Measures of model risk for continuous-time finance models

Article

Published Version

Lazar, E. ORCID: https://orcid.org/0000-0002-8761-0754, Qi, S. and Tunaru, R. (2024) Measures of model risk for continuous-time finance models. Journal of Financial Econometrics. ISSN 1479-8417 doi: 10.1093/jjfinec/nbae001 Available at https://centaur.reading.ac.uk/114615/

To link to this article DOI: http://dx.doi.org/10.1093/jjfinec/nbae001

Publisher: Oxford University Press

Journal of Financial Econometrics, 2024, 1–26
https://doi.org/10.1093/jjfinec/nbae001
Article

OXFORD

Measures of Model Risk for Continuous-Time Finance Models*

Emese Lazar [1], Shuyuan Qi [2], and Radu Tunaru [3]

[1]ICMA Centre, Henley Business School, University of Reading, Whiteknights, Reading, RG6 6BA, UK, [2]Chinese Academy of Finance and Development, Central University of Finance and Economics, Beijing, 100081, China, [3]University of Sussex Business School, University of Sussex, Brighton, BN1 9SL, UK

Address correspondence to Shuyuan Qi, Chinese Academy of Finance and Development, Central University of Finance and Economics, Beijing, 100081, China, or e-mail: shuyuan.qi@cufe.edu.cn

Abstract

Measuring model risk is required by regulators in financial and insurance markets. We separate model risk into parameter estimation risk (PER) and model specification risk (MSR), and we propose expected shortfall type model risk measures applied to Lévy jump, affine jump-diffusion, and multifactor models. We investigate the impact of PER and MSR on the models' ability to capture the joint dynamics of stock and option prices. Using Markov chain Monte Carlo techniques, we implement two methodologies to estimate parameters under the risk-neutral probability measure and the real-world probability measure jointly.

Keywords: Lévy models, MCMC, model specification risk, parameter estimation risk, stochastic volatility

JEL classifications: C11, C52, G12, G13

Model risk is currently considered one of the most overlooked risks faced by financial firms. The Basel Committee on Banking Supervision (2009), Federal Reserve Board of Governors (2011), and European Banking Authority (2012) require banks to measure and report model risk as for any other type of risk. The sources of model risk are parameter estimation risk (PER) and model specification risk (MSR). The latter is the risk associated with incorrectly identifying and modeling decisive factors that can jointly describe the dynamics of an economic asset, while PER denotes the risk of inaccurate estimation of parameters for a given model. The PER and MSR are the two components of the total model risk (TMR).

The majority of studies use point-wise estimation methods and consider model risk as model mispricing, thus not estimating the components of model risk. PER can be captured via Bayesian estimation methods. Jacquier and Jarrow (2000) study the PER of the Black and Scholes model. Furthermore, Jacquier, Polson, and Rossi (2002) apply Bayesian estimators for stochastic volatility (SV) models and find that the Bayesian approach produces more robust results compared with the moments and likelihood estimators. There are many advantages of using Markov chain Monte Carlo (MCMC) techniques in extracting

* We would like to thank Fabio Trojani (Editor) as well as two anonymous reviewers for their insightful comments and constructive suggestions. Also, we would like to thank Nicola Fusari for his Matlab code and Peter Gruber for useful discussions and suggestions to improve the paper, as well as the participants of the Southwestern Finance Association 2021 Annual Meeting, the 2021 Asian Meeting of the Econometric Society and 2022 AEA Poster Session for their useful comments and suggestions.

Received: September 14, 2021. **Revised:** October 30, 2023. **Editorial decision:** January 03, 2024. **Accepted:** January 08, 2024

inference on continuous-time models in finance that have been highlighted in a series of works by Eraker (2001), Polson and Stroud (2003), Jacquier, Johannes, and Polson (2007), Johannes and Polson (2010), and Yu, Li, and Wells (2011).

In this article, we propose an expected shortfall (ES)-based method to measure model risk. This ES-type model risk measurement is potentially superior in capturing model risk because it is able to capture model tail risk. We provide an applicable framework to separate and measure PER and MSR for a very competitive class of option pricing models. We disentangle the model risk for buyers and sellers and we highlight that the two parties in options contracts are exposed asymmetrically to model risk. We find that option sellers tend to be exposed to a higher PER than buyers, which is in line with the conclusions of Green and Figlewski (1999). Furthermore, we find that the option bid–ask spread is positively linked to the positional gap of model risk.

We implement two methodologies that estimate model parameters jointly under physical and risk-neutral probability measures. These two methodologies use different types of data and pricing error specifications. One estimation methodology introduces random effects to the autoregressive pricing error process to allow multiple options to be considered in the estimation process, while allowing for different estimation windows for the spot and option prices. The second estimation methodology relies on underlying spot volatility and option implied volatility (IV) data and assumes that the relative pricing errors are normally distributed. We apply our new methodology to several option pricing models with different abilities to explain S&P 500 index spot prices/volatility and option prices/IV. We find that the multifactor model has the lowest TMR.

The remainder of the article is organized as follows. In Section 1, we introduce the model risk measurement framework; then we revisit all models that are investigated in Section 2. Section 3 provides the details of the two estimation methodologies. Section 4 describes the data. The empirical analyses are presented in Section 5. The last section concludes.

1 Model Risk

This section discusses the theoretical background of model risk measures and then proceeds to define the PER and MSR. The ES-type model risk measures are introduced at the end of this section.

1.1 Theoretical Background

Most of the literature in the area of model risk of continuous-time models concerns measuring TMR. Hull and Suo (2002) introduce a methodology to measure model risk associated with mis-specified models in the evaluation of exotic options. Their focus is on the estimation of the IV surface from market option prices using three different models, Black-Scholes, IV function, and a two-factor SV model. Taking the latter as the benchmark for pricing exotic options, they quantify model risk by comparing the other two models with the benchmark model. The model performance is monitored separately for pricing and hedging. Another innovative work on model risk is Lindström et al. (2008), where parameter uncertainty is taken into account with a revised risk-neutral valuation formula.

Given a derivative written on an underlying asset $\{S_t\}_{t\geq 0}$, a pricing model $\mathcal{M}$ can be defined using a vector of parameters θ belonging to a compact domain Θ. Risk-neutral pricing theory dictates that the price at time t of any contingent claim C on the underlying asset S with maturity T and payoff $\Psi(S_T)$, such as a call option or a put option, is determined by $C(t, S_t, \theta) = e^{-r(T-t)} E_t^{\mathbb{Q}}[\Psi(S_T)]$ with $\mathbb{Q}$ a risk-neutral pricing measure and r the risk-free rate. If $\left\{\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}\right\}_{t\geq 0}$ is the Radon–Nikodym process that allows switching between the physical measure $\mathbb{P}$ and $\mathbb{Q}$ then the same price of the contingent claim C can be rewritten as

$$C(t, S_t, \theta) = e^{-r(T-t)} \frac{d\mathbb{P}}{d\mathbb{Q}}\bigg|_{\mathcal{F}_t} E_t^{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}\Psi(S_T)]. \tag{1}$$

For a fixed model $\mathcal{M}$, the investor or trader or risk manager is faced with uncertainty in the values of parameter θ. One way to account for this uncertainty is to integrate the contingent claim price, viewed as a function of individual parameter values, over the entire domain of Θ (Gzyl, ter Horst, and Malone 2008; Johannes and Polson, 2010; Rodríguez, Ter Horst, and Malone 2015). In that case, we have:

$$C(t, S_t) = e^{-r(T-t)} \int_{\theta\in\Theta} C(t, S_t, \theta) p(d\theta). \tag{2}$$

This can be rewritten as below:

$$C(t, S_t) = e^{-r(T-t)} \int_{\theta\in\Theta} \frac{d\mathbb{P}}{d\mathbb{Q}}\bigg|_{\mathcal{F}_t} E_t^{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}\Psi(S_T)]p(d\theta). \tag{3}$$

Applying the Bayes rule and denoting by $f(S_\tau : \tau \leq t)$, the pathway distribution of underlying S, being equal to $\int_{\theta\in\Theta} \frac{d\mathbb{P}}{d\mathbb{Q}}|_{\mathcal{F}_t} p(d\theta)$, leads to

$$C(t, S_t) = e^{-r(T-t)} \int_{\theta\in\Theta} f(S_\tau : \tau \leq t) E_t^{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}\Psi(S_T)]p(d\theta). \tag{4}$$

Formula (4) shows that the trader combines uncertainty on the pathway followed by the underlying asset S as governed by each set of parameter values θ.

1.2 Model Risk of Option Pricing Models

One major criticism of the above approach is that the integration over the entire domain Θ of the vector of parameters θ will combine favorable errors with unfavorable errors in the overall calculation of the probability distribution of the contingent claim. Also, buyers and sellers are affected differently by the various values of Θ.

For example, assuming that the trader is only selling call options, and for the sake of simplicity of exposition, that there is only one parameter $\theta \equiv \sigma$, the risk of loss for the seller is that they may sell call options at a value $\sigma^* < \sigma_0$; whereas if they sell European call options for $\sigma^* > \sigma_0$ then they are making a profit, with σ_0 being the true value of σ. The argument goes in the opposite direction for the buyer of the option. From a PER perspective, the buyer makes a loss when they buy the call option with $\sigma^* > \sigma_0$ and there is no loss if they buy the European call at $\sigma^* < \sigma_0$. This means that the position in the contract, whether long or short, should be recognized when computing model risk.

Hence, in our approach, we consider the two tails of the price distribution, keeping in mind that they affect the buyer and the seller differently. This is different from the so-called predictive approach that combines good scenarios with bad scenarios, making it difficult to disentangle the exact magnitude of model PER faced by the trader.[1] Our model risk measure is defined within a Bayesian inferential setup. Following Berger (1985), a posterior

[1] Alternative methods used in the literature are based on the conditional inferential approach whereby the trader will pick a value based on some criteria, usually related to the loss that they are facing. Pricing based on the posterior mean θ^{mean} is directly linked to a quadratic loss function while pricing based on the posterior median of θ^{med} is associated with an absolute loss function.

expected loss of an action a when the posterior distribution is $p(\theta|\mathcal{D}, a)$ can be written as $\int_{\Theta} L(\theta, a) dF^{p(\theta|\mathcal{D})}$, with $\mathcal{D}$ denoting the observed dataset. The Bayesian risk rule associated with the squared-error loss $L(\theta, a) = (\theta - a)^2$ is the posterior mean $E^{p(\theta|\mathcal{D})}[\theta]$. When the loss function is the absolute error $L(\theta, a) = |\theta - a|$ then the Bayes estimator corresponding to this risk is the posterior median. We define model risk at a desirable significance level, showing the potential losses the trader would incur due to model risk. As a first step, we fix η as the significance level for the model risk. Typical values for η are $1\%, 2.5\%$, or 5%, similar to market risk measures. Then the loss function

$$L(\theta, a) = \begin{cases} \theta - a, & \text{if } \theta - a \geq 0 \\ \left(\dfrac{1}{\eta} - 1\right)(\theta - a), & \text{otherwise.} \end{cases} \tag{5}$$

implies that the Bayes estimator is equal to the η quantile of the posterior density function $p(\theta|\mathcal{D})$.[2]

Model risk is generated by two types of risk. The PER assumes that the model specification is fixed but the exact value of parameter vector $\theta \in \Theta$ is uncertain, and it estimates a loss a trader would face due to incorrect estimation of the model parameters. The second source of model risk is the uncertainty in model specification, and this is called MSR. One common case is that of a set of nested models defined by expanding the vector of free parameters $\theta^{(1)} \subset \theta^{(2)} \subset \cdots \theta^{(m)}$, where $\theta^{(i)} \in \Theta$, $1 \leqslant i \leqslant m$, m being the number of models and we assume that the true value of the vector of parameter θ_0 coincides exactly with one of the $\theta^{(i)}$. In such a case, the MSR of the "true" model as well of the models which nest the true model should be equal to zero, while the nested models would generally have a nonzero MSR, due to having too few parameters. In practice, however, we also encounter the situation where there is a finite set of non-nested models each defined by different vector parameters of possibly different sizes. The most general case is when we have non-nested families of nested models, with different parameter sets across families and nested vectors of parameters for models within a given family. Without loss of generality, we can assume that the set of models operating in a given market is given by $\{\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_d\}$, these can be nested and non-nested. We also assume that the *true* data-generating process is unknown. Each model $\mathcal{M}_i$ will produce one price estimate for the targeted asset as well as a price distribution and the investor has to choose the most suitable model based on these outputs.

MSR is conceptualized in this article as additional to PER. It refers to the inability of a model used by the seller or buyer to generate prices that will match the market price of assets with sufficient Bayesian credibility. For a given critical level η, if the trader is a buyer, let CL denote the η quantile on the left side of the price distribution and correspondingly, if the trader is a seller, let CR be the $1-\eta$ quantile on the right side of the price distribution. These two quantiles will define a credibility interval akin to the confidence interval in a frequentist approach. The PER, if the trader is a seller, would be the difference between CR and the price estimate, and the PER, if the trader is a buyer, would be the difference between the estimated price and CL. If improved measures of tail risk such as ES type risk measures are sought, the theoretical construction remains the same. If the asset market price is above CL (buyer), or is below CR (seller), then MSR is zero because the model is able to capture the market price of the product, and the pricing error can be explained by PER.[3] If the market price is below CL (buyer) or is higher than CR (seller), however, then there is nonzero MSR for the trader.

[2] For proof see result 6 in Berger (1985), page 162.

[3] The concept of model risk is to some extent related to pricing error. In some special cases, the two can be equal. However, model risk can be substantial even if pricing error is small.

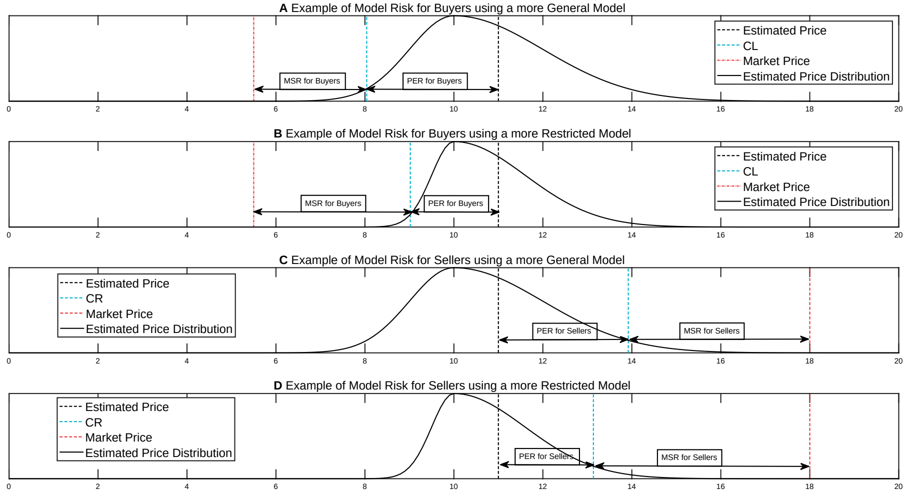

Figure 1. This figure provides an example to explain model risk measures. We assume that the EPD is constructed based on the estimated posterior distribution of the parameters of two models; a more general model and a restricted model; one gives a wider distribution, and the other leads to a narrower distribution. The blue lines mark the 5% and 95% cutoff points of the EPD, and the red lines mark two possible market prices for the option. Buyers fear overpricing, while sellers fear underpricing an option. Therefore, the left tail of the EPD is used to measure the model risk for buyers, and the right tail is used to measure model the risk for sellers. The black dashed line marks the estimated price, which can either be the posterior mean or median.

Figure 1 shows the intuition behind our measures and provides an example of model risk measures for a trader who is a buyer or a seller, which exemplifies the situation when the pricing error is not fully explained by PER. For example, if OP is the market price of a European call option such that $CR < OP$, then the MSR of a seller trader is the minimum value that the seller's price distribution should shift with so that the option price is inside the credibility interval. In this example, this value is clearly $OP-CR$; see Figure 1 (C) and (D). If, however, $OP < CL$, then the MSR of a buyer trader is the minimum shift in the model's estimated price distribution (EPD) so that the price is within the credibility interval; and this value is $CL-OP$; see Figure 1 (A) and (B). If a seller trader underestimates the option price, then they would bear losses caused by MSR because they will be prepared to sell even below the market price; whereas a buyer trader would bear losses caused by MSR when overestimating the price because they will be happy to buy at overinflated prices. So we measure the MSR for the buyer wanting to long (L) the assets as $\max(CL-OP, 0)$, and for the seller, who wants to short (S) the assets, as $\max(OP-CR, 0)$.

Figure 1 (A) and (B) compares the buyer's model risk for two different models, which can be nested. Figure 1 (A) shows the price distribution for a more general model, possibly with more free parameters, whereas Figure 1 (B) shows the price distribution obtained from a possibly nested model with a smaller number of free parameters. The more general model would typically have a higher PER, due to the higher number of free parameters, but a lower MSR, whereas the restricted model would have a lower PER, but a higher MSR. A similar comparison can be made for the model risk faced by the seller, shown in Figure 1 (C) and (D). If the market price is higher than CL (for the buyer) or lower than CR (for the seller), then the only model risk faced by the buyer or seller is the PER of the model, which is typically larger for the more general model.

1.3 Model Risk Measures

We now provide the definition of PER and MSR as follows:

Definition 1 *For a given contingent claim $\mathcal{C}$ and a given model $\mathcal{M}$ with vector of parameters θ, the* **PER** *refers to the risk of loss due to possible mispricing associated with erroneous estimates of parameters θ obtained via the estimation process $\mathcal{K}$ given dataset $\mathcal{D}$, for a given critical level.*

Definition 2 *For an option $\mathcal{C}$ and a model $\mathcal{M}$ with vector of parameters θ, the* **MSR** *of model $\mathcal{M}$ refers to the risk of mispricing due to the inability of the model to capture the market price of $\mathcal{C}$ based on dataset $\mathcal{D}$ and methodologies $\mathcal{K}$, for a given critical level.*

Our main objective is to measure PER and MSR, and then compare the model risk across different pricing models. Johannes and Polson (2010) state that the marginal posterior distribution through the Bayesian estimation characterizes the sample information regarding the objective and risk-neutral parameters and quantifies the PER. According to Bayes' theorem, for the model $\mathcal{M}(\theta)$, the posterior distribution of model parameters θ can be expressed as: $\pi(\theta|\mathcal{D}) \propto h(\mathcal{D}|\theta)\pi(\theta)$, where $\pi(\theta)$ is the so-called prior distribution of parameters and $h(\mathcal{D}|\theta)$ is the likelihood function. The posterior distribution of the parameters can further be used to construct the estimated posterior distribution of the asset price, $\tilde{F}_t(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K})$. Here $\mathcal{C}$ denotes the type of financial instrument such as an option (call or put) at time t conditional on model $\mathcal{M}$ with parameter vector θ, given an observed dataset $\mathcal{D}$. For clarity, we also insist on the notational $\mathcal{K}$ for different computational methodologies (including calibration and pricing). The posterior distribution of option prices is produced by the uncertainty in the value of parameters θ weighted by the combination of prior assumptions on θ and the likelihood coming out of historical data. Furthermore, we use the posterior mean as the point estimated of the price $\hat{F}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K})$. As discussed above, the buyer (seller) fears overpricing (underpricing) the product. Thus the PER for a trader seeking a long (short) position is measured based on the left (right) tail of $\tilde{F}_t(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K})$. We adopt ES-type risk measures in this article. ES has been shown to be a superior measure to Value-at-Risk (VaR) due to its coherence property. The regulators stipulate ES as the main measure of market risk. At the same time, they require model risk to be treated similarly to other types of risk (Basel Committee on Banking Supervision 2009). In the literature on model risk measurement, ES-type measures have been used to capture model risk (Barrieu and Scandolo 2015; Detering and Packham 2016). Let $VaR_\eta(\tilde{\Lambda})$ denote the VaR at critical level η, which is computed as the η quantile of the distribution $\tilde{\Lambda}$.[4] Then the ES can be calculated as $ES_\eta(\tilde{\Lambda}) = \frac{1}{\eta}\int_0^\eta VaR_x^{PER}(\tilde{\Lambda})dx$. The ES-type credibility interval for the estimated posterior distribution of the option price is based on the risk measures below in the left and right tail of the price distribution:

$$
\begin{aligned}
CL_{\eta,t}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) &= ES_\eta\left(\tilde{F}_t(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K})\right),\\
CR_{\eta,t}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) &= \left|ES_\eta\left(-\tilde{F}_t(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K})\right)\right|.
\end{aligned} \tag{6}
$$

[4] We consider $\eta\%$ to be 5% in this article.

The ES-type model risk measure for PER, at level η, for option $\mathcal{C}$, given model $\mathcal{M}$ with parameter vector θ, dataset $\mathcal{D}$, methodology $\mathcal{K}$, for a long or short position is defined as:

$$\rho^{PER}_{\eta,t}(\mathcal{C},L;\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) = \hat{F}_t(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) - CL_{\eta,t}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}),$$
$$\rho^{PER}_{\eta,t}(\mathcal{C},S;\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) = CR_{\eta,t}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) - \hat{F}_t(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}). \quad (7)$$

The MSR of the model for the buyer and the seller, respectively, is computed as:

$$\rho^{MSR}_{\eta,t}(\mathcal{C},L;\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) = \max\Big(CL_{\eta,t}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) - OP_t(\mathcal{C}),0\Big),$$
$$\rho^{MSR}_{\eta,t}(\mathcal{C},S;\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) = \max\Big(OP_t(\mathcal{C}) - CR_{\eta,t}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}),0\Big). \quad (8)$$

The TMR for long and short positions is defined as:

$$\rho^{TMR}_{\eta,t}(\mathcal{C},L;\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) = \rho^{PER}_{\eta,t}(\mathcal{C},L;\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) + \rho^{MSR}_{\eta,t}(\mathcal{C},L;\mathcal{M}(\theta),\mathcal{D},\mathcal{K}),$$
$$\rho^{TMR}_{\eta,t}(\mathcal{C},S;\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) = \rho^{PER}_{\eta,t}(\mathcal{C},S;\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) + \rho^{MSR}_{\eta,t}(\mathcal{C},S;\mathcal{M}(\theta),\mathcal{D},\mathcal{K}). \quad (9)$$

Furthermore, when comparing models, one might want to calculate the PER of the model itself without considering positions. In this case, we can assess the PER of a model $\mathcal{M}(\theta)$ as the maximum of the PER for long and short positions:

$$\rho^{PER}_{\eta,t}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) = \max[\rho^{PER}_{\eta,t}(\mathcal{C},L;\mathcal{M}(\theta),\mathcal{D},\mathcal{K}),\rho^{PER}_{\eta,t}(\mathcal{C},S;\mathcal{M}(\theta),\mathcal{D},\mathcal{K})]. \quad (10)$$

The MSR for the model regardless of positions is computed as:

$$\rho^{MSR}_{\eta,t}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) = \max[\rho^{MSR}_{\eta,t}(\mathcal{C},L;\mathcal{M}(\theta),\mathcal{D},\mathcal{K}),\rho^{MSR}_{\eta,t}(\mathcal{C},S;\mathcal{M}(\theta),\mathcal{D},\mathcal{K})]. \quad (11)$$

A good model should have both PER and MSR small. A model based on a large number of parameters might capture a market well and result in a very low MSR, but could be exposed to a large PER. On the other hand, a model with only a few parameters is easier to estimate with a low PER, but can be exposed to a large MSR. To directly compare models regardless of the trader's position, we define the (position-free) TMR of the model as the sum of the PER and MSR of the model:

$$\rho^{TMR}_{\eta,t}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) = \rho^{PER}_{\eta,t}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) + \rho^{MSR}_{\eta,t}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}). \quad (12)$$

2 Models

This section describes the set of models used in the empirical section. We consider five single-factor models: the SV model, the SV model with Merton jumps in returns (SVJ), the SV model with contemporaneous jumps in returns and volatility (SVCJ), the SV model with variance-gamma jumps in returns (SVVG), and the SV model with log-stable jumps in returns (SVLS). The model specification and the change of measure between $\mathbb{P}$ and $\mathbb{Q}$ for these single-factor models are given in Sections 2.1 and 2.2, respectively. We also study the model risk of a multifactor model, introduced in Section 2.3.

2.1 Single-Factor SV Models

Let $Y_t = \ln(S_t)$ denote the logarithm of the asset price. The dynamics, for all models, of the continuously compounded return on the asset price under the real-world measure $\mathbb{P}$ are:

$$\begin{aligned} dY_t &= \bar{\mu}dt + \sqrt{V_t}dW_t^Y(\mathbb{P}) + dJ_t^Y(\mathbb{P}), \\ dV_t &= \kappa(\vartheta - V_t)dt + \sigma_V\sqrt{V_t}dW_t^V(\mathbb{P}) + dJ_t^V(\mathbb{P}), \end{aligned} \tag{13}$$

where $W_t^Y(\mathbb{P})$ and $W_t^V(\mathbb{P})$ are standard Brownian motions under $\mathbb{P}$ with $dW_t^Y(\mathbb{P})dW_t^V(\mathbb{P}) = \rho dt$, with the correlation ρ contributing to the skewness of the returns' distribution. A negative ρ captures the leverage effect; $\bar{\mu}$ measures the mean return; V_t is the instantaneous variance of returns at time t; κ represents the speed of mean reversion; ϑ denotes the long-run mean of the variance process; and σ_V is the volatility of volatility.

The SVJ and SVCJ are AJD models (Duffie, Pan, and Singleton 2000) capturing continuous movements of assets with affine diffusions and large discontinuous jumps in asset returns with a Poisson process. The jump processes in AJD models are defined as $dJ_t^Y(\mathbb{P}) = \xi^Y dN_t^Y$ and $dJ_t^V(\mathbb{P}) = \xi^V dN_t^V$ where $\{N_t^Y\}_{t\geq 0}$ and $\{N_t^V\}_{t\geq 0}$ are Poisson processes as in Duffie, Pan, and Singleton (2000). The SVCJ model contains simultaneous correlated jumps, where $N_t^Y = N_t^V = N_t$, in both the return and volatility processes with a constant intensity[5] λ; the jump size in the variance process follows an exponential distribution, $\xi^V \sim \mathbb{EXP}(\mu_V)$, and the jump size in the asset log-prices is conditionally normally distributed with $\xi^Y|\xi^V \sim \mathbb{N}(\mu_J + \rho_J\xi^V, \sigma_J^2)$.[6] ρ_J is assumed to be zero for simplicity. The long-run mean–variance of the SVCJ model is $\vartheta + \mu_V\lambda/\kappa$ due to the jump component in the variance process. For SVJ, $J_t^V(\mathbb{P}) = 0$, and the process of the jump $J_t^Y(\mathbb{P})$ has the same specification as SVCJ. For SV, $J_t^Y(\mathbb{P}) = J_t^V(\mathbb{P}) = 0$.

The AJD model is constructed based on Brownian motions and compound Poisson processes, which are just special cases of Lévy processes. The AJD models we consider allow finite-activity jump processes and constant jump intensities only, while the Lévy processes are more flexible, allowing for infinite jump arrival rates. We consider two Lévy jump models, namely the SVVG model of Madan, Carr, and Chang (1998) and the SVLS model of Carr and Wu (2003).

The SVVG model is an infinite-activity but finite-variation jump model. The variance-gamma process can be described as:

$$X_t^{VG}(\sigma, \gamma, \nu) = \gamma G_t^\nu + \sigma W_{G_t^\nu}, \tag{14}$$

where $\{X^{VG}\}$ is an arithmetic Brownian motion with drift γ and volatility σ; $\{G_t^\nu\}_{t\geq 0}$ denotes the gamma process with unit mean rate and variance rate ν; and $\{W_t\}_{t\geq 0}$ is a standard Brownian motion independent of G_t^ν. Setting $J_t^Y(\mathbb{P}) = X_t^{VG}(\sigma, \gamma, \nu)$ and $J_t^V(\mathbb{P}) = 0$ reduces Equation (13) to the SVVG model.

The SVLS model is an example of infinite-activity and infinite-variation jump model. The log-stable process follows an α-stable distribution (S_α):

$$X_t^{LS}(\alpha, \sigma) - X_s^{LS}(\alpha, \sigma) \sim S_\alpha(\beta, \sigma^{\frac{1}{\alpha}}(t-s)^{\frac{1}{\alpha}}, \gamma), t > s, \tag{15}$$

[5] Bates (2000) finds that the model with state-dependent intensities is significantly misspecified while Andersen, Benzoni, and Lund (2002) state that there is no evidence to support time-varying intensity.

[6] $\mathbb{EXP}$ denotes the exponential distribution and $\mathbb{N}$ denotes the normal distribution.

where $\alpha \in (1,2]$ is the tail index of the α-stable distribution which determines the shape of the stable distribution; $\beta \in [-1,1]$ is the skew parameter determining the skewness of the distribution; $\sigma \geq 0$ is the scale parameter and $\gamma \in \mathbb{R}$ is the location parameter. We follow Carr and Wu (2003) and set $\beta = -1$ and $\gamma = 0$. Equation (13) reduces to SVLS, if $J_t^Y(\mathbb{P}) = X_t^{LS}(\alpha, \sigma)$ and $J_t^V(\mathbb{P}) = 0$.

2.2 Change of Measure and Option Pricing for Single-Factor Models

For Brownian motions, Pan (2002) proposes a standard practice for the change of measure:

$$\gamma_t^Y = \eta_s \sqrt{V_t}, \gamma_t^V = -\frac{1}{\sqrt{1-\rho^2}}\left(\rho\eta_s + \frac{\eta_v}{\sigma_V}\right)\sqrt{V_t}, \tag{16}$$

where γ_t^Y and γ_t^V represent the market prices of risk of Brownian shocks to returns and variance, respectively. The Brownian motions under $\mathbb{Q}$ in the return and variance processes are:

$$\begin{aligned} dW_t^Y(\mathbb{Q}) &= dW_t^Y(\mathbb{P}) + \gamma_t^Y dt, \\ dW_t^V(\mathbb{Q}) &= dW_t^V(\mathbb{P}) + \gamma_t^V dt. \end{aligned} \tag{17}$$

For the variance process, following Pan (2002), Bates (2000), and Broadie, Chernov, and Johannes (2007), we apply the following "time-series consistency" constraints in the change of measure such that both physical and risk-neutral probability densities are from the same family: $\kappa^{\mathbb{P}}\vartheta^{\mathbb{P}} = \kappa^{\mathbb{Q}}\vartheta^{\mathbb{Q}}$; $\rho^{\mathbb{P}} = \rho^{\mathbb{Q}}$ and $\sigma_V^{\mathbb{P}} = \sigma_V^{\mathbb{Q}}$.[7]

Moreover, for jump processes, the following restrictions are imposed: in SVJ, $(\lambda, \mu_J, \sigma_J)$ are able to change between $\mathbb{P}$ and $\mathbb{Q}$; in SVCJ, $(\lambda, \mu_J, \sigma_J, \mu_V)$ are able to change between $\mathbb{P}$ and $\mathbb{Q}$; in SVVG, γ and σ are able to change between $\mathbb{P}$ and $\mathbb{Q}$, while ν remains unchanged under $\mathbb{P}$ and $\mathbb{Q}$; in SVLS, α is allowed to change between $\mathbb{P}$ and $\mathbb{Q}$.[8] In AJD models, all parameters in the jump processes can be different under the physical and risk-neutral measures. However, this leads to some difficulties with the econometric identification, as shown in Pan (2002). To bypass this identification issue, in this article, we enable only λ to change between measures.[9] Finally, the jump parameters under both measures for SVJ, SVCJ, SVVG, and SVLS are $(\lambda^{\mathbb{P}}, \lambda^{\mathbb{Q}}, \mu_J, \sigma_J)$, $(\lambda^{\mathbb{P}}, \lambda^{\mathbb{Q}}, \mu_J, \sigma_J, \mu_V)$, $(\nu, \gamma^{\mathbb{P}}, \sigma^{\mathbb{P}}, \gamma^{\mathbb{Q}}, \sigma^{\mathbb{Q}})$, and $(\alpha^{\mathbb{P}}, \alpha^{\mathbb{Q}}, \sigma)$, respectively.

Under the framework described above, the Radon–Nikodym derivative is[10]:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_t = \exp\left\{-\int_0^t \gamma_s^Y dW_s^Y(\mathbb{P}) - \int_0^t \gamma_s^V dW_s^V(\mathbb{P}) - \frac{1}{2}\left[\int_0^t (\gamma_s^Y)^2 ds + \int_0^t (\gamma_s^V)^2 ds\right]\right\}\exp(U_t). \tag{18}$$

Then the return dynamics under the risk-neutral measure are expressed as:

[7] $\kappa^{\mathbb{Q}} = \kappa^{\mathbb{P}} - \eta_v$ and $\vartheta^{\mathbb{Q}} = \frac{\kappa^{\mathbb{P}}\vartheta^{\mathbb{P}}}{\kappa^{\mathbb{Q}}}$, we use κ and ϑ to represent $\kappa^{\mathbb{P}}$ and $\vartheta^{\mathbb{P}}$ in this article. Also, for simplicity, we use ρ to denote $\rho^{\mathbb{P}}$ and $\rho^{\mathbb{Q}}$, and use σ_V to represent $\sigma_V^{\mathbb{P}}$ and $\sigma_V^{\mathbb{Q}}$.

[8] In Yu, Li, and Wells (2011), no parameters in the SVLS model are allowed to change between measures. We relax this restriction to compare models under similar settings.

[9] In some previous studies (e.g., Pan, 2002; Yu, Li, and Wells 2011) μ_J is chosen to be the changeable parameter, besides, Hu and Liu (2022) allow both μ_J and λ to change between $\mathbb{P}$ and $\mathbb{Q}$; however, in many related papers, the estimates of μ_J under $\mathbb{P}$ tend to be insignificant (the posterior mean is almost 0 with large variance; see, e.g., Pan 2002; Asgharian and Bengtsson 2006; Li, Wells, and Yu 2008; and Yu, Li, and Wells 2011). In this article, we assume that μ_J remains unchanged between measures, so that information in both index returns and option prices can be considered.

[10] As jump processes are restricted to follow the same processes between measures, the expressions of U_t are different for models with different jump specifications. In this case, the models also differ in terms of Radon–Nikodym derivatives. Additionally, in this article, we use both e and exp to denote the exponential function.

$$dY_t = \left(r_t - \frac{1}{2}V_t + \Phi_J(-i)\right)dt + \sqrt{V_t}dW_t^Y(\mathbb{Q}) + dJ_t^Y(\mathbb{Q}),$$
$$dV_t = [\kappa(\vartheta - V_t) + \eta_v V_t]dt + \sigma_V\sqrt{V_t}dW_t^V(\mathbb{Q}) + dJ_t^V(\mathbb{Q}), \tag{19}$$

where r_t is the risk-free rate, $\Phi_J(-i)$ is the jump component, and the expressions of $\Phi_J(\cdot)$ for different models in this study are detailed in the Supplementary Appendix. Naturally, the drift term of the return process under $\mathbb{P}$ can be derived as $\mu = r_t - \frac{1}{2}V_t + \Phi_J(-i) + \eta_s V_t$.

When the risk-free interest rate is constant, the option price can be deduced from the closed-form solution to the characteristic function of the log stock price under $\mathbb{Q}$:

$$\phi(t,u) = \exp\left[iuY_0 + iu(r + \Phi_J(-i))t\right]\exp\left(-t\Phi_J(u)\right)\exp\left(-b(t,u)V_0 - c(t,u)\right), \tag{20}$$

where $\kappa^M(u) = \kappa - \eta_v - iu\sigma_V\rho$; $\delta(u) = \sqrt{(\kappa^M(u))^2 + (iu + u^2)\sigma_V^2}$; $Y_0 = \ln(S_0)$ denotes the log-spot price; V_0 represents the initial variance; $b(t,u) = \frac{(iu+u^2)(1-e^{-\delta(u)t})}{(\delta(u)+\kappa^M(u))+(\delta(u)-\kappa^M(u))e^{(-\delta(u)t)}}$; and $c(t,u) = \frac{\kappa\vartheta}{\sigma_V^2}\left[2\ln\frac{2\delta(u)-(\delta(u)-\kappa^M(u))(1-e^{-\delta(u)t})}{2\delta(u)} + (\delta(u) - \kappa^M(u))t\right]$.

Denoting by τ the time to expiration, we can price a European call option with strike K using the formula below (Yu, Li, and Wells 2011):

$$F(Y_0, V_0, \tau, K) = \mathrm{E}_0^{\mathbb{Q}}[e^{-r\tau}(S_\tau - K)^+] = \frac{e^{-r\tau}}{\pi} \times \mathrm{Re}\left(\int_0^\infty e^{-ix\ln(K)}\frac{\phi(\tau, x-i)}{-x^2 + ix}dx\right). \tag{21}$$

2.3 Three-Factor Double Exponential Model

Multifactor models have been shown to have superior performance in fitting the options market compared to single SV factor models (e.g., Christoffersen, Jacobs, and Ornthanalai 2012; Andersen, Fusari, and Todorov 2015b; Gruber, Tebaldi, and Trojani 2021). We also consider the three-factor model (SVTF hereafter) proposed by Andersen, Fusari, and Todorov (2015b):

$$\frac{dS_t}{S_{t-1}} = (r_t - \delta_t)dt + \sqrt{V_{1,t}}dW_{1,t}^{\mathbb{Q}} + \sqrt{V_{2,t}}dW_{2,t}^{\mathbb{Q}} + \eta\sqrt{U_t}dW_{3,t}^{\mathbb{Q}} + \int_{\mathbb{R}^2}(e^x - 1)\tilde{\mu}^{\mathbb{Q}}(dt, dx, dy),$$
$$dV_{1,t} = \kappa_1(\vartheta_1 - V_{1,t})dt + \sigma_{V,1}\sqrt{V_{1,t}}dB_{1,t}^{\mathbb{Q}} + \mu_1\int_{\mathbb{R}^2}x^2\mathbb{1}_{\{x<0\}}\mu(dt, dx, dy),$$
$$dV_{2,t} = \kappa_2(\vartheta_2 - V_{2,t})dt + \sigma_{V,2}\sqrt{V_{2,t}}dB_{2,t}^{\mathbb{Q}},$$
$$dU_t = -\kappa_u U_t dt + \mu_u\int_{\mathbb{R}^2}[(1-\rho_u)x^2\mathbb{1}_{\{x<0\}} + \rho_u y^2]\mu(dt, dx, dy). \tag{22}$$

$W_{1,t}^{\mathbb{Q}}, W_{2,t}^{\mathbb{Q}}, W_{3,t}^{\mathbb{Q}}, B_{1,t}^{\mathbb{Q}}, B_{1,t}^{\mathbb{Q}}$ are Brownian motions with $corr(W_{1,t}^{\mathbb{Q}}, B_{1,t}^{\mathbb{Q}}) = \rho_1$ and $corr(W_{2,t}^{\mathbb{Q}}, B_{2,t}^{\mathbb{Q}}) = \rho_2$, while the remaining Brownian motions are mutually independent. The number of jumps in the price and state factors is denoted by μ. The corresponding risk-neutral jump intensity is $dt \otimes \upsilon_t^{\mathbb{Q}}(dx, dy)$. The associated martingale jump measure can be expressed as $\tilde{\mu}^{\mathbb{Q}}(dt, dx, dy) = \mu(dt, dx, dy) - dt\upsilon_t^{\mathbb{Q}}(dx, dy)$, where x and y are two separate components that specify jumps. The co-jumps that occur simultaneously in S, V_1, and

U(if $\rho_u < 1$) are captured by x while y describes independent jumps in the U factor. Based on the above, the jump compensator is calculated as:

$$\frac{\upsilon_t^{\mathbb{Q}}(dx, dy)}{dxdy} = \begin{cases} \left(c^-(t) \mathbb{1}_{\{x<0\}} \lambda_- e^{-\lambda_-|x|} + c^+(t) \mathbb{1}_{\{x>0\}} \lambda^+ e^{-\lambda_+ x} \right) \text{ if } y = 0, \\ c^-(t) \lambda_- e^{-\lambda_-|y|} \text{ if } x = 0 \text{ and } y < 0. \end{cases} \tag{23}$$

The price jumps of the SVTF model are exponentially distributed, and the tail decay parameters for negative and positive jumps are separately specified as λ_- and λ_+. The time-varying jump intensities are given by:

$$\begin{aligned} c^-(t) &= c_0^- + c_1^- V_{1,t-1} + c_2^- V_{2,t-1} + U_{t-1}, \\ c^+(t) &= c_0^+ + c_1^+ V_{1,t-1} + c_2^+ V_{2,t-1} + c_u^+ U_{t-1}. \end{aligned} \tag{24}$$

We refer to Andersen, Fusari, and Todorov (2015b) for a more detailed description of the SVTF model and the option pricing approach. Also, we apply the same settings to SVTF as Andersen, Fusari, and Todorov (2015b).

3 Estimation Methodology

The models considered in this article are quite complex so parameter estimation may not be straightforward. For example, one problem is that the latent variables, such as SV and sizes and arrival rates of jump processes, are difficult to track. Moreover, Lévy processes are complex and many of them do not lead to closed-form option pricing formulae. We focus on joint estimation, which uses data under both physical and risk-neutral probability measures. Two joint estimation methodologies based on the Bayesian approach are developed in this study; the first one requires underlying spot prices and option prices (referred to as the SOP methodology), and the second one relies on underlying realized volatility and option IV (referred to as the RIV methodology). Additionally, these two methodologies assume different pricing error processes.

3.1 Joint Estimation with the SOP Methodology

Li, Wells, and Yu (2008) extend the application of the MCMC method to Lévy processes under the real-world probability measure ($\mathbb{P}$) for spot prices. Then, Yu, Li, and Wells (2011) further apply the MCMC method to Lévy processes under both $\mathbb{P}$ and $\mathbb{Q}$ probability measures with spot prices and option prices. Their setup has a groundbreaking contribution by jointly estimating parameters with both index returns and option prices under a Bayesian framework; however, these approaches have the following two limitations: first, the index returns and option prices data are restricted to be in the same time frame, which limits the estimation of the objective parameters. Generally, the estimation of risk-neutral parameters requires recent options data, while the estimation of real-world parameters demands index returns data over a longer period of time. The second limitation is that, only one at-the-money (ATM) call option per day is used in their estimation, which ignores out-of-the-money (OTM) options that are informative about jumps. The joint point-wise estimation with index prices and multiple options has been studied by Du and Luo (2019), who employ a maximum likelihood-based estimation method.[11] However, the parameter estimation with multiple options data using the MCMC approach has not been studied before in the literature.

[11] Durham (2013) also performs a joint analysis of the physical and risk-neutral models.

To address the above two issues, we propose extending the approach of Yu, Li, and Wells (2011). For model pricing errors, we still follow the autoregressive specification used in Yu, Li, and Wells (2011). The economic rationale is that if one day the option pricing error is high, then it is likely to be high on the following day as well. In order to allow for multiple options in the parameter estimation and set the pricing error updating equation in a more refined way, we consider a drift term a_{Δ_n} related to the pricing error process; a_{Δ_n} is a vector with the same length as the number of options used per day. We use Delta values to classify options because the option Delta measures the sensitivity of the option's value to changes in the underlying security price.[12] Let OP_{t,Δ_n} denote the market price at time t of the option with a Delta of Δ_n, while $F_{t,\Delta_n}(\mathcal{M}(\theta), Y, V)$ represents the model price at time t of model $\mathcal{M}$ with parameter vector θ for the option with a Delta of Δ_n, given Y and V, where the log stock prices are denoted by $Y = \{Y_t\}_{t=0}^{T}$, and the variance variables are $V = \{V_t\}_{t=0}^{T}$. Similarly to Yu, Li, and Wells (2011), we denote $PE_{t,\Delta_n}(\mathcal{M}(\theta), Y, V) = OP_{t,\Delta_n} - F_{t,\Delta_n}(\mathcal{M}(\theta), Y, V)$ henceforth. The joint dynamics of the daily spot and option prices upon discretization are summarized as follows:

$$
\begin{aligned}
Y_{t+1} &= Y_t + \left(r_t - \frac{1}{2}V_t + \psi_J^{\mathbb{Q}}(-i) + \eta_s V_t\right)\delta^* + \sqrt{V_t\delta^*}\epsilon_{t+1}^{Y} + J_{t+1}^{Y}, \\
V_{t+1} &= V_t + \kappa(\vartheta - V_t)\delta^* + \sigma_V\sqrt{V_t\delta^*}\epsilon_{t+1}^{V} + J_{t+1}^{V}, \\
PE_{t+1,\Delta_n}(\mathcal{M}(\theta), Y, V) &= a_{\Delta_n} + \rho_c PE_{t,\Delta_n}(\mathcal{M}(\theta), Y, V) + \sigma_c\epsilon_{t+1,\Delta_n}^{c}, \\
a_{\Delta_n} &\sim \mathbb{N}(a_m, a_s^2),
\end{aligned}
\tag{25}
$$

where δ^* is the one day time interval; ϵ_t^c, ϵ_{t+1}^Y and ϵ_{t+1}^V follow the standard normal distribution, ϵ_{t+1}^Y and ϵ_{t+1}^V are correlated with correlation ρ and are independent from ϵ_t^c. In addition, $a_{\Delta_n} \sim \mathbb{N}(a_m, a_s^2)$ provides random effects to the autoregressive pricing error process; a_m is the average size of a_{Δ_n} while a_s modulates the varying effects of the drift term across options with different strike prices as determined by the Delta values. The autoregressive specification also reflects the stochastic singularity argument mentioned in Johannes and Polson (2010); and since $\rho_c \in [-1, 1]$, the pricing error cannot get very large (otherwise the market will misprice the options for too long, allowing for arbitrage). Although discretization errors might impact the model risk, the discretization bias is quite small using daily data (Eraker 2004).

In addition to the Bayesian updating of the model, we also try to improve the estimation method computationally. Considering θ is the parameter vector under a given model, we split the parameters into four groups, that is $\theta = \left\{(\theta^{\mathbb{P}\mathbb{Q}}), (\theta^{uni}), (\theta^{risk\ \ premia}), (\theta^{pricing\ \ errors})\right\}$; the first group contains the parameters that remain unchanged between measures; the second one includes the parameters that are unique under $\mathbb{P}$ or $\mathbb{Q}$; the risk premium associated with "Brownian" return risk and volatility risk are in the third group, while the fourth part contains the parameters used to describe the option pricing errors of the models. For example, the SV model has no parameters that are unique under $\mathbb{P}$ or $\mathbb{Q}$, so $\theta = \{(\kappa\vartheta, \sigma_v, \rho), (), (\eta_s, \eta_v), (a_{\Delta_n}, a_m, a_s, \rho_c, \sigma_c)\}$; for SVJ, $\theta = \{(\kappa\vartheta, \sigma_v, \rho, \sigma_J, \mu_J), (\lambda^{\mathbb{P}}, \lambda^{\mathbb{Q}}), (\eta_s, \eta_v), (a_{\Delta_n}, a_m, a_s, \rho_c, \sigma_c)\}$; for SVCJ, $\theta = \{(\kappa\vartheta, \sigma_v, \rho, \sigma_J, \rho_J, \mu_J, \mu_V), (\lambda^{\mathbb{P}}, \lambda^{\mathbb{Q}}), (\eta_s, \eta_v), (a_{\Delta_n}, a_m, a_s, \rho_c, \sigma_c)\}$; for SVVG, $\theta = \{(\kappa\vartheta, \sigma_v, \rho, \nu), (\gamma^{\mathbb{P}}, \sigma^{\mathbb{P}}, \gamma^{\mathbb{Q}}, \sigma^{\mathbb{Q}}), (\eta_s, \eta_v), (a_{\Delta_n}, a_m, a_s, \rho_c, \sigma_c)\}$; and SVLS has $\theta = \{(\kappa\vartheta, \sigma_v, \rho, \sigma), (\alpha^{\mathbb{P}}, \alpha^{\mathbb{Q}}), (\eta_s, \eta_v), (a_{\Delta_n}, a_m, a_s, \rho_c, \sigma_c)\}$.

[12] Moreover, it is easy to determine the main information of an option by its Delta. We also select option data based on the Delta values in the empirical analysis.

Given the option prices $OP = \{OP_t\}_{t=T^*}^{T}$, where $0 \leq T^* \leq T$[13] and jumps times/sizes $J = \{J_t\}_{t=0}^{T}$, the posterior of parameters and latent variables can be decomposed into the product of individual conditionals:

$$p(\theta, V, J|Y, OP) \propto p(Y, OP, V, J, \theta) = p(OP|Y, V, \theta)p(Y, V|J, \theta)p(J|\theta)p(\theta). \tag{26}$$

Assuming n^* options are used per day, we have:

$$p(OP|Y, V, \theta) = \prod_{n=1}^{n^*}\prod_{t=T^*}^{T-1}\frac{1}{\sqrt{2\pi}\sigma_c}\exp\left\{-\frac{[PE_{t+1,\Delta_n}(\mathcal{M}(\theta), Y, V) - \rho_c PE_{t,\Delta_n}(\mathcal{M}(\theta), Y, V) - a_{\Delta_n}]^2}{2\sigma_c^2}\right\}. \tag{27}$$

For the SVVG model, a time-changing variable $G = \{G_t\}_{t=0}^{T}$, where[14] $G_{t+1} \sim \mathbb{G}(\frac{\delta^*}{\nu}, \nu)$, is introduced as a conditional latent variable on the jump process:

$$p(\theta, V, J, G|Y, OP) \propto p(Y, OP, V, J, G, \theta) = p(OP|Y, V, \theta)p(Y, V|J, \theta)p(J|G, \theta)p(G|\theta)p(\theta). \tag{28}$$

For the SVLS model, an auxiliary variable series $U = \{U_t\}_{t=0}^{T}$ is added:

$$p(\theta, V, J, U|Y, OP) \propto p(Y, OP, V, J, U, \theta) = p(OP|Y, V, \theta)p(Y, V|J, \theta)p(J, U|\theta)p(\theta). \tag{29}$$

It is difficult to simulate random draws directly from the joint posterior densities of the models shown above; instead, we estimate the parameters and latent variables by simulating from complete conditional distributions of each parameter and latent variable with the MCMC method. The complete conditional distributions of AJD models and Lévy processes under $\mathbb{P}$ can be found in several earlier studies, but few investigate the estimation of the parameters by a Bayesian approach under $\mathbb{Q}$. Broadie, Chernov, and Johannes (2007) simulate posterior distributions of $\theta^{\mathbb{P}}$ (parameters under $\mathbb{P}$) with complete conditional distributions and then they calibrate models with the estimated parameters under $\mathbb{P}$ (represented by $\hat{\theta}^{\mathbb{P}}$) to obtain the values of $\theta^{\mathbb{Q}}$ (parameters under $\mathbb{Q}$) based on the following objective function:

$$\theta^{\mathbb{Q}} = \operatorname{argmin}\sum_{t=T^*}^{T}\sum_{n=1}^{O_t}[IV_t(K_n, \tau_n, S_t, r_t, V_t) - IV_t(\theta^{\mathbb{Q}}|\hat{\theta}^{\mathbb{P}}, K_n, \tau_n, S_t, r_t, V_t)]^2, \tag{30}$$

here O_t is the number of options at time t; S_t, V_t, and r_t denote the spot price, instant variance, and risk-free rate at time t, respectively; K_n and τ_n represent the strike price and expiration of the n-th option; and IV is the IV. A disadvantage of this approach is that only the mean values of the posterior of the parameters under $\mathbb{P}$ are considered when calibrating the models, overlooking other possible values in the posterior distribution of parameters under $\mathbb{P}$. This two-step estimation method produces an interval estimation (under $\mathbb{P}$) plus a point estimation (under $\mathbb{Q}$), which is not ideal for taking PER into consideration. A different approach is presented by Yu, Li, and Wells (2011) who derive the complete conditional distribution of each individual parameter and latent variable under both measures, enabling the simulation of posterior samples of parameters and latent variables with the MCMC method. For objective parameters, they apply almost the same method with

[13] Y and OP are not necessarily in the same time frame. In this article, we only control for the two sets of data by making sure that they end on the same date.

[14] $\mathbb{G}$ denotes the Gamma distribution.

complete conditional distributions as Broadie, Chernov, and Johannes (2007), after which the random draws are accepted/rejected with the Damlen, Wakefild, and Walker method (Damlen et al., 1999) based on the likelihood value calculated with Equation (27) while the Metropolis–Hasting algorithm is used to estimate $\theta^{\mathbb{Q}}$. In the two-step method of Broadie, Chernov, and Johannes (2007), $\theta^{\mathbb{P}}$ is estimated from spot prices and $\theta^{\mathbb{Q}}$ is calibrated with $\theta^{\mathbb{P}}$ and option data, whereas Yu, Li, and Wells (2011) do these in one step, and the parameters are estimated with both spot price and option data.[15]

Combining the methods of Yu, Li, and Wells (2011) and Broadie, Chernov, and Johannes (2007), we estimate κ, ϑ, σ_v, and ρ with only spot prices based on the MCMC methods introduced in Li, Wells, and Yu (2008), while the other parameters are estimated with the method of Yu, Li, and Wells (2011) with both index returns and option prices. The corresponding risk-neutral values of κ and ϑ can be calculated with η_v as explained in Section 2.2. In order to speed up the estimation process, option data are also not considered when estimating σ_v and ρ; as explained in Broadie, Chernov, and Johannes (2007), the comparison of the evolution of Y and V under both measures implies that these parameters can be estimated using either equity index returns or option prices; besides, computationally, the determinant factor in estimating the size of $p(\{\sigma_v, \rho\}|Y, OP)$ is $p(Y|\{\sigma_v, \rho\})$. Even then, the estimation can be computationally expensive as it requires a recomputation of all option prices for each of the other parameters. Our Markov chain has 100,000 iterations in total. To speed up the estimation process, we first update κ, ϑ, σ_v, and ρ, which require only index returns, and take the first 90,000 runs as the burn-in period. The other parameters that need to be estimated with both index returns and option prices are started to update from the 80,001th iteration and discard the first 10,000 runs as the burn-in period. The option data contain more information, and 10,000 runs are sufficient to obtain convergence for the estimates.[16] Our method has less computational burden than the approach of Yu, Li, and Wells (2011); besides, compared with Broadie, Chernov, and Johannes (2007), our method estimates risk-neutral parameters and real-world parameters jointly, so that the PER can be captured fully.

However, the autoregressive specification of pricing errors may lead to unrealistic estimates of risk-neutral parameters. This is because the conditional probability (27) focuses on the autoregressive property between of $PE_{t,\Delta_n}(\mathcal{M}(\theta), Y, V)$ rather than on the size of the pricing error itself. By minimizing the pricing errors while capturing the autoregressive feature of the pricing errors, we introduce an extra condition for the magnitude of the pricing errors in the MCMC process when updating parameters whose posterior distributions contain the conditional probability (27), specifically: we accept the updated parameter $\tilde{\theta}^{(g+1)}$ only when the sum of squared errors associated with it is lower than the sum of squared errors computed with the previous value of the parameter, $\tilde{\theta}^{(g)}$:

$$\sum_{t=T^*}^{T}\sum_{n=1}^{n^*}\left[OP_{t,\Delta_n} - F_{t,\Delta_n}\left(\mathcal{M}\left(\theta|\left\{\tilde{\theta}^{(g+1)}\right\}\right), Y, V\right)\right]^2 \leqslant \sum_{t=T^*}^{T}\sum_{n=1}^{n^*}\left[OP_{t,\Delta_n} - F_{t,\Delta_n}\left(\mathcal{M}\left(\theta|\left\{\tilde{\theta}^{(g)}\right\}\right), Y, V\right)\right]^2, \tag{31}$$

where $F_{t,\Delta_n}(\mathcal{M}(\theta|\{\tilde{\theta}^{(g)}\}), Y, V)$ denotes the model price implied by the g-th updated parameter value; $\theta|\{\tilde{\theta}^{(g)}\}$ represents the parameter vector given the g-th updated $\tilde{\theta}$; and $\tilde{\theta}$ can be any model parameter that requires this extra condition for estimation.[17]

15 Candidate points of $\theta^{\mathbb{P}}$ are generated based on the posterior distribution with spot prices and only the points that fit the option prices are accepted.

16 Yu, Li, and Wells (2011) also discard the first 10,000 runs as "burn-in" period; Du and Luo (2019) sample 5000 times to estimate jump size/times and discard the first 2000 as burn-in.

17 This problem can be circumvented by disregarding the autoregressive component of pricing errors and using the size of pricing errors directly. However, as the empirical results highlight (ρ_c is about 0.8, see Figure D2 in the Supplementary Appendix), the autoregressive specification is a feature that should not be ignored when

More details of the MCMC methods for each model are provided in the Supplementary Appendix.

3.2 Joint Estimation with the RIV Methodology

The RIV estimation methodology builds on the approach used in Andersen, Fusari, and Todorov (2015b). We estimate the model with both option IV and realized volatility data. In order to capture PER, we estimate the parameters using a Bayesian approach. Following Andersen, Fusari, and Todorov (2015b), the implied and realized volatility fitting errors are adjusted with the 30-day ATM IV. As multiple options are used per day, we assume that the average fitting errors on day t follow a normal distribution:

$$\frac{1}{N_t}\sum_{j=1}^{N_t}\frac{\bar{IV}_{t,k_j,\tau_j}}{\sqrt{V_t^{ATM}}}-\frac{1}{N_t}\sum_{j=1}^{N_t}\frac{IV(k_j,\tau_j,Z_t,\theta)}{\sqrt{V_t^{ATM}}}=\sigma^{OF}\epsilon_t^{OF},$$
$$\frac{\sqrt{\hat{V}_t}}{\sqrt{V_t^{ATM}}}-\frac{\sqrt{V(Z_t,\theta)}}{\sqrt{V_t^{ATM}}}=\sigma^{VF}\epsilon_t^{VF}. \tag{32}$$

On day t, N_t options are used. Given $Z_t=\{V_{1,t},V_{2,t},U_t\}$, $IV(k_j,\tau_j,Z_t,\theta)$ is the model IV for option k_j, $\bar{IV}_{t,k_j,\tau_j}$ is the market price implied IV for option k_j; the model implied diffusive spot variance is $V(Z_t,\theta)=V_{1,t}+V_{2,t}+\eta^2U_t$; $\hat{V}_t$ is a nonparametric estimator of the diffusive spot variance constructed from the underlying intraday asset prices; and V_t^{ATM} is the squared short-term ATM Black-Scholes IV.

Let $PE_t^{OF}=\frac{1}{N_t}\sum_{j=1}^{N_t}\frac{\bar{IV}_{t,k_j,\tau_j}}{\sqrt{V_t^{ATM}}}-\frac{1}{N_t}\sum_{j=1}^{N_t}\frac{IV(k_j,\tau_j,Z_t,\theta)}{\sqrt{V_t^{ATM}}}=\sigma^{OF}\epsilon_t^{OF}$ and $PE_t^{VF}=\frac{\sqrt{\hat{V}_t}}{\sqrt{V_t^{ATM}}}-\frac{\sqrt{V(Z_t,\theta)}}{\sqrt{V_t^{ATM}}}=\sigma^{VF}\epsilon_t^{VF}$. We assume that the implied and realized volatility fitting errors are correlated and $corr(\epsilon^{OF},\epsilon^{VF})=\rho^{Fit}$. Following Andersen, Fusari, and Todorov (2015b), we also consider the tuning factor, represented by ξ; we get $PE_t^{OF}+\xi PE_t^{VF}=\sqrt{(\sigma^{OF})^2+(\xi\sigma^{VF})^2+2\rho^{Fit}\sigma^{OF}(\xi\sigma^{VF})}\epsilon_t^{Fit}$, where ϵ^{Fit} follows the standard normal distribution. We obtain the following posterior probability:

$$p(IV,Z|\hat{V}_t,V_t^{ATM},\theta)=\prod_{j=1}^{N_t}\prod_{t=1}^{T}\frac{1}{\sqrt{2\pi}\sqrt{(\sigma^{OF})^2+(\xi\sigma^{VF})^2+2\rho^{Fit}\sigma^{OF}\xi\sigma^{VF}}}$$
$$\times\exp\left\{-\frac{[PE_t^{OF}+\xi PE_t^{VF}]^2}{2((\sigma^{OF})^2+(\xi\sigma^{VF})^2+2\rho^{Fit}\sigma^{OF}\xi\sigma^{VF})}\right\}. \tag{33}$$

Let $(\sigma^{OF})^2+(\xi\sigma^{VF})^2+2\rho^{Fit}\sigma^{OF}\xi\sigma^{VF}=\sigma^S$, the log-likelihood function of Equation (33) is:

$$LLK(IV,Z|\hat{V}_t,V_t^{ATM},\theta)$$
$$=\sum_{j=1}^{N_t}\sum_{t=1}^{T}[-\frac{1}{2}ln(2\pi)-\frac{1}{2}ln(\sigma^S)-\frac{1}{2\sigma^S}[PE_t^{OF}+\xi PE_t^{VF}]^2]. \tag{34}$$

measuring model risk; that is, the pricing performance of a model to price an option today should relate to the model's performance the day before.

Given the log-likelihood function, all model parameters and latent variables are estimated using the Gibbs sampling method; σ^{OF}, σ^{VF}, and ρ^{Fit} are estimated with the Metropolis–Hasting method. The tuning factor is set to 0.2 following Andersen, Fusari, and Todorov (2015b).

4 Data

The two estimation methodologies we introduced specify different pricing error processes and require different data types. The SOP methodology relies on spot and option prices, and assumes an autoregressive pricing error process; in contrast, the RIV methodology requires implied and realized volatility data, and assumes that the pricing errors are normally distributed.

4.1 Data Used in the SOP Estimation Methodology

The empirical analysis around the SOP methodology is based on the S&P 500 index spot price from January 2, 1992 to December 29, 2017 and the corresponding S&P 500 index put option prices from January 3, 1996 to December 29, 2017. We use six daily series of option prices, and they are put options with 30 days to maturity with Delta values of −25, −30, −35, −40, −45, and −50, respectively. The option dataset is downloaded in the volatility surface file from Option Metrics. This file contains information on options with expiration ranges from 30 to 730 calendar days, and the 30-days-to-expiration options fully match the requirements of Yu, Li, and Wells (2011). The forward price and corresponding strike price of the underlying on the expiration date of the option are calculated with the zero-coupon yield curve and projected dividends; the option premium on these options are calculated daily using linear interpolation from the volatility surface, which is computed via kernel smoothing. Compared with picking options in the real market, the data in the volatility surface file have constant duration and Delta values, reducing measurement error arising from options that vary in maturity and moneyness. Also, the daily Treasury yield curve rates from the U.S. Department of the Treasury are used as the risk-free rate; these are referred to as constant duration Treasury rates, which provide the yield curve at fixed maturities.[18]

4.2 Data Used in the RIV Estimation Methodology

The RIV estimation methodology requires market option data, and the S&P 500 index option data are downloaded from OptionMetrics. We restrict our attention to out-of-the-money options with maturity between 7 and 180 calendar days.[19] We apply some commonly used option data filters to the raw data.[20] The realized volatility calculated with intraday data is downloaded from the Oxford-Man Institute's realized library. Moreover, we use the zero-coupon yield for the corresponding maturities (linear interpolation is used when necessary) to proxy the risk-free rate. The zero-coupon yield and the dividend yield are obtained from OptionMetrics. We consider two sample periods. Period one is the crisis

[18] One-month Treasury yield curve rates are not available prior to July 31, 2001; so the one-month risk free rate before this date is estimated with the three-month yield curve: (1) run the regression $rf_t^{one} = \beta_0 + \beta_1 rf_t^{three}$ with rf^{one} and rf^{three} denoting the one-month and three-month Treasury yields during July 31, 2001 to December, 29 2017, respectively. The estimated coefficients are $\hat{\beta}_0 = -0.0359$ and $\hat{\beta}_1 = 0.9822$, the adjusted R^2 is 99.5%. (2) the one-month risk free rate before July 31, 2001 is estimated as $rf_t^{one} = \hat{\beta}_0 + \hat{\beta}_1 rf_t^{three}$, where rf_t^{three} is the three-month yield before July 31, 2001. A more descriptive data summary is provided in the Supplementary Appendix.

[19] Several papers in the literature, such as Bardgett, Gourier, and Leippold (2019), rely on different option maturities to model the term structure of volatilities.

[20] More discussions about the data filter can be found in Johannes and Polson (2010) and Andersen, Fusari, and Todorov (2015b).

period ranging from January 2006 to December 2009, the other one is the post-crisis period spanning from January 2010 to December 2017.

5 Empirical Analysis

This section investigates the model risk of the models introduced in Section 2. Both the SOP and RIV estimation methodologies are used, and the corresponding results are presented in Sections 5.1 and 5.2.

5.1 Empirical Analysis Using the SOP Methodology

Using the SOP estimation methodology described in Section 3.1 and the data introduced in Section 4.1, we study the model risk of the single-factor models described in Section 2.1.[21] We employ MCMC to estimate $\theta = \{(\theta^{\mathbb{PQ}}), (\theta^{uni}), (\theta^{risk\ \ premia}), (\theta^{pricing\ \ errors})\}$. The empirical results are based on annual rolling window estimation. We use the current year option data in the estimation window. However, the estimation using 1-year returns can lead to very large PER, so we use 5-year returns; this is also needed because the objective parameters estimation needs a longer time period, as discussed in Section 3. For example, in 2011, we use index returns from the beginning of 2007 to the end of 2011 and option data in 2011 to estimate the parameters jointly. Applying the moving window estimation, we obtain 22 sets of estimates from 1996 to 2017.[22]

5.1.1 Model risk

This section provides an analysis of model risk. Supplementary Appendix Figure D4 presents the 5% to 95% quantiles of the estimated prices, showing that all models suffer from model risk. The sizes of model risk are reported in Table 1. TMR, PER, and MSR are defined in (12), (10), and (11), respectively. Jump models have lower TMR compared with SV for all Delta values; besides, SVJ, SVCJ, and SVLS always have smaller MSR than SV; SVVG has the largest MSR and the least TMR and PER, while SVLS has the lowest MSR. Moreover, we find that MSR increases with Delta (OTM options tend to have larger MSR), while PER and TMR decrease with Delta (OTM options tend to have lower PER and TMR). The deep OTM put options (high Delta) have low prices and have lower PER and higher MSR than options with lower Delta values. The reason for the higher MSR values for deep OTM options is that single-factor models are less able to capture the IV surface of deep OTM options (Andersen, Fusari, and Todorov 2015a).

In order to analyze the model risk of models during different market periods, we split the sample period into seven time windows by detecting abrupt changes in the IV of the standardized ATM options based on the method proposed by Killick, Fearnhead, and Eckley (2012).[23] Using average values of IV, we identify the following time windows: : January 3, 1996 to June 4, 1997; : June 5, 1997 to October 13, 2003; : October 14, 2003 to July 24, 2007; : July 25, 2007 to September 23, 2008; : September 24, 2008 to May 1, 2009; : May 4, 2009 to June 29, 2012; and : July 2, 2012 to December 29, 2017.

The values of the mean and standard deviation of the IV of the standardized ATM options during these seven time windows are reported in Panel A of Table 2. Period is the most tranquil period, followed by periods and , which are characterized by mean IV values below 0.2. During these three periods, the market is stable. In contrast, the market is turbulent during periods , , and . Period is very long and covers the 1997

21 Estimating the SVTF model using the SOP methodology can be challenging as the variance-related parameter estimates rely heavily on the spot returns; however, the two variance processes in the SVTF model are primarily specified to capture the implied volatility smile.

22 A comparison of model parameter inferences and a benchmark exercise with simulated data can be found in the Supplementary Appendix.

23 The ATM option dataset is downloaded in the Std Option Price file from Option Metrics.

Table 1. Model Risk under different values of delta

	SVLS	SVVG	SVCJ	SVJ	SV
Delta = −25					
PER_L	2.8512	1.6104	2.7691	2.7210	2.8454
PER_S	3.4434	1.8732	3.3379	3.2393	3.5582
PER	3.4637	1.8857	3.3841	3.2764	3.5768
MSR_L	0.0317	0.1444	0.0299	0.0373	0.0347
MSR_S	0.9932	2.1403	1.0679	1.0368	1.0872
MSR	1.0249	2.2846	1.0978	1.0740	1.1219
TMR	4.4886	4.1703	4.4820	4.3504	4.6987
Delta = −30					
PER_L	3.2802	1.8688	3.1939	3.1219	3.3082
PER_S	3.8790	2.1284	3.7575	3.6434	4.0269
PER	3.9096	2.1445	3.8186	3.6946	4.0539
MSR_L	0.0425	0.2046	0.0411	0.0518	0.0495
MSR_S	0.8079	1.9375	0.8826	0.8547	0.8873
MSR	0.8504	2.1422	0.9237	0.9066	0.9368
TMR	4.7600	4.2867	4.7423	4.6012	4.9907
Delta = −35					
PER_L	3.6294	2.0776	3.5669	3.4584	3.6800
PER_S	4.2170	2.3253	4.0903	3.9623	4.3842
PER	4.2606	2.3477	4.1718	4.0289	4.4203
MSR_L	0.0563	0.2701	0.0536	0.0684	0.0687
MSR_S	0.6688	1.7494	0.7390	0.714	0.7312
MSR	0.7251	2.0195	0.7926	0.7823	0.7998
TMR	4.9857	4.3672	4.9644	4.8113	5.2201
Delta = −40					
PER_L	3.9070	2.2402	3.8877	3.7372	3.9645
PER_S	4.4734	2.4712	4.3525	4.2115	4.6454
PER	4.5305	2.5021	4.4612	4.2944	4.6905
MSR_L	0.0723	0.3355	0.0683	0.0847	0.0923
MSR_S	0.5562	1.5753	0.6253	0.5998	0.6058
MSR	0.6286	1.9108	0.6936	0.6845	0.6981
TMR	5.1591	4.4130	5.1548	4.9789	5.3886
Delta = −45					
PER_L	4.1105	2.3555	4.1405	3.9522	4.1606
PER_S	4.6558	2.5701	4.5489	4.3963	4.8200
PER	4.7248	2.6097	4.6843	4.4943	4.8725
MSR_L	0.0887	0.4016	0.0846	0.1006	0.1209
MSR_S	0.4660	1.4167	0.5345	0.5092	0.5048
MSR	0.5547	1.8183	0.6191	0.6098	0.6257
TMR	5.2795	4.4280	5.3035	5.1042	5.4982

(continued)

Table 1. (continued)

	SVLS	SVVG	SVCJ	SVJ	SV
Delta = −50					
PER_L	4.2319	2.4204	4.3058	4.0903	4.2655
PER_S	4.7663	2.6234	4.6771	4.5161	4.9139
PER	4.8430	2.6702	4.8308	4.6244	4.9707
MSR_L	0.1066	0.4683	0.1005	0.1146	0.1524
MSR_S	0.3954	1.2718	0.4622	0.4396	0.4228
MSR	0.5020	1.7401	0.5626	0.5542	0.5752
TMR	5.3450	4.4103	5.3934	5.1786	5.5459

Notes: This table reports the average values of model risk measures for all models when pricing put options with Delta values of −25, −30, −35, −40, −45, and −50. PER, MSR, and TMR are defined in Equations (10), (11), and (12), respectively; the PER for long and short positions (PER_L and PER_S) are computed using Equation (7); the MSR for long and short positions (MSR_L and MSR_S) are computed using Equation (8). The results are based on annual rolling window estimation with daily spot prices of the S&P 500 from January 2, 1992 to December 29, 2017 and the corresponding S&P 500 index put option prices with Delta values of −25, −30, −35, −40, −45, and −50 from January 3, 1996 to December 29, 2017. SV is the stochastic volatility model; SVJ is the stochastic volatility model with Merton jumps in returns; SVCJ is the stochastic volatility model with contemporaneous jumps in returns and volatility; SVVG is the stochastic volatility model with variance-gamma jumps in returns; and SVLS is the stochastic volatility model with log-stable jumps in returns.

Asian financial crisis, the Russian financial crisis that hit on August 17, 1998, the Brazilian currency crisis from 1998 to 1999, the dot-com crash from March 11, 2000 to October 9, 2002, the 1998-to-2002 Argentine great depression, the 911 (September 11, 2001) and the WorldCom accounting scandal in 2002; period involves the subprime crisis; and period captures the European credit crisis since the end of 2009. Period marks the well-known Financial Crisis of 2008, when the mean IV reaches 0.4486 with a standard deviation of 0.0966.

Figure 2 illustrates the model risk estimates of our models for put options with a Delta of −45. The model risks of the put options with a Delta of −45 are used in the subsequent analysis, and the conclusions that can be drawn from the model risks of other options are similar. The black columns mark the PER of the models, while the gray columns are the TMR values; the differences between the gray and black columns represent the MSR. We find that MSR is the main source of TMR for SVVG, while PER is the dominant component of TMR for the other models. The TMR of each model has almost uniform trends and reflects the market conditions; the TMR is relatively low during stable periods , , and and relatively high during turbulent periods , , and ; TMR peaks during period , the Financial Crisis. PER dominates the model risk during quiet periods; there is almost no obvious MSR during period . In contrast, MSR becomes an important source of TMR when the market is unstable. Not surprisingly, a sudden and large MSR appears during the Financial Crisis for all models, indicating that all jump models fail to capture the market dynamics during this intense period. Model users should be aware of changes in model risk and replace the model if necessary. For example, SVVG has the lowest TMR in period , but its MSR increases dramatically during period and leads to a high TMR. By contrast, during the Financial Crisis, SVCJ has the lowest MSR with SVLS having the lowest TMR.

Panel B of Table 2 reports the average values of the differences between PER_L and PER_S during different periods, and significance is obtained based on the *t*-test.[24] A negative value indicates that a short position bears a higher PER. The results indicate that a short position

[24] A figure of the model risk of long and short positions of the models can be found in the Supplementary Appendix.

Table 2. Statistics on IV and the differences between the PER of long and short positions

Period							
Panel A: The mean and standard deviation of the IV of standardized ATM options							
Mean	0.1568	0.2265	0.1244	0.2182	0.4486	0.2080	0.1272
Std	0.0236	0.0484	0.0207	0.0351	0.0966	0.0557	0.0336
Panel B: Mean differences between the PER of long and short positions							
SV	−0.4985**	−0.5898**	−0.2329**	−0.6771**	−0.1266	−0.7017**	−1.1023**
SVJ	−0.3924**	−0.5441**	−0.1178**	−0.3356**	−0.9025**	−0.4824**	−0.5175**
SVCJ	−0.3994**	−0.5876**	−0.1531**	−0.4612**	0.4611**	−0.4878**	−0.4174**
SVVG	−0.0630**	−0.1375**	0.0138*	−0.1310**	0.0169**	−0.2865**	−0.5011**
SVLS	−0.4746**	−0.4879**	−0.1976**	−0.4598**	−0.3405**	−0.8413**	−0.7408**

Notes: Panel A reports the mean and standard deviation of the IV of the standardized ATM options during different periods; Panel B reports the mean values of the differences between the PER of long and short positions for put options with a Delta of −45 during different time periods. * and ** indicate values significant at 5% and 1% significance levels based on *t*-tests, respectively. SV is the stochastic volatility model; SVJ is the stochastic volatility model with Merton jumps in returns; SVCJ is the stochastic volatility model with contemporaneous jumps in returns and volatility; SVVG is the stochastic volatility model with variance-gamma jumps in returns; and SVLS is the stochastic volatility model with log-stable jumps in returns.

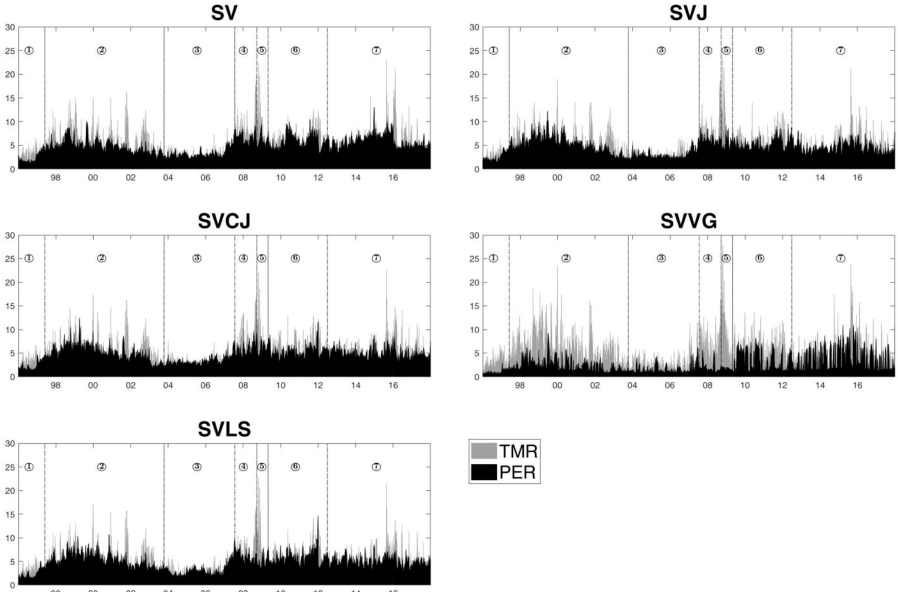

Figure 2. This figure presents the daily average TMR (gray columns) and PER (black columns) for S&P 500 index put options with a Delta of −45 between January 3, 1996 and December 29, 2017. SV is the stochastic volatility model; SVJ is the stochastic volatility model with Merton jumps in returns; SVCJ is the stochastic volatility model with contemporaneous jumps in returns and volatility; SVVG is the stochastic volatility model with variance-gamma jumps in returns; and SVLS is the stochastic volatility model with log-stable jumps in returns.

tends to have significantly higher PER for all models during normal periods. During the Financial Crisis (period), the difference for SV is negative but not significant, while a long position bears a greater model risk for SVCJ and SVVG. Overall, sellers are generally exposed to a higher model risk than buyers.

5.1.2 Analyzing the components of model risk

In this section, we highlight the necessity of measuring PER and MSR separately in terms of explaining absolute pricing errors.

Let $e_t(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) = |\hat{F}_t(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) - OP_t(\mathcal{C})|$ represent the absolute pricing error of option $\mathcal{C}$ conditional on model $\mathcal{M}$ with parameter vector θ, observed dataset $\mathcal{D}$, and methodology $\mathcal{K}$ at time t. We run the following regression to explore the explanatory power of the PER and MSR in explaining absolute pricing errors.

$$e_t(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) = \beta_0 + \beta_1\rho^{PER}_{\eta,t}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) + \beta_2\rho^{MSR}_{\eta,t}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) + \varepsilon_t. \quad (35)$$

Here, we focus on testing whether $\beta_1 = \beta_2$, as this would prove that it is not necessary to separate the PER and MSR from TMR. Taking $\beta_1 - \beta_2 = \alpha$ and using Equation (12), Equation (35) can be rewritten as:

$$e_t(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) = \beta_0 + \alpha\rho^{PER}_{\eta,t}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) + \beta_2\rho^{TMR}_{\eta,t}(\mathcal{C};\mathcal{M}(\theta),\mathcal{D},\mathcal{K}) + \varepsilon_t. \quad (36)$$

A test of $\alpha = 0$ in Equation (36) is a test of $\beta_1 = \beta_2$ in Equation (35). The results are reported in Table 3. All α's are statistically significant, which supports the necessity of measuring PER and MSR separately.

5.1.3 Out-of-sample analysis

In January of each year in our sample period, we use the estimated parameter values and forecast option prices for 1 month ahead, as well as for 1 year ahead. Our model risk measures are comparable with the absolute forecasting errors (AFEs). As described in Section 1, PER is one of the reasons behind forecasting error, and the part of the forecasting error that cannot be explained by PER, has to be explained by model mis-specification. Measuring model risk as described in Section 1, we decompose the forecasting errors into two parts: one is explained by PER, denoted by FE_{PER} and the other one captures the forecasting error stemming from MSR (denoted by FE_{MSR}). This second component is 0 if the AFE is fully explained by PER, and is equal with AFE − FE_{PER} if AFE is greater than PER. The results are presented in Table 4. The models with jumps no longer reveal substantially smaller forecasting errors than the SV model when the forecasting horizon is long; this is reasonable as it is difficult to forecast jumps for longer horizons. Only the forecasting errors of SVJ and SVLS are lower than those of the SV model for the one month forecasting horizon. AFE is greater than PER for all models, implying the existence of MSR. FE_{PER} reveals small increases when the forecasting horizon changes from 1 month to 1 year; in contrast, FE_{MSR} almost doubles, indicating that the forecasting error due to model specification increases with the length of the forecasting horizon.

5.2 Empirical Analysis Using the RIV Methodology

As defined in Section 1, model risk depends on the methodology and data used. In this subsection, we investigate the model risk of these option pricing models using the RIV methodology. As explained earlier, the RIV estimation methodology relies on underlying realized and option-IV data. Also, here we consider the SVTF model as well in addition to the models used in the previous section. The data used in this subsection are presented in Section 4.2.[25]

[25] Compared to the SOP estimation methodology, the RIV methodology is more time-consuming as all parameters are jointly determined by the option data, which means we need to calculate option prices when updating parameters in each iteration of the MCMC estimation. For example, for SVTF, following the setting of Andersen, Fusari, and Todorov (2015b), although several parameters are not used, we still need to calculate option prices 22 times in each iteration. The advantage of this method is that the parameters can converge faster;

Table 3. Explaining absolute pricing errors with PER and TMR

	β_0	α	β_2	**Adj. R^2 (%)**
SV	0.49^{**}	-1.09^{**}	1.47^{**}	72.76
SVJ	0.34^{**}	-1.03^{**}	1.45^{**}	73.35
SVCJ	0.28^{*}	-1.05^{**}	1.48^{**}	72.27
SVVG	0.43^{**}	-0.68^{**}	1.10^{**}	89.81
SVLS	0.46^{**}	-1.10^{**}	1.48^{**}	70.51

Notes: The regression results are based on Equation (36) with the model risk of the put options with a Delta of −45. * and ** indicate values significant at 5% and 1% significance levels, respectively. SV is the stochastic volatility model; SVJ is the stochastic volatility model with Merton jumps in returns; SVCJ is the stochastic volatility model with contemporaneous jumps in returns and volatility; SVVG is the stochastic volatility model with variance-gamma jumps in returns; and SVLS is the stochastic volatility model with log-stable jumps in returns.

Table 4. Forecasting error decomposition using model risk.

	SVLS	**SVVG**	**SVCJ**	**SVJ**	**SV**
Panel A: One-month ahead					
AFE	5.7169	7.3572	6.0926	5.8125	5.8642
FE_{PER}	4.8231	2.7252	5.0910	4.6471	5.0408
FE_{MSR}	0.8938	4.6320	1.0016	1.1654	0.8235
Panel B: One-year ahead					
AFE	7.7520	8.3288	8.1578	7.4252	7.6210
FE_{PER}	4.9867	2.9134	5.2733	4.8154	5.2464
FE_{MSR}	2.7652	5.4154	2.8846	2.6099	2.3747

Notes: This table reports the average values of AFE, FE_{PER}, and FE_{MSR} for all options introduced in Section 4.1. The out-of-sample period ranges from January 1993 to December 2017. AFE represents the absolute forecasting error; FE_{PER} is the forecast error due to the model PER_A calculated based on Equation (10); FE_{MSR} captures the forecasting error stemming from MSR, calculated as AFE - FE_{PER}. SVJ is the stochastic volatility model with Merton jumps in returns; SVCJ is the stochastic volatility model with contemporaneous jumps in returns and volatility; SVVG is the stochastic volatility model with variance-gamma jumps in returns; and SVLS is the stochastic volatility model with log-stable jumps in returns.

5.2.1 Model risk

Different from the rolling window estimation performed in Section 5.1, in this section, we study the model risk for two periods; specifically, these are the crisis period ranging from January 2006 to December 2009, and the post-crisis period from January 2010 January to December 2017. The model risk is reported in Table 5. The short position bears a greater PER than the long position; however, all single-factor models reveal much higher MSR for the long position during the post-crisis period. Additionally, the SV model reveals the largest MSR. The SVTF model always has the lowest TMR. It has the lowest PER and MSR during the post-crisis period. Although the MSR of SVTF is greater during the crisis period, it still reveals the lowest TMR thanks to the low PER.[26] For single-factor models, SVLS reveals the lowest TMR during the crisis period, but it becomes the second worst model during the post-crisis period; in contrast, SVVG has the lowest PER, MSR, and TMR

thus, following Du and Luo (2019), we set the total MCMC iteration number as 5000, and the first 2000 iterations are regarded as burn-in.

[26] For comparison, we transform the results for puts into equivalent results for call options using the put-call parity.

Table 5. Model risk estimated using the RIV methodology

	SV	SVJ	SVCJ	SVLS	SVVG	SVTF
Panel A: Crisis period						
PER_L	4.7537	5.1160	4.7978	3.3791	3.5069	0.0461
PER_S	5.4261	5.8897	5.1041	3.4736	3.6568	0.0491
PER	5.7479	6.1808	5.5435	3.7208	3.8807	0.0535
MSR_L	1.6256	1.5705	0.6547	0.3129	0.3252	1.3225
MSR_S	0.5988	0.4319	1.1581	0.6767	0.5848	0.8185
MSR	2.2244	2.0024	1.8128	0.9897	0.9100	2.1410
TMR_L	6.3793	6.6865	5.4525	3.692	3.8321	1.3686
TMR_S	6.0249	6.3215	6.2621	4.1503	4.2417	0.8676
TMR	7.9723	8.1832	7.3563	4.7105	4.7907	2.1945
Panel B: Post-crisis period						
PER_L	4.0029	3.7456	3.7998	3.1669	3.0798	0.1019
PER_S	4.7415	4.5826	4.7706	3.8318	3.7720	0.1058
PER	4.9109	4.7337	4.8919	3.9452	3.9581	0.1169
MSR_L	7.894	4.6234	3.9371	7.6638	2.5795	1.0484
MSR_S	0.016	0.1289	0.1369	0.0822	0.5539	1.0958
MSR	7.9100	4.7524	4.0740	7.7460	3.1334	2.1442
TMR_L	11.8969	8.3691	7.7369	10.8307	5.6593	1.1503
TMR_S	4.7575	4.7115	4.9075	3.914	4.3259	1.2016
TMR	12.8209	9.4861	8.9659	11.6912	7.0915	2.2611

Notes: This table reports the average values of TMR, PER, and MSR for all models; the corresponding values for long and short positions are also reported. PER, MSR and TMR are defined in Equations (10), (11), and (12), respectively; the PER for long and short positions (PER_L and PER_S) are computed using Equation (7); the MSR for long and short positions (MSR_L and MSR_S) are computed using Equation (8); the TMR for long and short positions (TMR_L and TMR_S) are computed using Equation (9). The crisis period ranges from January 2006 to December 2009, and the post-crisis period ranges from January 2010 to December 2017. SV is the stochastic volatility model; SVJ is the stochastic volatility model with Merton jumps in returns; SVCJ is the stochastic volatility model with contemporaneous jumps in returns and volatility; SVVG is the stochastic volatility model with variance-gamma jumps in returns; SVLS is the stochastic volatility model with log-stable jumps in returns; and SVTF is the three-factor double exponential model.

during the post-crisis period. The results suggest that there is a need to monitor model risk and changes in model risk might necessitate the replacement of the models.

5.2.2 Explaining the bid–ask spread with model risk

We analyze the impact of positional model risk gap on the bid–ask spread. We focus on the proportional bid–ask spread (PBAS), expressed as a fraction of the mid-price and typically used to measure liquidity (Wei and Zheng 2010), and run the following regression:

$$PBAS_i = \beta_0 + \beta_1 Gap_i^{PER} + \beta_2 Gap_i^{MSR} + \beta_3 Volume_i + \beta_4 M_i + \beta_5 \tau_i + \beta_6 IV_i + \epsilon_i, \qquad (37)$$

where $PBAS_i$ stands for the proportional bid–ask spread of option i, Gap is the absolute difference between the model risk of long and short positions divided by the option mid-price. We also consider the following factors used in Wei and Zheng (2010) as control variables. $Volume_i$ is the trading volume of options i scaled downward by 100,000, M represents option moneyness (option strike price over underlying spot price), τ is the time-to-maturity in years. IV_i, the IV of option i, is also included. The regression results are reported in Table 6. The estimated coefficient of Gap^{PER} and Gap^{MSR} are always significantly positive, suggesting that a higher positional model risk gap is accompanied by a wider PBAS, thus, lower liquidity. It is also worth noting that the trading volume and

Table 6. Explaining the bid-ask spread using positional model risk gap

Model	Gap^{PER}	Gap^{MSR}	Volume	M	τ	IV	Adj. R^2 (%)
Panel A: Crisis period							
SV	0.0996**	0.0825**					42.55
	0.0530**	0.0787**	−0.0065	0.4105**	−0.0312	0.0875**	54.85
SVJ	0.0859**	0.0845**					42.64
	0.0462**	0.0806**	−0.0059	0.4085**	−0.0367	0.0853**	54.94
SVCJ	0.1347**	0.0549**					44.59
	0.0804**	0.0528**	−0.0075	0.3639**	−0.0304	0.0635**	54.43
SVLS	0.3702**	0.0142**					35.56
	0.2704**	0.0134**	−0.0158	0.3504**	−0.0041	0.0425**	45.17
SVVG	0.2204**	0.0313**					33.89
	0.1569**	0.0271**	−0.0147	0.3599**	−0.0326	0.0424**	44.78
SVTF	3.0272**	0.0632**					51.57
	1.4220**	0.0552**	−0.0044	0.3518**	0.0025	0.0765**	60.52
Panel B: Post-crisis period							
SV	0.0935**	0.0300**					35.27
	0.0778**	0.0283**	−0.0412*	0.4906**	−0.1008**	0.2006**	41.96
SVJ	0.1391**	0.0490**					39.35
	0.1194**	0.0461**	−0.0424*	0.5055**	−0.0573**	0.2068**	45.77
SVCJ	0.1469**	0.0526**					33.07
	0.1250**	0.0489**	−0.0405*	0.5350**	−0.0437	0.2088**	40.29
SVLS	0.1383**	0.0456**					27.06
	0.1163**	0.0414**	−0.0533**	0.5521**	−0.0812**	0.2255**	35.01
SVVG	0.1439**	0.0647**					41.34
	0.1245**	0.0611**	−0.0288	0.4448**	−0.0948**	0.1491**	47.52
SVTF	0.1588**	0.0675**					32.5
	0.1240*	0.0605**	−0.0323	0.5775**	0.021	0.2285**	40.45

The regression results are based on Equation (37). * and ** indicate values significant at 5% and 1% significance levels, respectively. The crisis period ranges from January 2006 to December 2009, and the post-crisis period ranges from January 2010 to December 2017. Gap^{PER} and Gap^{MSR} are the absolute value of the PER and MSR difference between short and long positions divided by the option mid-price; Volume is the option trading volume; M is the option moneyness, which is the ratio of option strike to underlying spot price; τ is the option time-to-maturity in years; and IV denotes the option implied volatility. SV is the stochastic volatility model; SVJ is the stochastic volatility model with Merton jumps in returns; SVCJ is the stochastic volatility model with contemporaneous jumps in returns and volatility; SVVG is the stochastic volatility model with variance-gamma jumps in returns; SVLS is the stochastic volatility model with log-stable jumps in returns; and SVTF is the three-factor double exponential model.

option time-to-maturity are negatively linked to PBAS, but the effect is only significant during the post-crisis period.

6 Conclusions

In this article, we propose ES-type model risk measures that capture the PER and MSR of continuous-time finance models. We then apply this measure to Lévy jump, AJD, and multifactor models to investigate the MSR and PER of the models when explaining the joint dynamics of the underlying and option prices. To capture PER, we develop two estimation methodologies based on a Bayesian approach. The first one builds on the approaches of Broadie, Chernov, and Johannes (2007) and Yu, Li, and Wells (2011), we propose a random-effect specification for pricing errors and develop an effective MCMC method to jointly estimate parameters and latent variables with both stock and option prices. Our model specification and estimation method can consider multiple options and use index returns and option prices over different time periods. The second one builds on the methodology used in Andersen, Fusari, and Todorov (2015b). Assuming that pricing errors are normally distributed, we estimate the parameters and latent variables jointly with the realized and IV data using MCMC methods.

We find that modeling jumps is necessary as all jump models have smaller TMR compared with the SV model. Additionally, the multifactor model consistently reveals the lowest TMR compared to single-factor models. Our results highlight that it is necessary to measure PER and MSR separately as the two components of TMR. Furthermore, we show that the positional model risk gap is positively linked to the option bid–ask spread.

Supplemental Data

Supplemental data is available at https://www.datahostingsite.com.

References

Andersen, T. G., L. Benzoni, and J. Lund. 2002. An Empirical Investigation of Continuous-Time Equity Return Models. *The Journal of Finance* 57: 1239–1284.

Andersen, T. G., N. Fusari, and V. Todorov. 2015a. Parametric Inference and Dynamic State Recovery from Option Panels. *Econometrica* 83: 1081–1145.

Andersen, T. G., N. Fusari, and V. Todorov. 2015b. The Risk Premia Embedded in Index Options. *Journal of Financial Economics* 117: 558–584.

Asgharian, H., and C. Bengtsson. 2006. Jump Spillover in International Equity Markets. *Journal of Financial Econometrics* 4: 167–203.

Bardgett, C., E. Gourier, and M. Leippold. 2019. Inferring Volatility Dynamics and Risk Premia from the S&P 500 and VIX Markets. *Journal of Financial Economics* 131: 593–618.

Barrieu, P., and G. Scandolo. 2015. Assessing Financial Model Risk. *European Journal of Operational Research* 242: 546–556.

Basel Committee on Banking Supervision. 2009. *Revisions to the Basel II Market Risk Framework*. Basel, Switzerland: Bank for International Settlements.

Bates, D. S. 2000. Post-'87 Crash Fears in the S&P 500 Futures Option Market. *Journal of Econometrics* 94: 181–238.

Berger, J. O. 1985. *Statistical Decision Theory and Bayesian Analysis*. 2nd ed. New York, NY: Springer-Verlag.

Broadie, M., M. Chernov, and M. Johannes. 2007. Model Specification and Risk Premia: Evidence from Futures Options. *The Journal of Finance* 62: 1453–1490.

Carr, P., and L. Wu. 2003. The Finite Moment Log Stable Process and Option Pricing. *The Journal of Finance* 58: 753–777.

Christoffersen, P., K. Jacobs, and C. Ornthanalai. 2012. Dynamic Jump Intensities and Risk Premiums: Evidence from S&P 500 Returns and Options. *Journal of Financial Economics* 106: 447–472.

Damlen, P., J. Wakefield, and S. Walker. 1999. Gibbs Sampling for Bayesian Non-Conjugate and Hierarchical Models by Using Auxiliary Variables. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 61: 331–344.

Detering, N., and N. Packham. 2016. Model Risk of Contingent Claims. *Quantitative Finance* 16: 1357–1374.

Du, D., and D. Luo. 2019. The Pricing of Jump Propagation: Evidence from Spot and Options Markets. *Management Science* 65: 2360–2387.

Duffie, D., J. Pan, and K. Singleton. 2000. Transform Analysis and Asset Pricing for Affine Jump-Diffusions. *Econometrica* 68: 1343–1376.

Durham, G. B. 2013. Risk-Neutral Modeling with Affine and Nonaffine Models. *Journal of Financial Econometrics* 11: 650–681.

Eraker, B. 2001. MCMC Analysis of Diffusion Models with Applications to Finance. *Journal of Business & Economic Statistics* 19: 177–191.

Eraker, B. 2004. Do Stock Prices and Volatility Jump? Reconciling Evidence from Spot and Option Prices. *The Journal of Finance* 59: 1367–1403.

European Banking Authority. 2012. *Discussion Paper on Draft Regulatory Technical Standards on Prudent Valuation, Under Article 100 of the Draft Capital Requirements Regulation*. Paris, France: European Banking Authority.

Federal Reserve Board of Governors. 2011. *Supervisory Guidance on Model Risk Management. Federal Reserve SR Letter 11-7 Attachment*. Washington, DC: Federal Reserve Board of Governors.

Green, T. C., and S. Figlewski. 1999. Market Risk and Model Risk for a Financial Institution Writing Options. *The Journal of Finance* 54: 1465–1499.

Gruber, P. H., C. Tebaldi, and F. Trojani. 2021. The Price of the Smile and Variance Risk Premia. *Management Science* 67: 4056–4074.

Gzyl, H., E. ter Horst, and S. W. Malone. 2008. Bayesian Parameter Inference for Models of the Black and Scholes Type. *Applied Stochastic Models in Business and Industry* 24: 507–524.

Hu, G., and Y. Liu. 2022. The Pricing of Volatility and Jump Risks in the Cross-Section of Index Option Returns. *Journal of Financial and Quantitative Analysis* 57: 2385–2411.

Hull, J., and W. Suo. 2002. A Methodology for Assessing Model Risk and Its Application to the Implied Volatility Function Model. *The Journal of Financial and Quantitative Analysis* 37: 297–318.

Jacquier, E., and R. Jarrow. 2000. Bayesian Analysis of Contingent Claim Model Error. *Journal of Econometrics* 94: 145–180.

Jacquier, E., M. Johannes, and N. Polson. 2007. MCMC Maximum Likelihood for Latent State Models. *Journal of Econometrics* 137: 615–640.

Jacquier, E., N. G. Polson, and P. E. Rossi. 2002. Bayesian Analysis of Stochastic Volatility Models. *Journal of Business & Economic Statistics* 20: 69–87.

Johannes, M., and N. Polson. 2010. "MCMC Methods for Continuous-Time Financial Econometrics." In Y. Ait-Sahalia and L. P. Hansen (eds.), *Handbook of Financial Econometrics: Applications*, 1–72. Oxford: Elsevier.

Killick, R., P. Fearnhead, and I. A. Eckley. 2012. Optimal Detection of Changepoints with a Linear Computational Cost. *Journal of the American Statistical Association* 107: 1590–1598.

Li, H., M. T. Wells, and C. L. Yu. 2008. A Bayesian Analysis of Return Dynamics with Lévy Jumps. *Review of Financial Studies* 21: 2345–2378.

Lindström, E., J. Ströjby, M. Brodén, M. Wiktorsson, and J. Holst. 2008. Sequential Calibration of Options. *Computational Statistics & Data Analysis* 52: 2877–2891.

Madan, D. B., P. P. Carr, and E. C. Chang. 1998. The Variance Gamma Process and Option Pricing. *Review of Finance* 2: 79–105.

Pan, J. 2002. The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study. *Journal of Financial Economics* 63: 3–50.

Polson, N., and J. Stroud. 2003. "Bayesian Inference for Derivative Prices." In P. M. Lee (ed.), *Bayesian Statistics*, 7th ed., 641–650. Oxford: Oxford University Press.

Rodríguez, A., E. Ter Horst, and S. Malone. 2015. Bayesian Inference for a Structural Credit Risk Model with Stochastic Volatility and Stochastic Interest Rates. *Journal of Financial Econometrics* 13: 839–867.

Wei, J., and J. Zheng. 2010. Trading Activity and Bid–Ask Spreads of Individual Equity Options. *Journal of Banking & Finance* 34: 2897–2916.

Yu, C. L., H. Li, and M. T. Wells. 2011. MCMC Estimation of Lévy Jump Models Using Stock and Option Prices. *Mathematical Finance* 21: 383–422.