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VARIATIONAL PROBLEMS IN L^∞ INVOLVING SEMILINEAR SECOND ORDER DIFFERENTIAL OPERATORS

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Abstract. For an elliptic, semilinear differential operator of the form $S(u) = A : D^2u + b(x, u, Du)$, consider the functional $E_\infty(u) = \operatorname{ess\,sup}_\Omega |S(u)|$. We study minimisers of E_∞ for prescribed boundary data. Because the functional is not differentiable, this problem does not give rise to a conventional Euler–Lagrange equation. Under certain conditions, we can nevertheless give a system of partial differential equations that all minimisers must satisfy. Moreover, the condition is equivalent to a weaker version of the variational problem. The theory of partial differential equations therefore becomes available for the study of a large class of variational problems in L^∞ for the first time.

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1. Introduction

Variational problems involving an L^∞-norm tend to present challenges not shared by more conventional variational problems. Indeed, the underlying functionals are typically not differentiable, not even in the Gateaux sense, and therefore, the usual derivation of an Euler–Lagrange equation does not work. Sometimes it is possible to derive an associated partial differential equation nevertheless (the Aronsson equation [2] is an example) but such an equation is typically only degenerate elliptic and may have discontinuous coefficients [17]. Indeed, for higher order problems (as studied in this paper), the equations may be fully nonlinear and not elliptic at all [23]. Moreover, while the interesting functionals in the calculus of variations in L^∞ typically enjoy a certain degree of convexity, they are not strictly convex. Therefore, minimisers are not usually expected to be unique.

All these difficulties notwithstanding, there are some problems in the theory that are understood very well. This applies in particular to a class of problems involving first order derivatives of scalar functions. More precisely, let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain and consider functions $u\colon \Omega \to \mathbb{R}$ with fixed boundary data. For a function $F\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, consider the problem of minimising $\operatorname{ess\,sup}_{x\in\Omega} F(x, u(x), Du(x))$. Under certain conditions on F, there is a good theory giving, for example, existence of solutions with good properties [1, 3, 4], uniqueness [15, 16], and (for $F(x, y, z) = |z|^2$) regularity [9–11].

Recently, the authors have studied a different, second order variational problem in L^∞ and established good properties of its solutions [22]. Suppose now that we wish to minimise a quantity such as $\operatorname{ess\,sup}_{x\in\Omega} F(x, \Delta u(x))$. Assuming that $F\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies some growth and convexity conditions, and that Ω, F, and the boundary

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data are sufficiently regular, it turns out that there exists a unique minimiser, which satisfies a certain system of partial differential equations. Conversely, any solution of that system corresponds to a minimiser. Some of the underlying ideas go back to earlier work [25, 26], and similar tools have in the meantime also been used for other problems [12, 21, 24].

In the current paper, we study extensions of these results. This is one of a pair of works that examine two different types of generalisations. Here we replace Δu by more general, semilinear differential operators, while restricting our attention to $F(x,\xi) = |\xi|$ (but x-dependence is still included implicitly, because the coefficients of the differential operator need not be constant any more). As a result, the results discussed here have far more potential applications. It turns out that the inclusion of nonlinearities also changes the behaviour of the problem somewhat, and for this reason we require different arguments as well. In a companion paper [20], we study a quantity of the form $\operatorname{ess\,sup}_{x\in\Omega} F(x, u(x), \Delta u(x))$ for a fairly general class of functions $F\colon \Omega \times \mathbb{R}^2 \to \mathbb{R}$ (which again gives rise to more applications). There is of course some overlap between the two settings, and it would be interesting to have a common framework, but this appears to be difficult for technical reasons.

We consider the following situation. Let $\Omega \subseteq \mathbb{R}^n$, as above, be a bounded Lipschitz domain. (For some of our results we will need to impose additional regularity assumptions on Ω.) Let $A \in C^2(\overline{\Omega}; \mathbb{R}^{n\times n})$ and $b \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. We assume that there exists $\lambda > 0$ such that for every $x \in \Omega$, the matrix $A(x)$ is symmetric and satisfies $A(x) : \zeta \otimes \zeta \geq \lambda |\zeta|^2$ for all $\zeta \in \mathbb{R}^n$, where the colon denotes the Frobenius inner product. Define the semilinear differential operator

$$S(u) = A : D^2 u + b(x, u, Du) \tag{1.1}$$

for $u\colon \Omega \to \mathbb{R}$, where Du is the gradient and $D^2 u$ is the Hessian of u. We are interested in the functional

$$E_\infty(u) = \operatorname*{ess\,sup}_{\Omega} |S(u)|.$$

So far we have not mentioned the space on which this functional is defined. In order to obtain a good theory, we need to work in a Sobolev space that may appear unconventional, but is quite natural for the problem. We define

$$\mathcal{W}^{2,\infty}(\Omega) = \bigcap_{1<q<\infty} \left\{ u \in W^{2,q}(\Omega) \mid A : D^2 u \in L^\infty(\Omega) \right\}.$$

Furthermore,

$$\mathcal{W}^{2,\infty}_0(\Omega) = \mathcal{W}^{2,\infty}(\Omega) \cap W^{2,2}_0(\Omega).$$

Given $u_0 \in \mathcal{W}^{2,\infty}(\Omega)$, we wish to study minimisers of E_∞ in $u_0 + \mathcal{W}^{2,\infty}_0(\Omega)$. For other variational problems, the critical points would also be of interest, but in this case, the concept is not useful as E_∞ is not differentiable. We therefore work with the following idea instead.

Definition 1.1 (Almost-minimiser)**.** A function $u \in \mathcal{W}^{2,\infty}(\Omega)$ is called an *almost-minimiser* of E_∞ if there exists $M \in \mathbb{R}$ such that

$$E_\infty(u) \leq E_\infty(u + \phi) + M \|\phi\|^2_{W^{1,\infty}(\Omega)}$$

for every $\phi \in \mathcal{W}^{2,\infty}_0(\Omega)$.

Intuitively, the definition provides a substitute for the idea that the Taylor expansion of E_∞ has a vanishing first order term at u. Since E_∞ is not differentiable, it does of course not have a Taylor expansion. Instead, we use a quadratic form $Q\colon \mathcal{W}^{2,\infty}(\Omega) \to \mathbb{R}$ such that the graph of Q touches the graph of $E_\infty(\,\cdot\, + u) - E_\infty(u)$

from below. This is related to the notion of second order subjets or subdifferentials that appears in the theory of viscosity solutions of partial differential equations (discussed, *e.g.*, in a survey article by Crandall, Ishii, and Lions [6] or the introductory text by the first author [18]). In the case of Definition 1.1, the condition is formulated in terms of the norm of $W^{1,\infty}(\Omega)$, so it may be best to think of E_∞ as a functional defined on $W^{1,\infty}(\Omega)$ in this context, with $E_\infty(v) = \infty$ when $v \notin \mathcal{W}^{2,\infty}(\Omega)$. There is, however, some flexibility here. Our main results remain true if $\|\cdot\|_{W^{1,\infty}(\Omega)}$ is replaced, *e.g.*, by $\|\cdot\|_{W^{2,q}(\Omega)}$ for any sufficiently large $q < \infty$.

We show below that almost-minimisers can be characterised in terms of a system of partial differential equations. In order to formulate this, we use the formal linearisation of the operator S at a point $u \in \mathcal{W}^{2,\infty}(\Omega)$, denoted S'_u, and its formal L^2-adjoint, denoted S^*_u. We use the notation (x, y, z) for the variables in $\Omega \times \mathbb{R} \times \mathbb{R}^n$. We further write b_y for the partial derivative of b with respect to y, and b_z for the vector comprising the partial derivatives with respect to $z_1, \ldots, z_n$. Then

$$S'_u\phi = A : D^2\phi + b_z(x, u, Du) \cdot D\phi + b_y(x, u, Du)\phi$$

and

$$S^*_u f = \operatorname{div}\operatorname{div}(fA) - \operatorname{div}\big(f b_z(x, u, Du)\big) + f b_y(x, u, Du).$$

In the div–div term, the divergence is applied once column-wise and once row-wise.

We are interested in the equation

$$S^*_u f = 0. \tag{1.2}$$

We can make sense of this for $f \in L^1(\Omega)$: if

$$\int_\Omega f S'_u\phi \,\mathrm{d}x = 0 \tag{1.3}$$

holds for all $\phi \in C^\infty_0(\Omega)$, then we say that f is a *weak solution* of the equation.

Our first main result is as follows.

Theorem 1.2. *Let $u_\infty \in \mathcal{W}^{2,\infty}(\Omega)$ be such that there exist $e_\infty \geq 0$ and $f_\infty \in L^1(\Omega) \setminus \{0\}$ satisfying*

$$|f_\infty| S(u_\infty) = e_\infty f_\infty \tag{1.4}$$

almost everywhere in Ω and

$$S^*_{u_\infty} f_\infty = 0 \tag{1.5}$$

weakly. Then u_∞ is an almost-minimiser of E_∞.

The converse is also true, provided that we impose some additional regularity on $\partial\Omega$ and on the boundary data, and provided that we restrict our attention to differential operators that permit certain L^p-estimates.

Definition 1.3. For a differential operator S as in (1.1), we say that S is *admissible* if there exists $p_0 > 1$ such that for any $p \geq p_0$, the following holds true. Suppose that $u_0 \in \mathcal{W}^{2,\infty}(\Omega)$ and $\Lambda > 0$. Then there exists $C > 0$ such that for any $u \in u_0 + \mathcal{W}^{2,p}_0(\Omega)$, if

$$\|S(u)\|_{L^p(\Omega)} \leq \Lambda,$$

then

$$\|u\|_{W^{2,p}(\Omega)} \leq C.$$

Now we can formulate our second main result.

Theorem 1.4. *Suppose that $\partial\Omega$ is of class C^3 and $u_0 \in C^2(\overline{\Omega})$. Further suppose that S is admissible. If $u_\infty \in u_0 + \mathcal{W}_0^{2,\infty}(\Omega)$ is an almost-minimiser of E_∞, then there exist $f_\infty \in L^1(\Omega) \setminus \{0\}$ and $e_\infty > 0$ such that (1.4) holds almost everywhere in Ω and (1.5) holds weakly.*

For admissible operators, we can furthermore be certain that minimisers of E_∞ exist for prescribed boundary data. A minimiser is in particular an almost-minimiser, and thus we are guaranteed that the system (1.4), (1.5) has a non-trivial solution.

Proposition 1.5. *If S is admissible, then E_∞ attains its minimum in $u_0 + \mathcal{W}_0^{2,\infty}(\Omega)$ for any $u_0 \in \mathcal{W}^{2,\infty}(\Omega)$.*

The proof relies on standard arguments and in particular on the direct method. For completeness, we outline these arguments anyway.

Proof. Let $(u_k)_{k\in\mathbb{N}}$ be a minimising sequence. Then $\|S(u_k)\|_{L^\infty(\Omega)}$ is obviously bounded. Hence if p_0 is the number from Definition 1.3, then it follows that $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{2,p}(\Omega)$ for any $p \in [p_0, \infty)$. Therefore, we may assume (extracting a subsequence if necessary) that we have the convergence $u_k \rightharpoonup u_\infty$ weakly in $W^{2,p}(\Omega)$ for every $p < \infty$. Moreover, the limit belongs to $u_0 + W_0^{2,p}(\Omega)$.

Using the Sobolev embedding theorem, we further conclude that $(u_k)_{k\in\mathbb{N}}$ is bounded in $C^{1,\alpha}(\overline{\Omega})$ as well for every $\alpha \in (0,1)$. The Arzelà-Ascoli theorem implies that $u_k \to u_\infty$ in $C^1(\overline{\Omega})$. Hence $b(x, u_k, Du_k) \to b(x, u_\infty, Du_\infty)$ uniformly.

Since $(S(u_k))_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, we may assume that $S(u_k) \overset{*}{\rightharpoonup} \sigma$ weakly* in $L^\infty(\Omega)$ for some $\sigma \in L^\infty(\Omega)$. Using the above convergence, we conclude that $\sigma = S(u_\infty)$. Now it follows that

$$E_\infty(u_\infty) \leq \liminf_{k\to\infty} E_\infty(u_k),$$

and thus u_∞ is a minimiser. ∎

Summarising Theorem 1.2 and Theorem 1.4, we can say that the system comprising equations (1.4) and (1.5) is *equivalent* to the almost-minimising condition under certain assumptions. If we accept that the latter is a reasonable substitute for critical points, then we may think of (1.4) and (1.5) as a substitute for an Euler–Lagrange equation.

For a variety of other variational problems in L^∞, a corresponding differential equation has been identified by quite different methods. In the case of the optimal Lipschitz extension problem, the result is the Aronsson equation [2]. Formally, Aronsson's calculations can be carried out for the above problem as well. They give rise to the equation $S(u)D(S(u)) = 0$. The connections between this equation and the variational problem are not explored in this work, but we observe that the former is satisfied by solutions of (1.4).

The 'Aronsson equation' $S(u)D(S(u)) = 0$ is of third order, quasilinear, and in non-divergence form. It is not elliptic in any reasonable sense. It allows neither weak nor viscosity solutions, but there is another approach that does apply and has produced some results on equations such as this (see, *e.g.*, the papers of the first author and coauthors [7, 19, 23]). In this paper, however, we follow the alternative approach outlined above and consider solutions to the system (1.4), (1.5) instead. This can be seen as a divergence-form (or div–div form) alternative to the Aronsson equation.

The comparison with other variational problems in L^∞ makes another aspect of the above results remarkable. We note that the system (1.4), (1.5) is local in the sense that if it holds in Ω, then it is automatically satisfied in any open subdomain $\Omega' \subseteq \Omega$. Under the conditions of Theorem 1.4, it follows that the almost-minimising condition is also local in a similar sense. Indeed, if $\partial\Omega$ is of class C^3 and $u_0 \in C^2(\overline{\Omega})$, then for any almost-minimiser $u_\infty \in u_0 + \mathcal{W}_0^{2,\infty}(\Omega)$, Theorem 1.4 applies. Given a Lipschitz subdomain $\Omega' \subseteq \Omega$, we can then use Theorem 1.2 in Ω', concluding that u_∞ is an almost-minimiser in Ω' as well. This is in stark contrast to the optimal Lipschitz extension problem and similar variational problems, where locality must be imposed in order to obtain solutions with good properties. These solutions are then known as absolute minimisers.

For the special case $S(u) = \Delta u$ (and for certain other problems), a previous paper [22] gives stronger results. It is shown that solutions of (1.4), (1.5) are unique under the boundary condition $u_\infty \in u_0 + \mathcal{W}_0^{2,\infty}(\Omega)$ and correspond to unique minimisers of E_∞. For similar problems involving nonlinear operators, however, we do not have uniqueness in general [24]. For semilinear operators as in this paper, the question is open. Furthermore, for the problem studied here, we are led to the concept of almost-minimisers rather than minimisers. Because of these differences in the behaviour, we need different arguments in addition to the ideas from the earlier paper. They include a more sophisticated analysis of the functional and of the Sobolev spaces for the proof of Theorem 1.2 and the inclusion of a penalisation term for the proof of Theorem 1.4. The latter is related to an idea also used in a different paper [24], but takes a different form here and requires further arguments.

In the next section, we discuss second order elliptic, linear equations in div–div form, of which (1.5) is an example. We consider solutions in $L^1(\Omega)$ and derive some interior regularity that we will need for the proofs of our main results. We will also show that weak solutions can be tested with functions in $\mathcal{W}_0^{2,\infty}(\Omega)$. We then prove Theorem 1.2 and Theorem 1.4 in the following two sections. In the final section, we discuss some specific differential operators of the form $S(u) = \Delta u + g(u)$ for a function $g\colon \mathbb{R} \to \mathbb{R}$. In particular, we give some conditions that imply that the operator is admissible in the sense of Definition 1.3. The purpose of this section is not just to show that Theorem 1.4 is not vacuous, but also to give an idea of the nonlinearities allowed.

2. Elliptic equations in div–div form

In this section, we prove some properties of weak solutions of an equation of the form

$$\operatorname{div}\operatorname{div}(fA) + \operatorname{div}(fB) + fc = \operatorname{div} G + g, \tag{2.1}$$

where f is assumed to be in $L^1(\Omega)$ or even a Radon measure on Ω. The matrix A will be the same as in the introduction and will be fixed throughout. The properties of the coefficients B and c are described below. We eventually apply these results to equations such as $S_u^* f = \operatorname{div} G + g$ for some $u \in W^{2,p}(\Omega)$, or even to $S_{u_\infty}^* f = 0$ for a function $u_\infty \in \mathcal{W}^{2,\infty}(\Omega)$, but we formulate them more generally here.

First we prove some interior regularity for weak solutions of the equation. We duplicate some results from a more general theory here (see, *e.g.*, the survey article of Bogachev, Krylov, and Röckner [5]). In order to make the paper self-contained, we include a proof nevertheless.

Lemma 2.1. *For any $\Omega' \Subset \Omega$ and any $p \in (n, \infty)$, there exists a constant $C > 0$ such that the following holds true. Let $p' = \frac{p}{p-1}$ be the exponent conjugate to p. Suppose that $B \in L^\infty(\Omega; \mathbb{R}^n)$, $c \in L^\infty(\Omega)$, $g \in L^1(\Omega)$, and $G \in L^{p'}(\Omega; \mathbb{R}^n)$. Set*

$$\Gamma = \|B\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} + 1.$$

Suppose that $\mu \in (C_0^0(\Omega))^*$ is a distributional solution of

$$\operatorname{div}\operatorname{div}(A\mu) + \operatorname{div}(B\mu) + c\mu = \operatorname{div} G + g,$$

meaning that

$$\int_\Omega (A : D^2\phi - B \cdot D\phi + c\phi)\,\mathrm{d}\mu = \int_\Omega (g\phi - G \cdot D\phi)\,\mathrm{d}x \tag{2.2}$$

for every $\phi \in C_0^\infty(\Omega)$. Then μ is absolutely continuous with respect to the Lebesgue measure. Its Radon–Nikodym derivative f belongs to $W^{1,p'}_{\mathrm{loc}}(\Omega)$ and satisfies

$$\|f\|_{W^{1,p'}(\Omega')} \leq C\Gamma^2 \left(|\mu|(\Omega) + \|G\|_{L^{p'}(\Omega)} + \|g\|_{L^1(\Omega)}\right).$$

Proof. Consider a function $\chi \in C_0^\infty(\Omega)$. Define a functional $\alpha \in (C_0^1(\Omega))^*$ by

$$\alpha(\psi) = \int_\Omega \chi\psi \,\mathrm{d}\mu, \quad \psi \in C_0^1(\Omega).$$

(This means that α corresponds to the measure $\chi\mu$, but we regard it as a functional on $C_0^1(\Omega)$ at first.) Choose an open, precompact set $\Omega'' \Subset \Omega$ with smooth boundary and with $\operatorname{supp}\chi \subseteq \Omega''$. Given $\psi \in C_0^1(\Omega)$, we can solve the equation

$$A : D^2\phi = \psi$$

in $W^{2,p}(\Omega'') \cap W_0^{1,p}(\Omega'')$ by [13], Theorem 9.15. Moreover, [13], Theorem 9.19 implies that $\phi \in C^2(\Omega'')$. By approximation, we then see that (2.2) is satisfied for the test function $\chi\phi$. Thus we obtain

$$\begin{aligned}\alpha(\psi) &= \int_\Omega \chi A : D^2\phi \,\mathrm{d}\mu \\ &= \int_\Omega \big(\chi B \cdot D\phi + \phi B \cdot D\chi - c\chi\phi - A : (2D\chi \otimes D\phi + \phi D^2\chi)\big) \,\mathrm{d}\mu \\ &\quad + \int_\Omega (\chi g\phi - \chi G \cdot D\phi - \phi G \cdot D\chi) \,\mathrm{d}x.\end{aligned}$$

Note that [13], Lemma 9.17 implies that

$$\|\phi\|_{W^{2,p}(\Omega'')} \leq C_1 \|\psi\|_{L^p(\Omega)}$$

for a constant $C_1 = C_1(n, \Omega'', p, A)$. Hence

$$\|\phi\|_{C^1(\overline{\Omega''})} \leq C_2 \|\psi\|_{L^p(\Omega)}$$

for a constant C_2 with the same dependence. Therefore, we find a constant $C_3 = C_3(n, \Omega'', p, A, \chi)$ such that

$$|\alpha(\psi)| \leq C_3 \left(\Gamma|\mu|(\Omega) + \|g\|_{L^1(\Omega)} + \|G\|_{L^{p'}(\Omega)}\right) \|\psi\|_{L^p(\Omega)}.$$

In particular, the functional α has a continuous linear extension to $L^p(\Omega)$. It follows that there exists $\tilde{f} \in L^{p'}(\Omega)$ such that

$$\int_\Omega \chi\psi \,\mathrm{d}\mu = \int_\Omega \tilde{f}\psi \,\mathrm{d}x$$

for all $\psi \in C_0^1(\Omega)$. Since $C_0^1(\Omega)$ is dense in $C_0^0(\Omega)$, this means that $\chi\mu$ is absolutely continuous with respect to the Lebesgue measure and $\tilde{f}$ is the Radon–Nikodym derivative. Since these arguments work for any $\chi \in C_0^\infty(\Omega)$, the measure μ is absolutely continuous as well and has a Radon–Nikodym derivative $f \in L^{p'}_{\mathrm{loc}}(\Omega)$. Choosing χ such that $\chi \equiv 1$ in Ω', we also obtain the inequality

$$\|f\|_{L^{p'}(\Omega')} \leq C_3 \left(\Gamma|\mu|(\Omega) + \|G\|_{L^{p'}(\Omega)} + \|g\|_{L^1(\Omega)}\right). \tag{2.3}$$

We now conclude that (2.2) holds true for every $\phi \in W^{2,p}(\Omega)$ with compact support in Ω.

For $i \in \{1, \ldots, n\}$, we next consider the functional $\beta_i \in (C^1_0(\Omega))^*$ with

$$\beta_i(\psi) = \int_\Omega f\chi\psi_{x_i}\,\mathrm{d}x, \quad \psi \in C^1_0(\Omega)$$

(corresponding to a distributional derivative of χf). Given a fixed $\psi \in C^1_0(\Omega)$, solve

$$A : D^2\phi + (\operatorname{div} A) \cdot D\phi = \psi_{x_i}$$

in $W^{2,p}(\Omega'') \cap W^{1,p}_0(\Omega'')$. If we write e_i for the i-th standard unit vector in $\mathbb{R}^n$, then the equation can alternatively be represented in the form

$$\operatorname{div}(AD\phi) = \operatorname{div}(\psi e_i).$$

Standard L^p-estimates for weak solutions thus give the estimate

$$\|\phi\|_{W^{1,p}(\Omega'')} \leq C_4\|\psi\|_{L^p(\Omega)}$$

for a constant $C_4 = C_4(n, \Omega'', p, A)$. The function $\chi\phi$ is again a suitable test function for (2.2). We obtain

$$\begin{aligned}\beta_i(\psi) = &\int_\Omega f\big(\chi(\operatorname{div} A)\cdot D\phi - A : (2D\chi \otimes D\phi + \phi D^2\chi)\big)\,\mathrm{d}x \\ &+ \int_\Omega f\big(B\cdot(\chi D\phi + \phi D\chi) - c\chi\phi\big)\,\mathrm{d}x \\ &+ \int_\Omega (\chi g\phi - \chi G\cdot D\phi - \phi G\cdot D\chi)\,\mathrm{d}x.\end{aligned}$$

Hence there exists a constant $C_5 = C_5(n, \Omega'', p, A, \chi)$ such that

$$|\beta_i(\psi)| \leq C_5\left(\Gamma\|f\|_{L^{p'}(\Omega'')} + \|G\|_{L^{p'}(\Omega)} + \|g\|_{L^1(\Omega)}\right)\|\psi\|_{L^p(\Omega)}.$$

Therefore, there exists $h_i \in L^{p'}(\Omega)$ such that

$$\int_\Omega \chi f\psi_{x_i}\,\mathrm{d}x = \int_\Omega h_i\psi\,\mathrm{d}x$$

for all $\psi \in C^1_0(\Omega)$. This is true for $i = 1, \ldots, n$, so the function χf has weak derivatives in $L^{p'}(\Omega)$, which satisfy

$$\|(\chi f)_{x_i}\|_{L^{p'}(\Omega)} \leq C_5\left(\Gamma\|f\|_{L^{p'}(\Omega'')} + \|G\|_{L^{p'}(\Omega)} + \|g\|_{L^1(\Omega)}\right). \tag{2.4}$$

Since $\chi \in C^\infty_0(\Omega)$ can be chosen arbitrarily, it follows that $f \in W^{1,p'}_{\mathrm{loc}}(\Omega)$. Moreover, we obtain the desired inequality if we combine (2.3) with (2.4). □

We will also require the following statement, which says that a weak solution of an equation of the form

$$\operatorname{div}\operatorname{div}(fA) + \operatorname{div}(fB) + cf = 0 \tag{2.5}$$

can be tested with functions from the space $\mathcal{W}_0^{2,\infty}(\Omega)$.

Lemma 2.2. *Suppose that $f \in L^1(\Omega)$ is a weak solution of equation (2.5). Then*

$$\int_\Omega f(A : D^2\phi - B \cdot D\phi + c\phi)\,\mathrm{d}x = 0$$

for all $\phi \in \mathcal{W}_0^{2,\infty}(\Omega)$.

Proof. Given $\phi \in \mathcal{W}_0^{2,\infty}(\Omega)$, we first construct a family of approximations $(\phi_\epsilon)_{\epsilon\in(0,\epsilon_0]}$ in $C_0^\infty(\Omega)$ such that $\phi_\epsilon \to \phi$ in $W^{2,q}(\Omega)$ for any $q < \infty$ and, at the same time, such that $A : D^2\phi_\epsilon$ remains bounded in $L^\infty(\Omega)$.

For this purpose, we extend ϕ by 0 outside of Ω. Choose a finite open cover $\{G_1, \ldots, G_L\}$ of $\overline{\Omega}$ with the property that there exist $R > 0$ and there exist open cones $C_1, \ldots, C_L \subseteq \mathbb{R}^n$ such that for any $x \in \partial\Omega \cap G_\ell$,

$$C_\ell \cap B_R(0) \cap (x - \overline{\Omega}) = \emptyset.$$

(Hence $(x - C_\ell) \cap B_R(x)$ is an exterior cone to $\overline{\Omega}$.) This is possible, because Ω is a bounded Lipschitz domain.

For every $\ell = 1, \ldots L$, choose $\eta_\ell \in C_0^\infty(C_\ell \cap B_1(0))$ with $\eta_\ell \geq 0$ and

$$\int_{B_1(0)} \eta_\ell(x)\,\mathrm{d}x = 1.$$

For $\epsilon \in (0, R]$, set

$$\eta_{\ell\epsilon}(x) = \frac{1}{\epsilon^n}\eta_\ell\left(\frac{x}{\epsilon}\right)$$

and

$$\phi_{\ell\epsilon} = \phi * \eta_{\ell\epsilon}.$$

Then for $x \in \partial\Omega \cap G_\ell$,

$$\phi_{\ell\epsilon}(x) = \int_{C_\ell\cap B_\epsilon(0)\cap(x-\Omega)} \eta_{\ell\epsilon}(y)\phi(x - y)\,\mathrm{d}y = 0.$$

Moreover, the function $\phi_{\ell\epsilon}$ vanishes in a neighbourhood of any such point x.

Now choose a partition of unity $\chi_1, \ldots, \chi_L$ in Ω with $\chi_\ell \in C_0^\infty(G_\ell)$ for $\ell = 1, \ldots, L$. Set

$$\phi_\epsilon = \sum_{\ell=1}^L \chi_\ell\phi_{\ell\epsilon}.$$

Then it is clear that $\phi_\epsilon \in C_0^\infty(\Omega)$ and that $\phi_\epsilon \to \phi$ in $W^{2,q}(\Omega)$, for any $q < \infty$, as $\epsilon \searrow 0$.

For $x \in \Omega$, we compute

$$\begin{aligned} A(x) : D^2\phi_{\ell\epsilon}(x) &= \int_{B_\epsilon(0)} \eta_{\ell\epsilon}(y) A(x) : D^2\phi(x-y)\,\mathrm{d}y \\ &= \int_{B_\epsilon(0)} \eta_{\ell\epsilon}(y)(A(x) - A(x-y)) : D^2\phi(x-y)\,\mathrm{d}y \\ &\quad + \int_{B_\epsilon(0)} \eta_{\ell\epsilon}(y) A(x-y) : D^2\phi(x-y)\,\mathrm{d}y. \end{aligned}$$

Using an integration by parts, we find that

$$\begin{aligned} \int_{B_\epsilon(0)} \eta_{\ell\epsilon}(y)(A(x) - A(x-y)) : D^2\phi(x-y)\,\mathrm{d}y &= \int_{B_\epsilon(0)} \eta_{\ell\epsilon}(y) \operatorname{div} A(x-y) \cdot D\phi(x-y)\,\mathrm{d}y \\ &\quad + \int_{B_\epsilon(0)} (A(x) - A(x-y)) : D\eta_{\ell\epsilon}(y) \otimes D\phi(x-y)\,\mathrm{d}y. \end{aligned}$$

For $y \in B_\epsilon(0)$, we have the inequality

$$|A(x) - A(x-y)| \le \epsilon \|A\|_{C^1(\overline{\Omega})}.$$

Hence there exists a universal constant C_1 such that

$$\left| \int_{B_\epsilon(0)} \eta_{\ell\epsilon}(y)(A(x) - A(x-y)) : D^2\phi(x-y)\,\mathrm{d}y \right| \le C_1 \|A\|_{C^1(\overline{\Omega})} \big(1 + \|D\eta_\ell\|_{L^1(B_1(0))}\big) \|D\phi\|_{L^\infty(\Omega)}.$$

It is clear that

$$\left| \int_{B_\epsilon(0)} \eta_{\ell\epsilon}(y) A(x-y) : D^2\phi(x-y)\,\mathrm{d}y \right| \le \|A : D^2\phi\|_{L^\infty(\Omega)}.$$

Thus we obtain a uniform estimate for $\|A : D^2\phi_{\ell\epsilon}\|_{L^\infty(\Omega)}$. A similar estimate for ϕ_ϵ is then easy to prove.

It follows that there exists a sequence $\epsilon_k \searrow 0$ such that $A : D^2\phi_{\epsilon_k} \overset{*}{\rightharpoonup} g$, weakly* in $L^\infty(\Omega)$, for some $g \in L^\infty(\Omega)$. For any $\psi \in C_0^\infty(\Omega)$, we then compute

$$\begin{aligned} \int_\Omega \psi g\,\mathrm{d}x &= \lim_{k\to\infty} \int_\Omega \psi A : D^2\phi_{\epsilon_k}\,\mathrm{d}x \\ &= \lim_{k\to\infty} \int_\Omega \operatorname{div}\operatorname{div}(\psi A)\phi_{\epsilon_k}\,\mathrm{d}x \\ &= \int_\Omega \operatorname{div}\operatorname{div}(\psi A)\phi\,\mathrm{d}x \\ &= \int_\Omega \psi A : D^2\phi\,\mathrm{d}x. \end{aligned}$$

It follows that $g = A : D^2\phi$. It then also follows that $A : D^2\phi_\epsilon \overset{*}{\rightharpoonup} A : D^2\phi$, i.e, it is not necessary to take a subsequence.

Now the claim of the lemma is proved by approximation with ϕ_ϵ and with standard arguments. □

3. Sufficiency of the equations

We now show that a non-trivial solution of the system (1.4), (1.5) gives rise to an almost-minimiser of E_∞.

Proof of Theorem 1.2. We consider $u_\infty \in \mathcal{W}^{2,\infty}(\Omega)$ and assume that there exist $e_\infty \geq 0$ and $f_\infty \in L^1 \setminus \{0\}$ such that (1.4) is satisfied almost everywhere and (1.5) weakly in Ω. We are required to show that u_∞ is an almost-minimiser of E_∞. It suffices, however, to prove the inequality in Definition 1.1 under the assumption that $\|\phi\|_{W^{1,\infty}(\Omega)} \leq 1$, because otherwise,

$$E_\infty(u_\infty) \leq E_\infty(u_\infty + \phi) + M\|\phi\|^2_{W^{1,\infty}(\Omega)}$$

for the number $M = E_\infty(u_\infty)$.

We first note that $f_\infty \in W^{1,q}_{\mathrm{loc}}(\Omega)$ for some $q > 1$ by Lemma 2.1. Hence the equation $S^*_{u_\infty} f_\infty = 0$ can be written in the form

$$\operatorname{div}\big(ADf_\infty + f_\infty \operatorname{div} A - f_\infty b_z(x, u_\infty, Du_\infty)\big) + f_\infty b_y(x, u_\infty, Du_\infty) = 0.$$

With standard regularity theory for elliptic equations, we then obtain higher regularity, in particular $f_\infty \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for every $p < \infty$, and the results of Hardt and Simon [14] on the structure of the nodal set apply. It follows that $f_\infty \neq 0$ almost everywhere.

If $e_\infty = 0$, then (1.4) implies that $E_\infty(u_\infty) = 0$, and u_∞ is in fact a global minimiser. Thus it suffices to consider $e_\infty > 0$.

Fix $\phi \in \mathcal{W}^{2,\infty}_0(\Omega)$ with $\|\phi\|_{W^{1,\infty}(\Omega)} \leq 1$. For $t \in \mathbb{R}$, note that

$$\frac{\partial}{\partial t} S(u_\infty + t\phi) = S'_{u_\infty + t\phi}\phi$$

and

$$\begin{aligned}\frac{\partial^2}{\partial t^2} S(u_\infty + t\phi) = \phi^2 b_{yy}(x, u_\infty + t\phi, Du_\infty + tD\phi)& \\ + 2\phi D\phi \cdot b_{yz}(x, u_\infty + t\phi, Du_\infty + tD\phi)& \\ + (D\phi \otimes D\phi) : b_{zz}(x, u_\infty + t\phi, Du_\infty + tD\phi)&.\end{aligned}$$

Thus Taylor's theorem, applied to the function $t \mapsto S(u_\infty + t\phi)(x)$ for each $x \in \Omega$, implies there exists a function $\tau \colon \Omega \to [0, 1]$ such that

$$S(u_\infty + \phi) = S(u_\infty) + S'_{u_\infty}\phi + B_\tau \tag{3.1}$$

almost everywhere in Ω, where

$$\begin{aligned}B_\tau = \frac{1}{2}\phi^2 b_{yy}(x, u_\infty + \tau\phi, Du_\infty + \tau D\phi) + \phi D\phi \cdot b_{yz}(x, u_\infty + \tau\phi, Du_\infty + \tau D\phi)& \\ + \frac{1}{2}(D\phi \otimes D\phi) : b_{zz}(x, u_\infty + \tau\phi, Du_\infty + \tau D\phi)&.\end{aligned}$$

Therefore,

$$(S(u_\infty + \phi))^2 = (S(u_\infty))^2 + 2\big(S(u_\infty) + B_\tau\big)S'_{u_\infty}\phi + (S'_{u_\infty}\phi)^2 + 2S(u_\infty)B_\tau + B_\tau^2. \tag{3.2}$$

Formula (3.1) implies that B_τ is measurable. Since $\|\phi\|_{W^{1,\infty}(\Omega)} \leq 1$, we have the estimate

$$\|u_\infty + \tau\phi\|_{L^\infty(\Omega)} + \|Du_\infty + \tau D\phi\|_{L^\infty(\Omega)} \leq C_1$$

for a constant C_1 that is independent of ϕ. Hence there exists a constant C_2, also independent of ϕ, such that

$$\|B_\tau\|_{L^\infty(\Omega)} \leq C_2\|\phi\|^2_{W^{1,\infty}(\Omega)}. \tag{3.3}$$

We now claim that

$$E_\infty(u_\infty + \phi) \geq E_\infty(u_\infty) - 2C_2\|\phi\|^2_{W^{1,\infty}(\Omega)}. \tag{3.4}$$

Once this inequality is established, the proof is complete.

If $2C_2\|\phi\|^2_{W^{1,\infty}(\Omega)} > E_\infty(u_\infty)$, then (3.4) is obvious. Thus we assume that $2C_2\|\phi\|^2_{W^{1,\infty}(\Omega)} \leq E_\infty(u_\infty)$.

If $S'_{u_\infty}\phi = 0$ almost everywhere, then (3.2) and (3.3) imply that

$$(S(u_\infty + \phi))^2 \geq (S(u_\infty))^2 + 2S(u_\infty)B_\tau \geq (S(u_\infty))^2 - 2C_2E_\infty(u_\infty)\|\phi\|^2_{W^{1,\infty}(\Omega)}$$

almost everywhere. Therefore,

$$(E_\infty(u_\infty + t\phi))^2 \geq (E_\infty(u_\infty))^2 - 2C_2E_\infty(u_\infty)\|\phi\|^2_{W^{1,\infty}(\Omega)}. \tag{3.5}$$

If $S'_{u_\infty}\phi \neq 0$ in a set of positive measure, then we test equation (1.5) with ϕ. (This is possible in view of Lem. 2.2.) We obtain

$$\int_\Omega f_\infty\, S'_{u_\infty}\phi\, \mathrm{d}x = 0.$$

Recall that $f_\infty \neq 0$ almost everywhere. Therefore, there exists a set $\Omega_+ \subseteq \Omega$ of positive measure such that $f_\infty\, S'_{u_\infty}\phi > 0$ in Ω_+. As f_∞ has the same sign as $S(u_\infty)$ almost everywhere by (1.4), this means that $S(u_\infty)\, S'_{u_\infty}\phi > 0$ almost everywhere in Ω_+. Equation (1.4) also implies that $|S(u_\infty)| = e_\infty$ almost everywhere. As $\|\phi\|_{W^{1,\infty}(\Omega)} \leq \sqrt{e_\infty/(2C_2)}$ by the above assumption, inequality (3.3) implies that $S(u_\infty) + B_\tau$ has the same sign as $S(u_\infty)$ almost everywhere. Hence

$$\big(S(u_\infty) + B_\tau\big)S'_{u_\infty}\phi > 0$$

almost everywhere in Ω_+. With the help of (3.2) and (3.3), we conclude that

$$(S(u_\infty + \phi))^2 \geq e_\infty^2 - 2C_2e_\infty\|\phi\|^2_{W^{1,\infty}(\Omega)}$$

in Ω_+. Hence we obtain (3.5) in this case as well.

Finally, from (3.5) we now obtain the estimate

$$\begin{aligned} E_\infty(u_\infty + \phi) &\geq E_\infty(u_\infty)\sqrt{1 - \frac{2C_2\|\phi\|^2_{W^{1,\infty}(\Omega)}}{E_\infty(u_\infty)}} \\ &\geq E_\infty(u_\infty)\left(1 - \frac{2C_2\|\phi\|^2_{W^{1,\infty}(\Omega)}}{E_\infty(u_\infty)}\right) \\ &= E_\infty(u_\infty) - 2C_2\|\phi\|^2_{W^{1,\infty}(\Omega)}. \end{aligned}$$

This proves (3.4) and completes the proof. ☐

4. Necessity of the equations

In this section, we prove Theorem 1.4. For this purpose, we require the following lemma, which is an extension of a result proved by the authors in a previous paper [22], Lemma 8. This is where the extra regularity assumptions in Theorem 1.4 are used. For $r > 0$, we use the notation $\Omega_r = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < r\}$ here.

Lemma 4.1. *Suppose that $\partial\Omega$ is of class C^3 and $u_0 \in C^2(\overline{\Omega})$. Let $g \in C^0(\overline{\Omega})$ and $u \in C^1(\overline{\Omega})$. Given $\epsilon > 0$, there exist $r > 0$ and $v \in C^2(\overline{\Omega})$, with $v = u_0$ and $Dv = Du_0$ on $\partial\Omega$, such that $\|S'_u v - g\|_{L^\infty(\Omega_r)} \leq \epsilon$.*

Proof. Let $\delta > 0$. Choose $\tilde{u}_0 \in C^4(\overline{\Omega})$ such that

$$\|u_0 - \tilde{u}_0\|_{C^2(\overline{\Omega})} \leq \delta$$

and choose $\tilde{g} \in C^2(\overline{\Omega})$ such that

$$\left\|g - \tilde{g} - u_0 b_y(x, u, Du) - Du_0 \cdot b_z(x, u, Du)\right\|_{C^0(\overline{\Omega})} \leq \delta.$$

Consider a number $r_0 > 0$ such that the function $x \mapsto \operatorname{dist}(x, \partial\Omega)$ is of class C^3 in $\overline{\Omega}_{2r_0}$. We can construct a function $\rho \in C^3(\overline{\Omega})$ such that $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega_{r_0}$. Note that there exists $c > 0$ such that $A : D\rho \otimes D\rho \geq c$ in Ω_{r_0} by the fact that A is uniformly positive definite and $|D\rho| = 1$. Define $\lambda \in C^2(\overline{\Omega})$ such that $\lambda = 2A : D\rho \otimes D\rho$ in Ω_{r_0} and $\lambda > 0$ everywhere. Now define

$$h = \tilde{g} - A : D^2\tilde{u}_0$$

and

$$v = u_0 + \frac{\rho^2 h}{\lambda}.$$

Then

$$Dv = Du_0 + \frac{2\rho h}{\lambda} D\rho + \rho^2 D\left(\frac{h}{\lambda}\right)$$

and

$$D^2 v = D^2 u_0 + \frac{2h}{\lambda} D\rho \otimes D\rho + \frac{2\rho h}{\lambda} D^2\rho + 2\rho D\rho \otimes D\left(\frac{h}{\lambda}\right) + 2\rho D\left(\frac{h}{\lambda}\right) \otimes D\rho + \rho^2 D^2\left(\frac{h}{\lambda}\right).$$

Thus there exist $\Phi \in C^0(\overline{\Omega}; \mathbb{R}^n)$ and $\Psi \in C^0(\overline{\Omega}; \mathbb{R}^{n\times n})$ such that

$$Dv = Du_0 + \rho\Phi$$

and

$$D^2 v = D^2 u_0 + \frac{2h}{\lambda} D\rho \otimes D\rho + \rho\Psi.$$

In Ω_{r_0}, it follows that

$$\begin{aligned} S_u' v &= A : D^2 u_0 + \frac{2h}{\lambda} A : D\rho \otimes D\rho + \rho A : \Psi \\ &\quad + \left(u_0 + \frac{\rho^2 h}{\lambda}\right) b_y(x,u,Du) + (Du_0 + \rho\Phi)\cdot b_z(x,u,Du) \\ &= A : D^2 u_0 + \tilde{g} - A : D^2\tilde{u}_0 + u_0 b_y(x,u,Du) + Du_0 \cdot b_z(x,u,Du) + \rho X, \end{aligned}$$

where

$$X = A : \Psi + \frac{\rho h}{\lambda} b_y(x,u,Du) + \Phi \cdot b_z(x,u,Du).$$

By the choice of $\tilde{u}_0$ and $\tilde{g}$, we conclude that

$$\|S_u' v - g\|_{L^\infty(\Omega_r)} \le (\|A\|_{L^\infty(\Omega)} + 1)\delta + r\|X\|_{L^\infty(\Omega)}.$$

Choosing δ and r sufficiently small, we obtain the desired inequality.

The boundary conditions are readily checked as well. □

Proof of Theorem 1.4. We assume that $\partial\Omega$ is of class C^3 and $u_0 \in C^2(\overline{\Omega})$. We further assume that S is admissible. Suppose that $u_\infty \in u_0 + \mathcal{W}_0^{2,\infty}(\Omega)$ is an almost-minimiser of E_∞. We wish to show that there exist $e_\infty \ge 0$ and $f_\infty \in L^1(\Omega) \setminus \{0\}$ such that (1.4) is satisfied almost everywhere and (1.5) weakly.

If $E_\infty(u_\infty) = 0$, then we choose $e_\infty = 0$. By a consequence of the Fredholm alternative [8], Theorem 6.2.4, we can either find a nontrivial solution of $S_{u_\infty}^* f_\infty = 0$ in $W_0^{1,2}(\Omega)$, or we can solve the boundary value problem

$$\begin{aligned} S_{u_\infty}^* f_\infty &= 0 \quad \text{in } \Omega, \\ f_\infty &= 1 \quad \text{on } \partial\Omega, \end{aligned}$$

in $W^{1,2}(\Omega)$. In either case, we conclude that $f_\infty \in L^1(\Omega)$ and does not vanish identically. Hence the required conditions are satisfied.

We now assume that $E_\infty(u_\infty) > 0$. Because u_∞ is an almost-minimiser, there exists $M \in \mathbb{R}$ such that

$$E_\infty(u_\infty) \le E_\infty(v) + M\|u_\infty - v\|_{W^{1,\infty}(\Omega)}^2$$

for all $v \in u_0 + \mathcal{W}_0^{2,\infty}(\Omega)$. Choose $p_0 > n$ such that the statement from Definition 1.3 applies. Note that by [13], Lemma 9.17, there exists a constant C such that for all $\phi \in W_0^{2,p_0}(\Omega)$, the inequality

$$\|\phi\|_{W^{2,p_0}(\Omega)} \le C\|A : D^2\phi\|_{L^{p_0}(\Omega)}$$

holds true. In conjunction with the Sobolev embedding theorem, this implies that there exists $\mu > 0$ such that

$$E_\infty(u_\infty) \le E_\infty(v) + \mu\|A : D^2(u_\infty - v)\|_{L^{p_0}(\Omega)}^2$$

under the above assumptions.

For $p < \infty$, we consider the functionals

$$E_p(u) = \left(\fint_\Omega |S(u)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}.$$

Furthermore, we fix $\sigma > 0$ and define

$$E_p^\sigma(u) = E_p(u) + \sigma \|A : D^2(u_\infty - u)\|^2_{L^{p_0}(\Omega)}.$$

For every $p \geq p_0$, choose a minimiser $u_p \in u_0 + W_0^{2,p}(\Omega)$ of E_p^σ. (This can be found with the direct method. Coercivity of the functional is a consequence of the assumption that S is admissible, similarly to the proof of Proposition 1.5. Lower semicontinuity of E_p with respect to weak convergence in $W^{2,p}(\Omega)$ is also proved analogously to Proposition 1.5, and the lower semicontinuity of the additional term follows from its convexity.) For $p_0 \leq p \leq q$, the minimality of $E_p(u_p)$ and Hölder's inequality imply that

$$E_p^\sigma(u_p) \leq E_p^\sigma(u_q) \leq E_q^\sigma(u_q) \leq E_q^\sigma(u_\infty) = E_q(u_\infty) \leq E_\infty(u_\infty). \tag{4.1}$$

Because S is admissible, we infer that

$$\limsup_{p\to\infty} \|u_p\|_{W^{2,q}(\Omega)} < \infty \tag{4.2}$$

for any $q < \infty$. We may therefore choose a sequence $p_k \to \infty$ and find $w_\infty \in u_0 + \bigcap_{q<\infty} W_0^{2,q}(\Omega)$ such that $u_{p_k} \rightharpoonup w_\infty$ weakly in $W^{2,q}(\Omega)$ for every $q < \infty$. Then we also have the strong convergence $u_{p_k} \to w_\infty$ in $W^{1,\infty}(\Omega)$. It further follows that $S(u_{p_k}) \rightharpoonup S(w_\infty)$ weakly in $L^q(\Omega)$ for every $q < \infty$.

By the lower semicontinuity of the L^q-norm with respect to weak convergence and by (4.1), we have the inequalities

$$\begin{aligned} E_\infty^\sigma(w_\infty) &= \lim_{q\to\infty} E_q^\sigma(w_\infty) \\ &\leq \limsup_{q\to\infty} \liminf_{k\to\infty} E_q^\sigma(u_{p_k}) \\ &\leq \liminf_{k\to\infty} E_{p_k}^\sigma(u_{p_k}) \\ &\leq E_\infty(u_\infty). \end{aligned} \tag{4.3}$$

Hence $w_0 \in u_0 + \mathcal{W}_0^{2,\infty}(\Omega)$. As u_∞ is an almost-minimiser of E_∞, we also know that

$$E_\infty(u_\infty) \leq E_\infty^\mu(w_\infty).$$

Hence

$$(\sigma - \mu)\|A : D^2(w_\infty - u_\infty)\|_{L^{p_0}(\Omega)} \leq 0,$$

which implies that $w_\infty = u_\infty$ if $\sigma > \mu$. As this fixes the limit, we can in fact conclude that $u_p \rightharpoonup u_\infty$, as $p \to \infty$, weakly in $W^{2,q}(\Omega)$ for every $q < \infty$. It also follows from (4.3) and (4.1) that $E_p^\sigma(u_p) \to E_\infty(u_\infty)$ as $p \to \infty$.

We further observe that for any sequence $p_k \to \infty$,

$$\begin{aligned} E_\infty(u_\infty) + \sigma \liminf_{k\to\infty} \|A : D^2(u_{p_k} - u_\infty)\|^2_{L^{p_0}(\Omega)} &= \lim_{q\to\infty} E_q(u_\infty) + \sigma \liminf_{k\to\infty} \|A : D^2(u_{p_k} - u_\infty)\|^2_{L^{p_0}(\Omega)} \\ &\leq \limsup_{q\to\infty} \liminf_{k\to\infty} E_q(u_{p_k}) + \sigma \liminf_{k\to\infty} \|A : D^2(u_{p_k} - u_\infty)\|^2_{L^{p_0}(\Omega)} \\ &\leq \limsup_{q\to\infty} \liminf_{k\to\infty} E_q^\sigma(u_{p_k}). \end{aligned}$$

For every fixed $q < \infty$, Hölder's inequality gives $E_q^\sigma(u_{p_k}) \le E_{p_k}^\sigma(u_{p_k})$ whenever k is sufficiently large. Hence

$$\liminf_{k\to\infty} E_q^\sigma(u_{p_k}) \le \liminf_{k\to\infty} E_{p_k}^\sigma(u_{p_k}),$$

and we conclude that

$$E_\infty(u_\infty) + \sigma \liminf_{k\to\infty} \|A : D^2(u_{p_k} - u_\infty)\|_{L^{p_0}(\Omega)}^2 \le \liminf_{k\to\infty} E_{p_k}^\sigma(u_{p_k}) = E_\infty(u_\infty).$$

It follows that

$$\lim_{p\to\infty} \|A : D^2(u_p - u_\infty)\|_{L^{p_0}(\Omega)} = 0.$$

Set $e_\infty = E_\infty(u_\infty)$ and $e_p = E_p(u_p)$. Then it follows that

$$e_\infty = \lim_{p\to\infty} e_p.$$

Set furthermore

$$a_p = \|A : D^2(u_p - u_\infty)\|_{L^{p_0}(\Omega)}.$$

Then the Euler–Lagrange equation for u_p is

$$e_p^{1-p} S_{u_p}^*\big(|S(u_p)|^{p-2} S(u_p)\big) + 2\sigma|\Omega| a_p^{2-p_0} \operatorname{div}\operatorname{div}\big(|A : D^2(u_p - u_\infty)|^{p_0-2}(A : D^2(u_p - u_\infty))A\big) = 0.$$

Set

$$f_p = e_p^{1-p} |S(u_p)|^{p-2} S(u_p)$$

and

$$\phi_p = a_p^{2-p_0} |A : D^2(u_p - u_\infty)|^{p_0-2} A : D^2(u_p - u_\infty).$$

Then we have the system

$$|f_p|^{\frac{p-2}{p-1}} S(u_p) = e_p f_p, \tag{4.4}$$

$$S_{u_p}^* f_p + 2\sigma|\Omega| \operatorname{div}\operatorname{div}(\phi_p A) = 0. \tag{4.5}$$

We compute

$$⨍_\Omega |f_p|^{p/(p-1)} \, \mathrm{d}x = e_p^{-p} ⨍_\Omega |S(u_p)|^p \, \mathrm{d}x = 1. \tag{4.6}$$

Hence we can find a sequence $p_k \to \infty$ such that f_{p_k} converges, in the weak* sense in $(C^0(\overline{\Omega}))^*$, to a Radon measure F_∞ on $\overline{\Omega}$. Testing equation (4.5) with $\eta \in C_0^\infty(\Omega)$, we see that

$$\begin{aligned}\int_\Omega S'_{u_\infty}\eta\, dF_\infty &= \lim_{k\to\infty}\int_\Omega f_{p_k} S'_{u_\infty}\eta\,\mathrm{d}x\\ &= \lim_{k\to\infty}\int_\Omega f_{p_k} S'_{u_{p_k}}\eta\,\mathrm{d}x\\ &= -2\sigma|\Omega|\lim_{k\to\infty}\int_\Omega \phi_{p_k} A : D^2\eta\,\mathrm{d}x.\end{aligned}$$

(In the second step, we have used the fact that $b_y(x,u_p,Du_p) \to b_y(x,u_\infty,Du_\infty)$ and $b_z(x,u_p,Du_p) \to b_z(x,u_\infty,Du_\infty)$ uniformly as $p\to\infty$.) If p_0' is the exponent conjugate to p_0, then

$$\|\phi_p\|_{L^{p_0'}(\Omega)} = a_p^{2-p_0}\left(\int_\Omega |A : D^2(u_p - u_\infty)|^{p_0}\,\mathrm{d}x\right)^{\frac{p_0-1}{p_0}} = a_p \to 0 \tag{4.7}$$

as $p\to\infty$. Hence F_∞ is a distributional solution of $S^*_{u_\infty}F_\infty = 0$. According to Lemma 2.1, its restriction to Ω is absolutely continuous with respect to the Lebesgue measure, and the Radon–Nikodym derivative $f_\infty \in L^1(\Omega)$ belongs to $W^{1,q}_{\mathrm{loc}}(\Omega)$ for all $q\in(1,\frac{n}{n-1})$ (but F_∞ may have a part supported on $\partial\Omega$ as well). Obviously, we now have equation (1.5) in the weak sense.

Set $h_p = f_p + 2\sigma|\Omega|\phi_p$. Then (4.5) can be written in the form

$$S^*_{u_p}h_p = 2\sigma|\Omega|\left(\phi_p b_y(x,u_p,Du_p) - \operatorname{div}\big(\phi_p b_z(x,u_p,Du_p)\big)\right).$$

We already know that the functions $b_z(x,u_p,Du_p)$ and $b_y(x,u_p,Du_p)$ are uniformly bounded in $L^\infty(\Omega)$. Because of (4.7), we have uniform bounds for $\phi_p b_y(x,u_p,Du_p)$ and for $\phi_p b_z(x,u_p,Du_p)$ in $L^{p_0'}(\Omega)$. The coefficients of $S^*_{u_p}$ are also bounded in $L^\infty(\Omega)$. According to Lemma 2.1, this means that we have a uniform bound for $\|h_p\|_{W^{1,p_0'}(\Omega')}$ for any precompact open set $\Omega' \Subset \Omega$. In particular, we can choose the above sequence $p_k\to\infty$ such that $(h_{p_k})_{k\in\mathbb{N}}$ converges in $L^{p_0'}(\Omega')$. Since we have the convergence $\phi_p \to 0$ in the same space by (4.7), we must have $L^{p_0'}$-convergence for $(f_{p_k})_{k\in\mathbb{N}}$. The limit is of course f_∞.

We may assume that $f_{p_k}\to f_\infty$ almost everywhere in Ω' and, at the same time, that $\|f_{p_k} - f_\infty\|_{L^{p_0'}(\Omega')} \le 2^{-k}$ for every $k\in\mathbb{N}$ (otherwise we choose a further subsequence with this property). If we set

$$\theta = |f_\infty| + \sum_{k=1}^\infty |f_{p_k} - f_\infty|,$$

then this guarantees that $\theta\in L^{p_0'}(\Omega')$. Moreover, we see that $|f_{p_k}|\le\theta$ for every $k\in\mathbb{N}$. We therefore obtain the pointwise inequality

$$\left||f_{p_k}|^{\frac{p_k-2}{p_k-1}} - |f_\infty|\right|^{p_0'} \le (1+2\theta)^{p_0'}$$

(provided that $p_k > 2$), and the dominated convergence theorem implies that

$$|f_{p_k}|^{\frac{p_k-2}{p_k-1}} \to |f_\infty|$$

in $L^{p_0'}(\Omega')$. Since we have the weak convergence of $S(u_p)$ to $S(u_\infty)$ in $L^q(\Omega)$ for any $q < \infty$, we conclude that

$$|f_{p_k}|^{\frac{p_k-2}{p_k-1}} S(u_{p_k}) \rightharpoonup |f_\infty| S(u_\infty)$$

weakly in $L^1_{\mathrm{loc}}(\Omega)$. Recall that $e_p \to e_\infty$ as $p \to \infty$. Thus passing to the limit in (4.4), we obtain (1.4).

It remains to show that $f_\infty \not\equiv 0$.

For any $p \geq p_0$, we compute

$$|\Omega| e_p = e_p^{1-p} \int_\Omega |S(u_p)|^p \, \mathrm{d}x = \int_\Omega f_p S(u_p) \, \mathrm{d}x = \int_\Omega f_p \left(S'_{u_p} u_p - g_p \right) \mathrm{d}x,$$

where

$$g_p = -b(x, u_p, Du_p) + u_p b_y(x, u_p, Du_p) + Du_p \cdot b_z(x, u_p, Du_p).$$

We define g_∞ by the analogous formula as well. Given $\epsilon > 0$, we choose $r > 0$ and $v \in u_0 + \mathcal{W}^{2,\infty}_0(\Omega)$ with

$$\| S'_{u_\infty} v - g_\infty \|_{L^\infty(\Omega_r)} \leq \epsilon$$

with the help of Lemma 4.1. Then, by (4.5),

$$\begin{aligned}
|\Omega| e_p &= \int_\Omega f_p \left(S'_{u_p} u_p - g_p \right) \mathrm{d}x \\
&= \int_\Omega f_p \left(S'_{u_p} v - g_p \right) \mathrm{d}x - 2\sigma |\Omega| \int_\Omega \phi_p A : D^2(u_p - v) \, \mathrm{d}x \\
&= \int_{\Omega_r} f_p \left(S'_{u_\infty} v - g_\infty \right) \mathrm{d}x + \int_{\Omega_r} f_p \left(S'_{u_p} v - S'_{u_\infty} v - g_p + g_\infty \right) \mathrm{d}x \\
&\quad + \int_{\Omega \setminus \Omega_r} f_p \left(S'_{u_p} v - g_p \right) \mathrm{d}x - 2\sigma |\Omega| \int_\Omega \phi_p A : D^2(u_p - v) \, \mathrm{d}x \\
&\leq \epsilon |\Omega| + \int_{\Omega_r} f_p \left(S'_{u_p} v - S'_{u_\infty} v - g_p + g_\infty \right) \mathrm{d}x \\
&\quad + \int_{\Omega \setminus \Omega_r} f_p \left(S'_{u_p} v - g_p \right) \mathrm{d}x + 2\sigma |\Omega| a_p \, \| A : D^2(u_p - v) \|_{L^{p_0}(\Omega)}.
\end{aligned}$$

Letting $p \to \infty$, we note that $S'_{u_p} v \to S'_{u_\infty} v$ and $g_p \to g_\infty$ uniformly in Ω, while

$$\limsup_{p \to \infty} \| f_p \|_{L^1(\Omega)} < \infty$$

by (4.6). Furthermore, we know that $a_p \to 0$, while $A : D^2 u_p$ is uniformly bounded in $L^{p_0}(\Omega)$. We further know that $f_{p_k} \to f_\infty$ in $L^1(\Omega \setminus \Omega_r)$. Hence

$$|\Omega| (e_\infty - \epsilon) \leq \int_{\Omega \setminus \Omega_r} f_\infty (S'_{u_\infty} v - g_\infty) \, \mathrm{d}x.$$

Choosing $\epsilon < e_\infty$, we conclude that the integral on the right-hand side does not vanish. Hence $f_\infty \not\equiv 0$, and this concludes the proof. □

5. Examples

The condition for admissible operators in Definition 1.3 often follows from standard estimates for linear operators. For semilinear ones, some additional arguments are sometimes required. In this section, we consider two examples. We restrict our attention to operators of the form

$$S(u) = \Delta u + g(u)$$

for some function $g \in C^2(\mathbb{R})$ here. We show that S is admissible if either g has the correct sign or satisfies a suitable growth condition.

Proposition 5.1. *If $yg(y) \leq 0$ for all $y \in \mathbb{R}$, then S is admissible.*

Proof. Choose any $p_0 > n/2$. Given $u_0 \in \mathcal{W}^{2,\infty}(\Omega)$, let

$$\alpha = \sup_{x \in \partial\Omega} |u_0(x)|.$$

Suppose that $u \in u_0 + \mathcal{W}_0^{2,\infty}(\Omega)$ satisfies $\|S(u)\|_{L^p(\Omega)} \leq \Lambda$ for some $p \geq p_0$. Then $\|S(u)\|_{L^{p_0}(\Omega)} \leq |\Omega|^{1/p_0 - 1/p}\Lambda$ by Hölder's inequality. We now solve the boundary value problem

$$\begin{aligned} \Delta w &= -|S(u)| && \text{in } \Omega, \\ w &= \alpha && \text{on } \partial\Omega. \end{aligned}$$

Then $w \geq \alpha$ by the maximum principle. Furthermore, using L^p-estimates for the Laplacian and the Sobolev embedding theorem, we see that w is bounded by a constant that depends only on n, Ω, p_0, u_0, and Λ.

In the set $\{x \in \Omega \mid u(x) > 0\}$, we have the inequality

$$\Delta u = S(u) - g(u) \geq S(u) \geq \Delta w.$$

Hence the comparison principle implies that $u \leq w$ in Ω.

Similarly, we show that u is bounded from below by a constant depending only on n, Ω, p_0, u_0, and Λ. The condition from Definition 1.3 then follows with standard elliptic estimates. □

For our second example, we assume that

$$\lim_{y \to \pm\infty} \frac{yg(y)}{\int_0^y g(t)\,\mathrm{d}t} = \alpha. \tag{5.1}$$

That is, the function g has asymptotic growth at $\pm\infty$ like $y \mapsto cy^{\alpha-1}$ for some $c \in \mathbb{R}$.

Proposition 5.2. *Let $n \geq 3$. Suppose that g satisfies (5.1) for some $\alpha \in [2, \frac{2n}{n-2})$. Then S is admissible.*

Proof. Define

$$G(y) = \int_0^y g(t)\,\mathrm{d}t.$$

Given $\beta > \alpha$, inequality (5.1) implies that $yG'(y) < \beta G(y)$ when $|y|$ is sufficiently large. The Grönwall inequality implies that there exists $C_1 > 0$ with

$$G(y) \leq C_1(|y|^\beta + 1)$$

for all $y \in \mathbb{R}$. Using (5.1) again, we find another constant C_2 such that

$$|g(y)| \leq C_2(|y|^{\beta-1} + 1) \tag{5.2}$$

for all $y \in \mathbb{R}$.

Now suppose that $u_0 \in \mathcal{W}^{2,\infty}(\Omega)$ and consider $u \in u_0 + \mathcal{W}_0^{2,\infty}(\Omega)$. We write ν for the outer normal vector on $\partial\Omega$ and σ for the surface measure on $\partial\Omega$. Then an integration by parts yields the identity

$$\int_\Omega (|Du|^2 - ug(u))\,\mathrm{d}x = \int_{\partial\Omega} u_0\nu \cdot Du_0 \,\mathrm{d}\sigma - \int_\Omega uS(u)\,\mathrm{d}x. \tag{5.3}$$

We furthermore compute

$$\operatorname{div}\left((x\cdot Du)Du - \left(\frac{1}{2}|Du|^2 - G(u)\right)x\right) = (x\cdot Du)S(u) - \frac{n-2}{2}|Du|^2 + nG(u).$$

Hence

$$\begin{aligned}\int_\Omega \left(\frac{n-2}{2}|Du|^2 - nG(u)\right)\mathrm{d}x = \int_\Omega (x\cdot Du)S(u)\,\mathrm{d}x \\ - \int_{\partial\Omega}\left((x\cdot Du_0)(\nu\cdot Du_0) - \left(\frac{1}{2}|Du_0|^2 - G(u_0)\right)x\cdot\nu\right)\mathrm{d}\sigma.\end{aligned} \tag{5.4}$$

Fix $\beta \in (\alpha, \frac{2n}{n-2})$. Then the combination of (5.3) and (5.4) implies that

$$\begin{aligned}\left(\frac{1}{\beta} - \frac{n-2}{2n}\right)\int_\Omega |Du|^2\,\mathrm{d}x &= \int_\Omega \left(\frac{ug(u)}{\beta} - G(u)\right)\mathrm{d}x \\ &\quad - \int_\Omega \left(\frac{1}{\beta}uS(u) + \frac{1}{n}(x\cdot Du)S(u)\right)\mathrm{d}x + \frac{1}{\beta}\int_{\partial\Omega} u_0\nu\cdot Du_0\,\mathrm{d}\sigma \\ &\quad + \frac{1}{n}\int_{\partial\Omega}\left((x\cdot Du_0)(\nu\cdot Du_0) - \left(\frac{1}{2}|Du_0|^2 - G(u_0)\right)x\cdot\nu\right)\mathrm{d}\sigma.\end{aligned} \tag{5.5}$$

Under the assumptions of the proposition, we know that $yg(y) \leq \beta G(y)$ whenever $|y|$ is sufficiently large. Hence there exists some constant C_3, depending only on g, such that

$$\int_\Omega \left(\frac{ug(u)}{\beta} - G(u)\right)\mathrm{d}x \leq C_3|\Omega|.$$

The boundary integrals in (5.5) depend only on n, Ω, u_0, g, and β. Hence there exists $C_4 = C_4(n, \Omega, u_0, g, \alpha)$ such that

$$\int_\Omega |Du|^2\,\mathrm{d}x \leq C_4\left(\|S(u)\|_{L^2(\Omega)}\|u\|_{W^{1,2}(\Omega)} + 1\right).$$

From this and the Poincaré inequality, we derive the estimate

$$\|u\|_{W^{1,2}(\Omega)} \leq C_5\left(\|S(u)\|_{L^2(\Omega)} + 1\right)$$

for a constant C_5 with the same dependence.

Fix $p_0 > n$ and consider $p \geq p_0$. If we assume that

$$\|S(u)\|_{L^p(\Omega)} \leq \Lambda,$$

then we can now use standard bootstrapping arguments to derive higher estimates. Since the growth of g described in (5.2) is subcritical for the purpose of such estimates, we will eventually obtain a bound for $\|u\|_{W^{2,p}(\Omega)}$ that depends only on n, Ω, u_0, g, p, and Λ. Hence S is admissible. ☐

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