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Explaining abnormal returns in stock markets: An alpha-neutral version of the CAPM

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ABSTRACT

This paper develops a behavioural asset pricing model in which traders are not fully rational as is commonly assumed in the literature. The model derived is underpinned by the notion that agents' preferences are affected by their degree of optimism or pessimism regarding future market states. It is characterized by a representation consistent with the Capital Asset Pricing Model, augmented by a behavioural bias that yields a simple and intuitive economic explanation of the abnormal returns typically left unexplained by benchmark models. The results we provide show how the factor introduced is able to absorb the "abnormal" returns that are not captured by the traditional CAPM, thereby reducing the pricing errors in the asset pricing model to statistical insignificance.

1. Introduction

During the last 50 years, a substantial part of the research effort in both theoretical and empirical asset pricing has been focused on the disclosure of patterns in average stock returns which are not described by the Sharpe (1964), Lintner (1965), and Mossin (1966) capital asset pricing model (CAPM) and are thus referred to as "anomalies" in the asset pricing literature. Within this body of work, we might note the findings of patterns between stock returns and firms' characteristics,[1] long term reversals (De Bondt & Thaler, 1985) and momentum (Jegadeesh & Titman, 1993), the discovery of an excessively flat relationship between average returns and market beta,[2] the scarcity of explanatory power of the latter, which sometimes even manifests itself in a negative relationship (Fama & French, 1992; Lakonishok & Shapiro, 1986), and the instability of market beta over time (Guo, Wu, & Yu, 2017; Jagannathan & Wang, 1996). Moreover, the CAPM is fully rejected from a statistical point of view, in that the model intercepts generated from time series regressions on actual data (also known in the literature as Jensen's *alphas* after the seminal paper of

Jensen (1968), fail to result in parameter estimates that are jointly indistinguishable from zero.[3]

Despite all of the critiques cited above, the CAPM still remains a model most entrusted by both practitioners and academics (Fama & French, 1996a, 1996b). At the same time, however, such strong evidence against the CAPM, underlying the paucity of the explanatory power of a single-factor model, has driven scholars to engage in a huge effort to develop new multifactor models. In particular, developments in the asset pricing literature have given rise to two different approaches to the problem. The first, purely empirical, includes multi-factor models which can be seen as different specifications of Ross' asset pricing theory (Ross, 1976), such as the most praised Fama and French (1993) – henceforth FF – three-factor model, Carhart's 1997 four-factor model, the liquidity-adjusted CAPMs of Pástor and Stambaugh (2003) and Acharya and Pedersen (2005), and, more recently, the Fama and French (2015) five-factor model. As for the second approach, we have a stream of literature that collects all of the natural extensions of the classic CAPM through a relaxation of some of its underlying assumptions, such as Black's 1972 zero-beta CAPM, Merton's 1973

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[1] Relevant studies include those that relate expected returns to size (Banz, 1981), book-to-market-equity (Rosenberg, Reid, & Lanstein, 1985), the earnings-price ratio (Basu, 1977, 1983), debt-equity ratio (Bhandari, 1988), profitability (Fama & French, 2006; Novy-Marx, 2013) and investment (Fama & French, 2006; Titman, Wei, & Xie, 2004). A review of these anomalies can be found in Fama and French (2008).

[2] Friend and Blume (1970), Black, Jensen, and Scholes (1972), Blume and Friend (1973), Reinganum (1981), Stambaugh (1982), Fama and French (1992).

[3] See for instance Jensen (1968), Friend and Blume (1970), Black et al. (1972), Blume and Friend (1973), Fama and MacBeth (1973), Reinganum (1981), Stambaugh (1982), Gibbons, Ross, and Shanken (1989), Fama and French (1992, 1996b).

intertemporal (I)CAPM, Kraus and Litzenberger's 1976 three-moment CAPM, Jagannathan and Wang's 1996 conditional (C)CAPM, consumption/investment based CAPMs (Breeden, 1979; Cochrane, 1991) and Dittmar's 2002 four-moment CAPM. The connection between the two approaches lies in the interpretation of empirical multifactor models as different specifications of equilibrium models. For instance, Maio and Santa-Clara (2012, 2017) analyse the conditions that must be satisfied by a multifactor model in order for it to be justifiable by the ICAPM.

Whether we want to interpret multifactor models as equilibrium models or not, all these specifications have in common that they are mercilessly rejected from a statistical point of view in terms of Jensen's *alpha*, as shown in many empirical applications. Fama and French (2015), for instance, argue that their five-factor model performs better than their three-factor model (FF, 1993) but still shows *alphas* that are jointly significantly different from zero. Similarly, Harvey and Siddique (2000), Dittmar (2002), Messis, Alexandridis, and Zapranis (2021), Lewellen and Nagel (2006), and Maio and Santa-Clara (2012), present studies respectively on the conditional three-moment CAPM, four-moment CAPM, CAPM with asymmetric and constant systemic risk, conditional consumption CAPM, CCAPM, and ICAPM, finding similar results in terms of the significance of the intercepts.

Apart from the standard view stating that other risk factors are to be included in the evaluation, the "behaviouralist" interpretation argues that the return component left unexplained by the model should be attributed to some departure from the hypothesis of agents' full rationality (Barberis & Thaler, 2003). Common explanations that have been advanced include investors' over-reactions to bad economic news and market seasonality (De Bondt & Thaler, 1987), under- and over-reaction to public (Barberis, Shleifer, & Vishny, 1998) and private (Daniel, Hirshleifer, & Subrahmanyam, 2001) information, optimism/pessimism (Diether, Malloy, & Scherbina, 2002), narrow framing and loss aversion (Barberis, Huang, & Santos, 2001) and, more recently, ambiguity aversion (Guidolin & Liu, 2016).

Interestingly, another common feature of both multifactor models and equilibrium models, which in particular originates from considering the former as specifications of the latter, is that most are underpinned by the hypothesis of fully rational agents, represented by the usage of a von Neumann–Morgenstern (VNM) order of preferences. As showed by Cochrane (2009), in fact, the CAPM, and thus the market component in explaining the cross section of stocks' expected returns, can be derived directly by using different types of VNM utility functions. A serious issue, and one that in our view is still not suitably considered in the asset pricing literature, is that such preferences do not properly describe the actual behaviour of individuals. As shown in a large number of studies in decision-making under risk, in fact, VNM preferences are not able to capture a wide range of features that have been shown to characterize the behaviour of agents, including, just to name few, the under- and over-weighting of probabilities, loss aversion and narrow framing.[4]

In the light of these considerations, in this paper we introduce a different version of the CAPM in which agents are boundedly-rational in the sense that they behave not as they theoretically should but as the empirical evidence shows that they do. In particular, we focus our attention on the inclusion of probability weights and the extent to which agents are optimistic or pessimistic in the asset pricing model. These, in our view, represent the most compelling, and somehow encompassing, departures from rationality. It is now a commonly held view that the use of the prospect theory of Kahneman and Tversky (1979) and Tversky and Kahneman (1992) is warranted. However, the employment of such preferences in asset pricing leads to a considerable loss of analytical tractability, as one can appreciate from the attempts

made in this direction,[5] and such an approach results in specifications that are challenging to test on actual data.

In order to avoid such issues, we make use of an order of preferences adjusted for optimism proposed by Rocciolo, Gheno, and Brooks (2019), which is both simple and characterized by high descriptive power. We justify this choice as a compromise between the representativeness of agents' behaviour and analytical tractability in that the employment of such preferences permits the maintenance of the linearity of the asset pricing model and its expression in terms of the beta terminology of the original CAPM. This is a feature that is typically not achievable when other models such as prospect theory are employed. Moreover, the S-shaped value function typically assumed in prospect theory seems unqualified in describing agents' behaviour when they face "mixed" prospects — i.e., prospects characterized by both gains and losses (Levy & Levy, 2002). Conversely, optimism-adjusted preferences, accounting explicitly for the possible skewness of the prospects, describe these kinds of situations well. In this sense, the model that we are going to derive is similar in spirit to the three-moment CAPM, in which investors' attitude towards skewness is implicitly taken into account (as well as its extension to the fourth moment) in an optimism-adjustment to the utility function. The fundamental difference, however, with respect to the models cited above, is given by the fact that the latter inevitably end up as multifactor models while our specification, as we will show, preserves a single factor representation in terms of *beta* and consistency with the traditional CAPM.

Moreover, the CAPM derived provides a clear economic interpretation of Jensen's *alpha* that is also consistent with the empirical evidence reported in Diether et al. (2002). It also provides, through the introduction of market sentiment into the specification, new evidence concerning the empirical validity of the CAPM. The results shown are strongly consistent with the underlying theory, which, as we will demonstrate, outperforms the currently most celebrated asset pricing models such as the Fama–French three- and five-factor models. More specifically, the test that we conduct on a large sample of portfolios sorted by size, book-to-market, investment, and operating profitability, shows, independently from the asset considered, pricing errors that are jointly indistinguishable from zero. We thus provide new evidence that, contrary to the common view, when the CAPM is corrected for the departure from full rationality of agents' behaviour, it is still alive and well. The series of diagnostic tests we run for confirmation gives robustness to our findings.

The remainder of the paper is organized as follows. In the next section we outline the optimism-adjusted preferences framework used in the derivation of our behavioural capital asset pricing model. Section 3 explores the datasets and the econometric techniques employed in order to obtain the results summarized in Section 4. Finally, Section 5 concludes.

2. The model

In this section we proceed to the derivation of our asset pricing model under conditions departing from full rationality. We start by introducing the system of preferences that characterize the agents in our economy. This is necessary since the representation of how agents make choices in the market will act as the basic framework in the derivation of the model.

2.1. Optimism-adjusted preferences

Let us consider an agent characterized by a VNM utility function $u(x)$ and let X be a prospect represented by a finite number of outcomes

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[4] See, for instance, Allais (1953), Kahneman and Tversky (1979) and Tversky and Kahneman (1992) for probability weighting and loss aversion, and Thaler (1999) for mental accounting.

[5] See for instance Barberis and Huang (2008), He and Zhou (2011) and De Giorgi, Hens, and Levy (2011).

x_j, each of which has an assigned probability p_j. Then let γ be a positive real number in $[0, 1]$ representative of the agent's degree of optimism and $\phi(\sigma_X, \gamma) \in [0, 1]$ be a real positive function of the degree of optimism γ and of the standard deviation of the outcomes σ_X, such that $\phi(\sigma_X, \gamma) \in (\frac{1}{2}, 1]$ and increasing in the variance of the outcomes σ_X^2 if $\gamma \in (\frac{1}{2}, 1]$, $\phi(\sigma_X, \gamma) \in [0, \frac{1}{2})$ and decreasing in the variance of the outcomes σ_X^2 if $\gamma \in [0, \frac{1}{2})$, and $\phi(\sigma_X, \gamma) = \frac{1}{2}$ $\forall$ σ_X^2 if $\gamma = \frac{1}{2}$.

Following Rocciolo et al. (2019), an optimist, represented by a value of the parameter $\gamma \in (\frac{1}{2}, 1]$, can be described as an agent who assigns a bigger weight to the positive outcomes of the prospect with respect to an unbiased agent, and who sees in a larger variance an opportunity to earn more from the risky opportunity.[6] Conversely, the pessimist, represented by a value of the parameter $\gamma \in [0, \frac{1}{2})$, can be seen as an agent who assigns less weight to positive outcomes and who is scared of an increment in the variance. Finally, $\gamma = \frac{1}{2}$ represents a rational expected utility maximizer. Formally, by modelling these circumstances through the function $\phi(\gamma, \sigma_X)$, the subjective value of the prospect for an agent affected by an optimism/pessimism bias[7] can be represented as

$$U(X, \sigma_X, \gamma) = 2\left[\phi(\gamma, \sigma_X)\mathbb{E}[u(x)]_+ + [1 - \phi(\gamma, \sigma_X)]\mathbb{E}[u(x)]_-\right] \quad (1)$$

where $\mathbb{E}[u(x)]_+$ and $\mathbb{E}[u(x)]_-$ are the subjective expected values of respectively the gains and losses with respect to a reference point $\bar{x}$, and $\phi(\gamma, \sigma_X)$, which assumes the interpretation of an optimism weighting function. It determines the weight assigned to the gains (and thus to the losses) in the overall value function based on the degree of optimism of the agent.

In order to sketch out how the model works, let us consider three agents endowed with the same utility function $u(x)$ and level of absolute risk aversion ρ, and different degrees of optimism γ. In particular, let us assume that one of them is an optimist ($\frac{1}{2} < \gamma_1 \leq 1$), one a pessimist ($0 < \gamma_2 \leq \frac{1}{2}$) and the last one is a pure rational expected utility maximizer ($\gamma_3 = \frac{1}{2}$). With respect to a prospect X faced, the three agents, while sharing the same utility function and risk aversion, might end up with very different evaluations depending on the variance of the outcome. As shown in Fig. 1, in fact, the bigger the outcome's variance, the more the optimist will assign a greater (lower) weight $\phi(\gamma_1, \sigma_X)$ to the prospect's gains (losses), and the steeper (flatter) will be the adjusted utility function $u_*(x, \sigma_X, \gamma_1)$ (s)he employs in the evaluation of the positive (negative) outcomes of the prospect. Conversely, the bigger the outcome's variance, the more the pessimist will assign a lower (greater) weight to the prospect's gains (losses), the flatter (steeper) will be the adjusted utility function $u_*(x, \sigma_X, \gamma_2)$ that (s)he employs in the evaluation of the positive (negative) outcomes of the prospect. Thus, we have that, under such preferences and *ceteris paribus*, $U(X, \sigma_X, \gamma_1) > U(X, \sigma_X, \gamma_3) > U(X, \sigma_X, \gamma_2)$ if the prospect is risky, i.e. $\sigma_X > 0$, and $U(X, \sigma_X, \gamma_1) = U(X, \sigma_X, \gamma_3) = U(X, \sigma_X, \gamma_2)$ in the case of a risk-free opportunity, i.e. $\sigma_X = 0$.

The strength of this representation evidently lies in being a mere adjustment applicable to a wide range of existing models in the field. At the same time, it is able to reconcile one of the most widely acknowledged features in the decision-making literature – evidence that individuals make use of weighted probabilities (Kahneman &

Tversky, 1979) – with the expected utility paradigm and with the advantage of a very simple mathematical representation. In fact, since the weighting function $\phi(\gamma, \sigma_X)$ is deterministic and independent from the final outcomes of the prospect, we can rewrite Eq. (1) as

$$U(x, \sigma_x, \gamma) = 2\left[\sum_{x_j \leq \bar{x}} u(x_j)p_j[1 - \phi(\gamma, \sigma_x)] + \sum_{x_j > \bar{x}} u(x_j)p_j\phi(\gamma, \sigma_x)\right] \quad (2)$$

where $\phi(\gamma, \sigma_X)$ can be interpreted in this representation as a function which assigns different weights to the objective probabilities according to the degree of optimism of the agent and the standard deviation of the prospect's outcomes, p_j is the objective probability assigned to the outcome x_j in the prospect X, and $\bar{x}$ is the reference point.

In this kind of setting, the choice of a proper analytical expression for the weighting function $\phi(\gamma, \sigma_X)$ is needed in order to apply the model. We suggest the following

$$\phi(\gamma, \sigma_X) = \begin{cases} 1 - \frac{1}{2}\exp\left\{-\rho\left(\gamma - \frac{1}{2}\right)\sigma_X^2\right\} & \text{if} \quad \frac{1}{2} \leq \gamma < 1 \\ \left[1 - \frac{1}{2}\exp\left\{-\rho\left(\gamma - \frac{1}{2}\right)\sigma_X^2\right\}\right]\mathbb{1}_{\sigma_X^2 < \sigma_*^2} & \text{if} \quad 0 < \gamma < \frac{1}{2} \end{cases} \quad (3)$$

where, by defining σ_*^2 as the threshold variance beyond which a pessimistic agent will give up on the prospect faced, $\mathbb{1}_{\sigma_X^2 < \sigma_*^2}$ is an indicator function which assumes the value one if the variance of the prospect's outcomes is lower than the critical level σ_*^2 and conversely is equal to zero when $\sigma_X^2 \geq \sigma_*^2$.

Rocciolo et al. (2019) studied in detail how such preferences perform in terms of their descriptive power for many of the most acknowledged "counter-examples" of the expected utility criterion. Their tests show in particular how the adjustment for optimism, characterized through the use of an optimism weighting function such as that in Eq. (3), can adapt expected utility theory in order to allow the latter to better describe the empirical evidence collected in a wide number of empirical studies, such as Allais (1953) and Kahneman and Tversky (1979). Moreover, they showed how the latter form is convenient, especially when applied in a CARA-Normal assumptions setting, in that it allows the derivation of linear demand curves, as we will show in the next section. In this sense, our decision to make use of such an order of preferences finds justification in that improving the descriptive power of the expected utility criterion allows us to use the latter, which is still the currently preferred framework in the asset pricing literature. As shown in the next section, this preference ordering also allows us to derive an asset pricing model expressed in the usual beta language.

2.2. The alpha-neutral CAPM

As in the classic CAPM, let us consider as a basic framework an economy free of taxes and transaction costs, characterized by n risk-averse utility maximizing agents, N risky assets, each characterized by a normally distributed gross return R_j, and a risk-free asset with an exogenously determined gross risk-free return R^F.[8] The market is always in equilibrium and each agent i can invest any fraction of his/her capital in either the risk-free asset or any of the risky assets traded in the market, and can freely borrow and lend funds at the gross risk-free return R^F. All n agents are assumed to be price-takers and plan to trade over the same time horizon at prices that are determined as a consequence of the equilibrium condition. In addition, let us make the following further assumptions:

Assumption 1. All n agents have the same information and beliefs about the objective joint probability distribution of the returns of all individual stocks

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[6] A possible issue that arises from this definition of optimism is that one may suspect optimistic agents to be risk lovers. The authors analyse this possibility at length and show that an optimistic, risk-averse agent will not be a risk seeker unless a highly skewed prospect is considered.

[7] Here, the term rational is interpreted in the sense of VNM expected utility theory, as a characteristic of agents displaying preferences that do not violate expected utility theory. In this paper, we consider the degree of agents' optimism to be the sole source of non-rationality. Optimistic and pessimistic agents are not (fully) rational in the sense that they do not conform to the coherence paradigms of expected utility theory (EUT) and they display preference orderings that typically violate the latter.

[8] The inverse of the gross return and the gross risk-free return define respectively the stochastic discount factor $1/R_j$, specific to the asset j and the risk-free discount factor $1/R^F$.
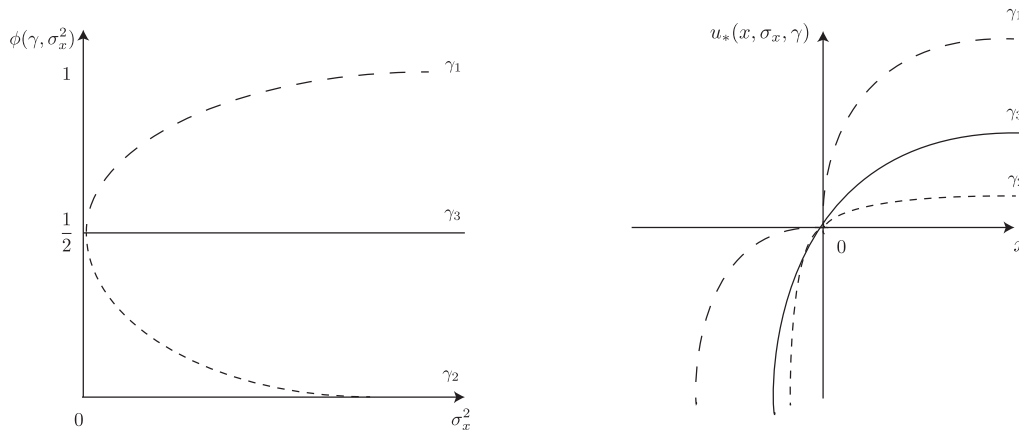
Fig. 1. The left-hand plot represents the optimism weighting $\phi(\gamma, \sigma_X)$ as a function of the prospect's variance σ_X^2 for the degrees of optimism $\frac{1}{2} < \gamma_1 \le 1$, $0 < \gamma_2 \le \frac{1}{2}$ and $\gamma_3 = \frac{1}{2}$. The right-hand plot represents the different distortions in the utility functions of the three agents, according to their degrees of optimisms and the optimism weighting function $\phi(\gamma, \sigma_X)$.

Assumption 2. Every agent i is equipped with an optimism-adjusted negative exponential utility function of the type $u_*(x, \sigma_X, \gamma_i) = -2[1 - \phi(\gamma, \sigma_X)]\exp\{-\rho_i x\}$, where the function $\phi(\gamma, \sigma_X)$ takes the form in (3), and where the parameters σ_X, ρ_i and γ_i are respectively the standard deviation of the prospect X, the absolute risk aversion coefficient, and the degree of optimism of the agent i. Risk aversion and agents' degree of optimism are assumed constant over time

The second assumption represents the actual breaking point with "rational" asset pricing theory through the introduction of a behavioural element in the evaluation of the assets, represented by the agent's degree of optimism. Being the unique difference with respect to the standard assumption set used in deriving the traditional CAPM, the asset pricing model we are going to derive makes a comparison with similar models in the literature an easy task. In particular, this greatly facilitates the study of where and how the original formulation of the CAPM fails and how it can be fixed simply by considering agents in the market as they actually are, i.e. not (fully) rational. Since we are using essentially the same framework as the traditional CAPM, the derivation of what follows traces the standard CARA-Normal procedure widely discussed in, amongst others, Cochrane (2009).

Let us start by considering the problem from the point of view of a single agent i characterized, at time $t - 1$, by an initial level of wealth $W_i(t - 1)$ that can be split how (s)he prefers between the risk-free and risky securities traded in the market, in order to maximize the utility of final level of wealth $W_i(t)$. Let x_i^f and $\mathbf{x}_i$ be respectively the amount of his(her) initial wealth invested in the risk-free asset and the $N \times 1$ vector of the amounts invested in the risky securities. His(her) maximization problem is given by

$$\underset{W_i}{\arg\max}\, U(W_i(t)) = \mathbb{E}[-2[1 - \phi(\gamma, \sigma_W)]\exp(-\rho_i W_i(t))] \tag{4}$$

subject to the following budget constraint

$$\mathbf{x}_i'\mathbf{1} + x_i^f = W_i(t - 1) \tag{5}$$

where the agent's final level of wealth is given by

$$W_i(t) = \mathbf{x}_i'\mathbf{R} + x_i^f R^F \tag{6}$$

$\mathbf{R}$ is the $N \times 1$ vector of gross risky returns and $\mathbf{1}$ an $N \times 1$ vector of ones.

Proposition 1. *Under* Assumption 2, *the $N \times 1$ vector of individual optimal demand schedules for the risky assets traded in the market, which solves the optimization problem in* (4) *subject to* (5)*, is given by*

$$\mathbf{x}_i = \boldsymbol{\Sigma}_R^{-1}\frac{\mathbb{E}[\mathbf{R}] - \mathbf{R}^F}{(\rho_i + \kappa(\gamma_i))} \tag{7}$$

where $\kappa(\gamma_i) = 1 - 2\gamma_i$, $\mathbf{R}^F = \mathbf{1}R^F$ and $\boldsymbol{\Sigma}_R$ is the $N \times N$ covariance matrix of risky asset gross returns. Under Assumptions 1 *and* 2*, every agent holds a portfolio characterized by different combinations, according to his/her risk aversion and degree of optimism, of the risk-free asset and the market portfolio so that, at an aggregate level and for each asset j traded in the market, the following relationship holds*

$$\mathbb{E}[R_j] - R^F = (\rho + \kappa(\gamma))COV(R_j, R^M) \tag{8}$$

where ρ and γ are aggregate measures of the agent's absolute risk aversion and degree of optimism respectively, and where R^M is the gross return on the market portfolio.

The first result in Eq. (7) is the optimal individual demand schedule, expressed in the usual hyperbolic form introduced in Grossman (1976), and which can be found in many other studies,[9] generalized for the case in which N risky assets are traded in the market, and adjusted for the behavioural bias implicit in the order of preferences used. As for the second result, Eq. (8) again represents the usual expression that ties the risky security excess returns to risk attitudes, adjusted through the agents' aggregate degree of optimism.

The term $\kappa(\gamma) = 1 - 2\gamma \in [-1, 1]$, contained in both equations (7) and (8), represents a quantification of the distance from rationality that characterizes typical agents who act in the economy. In particular, the term $\kappa(\gamma)$ in Eq. (7) identifies the mitigation, in the case that the agent i is an optimist, or the enhancement, in the case in which (s)he is a pessimist, on the total impact that the asset's risk has on the demand function.

Starting from the result in Eq. (8), the pricing equation can be rewritten in terms of the more commonly used beta language. Since, in fact, Eq. (8) holds for every agent i and every asset j, it also holds for the market portfolio. In particular, we have in this case that

$$\mathbb{E}[R^M] - R^F = (\rho + \kappa(\gamma))\sigma_M^2$$

and thus,

$$\rho + \kappa(\gamma) = \frac{\mathbb{E}[R^M] - R^F}{\sigma_M^2} \tag{9}$$

By plugging this last result into Eq. (8) and by defining the systematic risk component *beta* in a conventional way as $\beta_j = \frac{COV(R^j, R^M)}{\sigma_M^2}$, we end up with

$$\mathbb{E}[R_j] - R^F = (\mathbb{E}[R^M] - R^F)\beta_j, \tag{10}$$

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[9] The most recent works include Cochrane (2009), Mendel and Shleifer (2012) and Banerjee and Green (2015).

which is a pricing equation consistent with the original representation of the CAPM, with the difference that the systematic risk *beta* reflects, in this case, both the agents' risk aversion and their degree of optimism.

This last expression, which is the fruit of pure algebraic manipulation, is not so innocuous as it might first appear. By recalling that in the original derivation of the CAPM only the risk aversion ρ is taken into account in determining the market price of risk, it generates a clash between the model just derived, in which the behavioural bias is taken into account as well, and the traditional CAPM. To better understand this point, let us consider two types of asset pricing model, both with representation as in Eq. (10), which focus on two different conjectures of the market risk premium.

Assume that the market is not uniquely composed of fully rational expected utility maximizers, i.e., $\kappa(\gamma) \neq 0$.

Conjecture 1. *The market price of risk reflects not only the aggregate degree of risk aversion but also the aggregate degree of agents' optimism, i.e., Eq. (9) holds*

Conjecture 2. *The market price of risk reflects only agents' aggregate risk aversion without taking into consideration the potential presence of a behavioural bias in their decisions, resulting in the traditional version of the CAPM,*

$$\rho = \frac{\mathbb{E}[R^M] - R^F}{\sigma_M^2} \tag{11}$$

The two conjectures are clearly not compatible simultaneously in that they give rise to different expressions for the unitary market's risk premium. It is immediately clear that the only possible case in which the two expressions are equivalent is when $\kappa(\gamma) = 0$, i.e., all agents in the market are purely rational expected utility maximizers. As a result, we have that, under Conjecture 1 in which the model takes account of agents' behavioural biases in formulating asset prices, the representation in Eq. (10) holds for every agent i and for every security j traded in the market. Thus, under Conjecture 1, prices determined by the market and the model coincide. The same is evidently not true in the case of Conjecture 2, under which there will exist a misprice α between the market and the model, given by the fact that we are imposing a model which assumes rational agents (as the CAPM does) on the prices of assets which are traded by agents who are not rational. In particular, we have the following different result.

Proposition 2. *Let α_j be the misprice of asset j as a consequence of the assumption in Conjecture 2. Given the asset pricing model expressed by Eq. (8), under Conjecture 2 in which the model does not take into account agents' behavioural biases in formulating asset prices, Eq. (10) becomes*

$$\mathbb{E}[R_j] - R^F = \alpha_j + (\mathbb{E}[R^M] - R^F)\beta_j^* + \kappa(\gamma)COV(R_j, R^M) \tag{12}$$

which we will refer to from now on as an Alpha-Neutral CAPM, and where β_j^* is a measure of the systematic risk of asset j, which, as in the traditional CAPM and according to Conjecture 2, reflects only agents' risk aversion. Consistent with the name that we give to the model, we will refer to β_j^* as alpha-neutral betas in what follows.

A few comments are necessary on this last proposition. First, the expression in Eq. (12) has to be interpreted as a single-factor asset pricing model since, given our assumption-setting, we are still in an economy in which assets' prices are determined only according to the systematic risk of assets. The new element $\kappa(\gamma)COV(R_j, R^M)$ is actually a direct consequence of the fact that we are considering a mispricing of the traditional CAPM due to the non-fully rational behaviour of agents in the economy. In this sense, the factor $\kappa(\gamma)$ merely quantifies how much of the cross-sectional pricing error produced by the traditional CAPM is explained by the behavioural component $\kappa(\gamma)COV(R_j, R^M)$. This can be seen in the model as the portion of the covariance between the risky asset considered and the market left unexplained by the

traditional market β, and instead captured by the new factor. Notice that we have deliberately left the intercepts α_j in Eq. (12) in accordance with the idea of mispricing of the traditional version of the model as assumed in Conjecture 2. If the traditional CAPM completely explains the covariance between the asset considered and the market, the intercepts α_j as well as the coefficient $\kappa(\gamma)$ should not be distinguishable from zero since the latter would constitute an unnecessary explanatory variable in the regression of the excess returns against the market risk premium, since all of the co-movement between the asset and the market would be fully captured by the alpha-neutral *betas*, β^*, which would in this case be equivalent to the market β of the traditional CAPM.

Conversely, in the case in which the model's α_j estimate is significantly distinguishable from zero, and if, as we have conjectured, the pricing errors are fully generated by behavioural biases, we should expect for every α_j a model estimate $\kappa(\gamma)$ such that the net intercepts $\delta_j = \alpha_j + \kappa(\gamma)COV(R_j, R^M)$ are jointly indistinguishable from zero. In this sense, the model is an "alpha-neutral" version of the CAPM, in that the new factor, which exists because of the presence of a mispricing according to Conjecture 2, does not enter in the asset pricing equation as an explanatory variable for expected returns. Rather, it appears only as a counterbalance to the assumed misprice, which, if it works well, ends up "neutralizing" it. Moreover, if that is the case, such a result is consistent with the intuition behind the optimism-based order of preferences employed. According to Eq. (12) and the definition of the factor $\kappa(\gamma) = 1 - 2\gamma$, in fact, in the presence of a positive unexplained excess return, the CAPM holds only if $\kappa(\gamma) < 0$ in such a way that the net intercepts are nullified, and thus if agents are on average optimistic about returns on the asset under study. The contrary evidently applies in the case of negative *alphas* where we will have, on average, pessimistic traders with regard to the asset under consideration. Finally, by using the definition of net intercepts δ_j as above, the model in Eq. (12) can be rewritten as

$$\mathbb{E}[R_j] - R^F = \delta_j + (\mathbb{E}[R^M] - R^F)\beta_j^* \tag{13}$$

Eq. (13) tells the same story but from a different perspective. The main difference with respect to the previous representation in Eq. (12) is in that the absorption of the intercepts by the behavioural component $\kappa(\gamma)COV(R_j, R^M)$ is made explicit here, so that the expression recalls the traditional CAPM representation under conditions of non-full rationality and explicitly in a market where agents suffer from optimism/pessimism biases. In this sense, Eq. (13) defines a unique equilibrium characterized by an augmented security market line (SML*), which will, in general, be steeper with respect to the traditional SML defined by the traditional CAPM in Eq. (10). In fact, this change in the measurement of the intercept inevitably generates a change in the measurement of the systematic risk *beta*, which will result in a "purified", behaviourally driven part of the movement in the market which at the same time impacts positively on the slope of the SML. In a comparison between the traditional CAPM in Eq. (10) and the Alpha-Neutral CAPM in Eqs. (12) and (13) we will refer to β_j and β_j^* as, respectively, traditional *betas* and alpha-neutral *betas*.

In order to outline the intuition behind the model, let us consider a simplified version of our economy in which only three assets named A, B and C are traded. Let us suppose that the cross-sectional errors from the traditional CAPM are $\alpha_A^0 > 0, \alpha_B^0 > 0$ and $\alpha_C^0 < 0$ respectively for the three assets. Let us then imagine running the regression in Eq. (12) and finding the result that, consistent with the results previously obtained and with our Alpha-Neutral CAPM, the regressions on the assets A and C generate pricing errors $\alpha_A > 0$ and $\alpha_C < 0$ respectively and, consistent with these, the behavioural adjustments $\kappa_A < 0$ and $\kappa_C > 0$. Conversely, let us suppose that the asset B lies perfectly on the regression plane with $\alpha_B = 0$ and $\kappa_B = 0$.
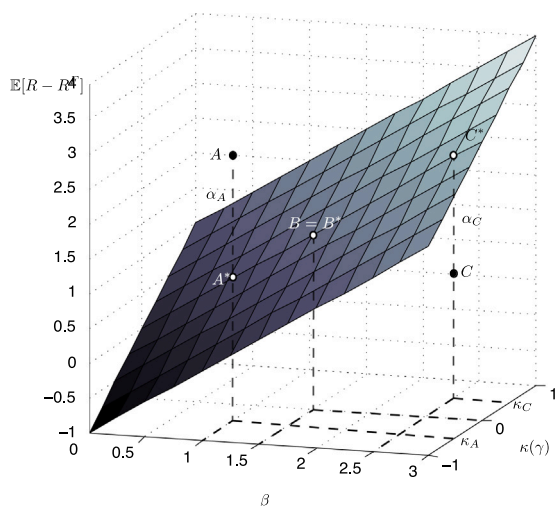
Fig. 2. The figure represents a hypothetical regression plane of the average excess returns on the three assets with respect to the *betas* of the latter and their associated behavioural bias factor $\kappa(\gamma)$.

Fig. 3. The figure represents, with respect to the example considered with three assets, the hypothetical security market lines respectively of the traditional CAPM (SML) and the Alpha-Neutral CAPM (SML*).

Fig. 2 represents the situation for the three assets. Consistent with the situation depicted, we have that the model describes well the excess returns of the assets A and C if respectively the segment $A - A^*$ is equal to $0 - \kappa_A$ and $C^* - C$ is equal to $\kappa_C - 0$. Regarding asset B, we have instead that the model does not help in explaining the abnormal return α_B^0 predicted by the traditional CAPM in that the asset lies, in equilibrium, on the plane with a coefficient κ_B equal to zero. As we will show in the next section, this situation is quite rare, at least in the dataset that we employ.

Assuming that the latter conditions on the behavioural factors of the three assets are satisfied, Fig. 3 shows the traditional SML in Eq. (10) and the augmented SML* in Eq. (13) for the example we consider with three assets. According to the previous results, assets A and C that were showing respectively positive and negative pricing errors under the traditional CAPM, result in equilibrium on the new SML* defined by the Alpha-Neutral model. In particular, as mentioned above, the augmented SML* will, in general, be steeper than the traditional SML and the *betas* associated with the assets' return reduced since, as argued above, the behavioural factor that we have included in the model also deadens the spurious component present in the *betas* when agents are not fully rational.

3. The playing field

3.1. Data description

Our empirical tests concern two main datasets: (a) average returns from Kenneth French's data library on 336 portfolios typically used in the literature to describe patterns in expected stock returns, and (b) the average returns on portfolios that are considered to mimic the patterns in the portfolios in (a), plus the covariances between returns to each of the assets in (a) and the proxy for the market portfolio. For both samples, the period considered is July 1963–December 2016, and the excess returns are observed at both a monthly and a daily frequency, where the former have been used in order to perform the main tests of the model, while the latter are employed only to compute the covariances that will be used as explanatory variables as in Eq. (12).

Sample (a) has been constructed by considering excess returns with respect to the one-month U.S. Treasury Bill rate on 36 one-way sorted portfolios (18 portfolios with stocks sorted on size quantiles and 18 portfolios with stocks sorted on book-to-market (BTM) ratio quantiles) and on three groups of 100 two-way sorted portfolios, which result
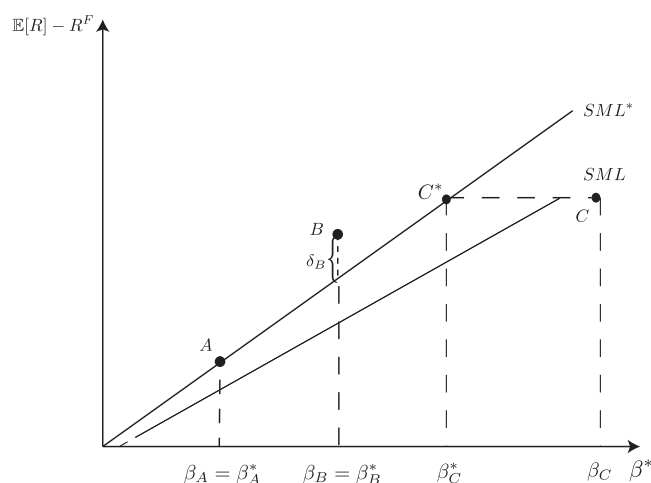
from the intersections of 10 portfolios of stocks sorted on size deciles and three groups of 10 portfolios in which the stocks have been independently sorted with respect to their BTM ratio, investment (INV) and operating profitability (OP) deciles. Consistent with Fama and French (1993, 1996a, 2015), the latter portfolios have been constructed at the end of each June using NYSE breakpoints and considering in the construction all NYSE, AMEX, and NASDAQ stocks for which returns and book values are available respectively on CRSP and COMPUSTAT.

Table 1 shows the monthly average excess returns for the portfolios considered in Sample (a). It is easy to recognize the typical patterns in the excess returns of the portfolios pointed out by Fama and French (1993, 1996a, 2015). The *size effect*, which is typically used to refer to the phenomenon characterized by a fall in the average returns from small stocks to big stocks is persistent in each panel of data analysed; exceptions are the first deciles of all three of the other firms' characteristics involved in the sorts — i.e., the BTM-Low (panel B), OP-Low (panel C) and INV-Low (panel D).

Panel B of Table 1 documents the *value effect* — i.e., the tendency of average returns to increase for higher values of the BTM ratio. This relationship shows up clearly in each row of the panel and, consistent with Fama and French (1993, 1996a, 2015), its effect is stronger for small size portfolios.

Panels C and Panel D of Table 1 instead provide evidence of the so called *profitability effect* (Fama & French, 2015; Novy-Marx, 2013) and the *investment effect* (Aharoni, Grundy, & Zeng, 2013; Fama & French, 2015) respectively. In particular, we observe that average returns typically increase for stocks of firms with higher operating profitability (Panel C) and decrease for stocks of firms that invest more (Panel D).

For Sample (b), we have considered monthly and daily excess returns with respect to the one-month Treasury bill rate on the portfolio of all sample stocks, which can be considered a proxy for the market portfolio, and the monthly returns on the portfolios typically used in order to mimic the risk factors acknowledged in the literature, represented by (i) size, (ii) value, (iii) momentum, (iv) operating profitability and (v) investment. The manner in which the latter portfolios have been constructed is described in detail in Fama and French (1993, 1996a) for portfolios (i) and (ii), Carhart (1997) for (iii), and Fama and French (2015) for (iv) and (v). In what follows, we provide a brief summary.

Portfolios (i) and (ii), named SMB (small minus big) and HML (high minus low), are constructed as the differences between, respectively, the average returns on three small-stock value-weighted portfolios and three big-stock value-weighted portfolios in the former case and

Table 1

Average monthly portfolio returns.

Average monthly percent returns for portfolios formed on size, book-to-market ratio, size and book-to-market ratio, size and operating profitability and size and investment. The returns are in excess of the one-month Treasury bill rate. Period: July 1963–December 2016, 642 months.

Panel A: 18 Size Portfolios + 18 Book-to-Market Portfolios									
Small	0.79		Size-10	0.76		BTM-Low	1.02	BTM-10	0.45
Size-2	0.73		Size-11	0.82		BTM-2	0.50	BTM-11	0.55
Size-3	0.52		Size-12	0.75		BTM-3	0.60	BTM-12	0.58
Size-4	0.77		Size-13	0.77		BTM-4	0.84	BTM-13	0.56
Size-5	0.78		Size-14	0.72		BTM-5	0.48	BTM-14	0.59
Size-6	0.74		Size-15	0.72		BTM-6	0.56	BTM-15	0.70
Size-7	0.69		Size-16	0.68		BTM-7	0.64	BTM-16	0.63
Size-8	0.50		Size-17	0.62		BTM-8	0.68	BTM-17	0.75
Size-9	0.78		Big	0.48		BTM-9	0.90	BTM-High	0.88

Panel B: 100 Size X *Book-to-Market Portfolios*									
	BTM-Low	BTM-2	BTM-3	BTM-4	BTM-5	BTM-6	BTM-7	BTM-8	BTM-9	BTM-High
Small	N/A	N/A	0.16	0.92	0.74	0.90	0.97	1.07	1.13	1.12
Size-2	0.25	0.78	0.82	0.74	0.85	0.71	1.04	0.98	1.07	1.16
Size-3	0.44	0.58	0.69	0.80	0.81	0.88	1.00	0.95	1.16	0.68
Size-4	0.28	0.53	0.57	0.81	0.55	0.78	0.77	0.96	1.10	0.64
Size-5	0.42	0.65	0.77	1.18	0.74	0.89	0.92	0.97	1.05	1.02
Size-6	0.41	0.61	0.75	0.72	0.68	0.75	0.93	0.83	1.01	1.15
Size-7	0.70	0.55	0.63	0.69	0.63	0.77	0.90	0.84	0.88	0.87
Size-8	0.60	0.55	0.65	0.55	0.88	0.65	0.82	0.82	0.82	0.80
Size-9	0.50	0.54	0.60	0.67	0.72	0.67	0.73	0.61	1.20	0.90
Big	0.49	0.55	0.54	0.51	0.48	0.61	0.32	0.93	1.03	0.86

Panel C: 100 Size X *Operating Profitability Portfolios*									
	OP-Low	OP-2	OP-3	OP-4	OP-5	OP-6	OP-7	OP-8	OP-9	OP-High
Small	0.44	0.85	1.02	0.99	1.02	0.91	1.13	0.99	1.03	0.76
Size-2	0.44	0.72	0.92	0.84	0.83	0.69	1.01	0.78	0.84	0.90
Size-3	0.57	0.88	0.93	0.86	0.75	1.05	0.78	0.77	0.93	0.99
Size-4	0.32	0.81	0.65	0.73	0.78	0.85	0.68	0.91	0.85	1.08
Size-5	0.37	0.70	0.86	0.82	0.93	0.85	0.76	0.73	1.00	0.97
Size-6	0.43	0.72	0.74	0.70	0.53	0.69	0.81	0.76	0.79	0.97
Size-7	0.41	0.47	0.88	0.82	0.65	0.71	0.62	0.69	0.82	0.83
Size-8	0.65	0.52	0.50	0.62	0.56	0.66	0.77	0.66	0.81	0.81
Size-9	1.05	0.62	0.46	0.61	0.69	0.49	0.62	0.52	0.70	0.61
Big	N/A	0.27	0.30	0.36	0.44	0.45	0.42	0.55	0.61	0.51

Panel D: 100 Size X *Investment Portfolios*									
	INV-Low	INV-2	INV-3	INV-4	INV-5	INV-6	INV-7	INV-8	INV-9	INV-High
Small	0.89	1.07	1.07	1.08	1.03	0.90	0.94	0.83	0.75	0.14
Size-2	0.84	0.98	0.90	0.78	1.11	0.91	0.93	0.83	0.63	0.23
Size-3	0.90	0.92	1.06	0.79	0.92	1.18	0.94	1.10	0.81	0.28
Size-4	0.97	0.83	0.89	0.78	0.99	0.70	0.91	0.77	0.64	0.31
Size-5	1.01	1.05	0.92	0.93	1.00	0.85	1.00	0.90	0.70	0.36
Size-6	0.75	0.84	0.84	0.94	0.74	0.76	0.69	0.78	0.76	0.31
Size-7	0.45	0.87	0.67	0.81	0.78	0.77	0.91	0.73	0.66	0.56
Size-8	1.02	0.75	0.66	0.78	0.67	0.76	0.68	0.72	0.76	0.26
Size-9	1.05	0.69	0.60	0.64	0.71	0.78	0.61	0.50	0.46	0.42
Big	1.02	0.69	0.56	0.50	0.49	0.43	0.55	0.42	0.54	0.35

the average returns on two high BTM stock value-weighted portfolios and two low BTM stock value-weighted portfolios in the latter case. Portfolio (iii), named UMD (up minus down), is computed by considering the difference between the average returns on two high prior (winner) stock value-weighted portfolios and two low prior (losers) stock value-weighted portfolios.

Finally, portfolios (iv) and (v), named RMW (robust minus weak) and CMA (conservative minus aggressive), are determined as the differences, respectively, between the average returns on two robust operating profitability stock value-weighted portfolios and the average returns on two weak operating profitability stock value-weighted portfolios; and between the average returns on two conservative investment stock value-weighted portfolios and two aggressive investment stock value-weighted portfolios.

Regarding portfolios (i), (ii), (iv), and (v), Fama and French (2015) consider different methods of construction that differ from the 2 × 3 sorts used in Fama and French (1993, 1996a). Although they find interesting insights from the different ways of constructing the risk factors, in this paper we focus our attention just on the standard construction since they show in their paper that different procedures

employed at this point do not affect the final result that is the principal objective of this paper.

Table 2 reports summary statistics for the monthly average returns on the portfolios that proxy for the risk factors. The extra three years of data with respect to the sample used in Fama and French (2015) do not significantly change the picture regarding the descriptive statistics of the risk factors. The only relevant change can be found with respect to SMB, which results in an average value of six basis points less and just 1.86 standard errors from zero. With respect to the remainder, we can still find a negative correlation between the value, profitability, and investment factors, and the market and size factors. An extremely high correlation between CMA and HML is still present, as well as evidence of non-correlation between RMW and HML.

To complete Sample (b), we have determined the covariances between the daily excess returns of all the portfolios in (a) and the daily returns on the market portfolio for each month. Formally, for each month of m days and by indicating with R_l^j and R_l^M the return on portfolio j and on the market portfolio for the lth day and with $\bar{R}^j$ and $\bar{R}^M$ the respective monthly averages, we have that

$$\sigma_{j,M} = \frac{1}{m} \sum_{l=1}^{m} (R_l^j - \bar{R}^j)(R_l^M - \bar{R}^M) \qquad (14)$$

Table 2

Descriptive statistics for the risk factors.

Summary statistics for the portfolios proxy for the risk factors: R^M-R^F are the monthly returns on the market portfolio proxy (portfolio of all the sample stocks) minus the one-month Treasury bill. SMB, HML, UMD, RMW and CMA are the value weighted monthly returns on the portfolios proxy for, respectively, size, value, momentum, operating profitability and investment. Panel A of the table shows the average returns and the standard deviations for each of the considered portfolios and the t-ratios. Panel B reports the correlations between each factor and the other five. Period: July 1963–December 2016, 642 months.

Panel A: Averages, standard deviations and t-statistics for risk factors						
	R^M-R^F	SMB	HML	UMD	RMW	CMA
Mean	0.510	0.227	0.373	0.664	0.242	0.310
Standard Deviation	4.424	3.087	2.819	4.228	2.234	2.007
t-statistic	2.922	1.863	3.350	3.980	2.742	3.910

Panel B: Correlations between risk factors						
	R^M-R^F	SMB	HML	UMD	RMW	CMA
R^M-R^F	1.000	0.295	−0.258	−0.132	−0.233	−0.384
SMB	0.295	1.000	−0.204	0.002	−0.404	−0.169
HML	−0.258	−0.204	1.000	−0.187	0.074	0.691
UMD	−0.132	0.002	−0.187	1.000	0.109	−0.013
RMW	−0.233	−0.404	0.074	0.109	1.000	−0.037
CMA	−0.384	−0.169	0.691	−0.013	−0.037	1.000

3.2. Estimation method

In order to test the performance of the Alpha-Neutral CAPM in Eqs. (12) and (13), we have made use of a two-step procedure that extends the usual time series testing approach for the purpose of making the latter suitable to test our model. The employment of this kind of testing approach is unusual in this context in that the behavioural component in Eq. (15) is not a traded asset and therefore, in general, a cross-sectional approach is usually favourable. Notwithstanding this, the particular kind of setting in which the Alpha-Neutral CAPM is conceived allows us the use of the GRS test provided by Gibbons et al. (1989) as in a normal setting with traded assets, without any consequences for the test's power or interpretation. In fact, we have the following result which we demonstrate in the paper's appendix.

Proposition 3. *Given the Alpha-Neutral CAPM, the average net cross-sectional pricing errors $\mathbb{E}[\delta_j]$ coincide with the average time-series net intercepts $\mathbb{E}[d_j]$. The same does not apply to the standard cross sectional pricing errors α_j, which will in general be different on average from the time series intercepts a_j given that the behavioural component is not a traded security.*

More specifically, the test will be structured in the following way: At the first step, for each portfolio j in Sample (a), we run the following 5-year rolling window time series regression of the type in Eq. (12), with the purpose of estimating the alpha-neutral *betas* and the behavioural factor $\kappa(\gamma)$, which represents the key element of our extension

$$R_t^j - R_t^F = a_j + b_j^*(R_t^M - R_t^F) + k_j\sigma(R_t^j, R_t^M) + e_{j,t} \quad (15)$$

where a_j, b_j^*, k_j and $e_{j,t}$ are respectively the intercepts, the slopes for the market risk factor, the behavioural factors which quantify the portions of the covariances left unexplained by the market and explained by agents' non-rational behaviour, and the regressions' residuals.

Then in the second step, we consider the restriction characterized by the definition of the model's net intercepts $d_j = a_j + k_j\sigma(R_t^j, R_t^M)$ as in the specification of the model given in Eq. (13). The restricted model will be

$$R_t^j - R_t^F = d_j + b_j^*(R_t^M - R_t^F) + e_{j,t} \quad (16)$$

which is not different from a CAPM adjusted for the hypothesis of agents' limited rationality.

If the traditional CAPM still works after adjusting for the limited rationality of agents in the market, we should find that all net intercepts are jointly indistinguishable from zero. In order to test this hypothesis, we have made use of the GRS statistic which, when applied to the

model expressed as in Eq. (13) can be used to perform a test of the null hypothesis $H_0 : d_j = 0, \quad \forall j \in [1, N]$ against the alternative hypothesis $H_1 : \exists d_i \neq 0, i \in [1, N]$ where N is the number of portfolios considered. The test statistic is given by

$$\text{GRS} = \left(\frac{T}{N}\right)\left(\frac{T - N - L}{T - L - 1}\right)\left(\frac{\mathbf{d}'\Sigma_\mathbf{e}^{-1}\mathbf{d}}{1 + \mathbb{E}[\mathbf{f}]'\Omega^{-1}\mathbb{E}[\mathbf{f}]}\right) \sim F_{N, T-N-L} \quad (17)$$

where T is the number of observations, L is the number of factors included in the regressions, $\mathbf{d}$ is the $N \times 1$ vector of estimated net intercepts from the time series regressions, $\mathbb{E}[\mathbf{f}]$ is the $L \times 1$ vector of factor averages, and Σ_e and Ω are respectively the unbiased $N \times N$ covariance matrix of time series regression residuals and the $L \times L$ matrix of covariances between the factors $\mathbf{f}$ employed.

In both steps, we analyse the performance of the model in describing the excess returns of the portfolios considered against the performance of the other most accredited asset pricing models. In particular, we consider the following alternatives to our model:

The traditional CAPM (Lintner, 1965; Mossin, 1966; Sharpe, 1964)

$$R_t^j - R_t^F = a_j + b_j(R_t^M - R_t^F) + e_{j,t} \quad (18)$$

The Fama–French three-factor model (Fama & French, 1993)

$$R_t^j - R_t^F = a_j + b_j(R_t^M - R_t^F) + s_j\text{SMB}_t + h_j\text{HML}_t + e_{j,t} \quad (19)$$

The Carhart four-factor model (Carhart, 1997)

$$R_t^j - R_t^F = a_j + b_j(R_t^M - R_t^F) + s_j\text{SMB}_t + h_j\text{HML}_t + u_j\text{UMD}_t + e_{j,t} \quad (20)$$

And the Fama–French five-factor model (Fama & French, 2015)

$$R_t^j - R_t^F = a_j + b_j(R_t^M - R_t^F) + s_j\text{SMB}_t + h_j\text{HML}_t + r_j\text{RMW}_t + c_j\text{CMA}_t + e_{j,t} \quad (21)$$

As a robustness check, we also run a performance test by considering the latter three models augmented for the behavioural bias measured by κ.

4. Results

4.1. Model performance summary

We now turn to the main empirical results. As widely discussed in the paper, our main target is to test the extent to which the Alpha-Neutral CAPM is able to explain the excess returns of portfolios of stocks, and to examine a comparison of the performance of our model against those of the Fama,-French and Carhart multifactor models. We

test the performance of the models by looking both at the time series regression-generated intercepts and at different measures of the overall explanatory power of the models involving the cross-sectional pricing errors.

Table 3 reports the GRS statistics of Gibbons et al. (1989) and the relative *p*-values, which test whether the models' net intercepts with respect to the behavioural bias, $A|d_j|$, are jointly statistically equal to zero – obviously, for models which do not consider the behavioural adjustment in the pricing equation, the net intercepts will coincide with the *alphas* – for the Alpha-Neutral CAPM and the seven alternative models considered. For every set of portfolios examined, the GRS test easily rejects the traditional CAPM along with all of the multifactor models that are not adjusted for the behavioural bias.

Conversely, the test never rejects the Alpha-Neutral CAPM, a result that is strongly robust across all samples as documented by the high level of the p-values (from 0.197 for the BTM portfolios to 0.995 for the Size X BTM portfolios). The conclusions from the Alpha-Neutral variations of the multifactor models are less obvious: the results from the augmented FF three-factor model are not robust for the size (Panel A) and BTM portfolios (Panel B) with GRS statistic p-values respectively equal to 0.115 and 0.125, while the augmented FF five-factor model is clearly rejected for the same portfolios with p-values equal to 0.045 and 0.005 respectively. On the contrary, with regard to the two-way sorted portfolios in panels C, D, and E, the two models cannot be rejected. In any case, it is interesting to observe that all the augmented multifactor models are, in terms of their GRS statistic, systematically outperformed by the Alpha-Neutral CAPM, except for the 100 Size X BTM portfolios (Panel C), where the best performance is achieved by the behaviourally augmented Carhart four-factor model.

Table 3 reports for each model and panel of data, along with the GRS test, the estimated average absolute intercepts $A|a_j|$, the estimated average absolute slope for the behavioural factor $A|\kappa_j|$, the percentage of sign reversals between the latter two, and the average absolute net intercepts $A|d_j|$, along with some descriptive statistics which characterize the empirical distribution of the latter: the maximum and minimum values for the net estimated intercepts, their standard deviation $\sigma(d_j)$, the skewness $Sk(d_j)$ and the kurtosis $Ku(d_j)$.

The intercepts a_j generated by the Alpha-Neutral variation of the models are always greater than those generated by the traditional models. However, by representing the pricing errors of the regressions' hyperplanes that consider the behavioural biases as independent variables, their magnitude is not relevant when testing model performance in that, as discussed in the previous sections, the tests are conducted in terms of the restriction applied to the models that the net intercepts equal zero. Moreover, the finding of higher standard intercepts in this setting is not necessarily bad news in that it is simply a consequence of estimating an asset pricing model that makes use of a non-traded asset, as explained in Proposition 3.

The behavioural coefficients k_j show up as always statistically significant in each sample. Consistent with the intuition of the model introduced, the behavioural adjustments generated by the products of the latter with the associated covariances display signs that are inverted with respect to the intercepts a_j in almost every portfolio analysed (the figure runs from 83% for the 18 BTM portfolios in Panel B to 100% for the 18 size portfolios in Panel A).

With respect to the Alpha-Neutral CAPM, for every sample considered, the average net intercept $A|d_j|$ is always significantly reduced in magnitude by the presence of the behavioural component with respect to the traditional CAPM in which the latter is not considered. This reduction is, however, never sufficient to generate net intercepts which are on average lower than the traditional multifactor model *alphas*. Nevertheless, although highly emphasized in the literature, the magnitude of the absolute average intercept is definitely not, on its own, an unquestionable measure of the performance of an asset pricing model, as highlighted by, among others, Barillas and Shanken (2016). In fact, it is highly informative to also look at the higher moments of

the net intercepts' distribution. Specifically, a skewness close to zero and a low kurtosis are good news since that would imply that the pricing errors will be distributed homogeneously on the equilibrium hyperplane and with a low frequency of values far from zero.

If, in general, the information contained in the descriptive statistics of the distribution of net intercepts is helpful in the interpretation of the GRS test, it is also true that it is not sufficient to fully describe the results. The Alpha-Neutral CAPM has a distribution of pricing errors clearly improved with respect to the traditional models for the 18 Size portfolios (Panel A), the 18 BTM portfolios (Panel B) and 100 Size X BTM portfolios (Panel C), with a skewness index that goes from −0.7 to 0.4 and kurtosis from 1.7 to 3.6. The same is not true for the 100 Size X OP portfolios (Panel C) and 100 Size X INV portfolios (Panel D) in which the statistics seem to contradict the GRS test result, showing a pricing error distribution for the Alpha-Neutral CAPM which is clearly outperformed by the traditional model and, in particular, by the FF five-factor model, which is instead rejected by the formal test.

4.1.1. Size portfolios

The CAPM, along with the FF three-factor and the Carhart four-factor models, are all easily rejected by the GRS test with *p*-values close to zero. The Alpha-Neutral version of the latter instead easily passes the test with *p*-values from 0.11 for the augmented three-factor model to 0.4 for the Alpha-Neutral CAPM. The traditional and the augmented five-factor model share *p*-values around the threshold values and thus the asset pricing test is inconclusive in these cases.

The average net intercept $A|d_j|$ produced by the Alpha-Neutral CAPM and the behaviourally augmented models are close in magnitude to the traditional multifactor models, which also share similar values for the descriptive statistics of the intercepts. Specifically, almost all models share a slightly skewed and platykurtic distribution of pricing errors. An interesting exception is represented by the high kurtosis displayed by the five-factor model (4.196), which identifies a higher frequency of values far from zero that is coherent with a rejection of the GRS test. The best possible distribution is achieved for this sample by the Alpha-Neutral variation of the FF three-factor model with a skewness index equal to −0.265 and a kurtosis of just 1.952, a result that is in contradiction with the rejection of the GRS test.

4.1.2. BTM portfolios

For the 18 BTM portfolios, the test easily rejects the FF three-factor model, the Carhart four factor model and the augmented five-factor model. With *p*-values from 0.1 and 0.2, the test is not able to reject the Alpha-Neutral variation of the CAPM, the three-factor model and the four-factor model. Again, the test is inconclusive for the five-factor asset pricing model. The average net intercept $A|d_j|$ values for the Alpha-Neutral CAPM are considerably higher than those of the traditional multifactor models and in particular show a magnitude similar to those of the traditional CAPM. The maximum value assumed by the net intercepts is equal to 0.32, which is again close to the 0.36 of the traditional CAPM and considerably greater than the maximum net intercept generated by the traditional multifactor models. However, the lower skewness (−0.2) and kurtosis (1.7) with respect to the other competing traditional models might justify the non-rejection of the GRS test. The best possible distribution is achieved this time by the traditional FF three-factor model with a skewness value tending towards a normal (0.06) and a kurtosis of just 1.861.

4.1.3. Size-BTM portfolios

The GRS test does not reject the null hypothesis that the net intercepts are jointly equal to zero for all of the Alpha-Neutral variations of the traditional models considered and conversely, it easily rejects the latter with *p*-values tending to zero. Again, the average absolute net intercepts for the Alpha-Neutral CAPM are lower than those of the traditional CAPM and higher than those of the traditional multifactor models. The descriptive statistics give strength to the non-rejection of

Table 3

Model performance summary.

Summary statistics for performance tests of the Alpha-Neutral CAPM (α-CAPM) against the performance of the traditional CAPM (equation 18), the Fama and French 3 Factor Model (equation 19), the Carhart 4 Factor Model (equation 20), the Fama and French 5 Factor Model (equation 21) and all the behavioural augmented versions of the latter three models. Sample: monthly excess returns on 18 Size portfolios (Panel A), 18 Book-to-Market portfolios (panel B), 100 Size and Book-to-Market portfolios (Panel C), 100 Size and Operating Profitability Portfolios (Panel D) and 100 Size and Investment Portfolios. The table shows, for each model and panel of data, the estimated average absolute intercepts $A|a_j|$, the estimated average absolute slope for the behavioural factor $A|k_j|$, the percentage of sign reversals between the latter two, the average absolute net intercepts $A|d_j|$, the standard deviation, the (absolute) minimum and the maximum estimated values, the skewness and the kurtosis of the distribution of the latter, the GRS statistic which test if the net intercepts are jointly equal to zero and the relative p-values. Period: July 1963–December 2016, 642 months.

	$A|a_j|$	$A|k_j|$	s.r.	$A|d_j|$	$\sigma(d_j)$	$\min(d_j)$	$\max(d_j)$	$Sk(d_j)$	$Ku(d_j)$	GRS	p(GRS)
Panel A: 18 Size Portfolios											
CAPM	0.127				0.072	0.030	0.234	−0.939	3.079	1.968	0.010
Fama French 3 Factor	0.033				0.044	0.103	0.042	−0.743	2.376	2.333	0.001
Carhart 4 Factor	0.038				0.047	0.097	0.042	−0.891	2.267	2.571	0.000
Fama French 5 Factor	0.023				0.029	0.053	0.076	−0.408	4.196	1.857	0.017
α-CAPM	0.477	0.019	1.0	0.034	0.046	0.064	0.102	0.400	2.460	1.047	0.404
α-Fama French 3 Factor	0.061	0.002	0.8	0.037	0.043	0.097	0.044	−0.265	1.952	1.421	0.115
α-Carhart 4 Factor	0.051	0.002	0.8	0.034	0.042	0.090	0.046	−0.488	2.005	1.354	0.148
α-Fama French 5 Factor	0.054	0.002	0.7	0.039	0.039	0.080	0.062	0.246	2.345	1.644	0.045
Panel B: 18 Book-to-Market Portfolios											
CAPM	0.160				0.151	0.137	0.358	0.019	2.018	1.770	0.025
Fama French 3 Factor	0.069				0.083	0.140	0.133	0.058	1.861	2.263	0.002
Carhart 4 Factor	0.064				0.079	0.103	0.137	0.310	1.768	2.116	0.005
Fama French 5 Factor	0.050				0.064	0.083	0.159	0.523	2.612	1.680	0.038
α-CAPM	0.268	0.005	0.8	0.153	0.138	0.106	0.320	−0.198	1.710	1.275	0.197
α-Fama French 3 Factor	0.103	0.004	0.7	0.074	0.094	0.145	0.193	0.454	2.361	1.398	0.125
α-Carhart 4 Factor	0.095	0.004	0.8	0.065	0.082	0.135	0.186	0.271	2.516	1.286	0.190
α-Fama French 5 Factor	0.089	0.005	0.8	0.095	0.146	0.133	0.522	2.143	8.279	2.115	0.005
Panel C: 100 Size X *Book-to-Market Portfolios*											
CAPM	0.263				0.259	0.601	0.721	−0.515	3.445	2.248	0.000
Fama French 3 Factor	0.122				0.182	0.752	0.382	−1.468	6.934	2.176	0.000
Carhart 4 Factor	0.122				0.179	0.781	0.390	−1.676	7.570	2.057	0.000
Fama French 5 Factor	0.116				0.156	0.644	0.384	−0.763	5.278	2.418	0.000
α-CAPM	0.512	0.017	0.9	0.214	0.252	0.708	0.575	−0.700	3.638	0.656	0.995
α-Fama French 3 Factor	0.177	0.006	0.8	0.128	0.173	0.660	0.364	−0.802	4.653	0.624	0.998
α-Carhart 4 Factor	0.164	0.005	0.8	0.125	0.164	0.659	0.365	−0.976	5.443	0.568	1.000
α-Fama French 5 Factor	0.169	0.007	0.7	0.161	0.194	0.485	0.413	−0.030	2.394	0.988	0.517
Panel D: 100 Size X *Operating Profitability Portfolios*											
CAPM	0.232				0.207	0.410	0.600	−0.597	3.274	1.894	0.000
Fama French 3 Factor	0.142				0.183	0.515	0.349	−0.890	3.772	1.807	0.000
Carhart 4 Factor	0.150				0.189	0.535	0.371	−0.965	4.191	1.881	0.000
Fama French 5 Factor	0.110				0.138	0.258	0.455	0.511	3.282	1.596	0.001
α-CAPM	0.520	0.017	0.9	0.168	0.200	0.662	0.407	−1.318	5.246	0.683	0.989
α-Fama French 3 Factor	0.198	0.006	0.7	0.160	0.207	0.628	0.476	−0.737	3.835	0.671	0.992
α-Carhart 4 Factor	0.203	0.005	0.7	0.166	0.210	0.627	0.523	−0.784	4.045	0.681	0.990
α-Fama French 5 Factor	0.155	0.005	0.7	0.129	0.165	0.355	0.649	0.704	3.951	0.638	0.996
Panel E: 100 Size X *Investment Portfolios*											
CAPM	0.295				0.256	0.506	0.640	−0.991	3.824	2.620	0.000
Fama French 3 Factor	0.142				0.189	0.706	0.473	−1.310	6.201	2.454	0.000
Carhart 4 Factor	0.144				0.185	0.684	0.519	−1.209	6.144	2.319	0.000
Fama French 5 Factor	0.122				0.152	0.438	0.511	−0.101	4.280	2.083	0.000
α-CAPM	0.547	0.017	0.9	0.232	0.264	0.702	0.507	−1.287	4.764	0.815	0.893
α-Fama French 3 Factor	0.167	0.005	0.8	0.140	0.176	0.596	0.417	−0.925	4.773	0.707	0.982
α-Carhart 4 Factor	0.166	0.005	0.8	0.141	0.173	0.555	0.457	−0.775	4.470	0.658	0.994
α-Fama French 5 Factor	0.158	0.005	0.8	0.131	0.158	0.331	0.454	0.212	2.929	0.639	0.996

the Alpha-Neutral CAPM despite the higher magnitudes of the intercepts. The maximum is of a lower magnitude than for the traditional CAPM, while the minimum is lower in magnitude with respect to the FF three-factor and the Carhart four-factor models. Skewness and kurtosis are the lowest among the competing traditional multifactor models, although, surprisingly, the values are higher than for the traditional CAPM. The best performance in terms of the distribution of intercepts is this time achieved by the augmented Fama and French five-factor model with a skewness that tends towards the normal (−0.03) and showing the only case of a platykurtic distribution amongst all the competing models ($Ku(d_j)$ = 2.115).

4.1.4. Size-OP portfolios and size-INV portfolios

As for the Size X BTM portfolios, the GRS test does not reject the null hypothesis that the net intercepts are jointly equal to zero

for all of the Alpha-Neutral variations of the traditional models and conversely, it easily rejects the latter with p-values tending to zero for both samples. The results for the portfolios formed from stocks sorted on size and operating profitability and on size and investment are, however, the most controversial for the Alpha-Neutral CAPM. Despite the clear non-rejection of the GRS test, the magnitude of the net average absolute intercepts, although inferior with respect to the traditional CAPM results, are again larger with respect to the traditional multifactor models. Moreover, contradicting the results with respect to the previous samples, the distribution of net intercepts for the Alpha-Neutral CAPM is in this case highly leptokurtic ($Ku(d_j)$ = 5.24 for Panel D and $Ku(d_j)$ = 4.76 for Panel E) and skewed ($Sk(d_j)$ = −1.32 for Panel D and $Ku(d_j)$= −1.29 for Panel E), identifying a high frequency of extreme values with respect to the competing models. Thus, the descriptive statistics regarding the distribution of net intercepts are in

contradiction to the result of the formal test, showing the necessity to delve deeper in analysing the results obtained.

4.2. Diagnostics

The apparent clash between the information obtained from the GRS test and the net average absolute value of the estimated intercepts highlights an important question. A possible controversy that may arise by observing the latter results concerns the extent to which the non-rejection of the GRS tests is due to chance rather than to an actual contraction of the (true) magnitude of the net intercepts.

This issue can be addressed through a dissection of the GRS statistic into the unexplained ex-post squared Sharpe ratio $\theta_u^2 = \mathbf{d}'\Sigma_\mathbf{e}^{-1}\mathbf{d}$ and the factors' Sharpe ratio $\theta_f^2 = \mathbb{E}[\mathbf{f}]'\Omega^{-1}\mathbb{E}[\mathbf{f}]$ according to the economic interpretation of the GRS statistic given in Gibbons et al. (1989), in which they show the possibility of rewriting the latter as

$$\text{GRS} = \left(\frac{T}{N}\right)\left(\frac{T-N-L}{T-L-1}\right)\left(\frac{\theta^2-\theta_f^2}{1+\theta_f^2}\right) \qquad (22)$$

where θ is the Sharpe ratio of the ex post tangency portfolio spanned by the N assets and the L factors. According to this interpretation, the less is the relative distance between the ex post tangency portfolio Sharpe ratio and the factor Sharpe ratio, the higher will be the unexplained Sharpe ratio θ_u, and thus the distance from

the intercepts to zero. In a recent study, Barillas and Shanken (2016) discuss this decomposition, showing that a comparison between competing models essentially relies on the magnitude of the factor Sharpe ratio while the test assets are shown as irrelevant unless one or more factors employed in the asset pricing model are not returns.

Table 4 reports the decomposition of the GRS statistics into unexplained Sharpe ratios θ_u^2 and Sharpe ratios of the factors θ_f^2 along with Shanken's 1987 efficiency ratio, $\rho = \theta_f/\theta$.

Ideally, if the portfolio given by the combination of factors is efficient, $\rho = 1$. Consistent with the result in Proposition 3 and with the findings of Barillas and Shanken (2016), the unexplained Sharpe ratio is approximately the same for every model in each of the samples considered. Thus, for each of the asset pricing models that have been considered, the actual explanatory power of the latter is wholly represented by the factor Sharpe ratio θ_f^2. The Alpha-Neutral models always display values considerably higher than those of the traditional asset pricing models (at least eight times higher than the FF five-factor model, which represents the best alternative amongst the traditional models). Notice also that the unexplained Sharpe ratios of the Alpha-Neutral models, although remaining very close to those obtained from the traditional models, benefit from a consistent reduction in three out of five of the samples.

Consistent with the findings of Fama and French (2015), the five-factor model always outperforms the three-factor model, but in terms of absolute efficiency, the combination of factors: MKT, SMB, HML, CMA and RMW, slightly exceeds 50% for the one-way sorted portfolios (panel A and B) and 30% for the two-way sorted portfolios (panel C, D and E). Conversely, the Alpha-Neutral CAPM, along with all the adjusted multifactor models, display an efficiency coefficient of around 70% for every sample, which is surprisingly robust across the samples.

Another important point that is not always well addressed in the empirical literature is represented by the fact that, more important than the magnitude of the estimated intercepts themselves, is the proportion in the estimation represented by the real unknown pricing errors and the estimation errors which naturally arise from the application of the econometric technique employed. The estimated intercepts d_j are in fact given by the true intercepts δ_j plus the sum of the estimation errors of the *alphas*, $\varepsilon_{j,\alpha}$, and of the behavioural bias $\varepsilon_{j,\kappa}$.

$$d_j = \delta_j + \varepsilon_{j,\alpha} + \varepsilon_{j,\kappa} \qquad (23)$$

Following Fama and French (2016), since δ_j is constant, the cross-sectional average over the expected value of d_j^2 is

$$\mathbb{E}[d_j^2] = \mathbb{E}[\delta_j^2] + \mathbb{E}[\varepsilon_{j,\alpha}^2] + \mathbb{E}[\varepsilon_{j,\kappa}^2] + 2\mathbb{E}[\varepsilon_{j,\alpha}\varepsilon_{j,\kappa}] = \mathbb{E}[\delta_j^2] + \text{VAR}[\varepsilon_j^2] \qquad (24)$$

where $\text{VAR}[\varepsilon_j^2]$ is the variance of the net intercepts δ_j due to estimation error which we estimate using the cross-sectional average standard error of d_j, $As^2(d_j)$. The ratio $As^2(d_j)/Ad_j^2$ thus measures the dispersion of net intercept estimates due to estimation error. Along with the latter ratio, Table 4 also reports some metrics, introduced in Fama and French (2015, 2016), which estimate the proportion of the cross-section of expected returns left unexplained by the models. Let $r_j = R_j - \bar{R}$, R_j be the time series average excess return on the portfolio j and $\bar{R}$ is the cross-section average of R_j, $A|d_j|/A|r_j|$ measures the dispersion of average expected returns left unexplained by the models while $A(d_j^2)/A(r_j^2)$ is the variance of the cross-sectional expected returns of the portfolios left unexplained by the models. As pointed out by Fama and French (2016), high values of the latter two ratios are bad in that this would suggest that the dispersion of intercepts is high relative to the dispersion of test assets. Conversely, high values of the ratio $As^2(d_j)/Ad_j^2$ are good in that it would tell us that a higher proportion of the dispersion is due to sampling error rather than to the dispersion of the true intercepts.

Except for the 18 Size portfolios (Panel A), the Alpha-Neutral CAPM displays dispersion coefficients that are always greater than one with intercepts that are thus more dispersed than the average returns. In terms of dispersion in particular, in this case, the best results are achieved by the five-factor model, although some of the Alpha-Neutral variations of the multifactor models produce results that are at least close to the three-factor model or to the five-factor model depending on the sample. Nevertheless, the ratio $As^2(d_j)/Ad_j^2$ for the Alpha-Neutral CAPM has a minimum equal to 58% in Panel (B) and around 1 for the other four samples. Thus, there is strong evidence that the larger dispersion is due predominantly to estimation error in the net intercepts, which in particular is higher than for the traditional model, according to Eqs. (23) and (24).

We conclude this section by reporting confidence intervals for the true unexplained Sharpe ratio θ_u as suggested in Lewellen, Nagel, and Shanken (2010). They show, in particular, that it is possible to find an exact confidence interval by representing the relative percentiles of the GRS statistic given by a non-centred Fisher F-distribution with non-centrality parameter $c = \theta_u^2/N$, as a function of the unexplained Sharpe ratio, and by studying the intersection with the observed value of the GRS statistic. In this last test, we focus only on a comparison between the traditional model with the best performance, i.e. the Fama–French five-factor model, and the Alpha-Neutral CAPM which is the primary interest of this paper.

Fig. 4 represents the 5th, 50th, and 95th percentiles of random extractions from a non-centred F-distribution with $N = 18$ (Panel A) and $N = 100$ (Panel B), which constitute the theoretical distributions that we need to compare with the observed GRS statistics for the 18 size and BTM portfolios (straight lines in Panel A) and the 100 size X BTM, size X OP and size X INV portfolios (straight lines in Panel B) respectively. In both panels, the blue lines are the GRS statistics produced by the Fama–French five-factor model while the red lines are produced by the Alpha-Neutral CAPM. Confidence intervals are formed taking the interceptions between the observed GRS statistics and the 95th percentiles for the left-hand side extreme and the 5th percentiles for the right-hand side of the confidence intervals.

From Fig. 4 it can be immediately noticed that for each sample, the confidence intervals for the FF five-factor model are always wider than that for the Alpha Neutral model. More specifically in Panel A, the five-factor model shows intervals that are approximately [0,0.5] for both size and BTM portfolios while for the Alpha-Neutral model they are around [0,0.05]. The evidence for the 100 two-way sorted portfolios in Panel B is even stronger. Confidence intervals for the five-factor

Table 4

Test diagnostics.

The table shows the summary statistics for the performance test's diagnostics of the Alpha-Neutral CAPM (α-CAPM) against the performance of the traditional CAPM (equation 18), the Fama and French three-factor model (equation 19), the Carhart four-factor model (equation 20), the Fama and French five-factor model (equation 21) and all the behavioural augmented versions of the latter three models. Sample: monthly excess returns on 18 Size portfolios (Panel A), 18 Book-to-Market portfolios (panel B), 100 Size and Book-to-Market portfolios (Panel C), 100 Size and Operating Profitability Portfolios (Panel D) and 100 Size and Investment Portfolios (Panel E). The table shows, for each model and panel of data, the dispersion's indexes of Fama and French (2015, 2016), the unexplained squared Sharpe ratio, the factor squared Sharpe ratio, Shanken's 1987 index of efficiency. Period: July 1963–December 2016, 642 months.

	$\frac{A|d_j|}{A|r_j|}$	$\frac{A|d_j^2|}{A|r_j^2|}$	$\frac{As^2|d_j|}{A|d_j^2|}$	θ_u^2	θ_f^2	ρ
Panel A: 18 Size Portfolios						
CAPM	1.589	1.915	0.267	0.053	0.015	0.350
Fama French 3 Factor	0.417	0.192	0.979	0.064	0.049	0.467
Carhart 4 Factor	0.482	0.204	1.051	0.075	0.105	0.542
Fama French 5 Factor	0.288	0.087	0.977	0.054	0.100	0.577
α-CAPM	0.431	0.202	1.006	0.051	0.868	0.804
α-Fama French 3 Factor	0.468	0.205	0.884	0.071	0.898	0.781
α-Carhart 4 Factor	0.427	0.176	1.003	0.072	1.026	0.790
α-Fama French 5 Factor	0.494	0.205	0.911	0.083	0.911	0.769
Panel B: 18 Book-to-Market Portfolios						
CAPM	1.274	1.618	0.594	0.047	0.015	0.362
Fama French 3 Factor	0.546	0.279	1.055	0.062	0.049	0.470
Carhart 4 Factor	0.512	0.252	1.041	0.061	0.105	0.566
Fama French 5 Factor	0.397	0.177	0.997	0.049	0.100	0.590
α-CAPM	1.215	1.403	0.576	0.060	0.788	0.784
α-Fama French 3 Factor	0.585	0.358	1.035	0.067	0.822	0.778
α-Carhart 4 Factor	0.517	0.285	0.989	0.067	0.977	0.793
α-Fama French 5 Factor	0.755	0.889	1.009	0.103	0.848	0.742
Panel C: 100 Size X Book-to-Market Portfolios						
CAPM	1.484	2.094	0.287	0.379	0.015	0.167
Fama French 3 Factor	0.689	0.662	0.466	0.379	0.049	0.265
Carhart 4 Factor	0.691	0.643	0.460	0.377	0.105	0.345
Fama French 5 Factor	0.654	0.490	1.122	0.442	0.100	0.323
α-CAPM	1.207	1.475	1.143	0.355	2.168	0.712
α-Fama French 3 Factor	0.721	0.601	1.275	0.343	2.224	0.718
α-Carhart 4 Factor	0.705	0.552	1.386	0.337	2.470	0.730
α-Fama French 5 Factor	0.907	0.757	1.086	0.547	2.245	0.669
Panel D: 100 Size X Operating Profitability Portfolios						
CAPM	1.465	1.891	0.412	0.354	0.013	0.162
Fama French 3 Factor	0.894	0.883	0.400	0.348	0.046	0.266
Carhart 4 Factor	0.945	0.962	0.337	0.379	0.093	0.331
Fama French 5 Factor	0.696	0.570	0.607	0.326	0.106	0.363
α-CAPM	1.061	1.179	1.764	0.397	2.047	0.694
α-Fama French 3 Factor	1.009	1.135	0.840	0.401	2.133	0.698
α-Carhart 4 Factor	1.049	1.203	0.756	0.425	2.271	0.698
α-Fama French 5 Factor	0.811	0.779	1.150	0.388	2.184	0.703
Panel E: 100 Size X Investment Portfolios						
CAPM	1.713	2.288	0.252	0.489	0.013	0.142
Fama French 3 Factor	0.824	0.771	0.331	0.473	0.046	0.237
Carhart 4 Factor	0.840	0.763	0.311	0.467	0.093	0.308
Fama French 5 Factor	0.709	0.529	0.485	0.425	0.106	0.333
α-CAPM	1.348	1.658	1.008	0.515	2.311	0.679
α-Fama French 3 Factor	0.811	0.690	0.906	0.455	2.371	0.695
α-Carhart 4 Factor	0.820	0.693	0.870	0.453	2.601	0.706
α-Fama French 5 Factor	0.761	0.564	1.148	0.414	2.394	0.706

model go from [0.02,0.2] for the 100 size and investment portfolios to [0.12,0.35] for the size and BTM portfolios. Conversely, the Alpha-Neutral CAPM displays an unexplained Sharpe ratio confidence interval of [0,0.02] for the size and investment portfolios and of [0,0.005] for the other two samples. Thus, this test gives further confirmation to the information contained by the ratio $As^2(d_j)/Ad_j^2$ and in particular to the intuition that much of the net intercepts generated constitutes estimation error and that it most likely, at least for the sample that we have analysed, that the GRS test does not make an error in not rejecting the null.

4.3. Factor spanning test

A common practice in the empirical asset pricing literature is to test whether a factor can be explained through a combination of the others and thus, if it is redundant as an explanator of the test assets considered. The nature of the model that we have introduced, the strength of our results and of the evidence about the contraction of the intercepts obtained in general compared with the three-factor and five-factor models (Fama & French, 1993, 1996a, 1996b, 2015, 2016) lead us to attempt to give an answer to an old controversy which usually characterizes the latter models, i.e. whether the "better" pricing attained by these model is rationally or irrationally driven (Fama & French, 1993, 2017); Titman, Wei and Xie, 2013).

The intuition behind our spanning test is the same as for the general test of the performance of competing models. A factor is not redundant in the model if and only if the other factors considered in the regression are insufficient to price the latter correctly. Formally, by considering for instance a standard CAPM and a factor f different from the market portfolio, the factor is important in order to explain average returns in
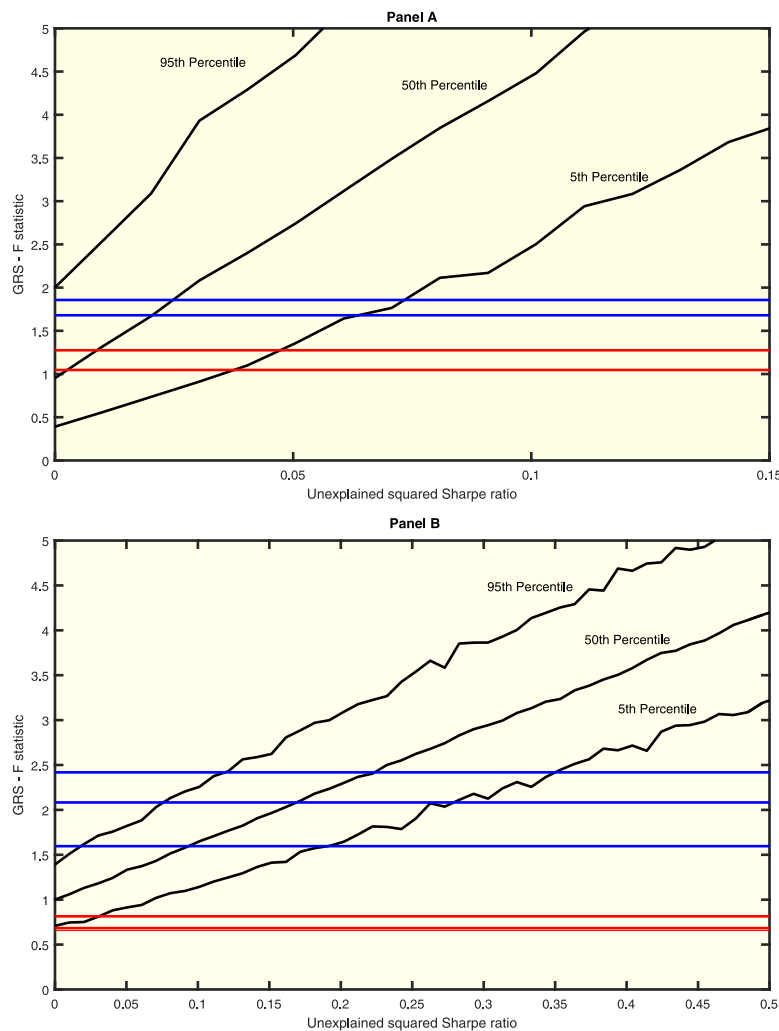
Fig. 4. The figure represents the sample distribution of the GRS statistic, with $N = 18$ (Panel A) and $N = 100$ (Panel B), and confidence intervals for the unexplained Sharpe ratios relative to the 336 test portfolios analysed. Sample period: July 1963–December 2016, 642 months. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

the test assets if, in a time-series regression of the type

$$f_t = a_f + b_f(R_t^M - R_t^F) + e_{f,t} \qquad (25)$$

the intercepts a_f are statistically different from zero.

However, if the market's mispricing of the factor is not due to the presence of a real effect that is not caught by the market but instead because of the presence of a behavioural bias, exactly as occurred for the test assets, we will have a behavioural bias coefficient statistically different from zero and a representation of the model in Eq. (25) as

$$f_t = a_f + b_f(R_t^M - R_t^F) + k_f\text{COV}(f_t, R_t^M) + e_{f,t} = d_f + b_f(R_t^M - R_t^F) + e_{f,t} \qquad (26)$$

where now the condition in order for the factor to not be discarded is d_f different from zero.

In each regression, the behavioural factor $\text{COV}(f_t, R_t^M)$ takes the form of the covariances, estimated from the daily returns, between the market and the factor that has to be explained. Panel A of Table 5 shows regressions in which six factors are used to explain the returns on the seventh. In terms of the estimated intercepts, our findings are similar to those in Fama and French (2015, 2017). Judging each of the different factors considered along with the market in terms of a_f, almost all seem to play a role in the explanation of the average returns of the test assets. In particular, the size, momentum and investment factors

show highly significant estimated intercepts with p-values less than 0.01. Again consistent with Fama and French (2015, 2017), we find that the value factor instead seems redundant with an estimated intercept of just 0.09 and a *p*-value of 0.27, while, different from them, the addition of the behavioural factor in the regression makes the profitability factor redundant as well with an intercept of 0.03 and an associated *p*-value of 0.06, although the result is less robust with respect to HML.

By instead judging the explanatory power of the factors under the logic introduced by the Alpha-Neutral framework, the situation is completely reversed. With the exception of the market factor, all of the remaining factors are replicable with combinations of the others within the Alpha-Neutral model, showing non-significant average net intercepts with p-values that run from 0.25 for SMB to 0.84 for CMA. These results are, however, not decisive, especially for the momentum factor, in that much of the non-rejection of the t-test is due to substantially increased standard errors obtained in the formation of the net intercepts.

The results shown in Panel B instead display more strength. In this case, we have used only the market and behavioural factors in order to explain the average monthly returns on each of the other factors. The results for the size factor are very interesting since, when all of the other factors are removed, we obtain estimated intercepts that are not significant with *p*-value 0.10. The behavioural bias represented by the coefficient κ_{SMB} has zero explanatory power, a result which is consistent with the logic of the Alpha-Neutral CAPM in which a

behavioural bias exists in the case of mispricing of the traditional model represented by a pricing error different from zero.

Regarding the HML factor, our test clearly rejects the null hypothesis that the value factor is redundant in an Alpha-Neutral framework which considers just optimism/pessimism as a departure from rationality. The intercepts ex ante and ex post netting are statistically different from zero, showing that the market factor alone cannot correctly price the HML factor and the latter is thus a necessary variable to include in the regressions to explain the average returns on the test asset.

More controversial is the result concerning the momentum factor. The estimated intercepts are statistically different from zero with p-values around 0.00 while, as for the previous cases, the net average intercepts from the t-test are equal to zero with a *p*-value of 0.34. Nevertheless, it is clear from the standard error that most of the non-rejection of the test is due to the magnitude of the latter, which is approximately five times the corresponding value associated with the estimated intercept. Thus we can easily see that the spanning test is not conclusive in this case.

The CMA and RMW factors show relevant results. The estimated intercepts in this case are both statistically different from zero, showing the presence of a consistent mispricing of the latter portfolios by the market. However, different from the value factor, this mispricing seems in both cases to be behaviourally driven, given that the average net intercepts are statistically insignificant with p-values of 0.15 and 0.65 respectively. An interesting insight is that the more a firm is characterized by a high level of operating profitability, the more the market is on average optimistic so that such a stock generates a misprice with a non-rational root. Conversely, and specifically for small firm stocks, the larger the level of a firm's investment, the less that firm will be seen as stable by the market which will interpret a larger variance as a bad signal, and which would be reflected in investors being pessimistic about that stock. Thus, under this logic, profitability and investment effects have non-zero impacts on the average excess returns of the test assets not because of a real effect from these two variables on the stocks' returns, but because of behavioural biases generated among investors regarding the firms' investment decisions and the characteristics of their profitability.

5. Conclusions

In this paper we have derived a capital asset pricing model in an economy in which traders, consistent with recently developed theories in the decision-making literature, do not behave rationally in the sense of von Neumann–Morgenstern expected utility theory. In particular, we have focused our attention on the inclusion of the agents' degrees of optimism in the capital asset pricing model. In our view, this represents the most compelling departure from rationality and at the same time is a crucial component in decision-making. The Alpha-Neutral CAPM that we derive provides an intuitive and analytically simple explanation of the abnormal returns left unexplained by the Sharpe (1964), Lintner (1965) and Mossin (1966) traditional CAPM and by many of the currently most accredited multifactor models by attributing the presence of these "anomalies" to the limited rationality of traders.

The results we present, both on the performances of competing models in Table 3 and on the spanning tests in Table 5, are consistent with the idea that the SMB, CMA and RMW factors are not necessary in order to explain the average returns of the test assets. Conversely, the parsimonious representation of the Alpha-Neutral CAPM, comprising just one risk factor augmented by the behavioural bias, seems sufficient to explain the variation in average returns.

Does this mean that all factors considered in the literature are just imperfect proxies for an effect that is purely behavioural and are thus not necessary? The answer to this question is not straightforward. First, from the spanning tests, not all of the factors are perfectly explicable in terms of just the behavioural bias characterized in terms of the degree of optimism that we introduce in this paper. A combination of the

market factor and the degree of optimism is in fact able to explain the cross-section of the size, profitability and investment effects but not the value or momentum effects, which seem instead to have a real impact on the average returns of the securities in the market. At the same time, it is also true that our representation of irrationality is limited in that we are considering just one, albeit somehow encompassing, departure from rationality. Moreover, the results we have obtained do not render the other factors studied in the literature outdated. We also have to deal with the problem that the behavioural factor is not a return. In this sense, the SMB, HML, UMD, RMW and CMA factors might remain essential in order to construct a traded portfolio whose returns mimic the behavioural factor.[10]

CRediT authorship contribution statement

Francesco Rocciolo: Conceptualization, Methodology, Software, Formal analysis, Writing – original draft. **Andrea Gheno:** Conceptualization, Supervision, Writing – review & editing. **Chris Brooks:** Project administration, Conceptualization, Supervision, Writing – review & editing.

Appendix. Proofs of propositions

Proof of Proposition 1

Let us start by considering the variable $W_i(t)$ in Eq. (6). If the vector of risky asset gross returns is distributed as a multivariate normal with mean $\mathbb{E}[\mathbf{R}]$ and covariance matrix $\boldsymbol{\Sigma_R}$, we will have that the final level of wealth $W_i(t)$ will be normally distributed as well with moments:

$$\mathbb{E}[W_i(t)] = \mathbf{x}_i'(\mathbb{E}[\mathbf{R}] - \mathbf{R^F}) + W_i(t-1)R^F \tag{A}$$

$$\mathrm{VAR}[W_i] = \mathbf{x}_i'\boldsymbol{\Sigma_R}\mathbf{x}_i \tag{B}$$

For a normally distributed variable $W_i(t)$ and a constant a, the following property holds

$$\mathbb{E}[\exp(aW_i(t))] = \exp(\mathbb{E}[aW_i(t)] + \frac{1}{2}\mathrm{VAR}[aW_i(t)]) \tag{C}$$

Applying this last expansion to Eq. (4) and by plugging in the results into Eqs. (A) and (B), we end up with the following objective function

$$-2[1-\phi(\gamma_i, \boldsymbol{\Sigma_R})]\left[\exp\left(-\rho_i(\mathbf{x}_i'(\mathbb{E}[\mathbf{R}]-\mathbf{R^F})+W_i(t-1)R^F)+\frac{\rho_i^2}{2}\mathbf{x}_i'\boldsymbol{\Sigma_R}\mathbf{x}_i\right)\right] \tag{D}$$

where, according to Eq. (3),

$$\phi(\gamma, \boldsymbol{\Sigma_R}) = \begin{cases} 1 - \frac{1}{2}\exp\left\{-\rho_i\left(\gamma - \frac{1}{2}\right)\mathbf{x}_i'\boldsymbol{\Sigma_R}\mathbf{x}_i\right\} & \text{if} \quad \frac{1}{2} \leq \gamma < 1 \\ 0 & \text{if} \quad 0 < \gamma < \frac{1}{2}, \boldsymbol{\Sigma_R} \geq \boldsymbol{\Sigma_*} \\ 1 - \frac{1}{2}\exp\left\{-\rho_i\left(\gamma - \frac{1}{2}\right)\mathbf{x}_i'\boldsymbol{\Sigma_R}\mathbf{x}_i\right\} & \text{if} \quad 0 < \gamma < \frac{1}{2}, \boldsymbol{\Sigma_R} < \boldsymbol{\Sigma_*} \end{cases} \tag{E}$$

As demonstrated by Rocciolo et al. (2019), the optimal demand function $\mathbf{x}_i$ is a solution to the optimization problem in Eq. (D), which will be the same for all three possible functional forms assumed by $\phi(\gamma, \boldsymbol{\Sigma_R})$ as in Eq. (E). Thus, we can solve the problem just for the case in which $\frac{1}{2} \leq \gamma < 1$. By plugging the explicit form of $\phi(\gamma, \boldsymbol{\Sigma_R})$ in Eq. (D), we can rewrite the latter as

$$-\left[\exp\left(-\rho_i(\mathbf{x}_i'(\mathbb{E}[\mathbf{R}]-\mathbf{R^F})+W_i(t-1)R^F)+\rho_i\left(\frac{\rho_i - 2\gamma_i + 1}{2}\right)\mathbf{x}_i'\boldsymbol{\Sigma_R}\mathbf{x}_i\right)\right] \tag{F}$$

This has a first order condition with respect to the demand $\mathbf{x}_i$ of:

Table 5
Factor spanning test.

Factor spanning test summary: MKT is the return on the market portfolio in excess of the one-month Treasury bill rate, SMB (small minus big) is the size factor, HML (high minus low) is the value factor, UMD (up minus down) is the momentum factor, CMA (conservative minus aggressive) is the investment factor and RMW (robust minus weak) is the profitability factor. a_f and $A(d_f)$ are respectively the time series regressions' intercepts and the Alpha-Neutral net average net intercepts. Finally, Cov(f,MKT) is the covariance between the market and the dependent variable. Panel A shows the slopes of the regressions which use six factors to explain the return of the seventh. Panel B shows the slopes of the regressions which use just the market and the behavioural factor to explain the returns on SMB, HML, UMD, CMA and RMW. Period: July 1963–December 2016, 642 months.

	a_f	$A(d_f)$	MKT	SMB	HML	UMD	CMA	RMW	Cov(f,MKT)	R^2
Panel A: Spanning test involving multiple factors										
MKT										
Coefficient	1.48	1.85		0.20	−0.09	−0.16	−0.69	−0.28	−0.62	0.30
Standard Error	0.17	0.19		0.05	0.08	0.04	0.11	0.07	0.08	
t-Statistic	(8.60)	(9.77)		(3.83)	(1.16)	(4.23)	(6.50)	(3.89)	(7.70)	
p-values	0.00	0.00		0.00	0.00	0.25	0.00	0.00	0.00	
SMB										
Coefficient	0.33	0.32	0.13		0.07	0.04	−0.13	−0.43	0.41	0.16
Standard Error	0.12	0.27	0.03		0.06	0.03	0.08	0.05	0.25	
t-Statistic	(2.79)	(1.15)	(4.43)		(1.18)	(1.28)	(1.64)	(8.35)	(1.67)	
p-values	0.01	0.25	0.00		0.24	0.20	0.10	0.00	0.10	
HML										
Coefficient	0.09	0.09	0.0	0.04		−0.12	0.98	0.16	−0.10	0.52
Standard Error	0.08	0.15	0.0	0.03		0.02	0.04	0.04	0.13	
t-Statistic	(1.09)	(0.60)	(0.18)	(1.31)		(6.70)	(23.29)	(4.24)	(0.76)	
p-values	0.27	0.55	0.86	0.19		0.00	0.00	0.00	0.45	
UMD										
Coefficient	0.73	0.78	−0.12	0.06	−0.52		0.40	0.26	−3.17	0.11
Standard Error	0.17	0.82	0.04	0.06	0.08		0.12	0.08	0.81	
t-Statistic	(4.42)	(0.94)	(3.03)	(1.12)	(6.62)		(3.43)	(3.34)	(3.93)	
p-values	0.00	0.34	0.00	0.26	0.00		0.00	0.00	0.00	
CMA										
Coefficient	0.03	0.04	−0.09	−0.03	0.47	0.05		−0.15	−0.59	0.56
Standard Error	0.06	0.18	0.01	0.02	0.02	0.01		0.03	0.17	
t-Statistic	(1.87)	(0.20)	(7.00)	(1.61)	(23.64)	(3.70)		(5.80)	(3.41)	
p-values	0.06	0.84	0.00	0.11	0.00	0.00		0.00	0.00	
RMW										
Coefficient	0.08	−0.07	−0.08	−0.23	0.18	0.09	−0.34		−1.39	0.23
Standard Error	0.08	0.24	0.02	0.03	0.04	0.02	0.06		0.23	
t-Statistic	(2.61)	(0.30)	(4.17)	(8.54)	(4.45)	(4.38)	(5.96)		(6.07)	
p-values	0.01	0.76	0.00	0.00	0.00	0.00	0.00		0.00	

	a_f	$A(d_f)$	MKT	Cov(f,MKT)	R^2
Panel B: Spanning test involving just the market and the behavioural bias					
SMB					
Coefficient	0.19	0.18	0.19	0.32	0.08
Standard Error	0.12	0.27	0.03	0.26	
t-Statistic	(1.63)	(0.67)	(7.07)	(1.24)	
p-values	0.10	0.50	0.00	0.21	
HML					
Coefficient	0.42	0.41	−0.16	−0.30	0.07
Standard Error	0.11	0.15	0.02	0.17	
t-Statistic	(3.84)	(2.75)	(6.60)	(1.73)	
p-values	0.00	0.01	0.00	0.09	
UMD					
Coefficient	0.74	0.79	−0.13	−3.26	0.04
Standard Error	0.16	0.82	0.04	0.83	
t-Statistic	(4.52)	(0.96)	(3.40)	(3.91)	
p-values	0.00	0.34	0.00	0.00	
CMA					
Coefficient	0.31	0.27	−0.17	−0.58	0.15
Standard Error	0.08	0.18	0.02	0.24	
t-Statistic	(3.89)	(1.46)	(9.81)	(2.45)	
p-values	0.00	0.15	0.00	0.01	
RMW					
Coefficient	0.21	0.11	−0.11	−1.11	0.08
Standard Error	0.09	0.24	0.02	0.24	
t-Statistic	(2.44)	(0.46)	(5.70)	(4.60)	
p-values	0.02	0.65	0.00	0.00	

$$\rho_i[(\mathbb{E}[\mathbf{R}] - \mathbf{R}^F) - (\rho_i - 2\gamma_i + 1)\boldsymbol{\Sigma}_R\mathbf{x}_i]$$
$$\times \exp\Bigg(-\rho_i\big((\mathbf{x}_i'(\mathbb{E}[\mathbf{R}] - \mathbf{R}^F) + W_i(t-1)R^F) +$$
$$+ \rho_i\left(\frac{\rho_i - 2\gamma_i + 1}{2}\right)\mathbf{x}_i'\boldsymbol{\Sigma}_R\mathbf{x}_i\big)\Bigg) = 0 \qquad \text{(G)}$$

In Eq. (G), the exponential term is always positive so that the latter reduces to a concave programming problem which is solved by

$$\mathbf{x}_i = \boldsymbol{\Sigma}_\mathbf{R}^{-1}\frac{\mathbb{E}[\mathbf{R}] - \mathbf{R}^F}{(\rho_i + \kappa(\gamma_i))} \qquad \text{(H)}$$

which is the first result of Proposition 1 embodied in Eq. (7).

With regard to the result in Eq. (8), we start by computing the aggregate demand for risky assets as

$$\mathbf{x} = \sum_{i=1}^{n} \mathbf{x}_i = \sum_{i=1}^{n} \left(\boldsymbol{\Sigma}_{\mathbf{R}}^{-1} \frac{\mathbb{E}[\mathbf{R}] - \mathbf{R}^{\mathbf{F}}}{(\rho_i + \kappa(\gamma_i))} \right) = \boldsymbol{\Sigma}_{\mathbf{R}}^{-1} (\mathbb{E}[\mathbf{R}] - \mathbf{R}^{\mathbf{F}}) \sum_{i=1}^{n} (\rho_i + \kappa(\gamma_i))^{-1} \quad \text{(I)}$$

By defining $(\rho + \kappa(\gamma))^{-1} = \sum_{i=1}^{n} (\rho_i + \kappa(\gamma_i))^{-1}$, where ρ and $\kappa(\gamma)$ might be interpreted as aggregate measures of the absolute risk aversion and of the distance from rationality respectively, we have that, by inverting the first order condition in (I)

$$\mathbb{E}[\mathbf{R}] - \mathbf{R}^{\mathbf{F}} = (\rho + \kappa(\gamma)) \boldsymbol{\Sigma}_{\mathbf{R}} \mathbf{x} \quad \text{(J)}$$

where $\boldsymbol{\Sigma}_{\mathbf{R}} \mathbf{x}$ is the $N \times 1$ vector of covariances between each asset's return R_j and the return on the "comprehensive" portfolio obtained through the aggregation of all the individual portfolios held by the n agents.

In fact, $\boldsymbol{\Sigma}_{\mathbf{R}} \mathbf{x}$ is given by

$$\begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \cdots\cdots\cdots\cdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_N^2 \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{n} x_{i,1} \\ \sum_{i=1}^{n} x_{i,2} \\ \vdots \\ \sum_{i=1}^{n} x_{i,N} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{n} x_{i,1}\sigma_1^2 + \sum_{i=1}^{n} x_{i,2}\sigma_{12} + \cdots \sum_{i=1}^{n} x_{i,n}\sigma_{1n} \\ \sum_{i=1}^{n} x_{i,1}\sigma_{12} + \sum_{i=1}^{n} x_{i,2}\sigma_2^2 + \cdots \sum_{i=1}^{n} x_{i,n}\sigma_{2n} \\ \vdots \\ \sum_{i=1}^{n} x_{i,1}\sigma_{1n} + \sum_{i=1}^{n} x_{i,2}\sigma_{2n} + \cdots \sum_{i=1}^{n} x_{i,n}\sigma_n^2 \end{bmatrix}$$

while the return on the market portfolio is,

$$R^W = \mathbf{x}'\mathbf{R} + \sum_{i=1}^{n} x_i^f R^F = \sum_{i=1}^{n} \sum_{j=1}^{N} x_{i,j} R_j + \sum_{i=1}^{n} x_i^f R^F$$

Considering for instance a single asset $j = 1$, we have

$$\text{COV}(R_1, R^W) = \text{COV}\left(R_1, \sum_{i=1}^{n}\sum_{j=1}^{N} x_{i,j} R_j + \sum_{i=1}^{n} x_i^f R^F\right)$$

$$= \text{COV}\left(R_1, \sum_{i=1}^{n}\sum_{j=1}^{N} x_{i,j} R_j\right) =$$

$$= \text{COV}\left(R_1, \sum_{i=1}^{n} x_{i,1} R_1\right) + \text{COV}\left(R_1, \sum_{i=1}^{n} x_{i,2} R_2\right) + \cdots$$

$$+ \text{COV}\left(R_1, \sum_{i=1}^{n} x_{i,N} R_N\right) =$$

$$= \sum_{i=1}^{n} x_{i,1}\text{COV}(R_1, R_1) + \sum_{i=1}^{n} x_{i,2}\text{COV}(R_1, R_2) + \cdots$$

$$+ \sum_{i=1}^{n} x_{i,N}\text{COV}(R_1, R_N)$$

which means that $\sigma_j x_j = \text{COV}(R_j, R^M)$.

Thus, at a single asset level, (J) coincides with

$$\mathbb{E}[R_j] - R^F = (\rho + \kappa(\gamma))\text{COV}(R_j, R^M) \quad \text{(K)}$$

which is the result in Eq. (8).

Proof of Proposition 2

Let us consider the final result in Eq. (K). By plugging in the implication of Conjecture 2 that

$$\rho = \frac{\mathbb{E}[R^M] - R^F}{\sigma_M^2} \quad \text{(L)}$$

we have that

$$\mathbb{E}[R_j] - R^F = \left(\frac{\mathbb{E}[R^M] - R^F}{\sigma_M^2} + \kappa(\gamma)\right)\text{COV}(R_j, R^M) \quad \text{(M)}$$

and eventually, by defining *beta* in the usual way as $\beta_j = \frac{\text{COV}(R^j, R^M)}{\sigma_M^2}$, we end up with

$$\mathbb{E}[R_j] - R^F = \alpha_j + (\mathbb{E}[R^M] - R^F)\beta_j^* + \kappa(\gamma)\text{COV}(R_j, R^M) \quad \text{(N)}$$

which is the result in Proposition 2.

Proof of Proposition 3

For the Alpha-Neutral CAPM in Eq. (12), the time series regression is:

$$R_t^j - R_t^F = a_j + b_j^*\text{MKT}_t + k_j\sigma(R_t^j, R_t^M) + e_{j,t} \quad \text{(O)}$$

where $\text{MKT}_t = (R_t^M - R_t^F)$.

The correct asset pricing model is

$$\mathbb{E}_t[R_j] - R^F = \mathbb{E}_t[\delta_j] + b_j^*\lambda_{\text{MKT}} \quad \text{(P)}$$

where δ_j are the pricing errors, theoretically equal to zero, and λ_{MKT} is the market factor premium.

Contrasting the correct model (P) with the expected value of the time series regression in (O) we have

$$\mathbb{E}_t[\delta_j] = \mathbb{E}_t[a_j] + b_j^*(\mathbb{E}_t[\text{MKT}_t] - \lambda_{\text{MKT}}) + k_j\mathbb{E}_t[\sigma(R_t^j, R_t^M)] \quad \text{(Q)}$$

The market factor is represented by the excess returns on the market portfolio so that $\mathbb{E}[\text{MKT}_t] = \lambda_{\text{MKT}}$, while, by definition, $\mathbb{E}_t[a_j] + k_j\mathbb{E}_t[\sigma(R_t^j, R_t^M)] = \mathbb{E}_t[d_j]$. Thus, we end up with

$$\mathbb{E}_t[\delta_j] = \mathbb{E}_t[d_j] \quad \text{(R)}$$

which is the result in Proposition 3.

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