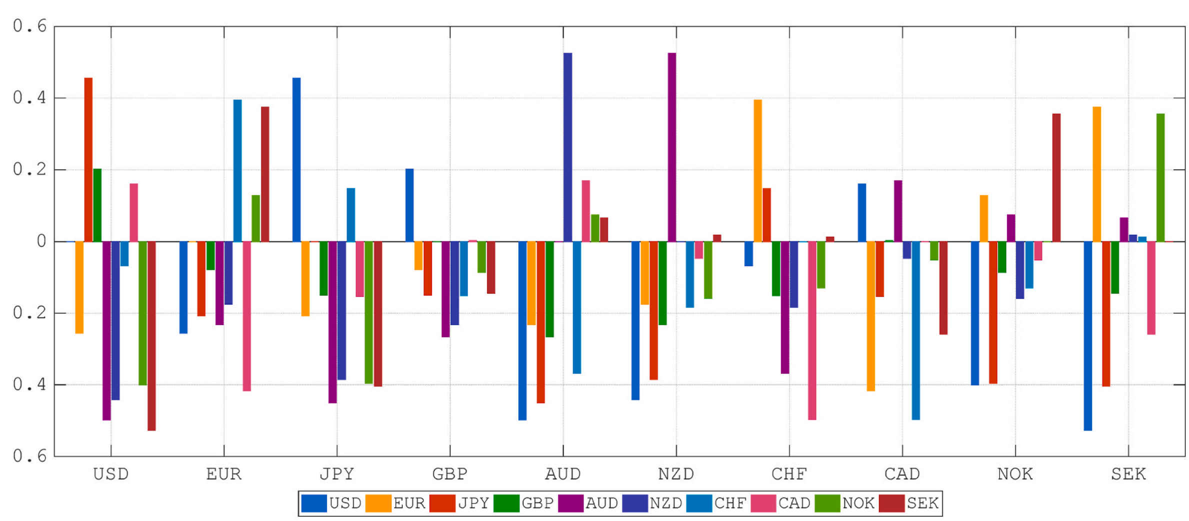

Fig. 3. Correlation coefficients of the multilateral exchange rates. *Notes:* Fig. 3 displays the off-diagonal correlation coefficients of the observed correlation matrix between the movements of the multilateral exchange rates.

off-diagonal correlation coefficients of the observed correlation matrix between the movements of each multilateral exchange rate and the movements of the other multilateral exchange rates.

The average of the off-diagonal observed correlations is −0.105. In addition, the average observed correlations between the movements in each multilateral exchange rate and the movements in the other multilateral exchange rates are all negative. Thus, the interesting observed correlations are positive, which are driven by integrated economies from being geographically close, exporting commodities, and risk-off/risk-on currencies. For example, some geographically close currencies have positive observed correlation, such as 0.525 between the Australian dollar (AUD) and the New Zealand dollar (AUD), 0.395 between the Eurozone euro (EUR) and the Swiss franc (CHF), 0.375 between the Eurozone euro (EUR) and the Swedish krona (SEK), 0.356 between the Norwegian krone (NOK) and the Swedish krona (SEK), 0.161 between the US dollar (USD) and the Canadian dollar (CAD). In contrast, the significant positive observed correlation of 0.170 between the Canadian dollar (CAD) and the Australian dollar (AUD) is driven by Canada and

Australia exporting commodities (see Chen & Rogoff, 2003). Finally, the significant positive observed correlation of 0.456 between the US dollar (USD) and the Japanese yen (JPY) is driven by both currencies being risk-off, or flight-to-safety, currencies.

3.6. Skewness

Skewness is the third central normalised sample moment of a data sample and measures the symmetry of the data sample relative to the mean and normalised by the standard derivation to the power of three. Table 6 reports the observed skewness for the log returns of the US dollar bilateral exchange rates, together with all the skewness and co-skewness terms that are used in the decomposition in Eq. (18). Fig. 4 displays the observed skewness terms for both the US dollar bilateral exchange rates and the multilateral exchange rates. The observed skewness of the movements of the bilateral exchange rates are much less intuitive. For example, the observed skewness of the movements of the Japanese yen/US dollar (JPY/USD) bilateral exchange rate is −0.071 compared to the observed skewness of 1.210 for the movements of the Japanese yen (JPY) multilateral exchange rate and the observed skewness of 0.251 for the movements of the US dollar (USD) multilateral exchange rate. The decomposition of the skewness in Eq. (18) can be used to understand this result by:

$$\begin{aligned} s_{JPY/USD} &= w_{JPY}s_{JPY} - w_{JPY,JPY,USD}cs_{JPY,JPY,USD} \\ &\quad + w_{JPY,USD,USD}cs_{JPY,USD,USD} - w_{USD}s_{USD} \\ &= 1.162 \times 1.210 - 2.756 \times 0.934 + 2.179 \times 0.570 - 0.574 \times 0.251 \\ &= 1.407 - 2.574 + 1.241 - 0.144 \\ &= -0.071 \end{aligned} \tag{25}$$

where $w_{JPY} = 1.162$; $w_{JPY,JPY,USD} = 2.756$; $w_{JPY,USD,USD} = 2.179$; and $w_{USD} = 0.574$. The large co-skewness of the US dollar with the Japanese yen of $c_{JPY,JPY,USD} = 0.934$, together with the associated large weight of $w_{JPY,JPY,USD} = 2.756$ pulls the values of the skewness in the movements of the JPY/USD downwards. Thus, both the co-skewness and weight terms play important roles in the magnitude and sign of the skewness of the movements in bilateral exchange rates.

3.6.1. Co-skewness

The co-skewness between the log returns of *base-currency* multilateral exchange rate and the log returns of the *quote-currency* multilateral exchange rate plays an integral role in the magnitude of the skewness of the associated bilateral exchange rate. Table 7 reports the observed co-skewness for the multilateral exchange rates. Fig. 5 displays the observed co-skewness between the movements of each multilateral exchange rate and the movements of the other multilateral exchange rates. The Japanese yen and the Swiss franc have positive observed co-skewness against all other currencies, where the co-skewness terms are $c_{JPY,j,j}$ for the Japanese yen and $c_{CHF,j,j}$ for the Swiss franc. In contrast, only the Canadian dollar has negative observed co-skewness against all other currencies, where co-skewness terms are $c_{CAD,j,j}$.

Table 6
Decomposition of the skewness.

i	$s_{i/USD}$	s_i	$cs_{i,i,USD}$	$cs_{i,USD,USD}$	s_{USD}
USD		0.251			0.251
EUR	−0.267	−0.006	0.131	−0.101	0.251
JPY	−0.071	1.210	0.934	0.570	0.251
GBP	−0.349	−1.084	−0.155	−0.001	0.251
AUD	−0.708	−0.654	0.502	−0.441	0.251
NZD	−0.378	−0.322	0.160	−0.267	0.251
CHF	0.148	1.482	0.367	0.222	0.251
CAD	−0.921	−0.416	0.113	−0.339	0.251
NOK	−0.344	−0.396	0.250	−0.092	0.251
SEK	−0.115	−0.027	0.006	−0.061	0.251

Notes: Table 6 reports the decomposition of observed skewness for the log returns of the US dollar bilateral exchange rates, together with the observed skewness for the log returns of the multilateral exchange rates.

3.7. Kurtosis

Kurtosis is the fourth central normalised sample moment of a data sample and measures the extremity of the data sample relative to the mean and normalised by the standard derivation to the power of three. Table 8 reports both the observed kurtosis for the log returns of the US dollar bilateral exchange rates and the observed kurtosis for the log returns of the multilateral exchange rates. Fig. 6 displays the observed kurtosis in the movements for all ten multilateral exchange rates. The observed kurtosis in the movements of the bilateral exchange rates are less intuitive. For example, the observed kurtosis in the movements of the Canadian dollar/US dollar (CAD/USD) bilateral exchange rate 8.07 compared to the observed kurtosis of 3.78 for the movements of the Canadian dollar (CAD) multilateral exchange rate and the observed kurtosis of 4.38 for the movements of the US dollar (USD) multilateral exchange rate. The decomposition of the kurtosis in Eq. (22) can be used to understand this result by:

$$\begin{aligned} k_{CAD/USD} &= w_{CAD}k_{CAD} \\ &\quad - w_{CAD,CAD,CAD,USD}ck_{CAD,CAD,CAD,USD} \\ &\quad + w_{CAD,CAD,USD,USD}ck_{CAD,CAD,USD,USD} \\ &\quad - w_{CAD,USD,USD,USD}ck_{CAD,USD,USD,USD} \\ &\quad + w_{USD}k_{USD} \\ &= 0.22 \times 3.78 - 1.10 \times (-0.18) + 2.03 \times 1.88 - 1.66 \times (-0.59) + 0.51 \times 4.38 \\ &= 0.84 - (-0.20) + 3.82 - (-0.98) + 2.24 \\ &= 8.07 \end{aligned} \tag{26}$$

where $w_{CAD} = 0.22$; $w_{CAD,CAD,CAD,USD} = 1.10$; $w_{CAD,CAD,USD,USD} = 2.03$; $w_{CAD,USD,USD,USD} = 1.66$; and $w_{USD} = 0.51$. The large co-kurtosis of the Canadian dollar (CAD) with the US dollar (USD) of $ck_{CAD,CAD,CAD,USD} = 1.88$, together with the associated large weight of $w_{CAD,CAD,USD,USD} = 2.03$ pushes the values of the kurtosis in the movements of the CAD/USD bilateral exchange rate upwards. Thus, both the co-kurtosis and weight terms play important roles in the magnitude of the kurtosis of the movements in bilateral exchange rates.

3.7.1. Co-kurtosis

The co-kurtosis between the log returns of *base-currency* multilateral exchange rate and the log returns of the *quote-currency* multilateral exchange rate plays an integral role in the magnitude of the kurtosis of the associated bilateral exchange rate. For completeness, Table 9 reports the observed co-kurtosis ($ck_{i,j,j,j}$) matrix for the multilateral exchange rates and Table 10 reports the observed co-kurtosis ($ck_{i,i,j,j}$) matrix for the multilateral exchange rates.

3.8. Currency hedging

International investors and multinational corporations typically have exposure to exchange rate risk from foreign investments, such as currencies, equities, and bonds. For example, when US dollar investors buy foreign currencies they are simultaneously selling the US dollar, which can be seen from the decomposition in Eq. (6) by:

$$\Delta p_{i/USD}(t) = \Delta p_i(t) - \Delta p_{USD}(t) \tag{27}$$

where $i = 1,\ldots,N$; $t = 1,\ldots,T$; $\Delta p_{i/USD}(t)$ is the log return of the ith/USD bilateral exchange rate; $\Delta p_i(t)$ is the log return of the *base-currency* the (ith) multilateral exchange rate; and $\Delta p_{USD}(t)$ is the log return of the *quote-currency* (US dollar) multilateral exchange rate.

The standard approach to hedge bilateral exchange rate risk is by trading the associated bilateral exchange rate, which reduces the exposure to the bilateral exchange rate and consequently, the bilateral exchange rate risk. More specifically for US dollar investors, bilateral exchange rate risk is hedged by trading the associated US dollar bilateral

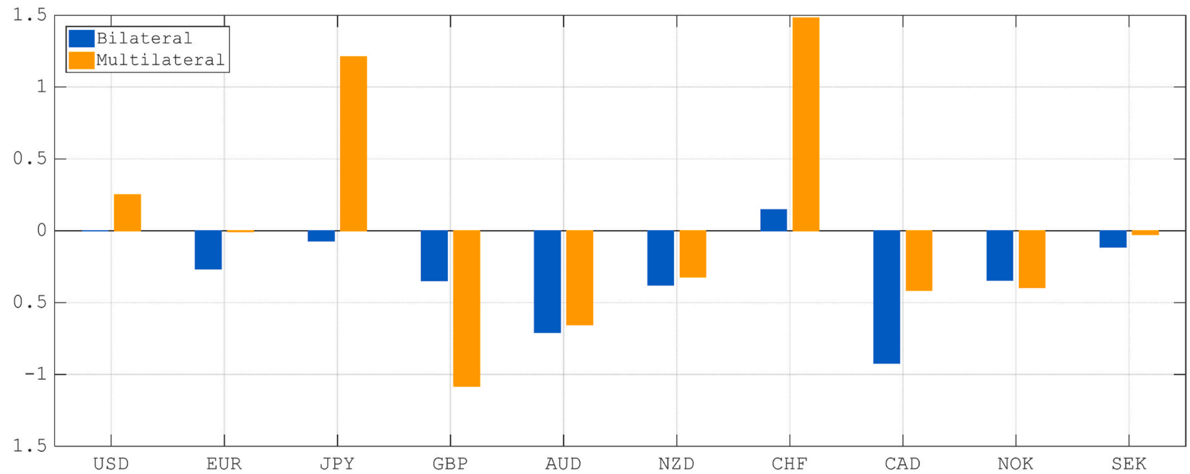

Fig. 4. Skewness of both the US dollar bilateral, and the multilateral, exchange rates. *Notes:* Fig. 4 displays the observed skewness in the movements of both the US dollar bilateral exchange rates and the multilateral exchange rates.

Table 7
Co-skewness ($cs_{i,j,j}$) matrix for the movements of the multilateral exchange rates.

	USD	EUR	JPY	GBP	AUD	NZD	CHF	CAD	NOK	SEK	AVG
USD		0.131	0.934	−0.155	0.502	0.160	0.367	0.113	0.249	0.006	0.256
EUR	−0.101		−0.372	0.394	−0.111	0.030	−0.092	−0.097	−0.009	0.080	−0.031
JPY	0.570	0.385		0.291	0.703	0.238	0.828	0.405	0.412	0.024	0.428
GBP	−0.001	−0.347	−0.148		0.044	0.043	−0.431	−0.029	−0.032	−0.048	−0.105
AUD	−0.441	−0.122	−0.917	0.155		−0.298	−0.356	−0.238	−0.227	0.003	−0.271
NZD	−0.267	−0.247	−0.369	0.240	−0.400		−0.502	−0.085	−0.146	−0.046	−0.202
CHF	0.222	0.453	0.453	0.567	0.324	0.110		0.369	0.145	0.094	0.304
CAD	−0.339	−0.124	−0.428	−0.119	−0.312	−0.201	−0.646		−0.018	−0.165	−0.261
NOK	−0.092	−0.203	−0.559	−0.284	−0.278	0.091	−0.501	−0.144		0.089	−0.209
SEK	−0.061	−0.052	−0.552	−0.088	−0.136	0.129	−0.502	−0.089	−0.177		−0.170
AVG	−0.057	−0.014	−0.218	0.111	0.038	0.033	−0.204	0.023	0.022	0.004	−0.026

Notes: Table 7 reports both the observed co-skewness for the movements of the multilateral exchange rates, where the ith row and the jth column contains the observed co-skewness of $cs_{i,j,j}$.

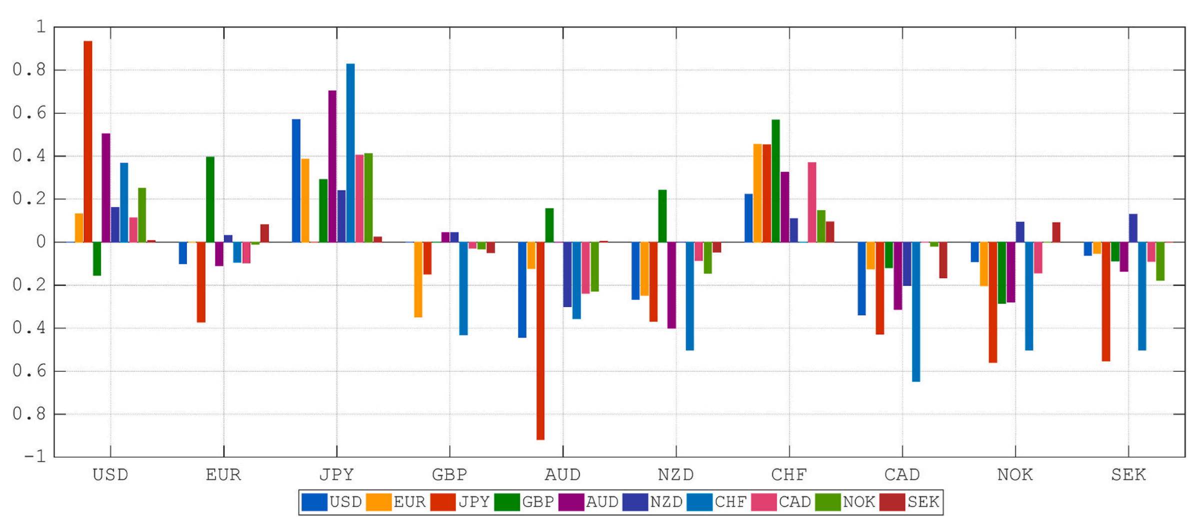

Fig. 5. Multilateral co-skewness. *Notes:* Fig. 5 displays the observed co-skewness between the movements of each multilateral exchange rate and the movements of the other multilateral exchange rates.

exchange rate by:

$$\Delta p_{i/USD}(t) - h_{i/USD}\Delta p_{i/USD}(t) \tag{28}$$

where $i = 1,\ldots,N$; $t = 1,\ldots,T$; $\Delta p_{i/USD}(t)$ is the log return of the ith/ USD bilateral exchange rate; and $h_{i/USD}$ is the *exchange-rate hedge ratio* for the ith/USD bilateral exchange rate. An exchange-rate hedge ratio of zero ($h_{i/USD} = 0$) represents a no-hedge strategy, where investors are fully exposed to bilateral exchange rate risk. In contrast, a hedge ratio of one

Table 8
Decomposition of the kurtosis.

i	$k_{i/USD}$	k_i	$ck_{i,i,i,USD}$	$ck_{i,i,USD,USD}$	$ck_{i,USD,USD,USD}$	k_{USD}
USD		4.38				4.38
EUR	4.55	4.15	−1.302	1.491	−1.410	4.38
JPY	3.67	8.98	5.701	4.673	3.636	4.38
GBP	4.59	7.35	1.443	1.196	0.327	4.38
AUD	6.01	4.81	−3.155	3.291	−3.124	4.38
NZD	4.30	4.17	−1.923	1.891	−2.054	4.38
CHF	4.74	10.22	0.587	2.266	0.497	4.38
CAD	8.07	3.78	−0.179	1.884	−0.587	4.38
NOK	4.25	3.86	−1.391	1.959	−1.842	4.38
SEK	3.97	3.89	−1.920	1.884	−2.245	4.38

Notes: Table 8 reports the decomposition of observed kurtosis for the log returns of the US dollar bilateral exchange rates, together with the observed kurtosis for the log returns of the multilateral exchange rates.

$(h_{i/USD} = 1)$ represents a full-hedge strategy, where investors are not exposed to bilateral exchange rate risk. Furthermore, a hedge ratio between zero and one $(0 < h_{i/USD} < 1)$ represents a fractional-hedge strategy, where investors are partially exposed to bilateral exchange rate risk.

The decomposition of bilateral exchange rates in Eq. (6) provides an alternative approach to hedge bilateral exchange rate risk by trading the two associated multilateral exchange rates: the base-currency (ith) multilateral exchange rate and the quote-currency (US dollar) multilateral exchange rate (see Kunkler, 2021). This can be seen by substituting the decomposition of US dollar bilateral exchange rates in Eq. (27) into Eq. (28) and allowing for separate *currency hedge ratios* to give:

$$\Delta p_{i/USD}(t) - h_i \Delta p_i(t) + h_{USD} \Delta p_{USD}(t) \tag{29}$$

where $i = 1, \ldots, N$; $t = 1, \ldots, T$; $\Delta p_{i/USD}(t)$ is the log return of the ith$/USD$ bilateral exchange rate; h_i is the *base-currency hedge ratio* for the base-currency (ith) multilateral exchange rate; $\Delta p_i(t)$ is the log return of the *base-currency* the (ith) multilateral exchange rate; h_{USD} is the *quote-currency hedge ratio* for the quote-currency (US dollar) multilateral exchange rate; and $\Delta p_{USD}(t)$ is the log return of the *quote-currency* (US dollar) multilateral exchange rate. It should be noted that Eq. (29) is equivalent to Eq. (28) when the two currency hedge ratios are equal $(h_i = h_{USD})$.

The alternative approach to hedging bilateral exchange rate risk in Eq. (29) provides extra flexibility by having two currency hedge ratios, rather than the standard approach that restricts investors to having a *common* exchange rate hedge ratio. Thus, investors can hedge either the *base currency risk* or the *quote currency risk*, or both. The decision on whether or not to hedge bilateral exchange rate risk could depend on many factors, some of which could be the mean, the standard deviation, skewness, and kurtosis of the underlying movements in the bilateral exchange rate. For example, risk-averse investors prefer positive skew to reduce exposure to extreme left-tail events and lower kurtosis to reduce exposure to extreme events generally (Kim et al., 2014). Thus, currency attribution can provide insight on whether to hedge the *base currency*

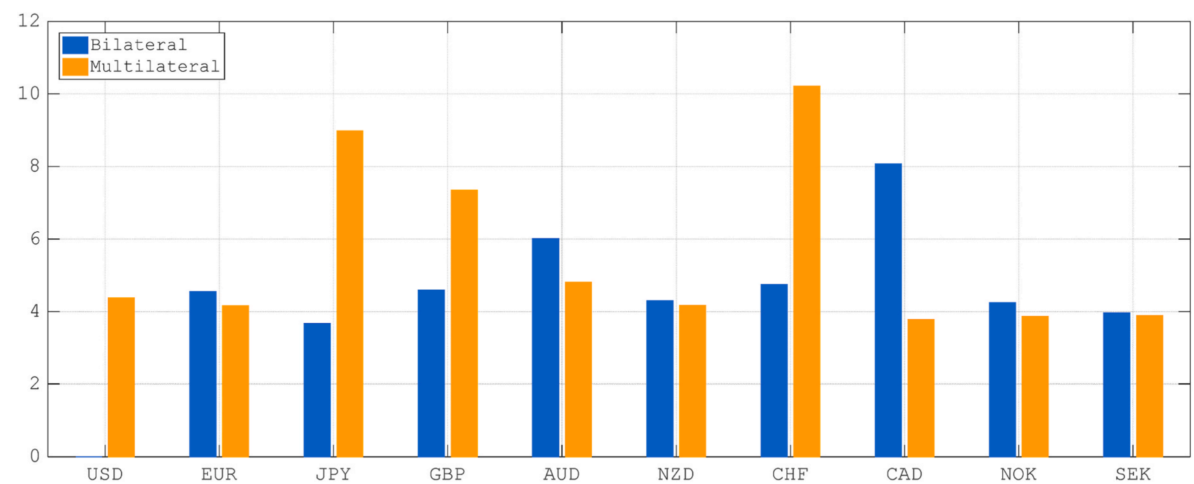

Fig. 6. Kurtosis of both the US dollar bilateral, and the multilateral, exchange rates. *Notes:* Fig. 6 displays the observed kurtosis in the movements of both the US dollar bilateral exchange rates and the multilateral exchange rates.

Table 9
Co-kurtosis ($ck_{i,j,j,j}$) matrix for the movements of the multilateral exchange rates.

	USD	EUR	JPY	GBP	AUD	NZD	CHF	CAD	NOK	SEK	AVG
USD		−1.302	5.701	1.443	−3.155	−1.923	0.587	−0.179	−1.391	−1.920	−0.238
EUR	−1.410		−2.461	−1.787	−0.025	−0.175	0.736	−0.923	−0.229	0.864	−0.601
JPY	3.636	−0.630		−1.971	−3.970	−1.726	2.693	−1.490	−1.445	−0.755	−0.629
GBP	0.327	−1.760	−0.845		−0.380	−1.039	−2.845	0.073	−0.043	−0.790	−0.811
AUD	−3.124	−0.163	−5.994	−1.464		2.214	−1.609	1.194	0.259	0.238	−0.939
NZD	−2.054	0.061	−3.354	−1.612	2.412		−1.641	0.051	−0.597	0.253	−0.720
CHF	0.497	2.153	2.760	−3.030	−2.384	−0.701		−2.192	−0.822	−0.429	−0.461
CAD	−0.587	−1.437	−3.121	0.486	1.868	−0.506	−4.497		0.092	−0.880	−0.954
NOK	−1.842	−0.340	−3.523	1.045	1.172	−0.912	−2.378	0.411		0.807	−0.618
SEK	−2.245	1.135	−3.256	0.326	1.234	0.551	−1.830	−0.342	1.334		−0.344
AVG	−0.756	−0.254	−1.566	−0.729	−0.359	−0.469	−1.198	−0.377	−0.316	−0.290	−0.631

Notes: Table 9 reports the observed co-kurtosis for the movements of the multilateral exchange rates, where the ith row and the jth column contains the observed co-kurtosis of $ck_{i,j,j,j}$.

Table 10
Co-kurtosis ($ck_{i,i,j,j}$) matrix for the movements of the multilateral exchange rates.

	USD	EUR	JPY	GBP	AUD	NZD	CHF	CAD	NOK	SEK	AVG
USD		1.491	4.673	1.196	3.291	1.891	2.266	1.884	1.959	1.884	2.282
EUR	1.491		1.756	2.553	1.494	1.769	4.103	1.636	1.859	1.316	1.997
JPY	4.673	1.756		1.413	4.997	2.236	2.768	2.637	2.215	2.231	2.770
GBP	1.196	2.553	1.413		1.025	1.503	3.458	0.986	1.339	1.128	1.622
AUD	3.291	1.494	4.997	1.025		2.052	2.160	2.101	1.881	1.632	2.293
NZD	1.891	1.769	2.236	1.503	2.052		1.907	1.537	1.444	1.521	1.762
CHF	2.266	4.103	2.768	3.458	2.160	1.907		2.832	2.022	1.552	2.563
CAD	1.884	1.636	2.637	0.986	2.101	1.537	2.832		1.343	1.638	1.844
NOK	1.959	1.859	2.215	1.339	1.881	1.444	2.022	1.343		1.323	1.709
SEK	1.884	1.316	2.231	1.128	1.632	1.521	1.552	1.638	1.323		1.580
AVG	2.282	1.997	2.770	1.622	2.293	1.762	2.563	1.844	1.709	1.580	2.042

Notes: Table 10 reports the observed co-kurtosis for the movements of the multilateral exchange rates, where the *i*th row and the *j*th column contains the observed co-kurtosis of $ck_{i,i,j,j}$.

risk or the *quote currency risk*, or both.

To simplify the analysis, it is assumed that US investors are happy with the exposure to the base (foreign) currency ($h_i = 0$) but want to hedge the exposure to the quote currency (US dollar). In this situation, Eq. (29) can be rewritten by setting $h_i = 0$ and by substituting the decomposition in Eq. (27) to give:

$$\Delta p_{i/USD}(t) + h_{USD}\Delta p_{USD}(t) = \Delta p_i(t) - (1 - h_{USD})\Delta p_{USD}(t) \tag{30}$$

where $i = 1, \ldots, N$; $t = 1, \ldots, T$; $\Delta p_{i/USD}(t) = \Delta p_i(t) - \Delta p_{USD}(t)$ is the log return of the ith$/USD$ bilateral exchange rate; h_{USD} is the quote-currency (US dollar) hedge ratio; and $\Delta p_{USD}(t)$ is the log return of the quote-currency (US dollar) multilateral exchange rate.

Table 11 reports the mean, standard deviation, skewness, and kurtosis for both a zero-hedge strategy ($h_{USD} = 0$) and full-hedge strategy ($h_{USD} = 1$). A hedge ratio of zero ($h_{USD} = 0$) represents a no-hedge strategy, where the US investor is fully exposed to bilateral exchange rate risk, where Eq. (30) is equal to $\Delta p_{i/USD}(t)$. In contrast, a hedge ratio of one ($h_{USD} = 1$) represents a full-hedge strategy, where US investors are fully exposed to base currency (foreign) rate risk and no exposure to the quote currency (US dollar), where Eq. (30) is equal to $\Delta p_i(t)$. Fig. 7 displays the mean, standard deviation, skewness, and kurtosis for fractional-hedge strategies, where the range of quote-currency (US dollar) hedge ratios is between zero and one ($0 \le h_{USD} \le 1$).

When US investors buy a foreign currency, they are also selling the US dollar. However, the annualised mean for the US dollar multilateral exchange rate is −0.33% (see Tables 1 and 3). Consequently, the US dollar adds 0.33% to the annualised mean of all US dollar bilateral exchange rates (see Section 3.4). Thus, a larger US dollar hedge ratio corresponds to a smaller annualised mean for a fractional-hedge strategy.

Most of the annualised standard deviations are reduced as the US dollar hedge ratio increases. The exception is the Japanese yen (JPY), which has an annualised standard deviation of 9.40% for the fully hedged ($h_{USD} = 1$) and 8.92% for the unhedged hedged ($h_{USD} = 0$). The positive correlation of 0.456 between the comovements in the Japanese yen multilateral exchange rate (JPY) and the US dollar multilateral exchange rate (USD) shifts the observed standard deviation of the Japanese yen/US dollar (JPY/USD) bilateral exchange down (see Section 3.6). Thus, the positive correlation between the Japanese yen and the US dollar diversifies the bilateral exchange rate risk and reduces the need for hedging.

Risk-averse investors prefer positive skew to reduce exposure to extreme left-tail events (Kim et al., 2014). A full-hedge strategy for both the Japanese yen (JPY) and the Swiss franc (CHF) significantly increases the observed skewness from −0.071 to 1.210 for the Japanese yen and from 0.148 to 1.482 for the Swiss franc. In contrast, a full-hedge strategy for the British pound (GBP) decreases the observed skewness from −0.349 to −1.084.

Risk-averse investors also prefer lower kurtosis to reduce exposure to extreme events generally (Kim et al., 2014). A full-hedge strategy for both the Japanese yen (JPY) and the Swiss franc (CHF) results in a higher observed kurtosis from 3.67 to 8.98 for the Japanese yen and from 4.74 to 10.22 for the Swiss franc. In contrast, a full hedge strategy for the Canadian dollar (CAD) results in a lower observed kurtosis from 8.07 to 3.78.

In summary, the decomposition of the bilateral exchange rates provides an alternative approach to hedging bilateral exchange rate risk. Investors change choose to hedge the base currency, the quote currency, or both. Currency attribution can act as a guide in the decision-making about whether or not to hedge bilateral exchange rate risk. More

Table 11
No-hedge and full-hedge strategies for the quote currency (US dollar).

i	$\mu_{i/USD}$	μ_i	$\sigma_{i/USD}$	σ_i	$s_{i/USD}$	s_i	$k_{i/USD}$	k_i
EUR	0.53%	0.20%	9.63%	4.54%	−0.267	−0.006	4.55	4.15
JPY	−0.53%	−0.86%	8.92%	9.38%	−0.071	1.210	3.67	8.98
GBP	−0.80%	−1.14%	8.62%	6.13%	−0.349	−1.084	4.59	7.35
AUD	0.48%	0.15%	12.32%	6.81%	−0.708	−0.654	6.01	4.81
NZD	1.24%	0.91%	12.96%	7.84%	−0.378	−0.322	4.30	4.17
CHF	2.53%	2.19%	9.82%	5.96%	0.148	1.482	4.74	10.22
CAD	0.61%	0.27%	8.77%	6.03%	−0.921	−0.416	8.07	3.78
NOK	−0.43%	−0.77%	11.66%	6.51%	−0.344	−0.396	4.25	3.86
SEK	−0.28%	−0.62%	11.30%	5.47%	−0.115	−0.027	3.97	3.89
h_{USD}	0	1	0	1	0	1	0	1

Notes: Table 11 reports the observed mean $\mu_{i/USD}$, standard deviation $\sigma_{i/USD}$, skewness $s_{i/USD}$, and kurtosis $k_{i/USD}$ for the no-hedge strategy ($h_{USD} = 0$) for the quote currency (US dollar) and the observed mean μ_i, standard deviation σ_i, skewness s_i, and kurtosis k_i for the full-hedge strategy ($h_{USD} = 1$) for the quote currency (US dollar).

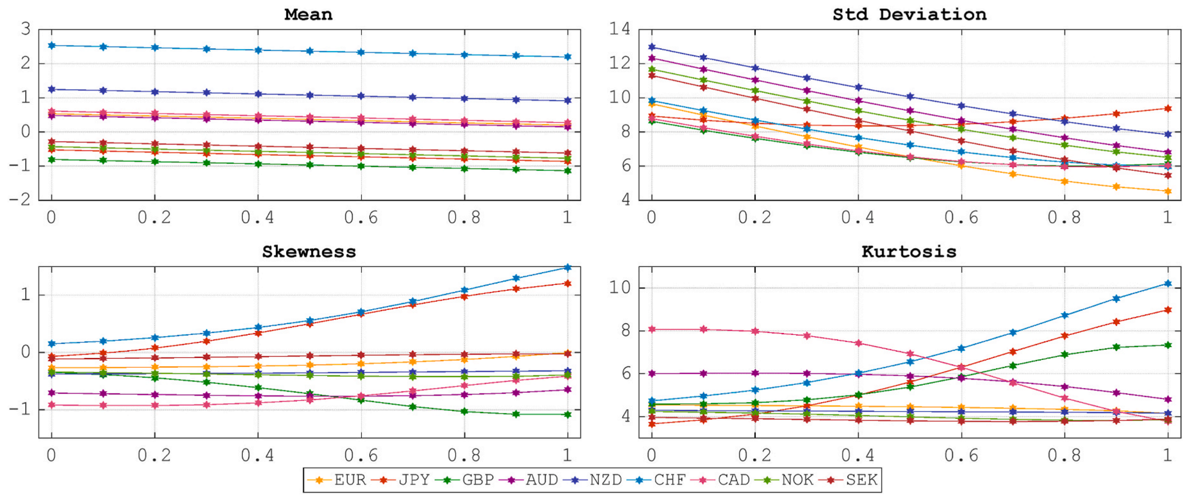

Fig. 7. Fractional-hedge strategies for the quote currency (US dollar). *Notes:* Fig. 7 displays the observed mean, standard deviation, skewness, and kurtosis for fractional-hedge strategies for a range of quote-currency (US dollar) hedge ratios ($0 \leq h_{USD} \leq 1$).

sophisticated optimal currency hedging techniques are possible but are beyond the scope of this paper. The interested reader is referred to Campbell et al. (2010), Kim et al. (2014), and Boudoukh et al. (2019).

4. Limitations

A limitation of this paper is that it focused solely on a *multicurrency numéraire* that consisted of an equally-weighted basket of currencies, rather than considering other multicurrency numéraires. For example, a stable aggregate currency is an optimal multicurrency numéraire, which is estimated using a minimum-variance portfolio optimisation, subject to the index weights sum to one, and that all the index weights are positive (Hovanov et al., 2004). The International Monetary Fund's (IMF) Special Drawing Right (SDR) is another multicurrency numéraire that is an international reserve asset and consists of a weighted basket of international reserve currencies.

Another limitation of this paper is that only one universe of currencies was considered in the *Results* section. The movements in a multilateral exchange rate are calculated based on the universe of currencies. Thus, any modification of the universe of currencies will impact the decompositions. In other work associated with the Frankel-Wei regression framework, it was recommended that large a well-diversified universe of currencies be chosen for the multicurrency numéraire (Kunkler, 2022a).

5. Conclusion

This paper presented a quantitative methodology to perform currency attribution, which decomposes the mean, variance, skewness, and kurtosis of the movements in a bilateral exchange rate into the base currency and the quote currency of the bilateral exchange rate. It is shown that the mean can be decomposed into the difference between two terms. In addition, the variance can be decomposed into a weighted sum of three terms. Furthermore, the skewness can be decomposed into a weighted sum of four terms. Finally, the kurtosis can be decomposed into a weighted sum of five terms.

CRediT authorship contribution statement

Michael Kunkler: Conceptualization, Methodology, Software, Writing, Visualization, Investigation.

Declaration of competing interest

We the undersigned declare that this manuscript is original, has not been published before and is not currently being considered for publication elsewhere.

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us.

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