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Volterra and Composition Inner Derivations on the Fock–Sobolev Spaces

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Abstract

On the Fock–Sobolev spaces, we study the range of Volterra inner derivations and composition inner derivations. The Volterra inner derivation ranges in the ideal of compact operators if and only if the induced function g is a linear polynomial. The composition inner derivation ranges in the ideal of compact operators if and only if the induced function φ is either identity or a contractive linear self-mapping of $\mathbb{C}$. Moreover, we describe the compact intertwining relations for composition operators and Volterra operators between different Fock–Sobolev spaces. In this paper, our results are complement and in a sense extend some aspects of Calkin's result (Ann Math 42:839–873, 1941) to the algebras of bounded linear operators on Fock–Sobolev spaces.

Keywords Fock–Sobolev space · Inner derivation · Volterra operator · Composition operator · Compact intertwining relation

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1 Introduction

Let $\mathscr{A}$ be a Banach algebra over the complex field. A linear map $D : \mathscr{A} \to \mathscr{A}$ is a derivation if $D(xy) = xD(y) + D(x)y$ for all $x, y \in \mathscr{A}$. Over the last half century, there have been lots of results giving conditions on a derivation of a Banach algebra implying that its range is contained in some ideal. One of the most famous result given by Singer and Wermer [19] says that every continuous derivation of a commutative Banach algebra maps has into Jacobson radical of the algebra. Previously, In Calkin [3], Calkin proved that an *inner derivation* $X \mapsto [T, X] := TX - XT$ maps the algebra of all bounded operators on a Hilbert space to the ideal of all compact operators if and only if T is a compact perturbation of a scalar operator. In general, this conclusion fails to hold true on Banach spaces, see [18, p. 288].

In this paper, we are interested in Volterra-type inner derivations on Fock–Sobolev spaces, in particular, we give characterizations which complement and in a sense extend some aspects of Calkin's result to the algebras of bounded linear operators on Fock–Sobolev spaces. To reach this goal, we use the compact intertwining relations for Volterra and composition operators and some results of the bounded and compact Volterra and composition operators between different Fock–Sobolev spaces.

To state our main results, we recall some basic definitions. Let $H(\mathbb{C})$ be the class of all entire functions on the complex plane $\mathbb{C}$. For $0 < p < \infty$ and a nonnegative integer m, the Fock–Sobolev spaces F_m^p consist of entire $f \in H(\mathbb{C})$ for which

$$\int_{\mathbb{C}} \left| f^{(m)}(z) \right|^p e^{-\frac{p}{2}|z|^2} dA(z) < \infty,$$

where dA denotes the Lebesgue area measure on $\mathbb{C}$. The Fock–Sobolev spaces were introduced in [7] where it was proved that $f \in F_m^p$ if and only if

$$\|f\|_{(m,p)} := \left(\frac{p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-p\psi_m(z)} dA(z) \right)^{\frac{1}{p}} < \infty,$$

where $\psi_m(z) = \frac{1}{2}|z|^2 - m\log(1 + |z|)$. It is clear that F_0^p is the classic Fock spaces. Interested reader in this topic can refer [26] for more details.

Furthermore, the Fock–Sobolev space F_m^∞ also has the following equivalent definition

$$F_m^\infty := \left\{ f \in H(\mathbb{C}) : \|f\|_{(m,\infty)} = \sup_{z\in\mathbb{C}} |f(z)|e^{-\psi_m(z)} < \infty \right\}.$$

Note that for each nonnegative integer m, the space F_m^2 is a reproducing kernel Hilbert space with the reproducing kernel function $K_m(z, w)$, for $w \in \mathbb{C}$. An explicit expression for $K_m(z, w)$ is still unknown. For each $w \in \mathbb{C}$, by Proposition 2.7 in [6], we have the following asymptotic properties

$$\|K_m(\cdot, w)\|_{(m,2)}^2 \approx e^{2\psi_m(w)}.$$

For other values of p, by Theorem 14 of [7], we have an upper estimate

$$\|K_m(\cdot, w)\|_{(m,p)}^2 \lesssim e^{\psi_m(w)}.$$

Note that when $m = 0$, the sapce F_m^2 reduces to the classical Fock space F^2, in particular, F^2 is a reproducing kernel Hilbert space with normalized kernel function $k_w(z) = e^{-\frac{1}{2}|w|^2 + \bar{w}z}$.

For $f \in H(\mathbb{C})$, every $\varphi \in H(\mathbb{C})$ induces a composition operator C_φ by $C_\varphi f = f \circ \varphi$. The bounded and compact composition operators on various holomorphic functions spaces have been studied intensively in the past few decades. Interested readers may refer to books [10, 17] and recent papers [1, 2, 4, 12, 21, 22] on the Fock spaces and the references therein.

If $g \in H(\mathbb{C})$, the Volterra operator V_g is defined by

$$V_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta$$

where $z \in \mathbb{C}$ and $f \in H(\mathbb{C})$.

The discussion of Volterra-type operators first arose in connection with semigroup of composition operators, and readers can refer to [20] for further details and background. On the Fock type spaces, Constantin [8] and Peleáz [9] firstly studied the bounded and compact Volterra type operators. Later, Mengestie [13] characterized the products of integral type operators and composition operators between different Fock spaces.

Let $\mathscr{B}(F_m^p)$ be the Banach algebra of bounded linear operators on the Fock–Sobolev spaces F_m^p, where $0 < p \leq \infty$. The *Volterra inner derivation* induced by $g \in H(\mathbb{C})$ on $\mathscr{B}(F_m^p)$ is defined by

$$D(V_g) : \mathscr{B}(F_m^p) \to \mathscr{B}(F_m^p) \qquad T \mapsto [V_g, T], \quad \forall T \in \mathscr{B}(F_m^p).$$

We can now state our main results.

Theorem A *Let $0 < p \leq \infty$, the Volterra inner derivation $D(V_g)$ on $\mathscr{B}(F_m^p)$ maps into the ideal of compact operators if and only if $g(z) = az + b$ with $a, b \in \mathbb{C}$.*

The *composition inner derivation* induced by $\varphi \in H(\mathbb{C})$ on $\mathscr{B}(F_m^p)$ is defined by

$$D(C_\varphi) : \mathscr{B}(F_m^p) \to \mathscr{B}(F_m^p) \quad T \mapsto [C_\varphi, T], \quad \forall T \in \mathscr{B}(F_m^p).$$

Theorem B *Let $0 < p \leq \infty$, the composition inner derivation $D(C_\varphi)$ on $\mathscr{B}(F_m^p)$ maps into the ideal of compact operators if and only if $\varphi = \mathrm{id}$ or $\varphi(z) = az + b$ with $a, b \in \mathbb{C}$ and $|a| < 1$.*

The proofs of Theorems A and B are given in Sects. 3 and 4, respectively. In addition, at the end of this paper, we study the unbounded composition operators $C_\varphi : F_m^p \to F_m^q$

when $0 < q < p \leq \infty$ proving that there are some unbounded composition operators that compactly interwine all bounded Volterra operators.

Throughout the paper, we use the following notations: $A \lesssim B$ means that there is a positive constant C such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2 Preliminaries

2.1 Compact Intertwining Relations

Let X and Y be two metric linear spaces, we denote by $\mathscr{B}(X, Y)$ the collection of all continuous linear operators from X to Y and by $\mathscr{K}(X, Y)$ the collection of all compact elements of $\mathscr{B}(X, Y)$, and by $\mathscr{Q}(X, Y)$ the quotient space $\mathscr{B}(X, Y)/\mathscr{K}(X, Y)$.

For $A \in \mathscr{B}(X, X)$, $B \in \mathscr{B}(Y, Y)$ and $T \in \mathscr{B}(X, Y)$, we say that T intertwines A and B in $\mathscr{Q}(X, Y)$ (or T intertwines A and B compactly) if

$$TA - BT \in \mathscr{K}(X, Y) \quad \text{where} \quad T \neq 0.$$

More intuitively, the compact intertwining relation is explained by the following commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{A} & X \\ \big\downarrow{\scriptstyle T} & & \big\downarrow{\scriptstyle T} \\ Y & \xrightarrow{B} & Y \end{array} \qquad \text{mod } \mathscr{K}(X, Y).$$

When $X = Y$ and $A = B$ it is easy to see the following two assertions are equivalent:

(a) T intertwines every $A \in \mathscr{B}(X)$ compactly.
(b) The inner derivation $D(T) : \mathscr{B}(X) \to \mathscr{B}(X)$ ranges in the compact ideal.

From this point of view, we will study the compact intertwining relations for composition operators and Volterra operators between different Fock–Sobolev spaces, which are then used to obtain our two main results (Theorems A and B) as direct consequences.

In the series papers [23–25], Yuan, Tong and Zhou firstly investigate the intertwining relations for Volterra operators and composition operators on the Bergman spaces, bounded analytic function spaces and Bloch spaces in the unit disk. By continuing this line of work, we characterize the compact intertwining relations for composition operators and Volterra operators between different Fock–Sobolev spaces. Our main results on the Volterra and composition inner derivation on $\mathscr{B}(F^p_m)$ then follow immediately.

2.2 Background on Volterra and Composition Operators

In this subsection, we present some preliminary lemmas give characterizations of the bounded and compact Volterra and composition operators on the Fock–Sobolev spaces whether $0 < p \le q \le \infty$ or $0 < q < p < \infty$. Combining Lemma 2.2 in [14] and Lemma 2.1 in [15], we conclude the following lemma.

Lemma 2.1 *If $f \in H(\mathbb{C})$, the following inequalities hold.*

(a): *If $0 < p < \infty$, then*

$$\|f\|_{(m,p)}^p \approx |f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p}{(1+\psi_m'(z))^p} e^{-p\psi_m(z)} dA(z).$$

(b): *If $p = \infty$, then*

$$\|f\|_{(m,\infty)} \approx |f(0)| + \sup_{z\in\mathbb{C}} \frac{|f'(z)|}{1+\psi_m'(z)} e^{-\psi_m(z)}.$$

The bounded and compact Volterra operators $V_g : F_m^p \to F_m^q$ were characterized in [14, 15], and we summarize them as follows.

Lemma 2.2 *Let $0 < p, q \le \infty$, $g \in H(\mathbb{C})$.*

(a): *If $0 < p \le q \le \infty$, then*

- (i): *$V_g : F_m^p \to F_m^q$ is bounded if and only if $g(z) = az^2 + bz + c$, where $a, b, c \in \mathbb{C}$;*
- (ii): *$V_g : F_m^p \to F_m^q$ is compact if and only if $g(z) = az + b$, where $a, b \in \mathbb{C}$.*

(b): *If $0 < q < p < \infty$, then $V_g : F_m^p \to F_m^q$ is bounded if and only if it is compact and if and only if $g(z) = az+b,\ a, b \in \mathbb{C}$ whenever $\frac{q}{2} > \frac{p-q}{p}$, and g is constant otherwise.*

(c) *If $0 < q < \infty$, then $V_g : F_m^\infty \to F_m^q$ is bounded if and only if it is compact and if and only if $g(z) = az + b,\ a, b \in \mathbb{C}$ whenever $q > 2$, and g is constant otherwise.*

The bounded and compact composition operators on the Fock–Sobolev spaces are characterized in [16]. The following lemma summarizes those characterizations as follows whenever $0 < p, q \le \infty$.

Lemma 2.3 *Let $0 < p, q \le \infty$, $\varphi \in H(\mathbb{C})$.*

(a): *If $0 < p \le q \le \infty$, then*

- (i): *$C_\varphi : F_m^p \to F_m^q$ is bounded if and only if $\varphi(z) = az+b$ where $|a| < 1, b \in \mathbb{C}$ or $\varphi(z) = az$ where $|a| = 1$;*
- (ii): *$C_\varphi : F_m^p \to F_m^q$ is compact if and only if $\varphi(z) = az+b$ where $|a| < 1, b \in \mathbb{C}$.*

(b): *If $0 < q < p < \infty$, then $C_\varphi : F_m^p \to F_m^q$ is bounded if and only if it is compact and if and only if $\varphi(z) = az + b$ for $|a| < 1$, $b \in \mathbb{C}$.*

(c): *If $0 < q < \infty$ and $p = \infty$, then $C_\varphi : F_m^\infty \to F_m^q$ is bounded if and only if it is compact and if and only if $\varphi(z) = az + b$ where $|a| < 1$, $b \in \mathbb{C}$.*

2.3 Carleson Measures

The Carleson measure theorems will play an important role in our proofs. So, let us give the definitions of Carleson and vanishing Carleson measures for Fock–Sobolev spaces.

Let $0 < p \le \infty$ and $0 < q < \infty$. We say that a nonnegative Borel measure μ on $\mathbb{C}$ is a (F_m^p, q)-Carleson measure if

$$\int_{\mathbb{C}} |f(z)|^q e^{-\frac{q}{2}|z|^2} d\mu(z) \lesssim \|f\|_{(p,m)}^q, \quad \text{for every} \quad f \in F_m^p.$$

In other words, the measure μ is a (F_m^p, q)-Carleson measure if and only if the embedding map $I_\mu : F_m^p \to L^q(\sigma_q)$ is bounded where $d\sigma_q(z) = e^{-\frac{q}{2}|z|^2} d\mu(z)$.

We say that the measure μ is a vanishing (F_m^p, q)-Carleson measure if

$$\lim_{j\to\infty} \int_{\mathbb{C}} |f_j(z)|^q e^{-\frac{q}{2}|z|^2} d\mu(z) = 0,$$

whenever f_j is a bounded sequence in F_m^p that converges uniformly to zero on compact subsets of $\mathbb{C}$ as $j \to \infty$.

For $s, t > 0$, we define the (t, s)-Berezin type transform of μ by

$$\widetilde{\mu}_{(t,s)}(w) = \int_{\mathbb{C}} (1 + |z|)^{-s} e^{-\frac{t}{2}|z-w|^2} d\mu(z).$$

The following lemma is the main result in [16].

Lemma 2.4 *Let $0 < p, q < \infty$ and μ be a nonnegative measure on $\mathbb{C}$.*

(a): *If $0 < p \le q < \infty$, then μ is a vanishing (F_m^p, q)-Carleson measure if and only if $\widetilde{\mu}_{(t,mq)}(z) \to 0$ as $|z| \to \infty$ for some (or any) $t > 0$.*

(b): *If $0 < p \le q < \infty$, then μ is a (F_m^p, q)-Carleson measure if and only if $\widetilde{\mu}_{(t,mq)}(z) \in L^\infty$ for some (or any) $t > 0$.*

(c): *If $0 < q < p < \infty$, then μ is a (F_m^p, q)-Carleson measure if and only if μ is a vanishing (F_m^p, q)-Carleson measure, and if and only if $\widetilde{\mu}_{(t,mq)} \in L^{\frac{p}{p-q}}$ for some (or any) $t > 0$.*

(d): *If $0 < q < \infty$ and $p = \infty$, then μ is a (F_m^∞, q)-Carleson measure if and only if μ is a vanishing (F_m^∞, q)-Carleson measure, and if and only if $\widetilde{\mu}_{(t,mq)} \in L^1$ for some (or any) $t > 0$.*

3 Proof of Theorem A

In this section we study the boundedness and compactness of the operator $T_{\varphi,g}$, we define below. Then, we characterize the compact intertwining relation for Volterra operators V_g and C_φ from F_m^p to F_m^q for $0 < p, q \le \infty$. Using this fact, we prove the first main theorem of this paper and at the end of this section we study the connection between the operators V_g and $T_{\varphi,g}$. To prove Theorem A, $\varphi, g \in H(\mathbb{C})$, we consider the following operator

$$T_{\varphi,g} f(z) = \int_0^{\varphi(z)} f(w)g'(w)dw - \int_0^{z} f(\varphi(w))g'(w)dw,$$

for $f \in F_m^p$ and $z \in \mathbb{C}$.

To characterize the properties of $T_{\varphi,g}$ we define another integral operator as follows:

$$I_{g,\varphi}^{p,q}(w) := \int_{\mathbb{C}} \frac{|(g \circ \varphi - g)'(z)|^q (1+|z|)^{mq}}{(1+|\varphi(z)|)^{mq}(1+\psi_m'(z))^q} |k_w(\varphi(z))|^q \, e^{-\frac{q|z|^2}{2}} dA(z).$$

The following Propositions give some necessary and sufficient conditions for the bounded and compact $T_{\varphi,g}$ between different Fock–Sobolev spaces F_m^p and F_m^q whether $0 < p \le q \le \infty$ or $0 < q < p \le \infty$.

Proposition 3.1 *Let $\varphi, g \in H(\mathbb{C})$ and $0 < p, q \le \infty$.*

(a): *If $0 < p \le \infty$, then $T_{\varphi,g}$ is bounded from F_m^p to F_m^∞ if and only if*

$$\sup_{z\in\mathbb{C}} \frac{|(g \circ \varphi - g)'(z)|}{1+\psi_m'(z)} e^{\psi_m(\varphi(z))-\psi_m(z)} < \infty.$$

(b): *If $0 < p \le q < \infty$, then $T_{\varphi,g}$ is bounded from F_m^p to F_m^q if and only if*

$$\sup_{z\in\mathbb{C}} I_{g,\varphi}^{p,q}(w) < \infty.$$

(c): *If $0 < q < p < \infty$, then $T_{\varphi,g}$ is bounded from F_m^p to F_m^q if and only if*

$$I_{g,\varphi}^{p,q}(w) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA).$$

(d): *If $0 < q < \infty$, then $T_{\varphi,g}$ is bounded from F_m^∞ to F_m^q if and only if*

$$I_{g,\varphi}^{\infty,q}(w) \in L^1(\mathbb{C}, dA).$$

Proof To prove the sufficient condition of **(a)**, we apply Lemma 2.1 to have

$$\begin{aligned}
\|T_{\varphi,g} f\|_{(m,\infty)} &\approx \sup_{z\in\mathbb{C}} \frac{|(g \circ \varphi - g)'(z)||f(\varphi(z))|}{1+\psi_m'(z)} e^{-\psi_m(z)} \\
&\le \sup_{z\in\mathbb{C}} \frac{|(g \circ \varphi - g)'(z))|}{1+\psi_m'(z)} e^{\psi_m(\varphi(z))-\psi_m(z)} \sup_{z\in\mathbb{C}} |f(\varphi(z))| e^{-\psi_m(\varphi(z))} \\
&= \|f\|_{(m,\infty)} \sup_{z\in\mathbb{C}} \frac{(g \circ \varphi - g)'(z)}{1+\psi_m'(z)} e^{\psi_m(\varphi(z))-\psi_m(z)} \\
&\lesssim \|f\|_{(m,\infty)} \lesssim \|f\|_{(m,p)},
\end{aligned}$$

where the last inequality follows from the monotonicity property $F_m^p \subseteq F_m^\infty$.

Conversely, for each $w \in \mathbb{C}$, let $\xi_{(w,m)}(z) = e^{-\psi_m(w)} K_{(w,m)}(z)$. By Corollary 14 in [7] for $p < \infty$ and a direct computation for $p = \infty$, we have

$$\|\xi_{(w,m)}\|_{(m,p)} \lesssim 1,$$

where the constant involved is independent of p and w. Applying $T_{\varphi,g}$ to $\xi_{(w,m)}$ yields

$$\begin{aligned}\|T_{\varphi,g}\xi_{(w,m)}\|_{(m,\infty)} &\approx \sup_{z\in\mathbb{C}} \frac{|(g\circ\varphi - g)'(z)||\xi_{(w,m)}(\varphi(z))|}{1+\psi_m'(z)} e^{-\psi_m(z)} \\ &\geq \frac{|(g\circ\varphi - g)'(z)||\xi_{(w,m)}(\varphi(z))|}{1+\psi_m'(z)} e^{-\psi_m(z)},\end{aligned}$$

for all points w and z in $\mathbb{C}$. In particular, by setting $w = \varphi(z)$, we have

$$\begin{aligned}\|T_{\varphi,g}\|_{(m,\infty)} &\gtrsim \frac{|(g\circ\varphi - g)'(z)|}{1+\psi_m'(z)} e^{\psi_m(\varphi(z))-\psi_m(z)} |\xi_{(\varphi(z),m)}(\varphi(z))| e^{-\psi_m(\varphi(z))} \\ &\approx \frac{(g\circ\varphi - g)'(z)}{1+\psi_m'(z)} e^{\psi_m(\varphi(z))-\psi_m(z)}.\end{aligned}$$

Since $T_{\varphi,g}$ is bounded from F_m^p to F_m^∞, the proof of **(a)** is complete.

Next, we prove **(b)** for the case $0 < p \leq q < \infty$. By setting

$$dV(z) = \frac{|(g\circ\varphi - g)'(z)|^q}{(1+\psi_m'(z))^q} e^{-q\psi_m(z)+\frac{q}{2}|\varphi(z)|^2} dA(z) \quad \text{and} \quad d\theta(z) = dV(\varphi^{-1}(z)),$$

we estimate the norm of $T_{\varphi,g} f$ as follows

$$\begin{aligned}\|T_{\varphi,g} f\|_{(m,q)}^q &\approx \int_{\mathbb{C}} \frac{|(g\circ\varphi - g)'(z)|^q |f(\varphi(z))|^q}{(1+\psi_m'(z))^q} e^{-q\psi_m(z)} dA(z) \\ &= \int_{\mathbb{C}} |f(\varphi(z))|^q e^{-\frac{q}{2}|\varphi(z)|^2} dV(z) \\ &= \int_{\mathbb{C}} |f(z)|^q e^{-\frac{q}{2}|z|^2} d\theta(z).\end{aligned}$$

Hence, the operator $T_{\varphi,g} : F_m^p \to F_m^q$ is bounded if and only if θ is a (F_m^p, q)-Carleson measure. By **(b)** of Lemma 2.4, it follows that the desire result follows if and only if

$$\widetilde{\theta}_{(q,mq)}(w) = \int_{\mathbb{C}} \frac{1}{(1+|z|)^{mq}} e^{-\frac{q}{2}|z-w|^2} d\theta(z) \in L^\infty.$$

Substituting back $d\theta$ and dV in terms of dA, we obtain

$$\begin{aligned}\widetilde{\theta}_{(q,mq)}(w) &= \int_{\mathbb{C}} \frac{|(g\circ\varphi-g)'(z)|^q}{(1+|\varphi(z)|)^{mq}(1+\psi_m'(z))^q} e^{\frac{q}{2}|\varphi(z)|^2-q\psi_m(z)-\frac{q}{2}|\varphi(z)-w|^2} dA(z)\\ &= \int_{\mathbb{C}} \frac{|(g\circ\varphi-g)'(z)|^q(1+|z|)^{mq}}{(1+|\varphi(z)|)^{mq}(1+\psi_m'(z))^q} \left|k_w(\varphi(z))e^{-\frac{|z|^2}{2}}\right|^q dA(z)\\ &< \infty,\end{aligned}$$

which completes the proof of **(b)**.

The proofs of **(c)** and **(d)** are similar to **(b)** and we omit it. □

Proposition 3.2 *Let $\varphi, g \in H(\mathbb{C})$ and $0 < p, q \le \infty$.*

(a): *If $0 < p \le \infty$, then $T_{\varphi,g}$ is compact from F_m^p to F_m^∞ if and only if $T_{\varphi,g}$ is bounded and*

$$\lim_{|\varphi(z)|\to\infty} \frac{|(g\circ\varphi-g)'(z)|}{1+\psi_m'(z)} e^{\psi_m(\varphi(z))-\psi_m(z)} = 0. \tag{3.1}$$

(b): *If $0 < p \le q < \infty$, then $T_{\varphi,g}$ is compact from F_m^p to F_m^q if and only if*

$$\lim_{|w|\to\infty} I_{g,\varphi}^{p,q}(w) = 0.$$

(c): *If $0 < q < p < \infty$, then $T_{\varphi,g}$ is compact from F_m^p to F_m^q if and only if*

$$I_{g,\varphi}^{p,q}(w) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA).$$

(d): *If $0 < q < \infty$, then $T_{\varphi,g}$ is compact from F_m^∞ to F_m^q if and only if*

$$I_{g,\varphi}^{p,q}(w) \in L^1(\mathbb{C}, dA).$$

Proof To prove **(a)**, we first assume that the operator $T_{\varphi,g}$ is compact. We observe that the sequence $\{\xi_{(w,m)}\}$ converges to zero uniformly on compact subsets of $\mathbb{C}$ as $|w| \to \infty$. Then, the compactness of $T_{\varphi,g}$ and Lemma 2.1 give

$$\begin{aligned}0 &= \lim_{|w|\to\infty} \|T_{\varphi,g}\xi_{(w,m)}\|_{(m,\infty)}\\ &\approx \lim_{|w|\to\infty} \sup_{z\in\mathbb{C}} \frac{|(g\circ\varphi-g)'(z)||\xi_{(w,m)}(\varphi(z))|}{1+\psi_m'(z)} e^{-\psi_m(z)}\\ &\gtrsim \lim_{|w|\to\infty} \frac{|(g\circ\varphi-g)'(z)||\xi_{(w,m)}(\varphi(z))|}{1+\psi_m'(z)} e^{\psi_m(w)-\psi_m(z)} e^{-\psi_m(w)},\end{aligned}$$

for every z in $\mathbb{C}$. In particular,, putting $w = \varphi(z)$, we have

$$0 \gtrsim \lim_{|\varphi(z)|\to\infty} \frac{|(g\circ\varphi - g)'(z)| e^{\psi_m(\varphi(z))-\psi_m(z)}}{1+\psi_m'(z)} |\xi_{(\varphi(z),m)}(\varphi(z))| e^{-\psi_m(\varphi(z))}$$
$$\approx \lim_{|\varphi(z)|\to\infty} \frac{|(g\circ\varphi - g)'(z)|}{1+\psi_m'(z)} e^{\psi_m(\varphi(z))-\psi_m(z)}.$$

Conversely, let $\{f_j\}$ be a bounded sequence of functions in F_m^p and $\{f_j\}$ converges uniformly to zero on compact subsets of $\mathbb{C}$ as $j \to \infty$. It is easy to obtain

$$\|T_{\varphi,g} f_j\|_{(m,\infty)} \approx \sup_{z\in\mathbb{C}} \frac{|(g\circ\varphi - g)'(z)||f_j(\varphi(z))|}{1+\psi_m'(z)} e^{-\psi_m(z)}$$
$$\leq \max\left\{ \sup_{|\varphi(z)|>N_1} G(z), \sup_{|\varphi(z)|\leq N_1} G(z) \right\},$$

where $G(z) = \frac{|(g\circ\varphi-g)'(z)||f_j(\varphi(z))|}{1+\psi_m'(z)} e^{-\psi_m(z)}$. Since (3.1) holds, for each $\epsilon > 0$ there exists a positive N_1 such that

$$\frac{|(g\circ\varphi - g)'(z)|}{1+\psi_m'(z)} e^{\psi_m(\varphi(z))-\psi_m(z)} < \epsilon,$$

whenever $|\varphi(z)| > N_1$. Hence,

$$\sup_{|\varphi(z)|>N_1} G(z) \leq \sup_{|\varphi(z)|>N_1} \frac{|(g\circ\varphi - g)'(z)|}{1+\psi_m'(z)} e^{\psi_m(\varphi(z))-\psi_m(z)} \sup_{z\in\mathbb{C}} |f_j(\varphi(z))| e^{-\psi_m(\varphi(z))}$$
$$\leq \epsilon \|f_j\|_{(m,\infty)} \leq \epsilon \|f_j\|_{(m,p)},$$

for every positive integer j.

Because of $\{f_j\}$ converging to zero uniformly on compact subsets of $\mathbb{C}$, we

$$\sup_{|\varphi(z)|\leq N_1} G(z) \lesssim \sup_{|\varphi(z)|\leq N_1} |f_j(\varphi(z))| \to 0 \quad \text{as } j \to \infty,$$

which completes the proof of **(a)**.

Next, we use the vanishing Carleson embedding theorem to prove **(b), (c)** and **(d)** by following the same arguments used in Proposition 3.1. Hence, we omit this. □

Now, we are ready to characterize the compact intertwining relation for Volterra operators between Fock–Sobolev spaces.

Theorem 3.3 *Let $0 < p \leq q \leq \infty$. The Volterra operators $V_g : F_m^p \to F_m^q$ compactly intertwines all composition operators C_φ which are bounded both on F_m^p and F_m^q if and only if $g(z) = az + b$ for $a, b \in \mathbb{C}$.*

Proof Let $0 < p \le q \le \infty$. If $g(z) = az + b$ for $a, b \in \mathbb{C}$, we can see V_g is compact from F_m^p to F_m^q by Lemma 2.2. Hence

$$C_\varphi|_{F_m^q} V_g|_{F_m^p \to F_m^q} - V_g|_{F_m^p \to F_m^q} C_\varphi|_{F_m^p}$$

is compact for every C_φ bounded on F_m^p and F_m^q.

For the necessary, we just give the proof for the case $0 < p \le q < \infty$, because the proof when $0 < p \le \infty$ and $q = \infty$ is highly similar.

By **(a)** of Lemma 2.2, $V_g : F_m^p \to F_m^q$ is bounded if and only if $g(z) = az^2 + bz + c$ for $a, b, c \in \mathbb{C}$. Putting $\varphi(z) = \lambda z$ with $|\lambda| = 1$, by **(b)** of Proposition 3.2, we have

$$\begin{aligned} 0 &= \lim_{|w| \to \infty} \int_{\mathbb{C}} \frac{|(g \circ \varphi - g)'(z)|^q (1 + |z|)^{mq}}{(1 + |\varphi(z)|)^{mq} (1 + \psi_m'(z))^q} |k_w(\varphi(z)) e^{-\frac{|z|^2}{2}}|^q dA(z) \\ &= \lim_{|w| \to \infty} \int_{\mathbb{C}} \frac{|2(\lambda^2 - 1)az + b(\lambda - 1)|^q (1 + |z|)^q}{(1 + |z| + ||z|^2 + |z| - m|)^q} e^{-\frac{q}{2}|\lambda z - w|^2} dA(z) \\ &\gtrsim \lim_{|w| \to \infty} \int_{D(w,1)} \frac{|2(\lambda^2 - 1)az + b(\lambda - 1)|^q (1 + |z|)^q}{(1 + |z| + ||z|^2 + |z| - m|)^q} dA(z) \\ &\gtrsim \lim_{|w| \to \infty} \frac{|2(\lambda^2 - 1)aw + b(\lambda - 1)|^q (1 + |w|)^q}{(1 + |w| + \big||w|^2 + |w| - m\big|)^q}. \end{aligned}$$

Thus, we must have $a = 0$. Therefore, g has the form $bz + c$ for some $b, c \in \mathbb{C}$, which completes the proof. □

We now prove our first main theorem

Proof of Theorem A Let $p = q$ and use Theorem 3.3 to have $[C_\varphi, V_g] \in \mathcal{K}(F_m^p)$ for every $C_\varphi \in \mathcal{B}(F_m^p)$ if and only if $g(z) = az + b$ with $a, b \in \mathbb{C}$. According to Lemma 2.2, it is equivalent to $V_g \in \mathcal{K}(F_m^p)$. Hence, $D(V_g)$ maps into $\mathcal{K}(F_m^p)$ if and only if V_g is a compact operator. □

Remark 3.4 In Theorem 3.3, we characterize the compact intertwining relations

$$C_\varphi|_{F_m^q} V_g|_{F_m^p \to F_m^q} - V_g|_{F_m^p \to F_m^q} C_\varphi|_{F_m^p} \tag{3.2}$$

for $0 < p \le q \le \infty$.

The compact intertwining relations (3.2) in cases $0 < q < p \le \infty$ are trivial because we can see the boundedness and compactness of $V_g : F_m^p \to F_m^q$ are equivalent by Lemma 2.2.

At the end of this section, we study the connection between operators V_g and $T_{\varphi,g}$.

Theorem 3.5 *Let $0 < q < p \le \infty$. If either $\varphi(z) = az + b$ for $|a| < 1, b \in \mathbb{C}$ or $\varphi(z) = az$ for $|a| = 1$, the operator $V_g : F_m^p \to F_m^q$ is bounded if and only if $T_{\varphi,g} : F_m^p \to F_m^q$ is bounded.*

In this point of view, we conclude that there is no unbounded V_g acting from F_m^p to F_m^q such that V_g compactly intertwines all composition operators C_φ which are bounded on F_m^p and F_m^q.

Proof The necessity is trivial by the fact that $V_g : F_m^p \to F_m^q$ is bounded if and only if it is compact when $0 < q < p \le \infty$, see **(b)** and **(c)** of Lemma 2.2.

For the sufficiency, we just prove the case $0 < q < p < \infty$ by **(b)** of Lemma 2.3, because the proof of the case $0 < q < \infty$ and $p = \infty$ will be the same by **(c)** of Lemma 2.3.

Note that the following estimates are true whenever $z \in D(w, 1)$:

$$\begin{aligned}
&1 + |z| \approx 1 + |w|;\\
&1 + |az + b| \approx 1 + |aw + b|;\\
&1 + |z| + \Big||z|^2 + |z| - m\Big| \approx 1 + |w| + \Big||w|^2 + |w| - m\Big|.
\end{aligned}$$

By the subharmonicity of $|(g \circ \varphi - g)'|^{\frac{pq}{p-q}}$, we have

$$\begin{aligned}
&\int_{\mathbb{C}} \left(\frac{|(g\circ\varphi - g)'(w)|^q (1+|w|)^{mq}}{(1+\psi_m'(w))^q (1+|\varphi(w)|)^{mq}} \right)^{\frac{p}{p-q}} dA(w)\\
&= \int_{\mathbb{C}} \left(\frac{|(g\circ\varphi - g)'(w)|^q (1+|w|)^{mq+q}}{(1+|w|+||w|^2+|w|-m|)^q (1+|\varphi(w)|)^{mq}} \right)^{\frac{p}{p-q}} dA(w)\\
&\lesssim \int_{\mathbb{C}} \left(\int_{D(w,1)} \frac{|(g\circ\varphi - g)'(z)|^q (1+|z|)^{mq+q}}{(1+|z|+||z|^2+|z|-m|)^q (1+|\varphi(z)|)^{mq}} dA(z) \right)^{\frac{p}{p-q}} dA(w)\\
&\lesssim \int_{\mathbb{C}} \left(\int_{D(w,1)} \frac{|(g\circ\varphi - g)'(z)|^q (1+|z|)^{mq+q} e^{\frac{q}{2}(|\varphi(z)|^2 - |\varphi(z)-w|^2) - \frac{q}{2}|z|^2}}{(1+|z|+||z|^2+|z|-m|)^q (1+|\varphi(z)|)^{mq}} dA(z) \right)^{\frac{p}{p-q}} dA(w)\\
&\le \int_{\mathbb{C}} \left(\int_{\mathbb{C}} \frac{|(g\circ\varphi - g)'(z)|^q (1+|z|)^{mq}}{(1+|\varphi(z)|)^{mq} (1+\psi_m'(z))^q} |k_w(\varphi(z)) e^{-\frac{|z|^2}{2}}|^q dA(z) \right)^{\frac{p}{p-q}} dA(w),
\end{aligned}$$

where $\varphi(z) = az + b$ for $|a| < 1, b \in \mathbb{C}$ or $\varphi(z) = az$ for $|a| = 1$. Since $T_{\varphi,g}$ is bounded from F_m^p to F_m^q,

$$\int_{\mathbb{C}} \left(\frac{|(g\circ\varphi - g)'(w)|^q (1+|w|)^{mq}}{(1+\psi_m'(w))^q (1+|\varphi(w)|)^{mq}} \right)^{\frac{p}{p-q}} dA(w) < \infty.$$

From which we conclude that $(g \circ \varphi - g)'$ must be a constant. In addition, if $(g \circ \varphi - g)'$ is a nonzero constant, the above holds only if $\frac{pq}{p-q} > 2$. Then, the desired result follows from **(b)** of Lemma 2.2. □

4 Proof of Theorem B

In this section, we characterize the compact intertwining relations for composition operators and Volterra operators between different Fock–Sobolev spaces $\mathscr{B}(F_m^p)$ and

$\mathscr{B}(F_m^q)$ when $0 < p \le q \le \infty$ or $0 < q \le p \le \infty$, which leads us to prove our main Theorem B.

Theorem 4.1 *Let $\varphi \in H(\mathbb{C})$ and $g \in H(\mathbb{C})$. In either case $0 < p \le q < \infty$ or $0 < p < \infty$ and $q = \infty$, the bounded composition operator $C_\varphi : F_m^p \to F_m^q$ compactly intertwines all Volterra operators V_g which are bounded both on F_m^p and F_m^q if and only if either $\varphi(z) = az + b$ for $|a| < 1$ or $\varphi(z) = \pm z$.*

Proof Since V_g is bounded both on F_m^p and F_m^q, it means that g is a quadratic polynomial on $\mathbb{C}$ by **(a)** of Lemma 2.2.

We first prove the theorem whenever $0 < p \le q < \infty$. For $\varphi(z) = az + b$ with $|a| < 1$, the composition operator C_φ is a compact operator from F_m^p to F_m^q by **(a)** of Lemma 2.3. Thus, $T_{\varphi,g}$ is a compact operator from F_m^p to F_m^q for every quadratic polynomial g.

If $\varphi(z) = z$, for any entire g, it is obvious that $T_{\varphi,g}$ is a zero operator and hence compact. If $\varphi(z) = -z$, we have

$$\begin{aligned}
&\int_{\mathbb{C}} \frac{|(g\circ\varphi - g)'(z)|^q (1+|z|)^{mq}}{(1+|\varphi(z)|)^{mq}(1+\psi_m'(z))^q} |k_w(\varphi(z)) e^{-\frac{|z|^2}{2}}|^q \, dA(z) \\
&= \int_{\mathbb{C}} \frac{|2b_1|^q (1+|z|)^q}{(1+|z|+||z|^2+|z|-m|)^q} e^{-\frac{q|z+w|^2}{2}} \, dA(z) \\
&\lesssim \int_{\mathbb{C}} e^{-\frac{q|z+w|^2}{2}} \, dA(z) < \infty,
\end{aligned}$$

where $g(z) = a_1 z^2 + b_1 z + c_1$, with $a_1, b_1, c_1 \in \mathbb{C}$. By the dominating convergence theorem, we have

$$\lim_{|w|\to\infty} \int_{\mathbb{C}} \frac{|(g\circ\varphi - g)'(z)|^q (1+|z|)^{mq}}{(1+|\varphi(z)|)^{mq}(1+\psi_m'(z))^q} |k_w(\varphi(z)) e^{-\frac{|z|^2}{2}}|^q \, dA(z) = 0.$$

Then, by **(b)** of Proposition 3.2, $T_{\varphi,g}$ is compact from F_m^p to F_m^q.

On the other hand, the boundedness of composition operator $C_\varphi : F_m^p \to F_m^q$ implies that either $\varphi(z) = az + b$ with $|a| < 1$, $b \in \mathbb{C}$ or $\varphi(z) = az$ with $|a| = 1$ by Lemma 2.3.

If $\varphi(z) = az$ with $|a| = 1$, we have

$$\begin{aligned}
&\int_{\mathbb{C}} \frac{|(g\circ\varphi - g)'(z)|^q (1+|z|)^{mq}}{(1+|\varphi(z)|)^{mq}(1+\psi_m'(z))^q} |k_w(\varphi(z)) e^{-\frac{|z|^2}{2}}|^q \, dA(z) \\
&= \int_{\mathbb{C}} \frac{|2a_1 z(a^2-1) + b_1(a-1)|^q (1+|z|)^q}{(1+|z|+||z|^2+|z|-m|)^q} e^{-\frac{|az-w|^2}{2}} \, dA(z) \\
&\gtrsim \int_{D(w,1)} \frac{|2a_1 z(a^2-1) + b_1(a-1)|^q (1+|z|)^q}{(1+|z|+||z|^2+|z|-m|)^q} \, dA(z) \\
&\gtrsim \frac{|2a_1 w(a^2-1) + b_1(a-1)|^q (1+|w|)^q}{(1+|w|+||w|^2+|w|-m|)^q}.
\end{aligned}$$

By Proposition 3.2, we have

$$\lim_{|w|\to\infty} \frac{|2a_1 w(a^2-1)+b_1(a-1)|^q(1+|w|)^q}{(1+|w|+||w|^2+|w|-m|)^q}=0.$$

Thus, we have $a^2=1$. That is $\varphi(z)=\pm z$.

Now, we study the case $0<p<\infty$ and $q=\infty$. If $\varphi(z)=az+b$ with $|a|<1$, $b\in\mathbb{C}$, by **(a)** of Lemma 2.3, it means that C_φ is a compact operator from F_m^p to F_m^∞. So, $T_{\varphi,g}$ is a compact operator from F_m^p to F_m^∞, for any quadratic polynomial g. If $\varphi(z)=az$ with $a=\pm1$, we get

$$\begin{aligned}&\lim_{|\varphi(z)|\to\infty} \frac{\left|(g\circ\varphi-g)'(z)\right|}{1+\psi_m'(z)} e^{\psi_m(\varphi(z))-\psi_m(z)}\\ &\quad=\lim_{|z|\to\infty}\frac{(2a_1 z(a^2-1)+b_1(a-1))(1+|z|)}{(1+|z|+||z|^2+|z|-m|)}=0,\end{aligned}$$

where $g(z)=a_1z^2+b_1z+c_1$ with $a_1,b_1,c_1\in\mathbb{C}$. It then follows from **(a)** of Proposition 3.2 that $T_{\varphi,g}$ is compact for any quadratic polynomial g.

Conversely, by a similar computation as above and **(a)** of Proposition 3.2, we have $a=\pm1$ if $\varphi(z)=az$ with $|a|=1$, which completes the proof. □

Proof of Theorem B The sufficient is trivial. For the necessity, let $p=q$ and use Theorem 4.1. It remains to check the case when $\varphi(z)=-z$.

We let $M(f)(z):=zf(z)$ for $f\in H(\mathbb{C})$. From Theorem 3.1 in [16], we get M is bounded on F_m^p. It follows by a direct computation that

$$[C_{-z},M]f(z)=-2MC_{-z}f(z).$$

Using Theorem 3.1 in [16] again, we get $[C_{-z},M]$ is bounded and noncompact. This completes the proof. □

Remark 4.2 In Theorem 4.1, we characterize the compact intertwining relations

$$V_g|_{F_m^q}C_\varphi|_{F_m^p\to F_m^q}-C_\varphi|_{F_m^p\to F_m^q}V_g|_{F_m^p} \tag{4.3}$$

whenever in the case $0<p\le q<\infty$ or $0<p<\infty$ and $q=\infty$.

The compact intertwining relations (4.3) in cases $0<q<p\le\infty$ are trivial because the boundedness and compactness of $C_\varphi: F_m^p\to F_m^q$ are equivalent by **(b)** and **(c)** of Lemma 2.3.

At the end of this paper, we study the unbounded composition operator $C_\varphi: F_m^p\to F_m^q$ so that $T_{\varphi,g}: F_m^p\to F_m^q$ is compact for every quadratic polynomials g.

Proposition 4.3 *Let $0<q<p\le\infty$. Let g be a quadratic polynomial. Then, the operator $T_{\varphi,g}: F_m^p\to F_m^q$ is compact if $\varphi(z)=az+b$ with $a^2=1, b\in\mathbb{C}$ whenever $\frac{pq}{p-q}>2$, and $\varphi(z)=z$ otherwise.*

Proof It is easy to see that $(g \circ \varphi - g)'$ is a constant. Denote by $(g \circ \varphi - g)' \equiv \lambda \in \mathbb{C}$, and we have

$$\begin{aligned}
&\int_{\mathbb{C}} \left(\int_{\mathbb{C}} \frac{|(g \circ \varphi - g)'(z)|^q (1+|z|)^{mq}}{(1+|\varphi(z)|)^{mq}(1+\psi_m'(z))^q} \left| k_w(\varphi(z)) e^{-\frac{|z|^2}{2}} \right|^q dA(z) \right)^{\frac{p}{p-q}} dA(w) \\
&= \int_{\mathbb{C}} \left(\int_{\mathbb{C}} \frac{|\lambda|^q (1+|z|)^{mq+q} e^{\frac{q}{2}(|\varphi(z)|^2 - |\varphi(z)-w|^2) - \frac{q}{2}|z|^2} dA(z)}{(1+|z|+||z|^2+|z|-m|)^q (1+|\varphi(z)|)^{mq}} \right)^{\frac{p}{p-q}} dA(w) \\
&\lesssim \int_{\mathbb{C}} \left(\int_{\mathbb{C}} \frac{|\lambda|^q}{(1+|z|)^q} e^{\frac{q}{2}(|\varphi(z)|^2 - |\varphi(z)-w|^2) - \frac{q}{2}|z|^2} dA(z) \right)^{\frac{p}{p-q}} dA(w) \\
&\approx \int_{\mathbb{C}} |\lambda|^q (1+|w|)^{-\frac{pq}{p-q}} dA(w) < \infty,
\end{aligned}$$

where the last integral converges since either $\frac{pq}{p-q} > 2$ or $\lambda = 0$.

If $0 < q < \infty$ and $p = \infty$, then $T_{\varphi,g} : F_m^\infty \to F_m^q$ is compact if $\varphi(z) = az + b$ with $a^2 = 1, b \in \mathbb{C}$ whenever $q > 2$, and $\varphi(z) = z$ otherwise. Hence, the conclusion is the same as above. □

Remark 4.4 From Proposition 4.3, we notice that there are unbounded composition operators which compactly intertwine all bounded Volterra operators when $0 < q < p \leq \infty$.

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Declarations

Conflict of interest The authors declare no conflict of interest.

References

1. Arroussi, H., Tong, C.: Weighted composition operators between large Fock spaces in several complex variables. J. Funct. Anal. **277**, 3436–3466 (2019)

2. Arroussi, H.: Weighted composition operators on Bergman spaces A^p_ω. Math. Nachr. **295**(4), 631–656 (2022)
3. Calkin, J.W.: Two-sided ideals and congruences in the ring of bounded operator in Hilbert space. Ann. Math. **42**(2), 839–873 (1941)
4. Carswell, B.J., MacCluer, B.D., Schuster, A.: Composition operators on the Fock space. Acta Sci. Math. (Szeged) **69**, 871–887 (2003)
5. Constantin, O., Peláez, José Angel.: Integral operators, embedding theorems and a Littlewood-Paley formula on weighted Fock spaces. J. Geom. Anal. **26**(2), 1109–1154 (2015)
6. Cho, H.R., Choe, B.R., Koo, H.: Linear combinations of composition operators on the Fock–Sobolev spaces. Potential Anal. **4**(41), 1223–1246 (2014)
7. Cho, R., Zhu, K.: Fock–Sobolev spaces and their Carleson measures. J. Funct. Anal. **263**(8), 2483–2506 (2012)
8. Constantin, O.: A Volterra-type integration operator on Fock spaces. Proc. Amer. Math. Soc. **140**(12), 4247–4257 (2012)
9. Constantin, O., Peláez, J.A.: Integral operators, embedding theorems and a Littlewood-Paley formula on weighted Fock spaces. J. Geom. Anal. **26**(2), 1109–1154 (2016)
10. Cowen, C.C., MacCluer, B.D.: Composition operators on spaces of analytic functions. CRC Press, Boca Raton (1995)
11. Čučković, Ž, Zhao, R.H.: Essential norm estimates of weighted composition operators between Bergman spaces on strongly pseudoconvex domains. Math. Proc. Camb. Phil. Soc. **142**, 525–534 (2007)
12. Khoi, L.H., Thom, L.T.H., Tien, P.T.: Weighted composition operators between Fock spaces $\mathcal{F}^\infty(\mathbb{C})$ and $\mathcal{F}^p(\mathbb{C})$, Inter. J. Math., (2019)
13. Mengestie, T.: Volterra type and weighted composition operators on weighted Fock spaces. Integ. Eq. Oper. Theory **76**(1), 81–94 (2013)
14. Mengestie, T.: Spectral properties of Volterra-type integral operators on Fock–Sobolev spaces. J. Kor. Math. Soc. **54**(6), 1801–1816 (2017)
15. Mengestie, T.: On the spectrum of volterra-type integral operators on Fock–Sobolev spaces. Complex Anal. Oper. Theory **11**(6), 1451–1461 (2017)
16. Mengestie, T.: Carleson type measures for Fock–Sobolev spaces. Complex Anal. Oper. Theory **8**, 1225–1256 (2014)
17. Shapiro, J.H.: Composition operators and classical function theory. Spriger, Berlin (1993)
18. Shulman, V.S., Turovskii, Yu.V.: Topological radicals and joint spectral radius, Funktsional. Anal. i Prilozhen. 46(2012), 61-82; translation in Funct. Anal. Appl., 46(4), 287-304 (2012)
19. Singer, I.M., Wermer, J.: Derivations on commutative normed algebras. Math. Ann. **129**, 260–264 (1955)
20. Siskakis, A.G., Zhao, R.H.: A Volterra type operator on spaces of analytic functions. Function spaces (Edwardsville, IL, 1998). Contemp. Math. **232**, 299–311 (1999)
21. Tien, P.T., Khoi, L.H.: Weighted composition operators between different Fock spaces. Potential Anal. **50**, 171–195 (2019)
22. Tien, P.T.: Products of Volterra Type Operators and Composition Operators Between Fock Spaces. Results Math. **75**(3), 104 (2020)
23. Tong, C., Yuan, C., Zhou, Z.: Compact intertwining relations for composition operators on H^∞ and the Bloch space. New York J. Math. **24**, 611–629 (2018)
24. Tong, C., Zhou, Z.: Intertwining relations for Volterra operators on the Bergman space. Illinois J. Math. **57**(1), 195–211 (2013)
25. Tong, C., Zhou, Z.: Compact intertwining relations for composition operators between the weighted Bergman spaces and the weighted Bloch spaces. J. Korean Math. Soc. **51**(1), 125–135 (2014)
26. Zhu, K.: Analysis on Fock spaces. Springer, New York (2012)