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International Mathematics Research Notices, 2025, **2025**(1), 1–20
https://doi.org/10.1093/imrn/rnae259
Advance access publication date 2 December 2024
Article

Spectral Instability of Coverings

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Communicated by Toby Colding

We study the behaviour of eigenvalues, below the bottom of the essential spectrum, of the Laplacian under finite Riemannian coverings of complete and connected Riemannian manifolds. We define spectral stability and instability of such coverings. Among others, we provide necessary conditions for stability or, equivalently, sufficient conditions for instability.

1 Introduction

Recently, Magee et al. [15, 20] have initiated a study of the spectrum of the Laplacian of a random Riemannian cover of a fixed hyperbolic (i.e., curvature $= -1$) surface. Broadly speaking, the main results obtained by them say that *asymptotically almost surely* the spectrum of a cover does not acquire new eigenvalues below a specific threshold $< 1/4$; that is, among n-sheeted covers, the percentage of those acquiring a new eigenvalue below that threshold tends to zero as n tends to infinity. What is even more interesting is that the threshold is independent of the bottom surface, but only depends on its type; see below.

A classical result of Randol [26] is quite opposite to the results of Magee et al. Namely, for any hyperbolic metric on a *closed* (i.e., *compact and connected with empty boundary*) surface S of genus $g \geq 2$, any natural number ℓ and any $\varepsilon > 0$, there is a finite Riemannian cover $p: S' \to S$ such that S' has at least ℓ eigenvalues in $[0, \varepsilon)$. (See [3, Theorem 4.1] for an elementary proof.)

One motivation for our studies in this paper is that, although the above mentioned results of Magee et al. show that asymptotically almost all n-sheeted covers of the surface in question are *spectrally stable* in a specific range, they do not provide any necessary or sufficient condition for this to happen. Among others, we provide, in this paper, some necessary conditions.

To set the stage, let M be a complete and connected Riemannian manifold of dimension m. Denote by $\tilde{M}$ the universal covering space of M, endowed with the lifted Riemannian metric, and let Γ be the fundamental group of M, viewed as the group of covering transformations on $\tilde{M}$.

Denote by $\lambda_0(M) \leq \lambda_{\mathrm{ess}}(M)$ the bottom of the spectrum and the essential spectrum (of the Laplacian Δ) of M, respectively. Recall that $\lambda_0(M)$ need not vanish in general, that $\lambda_0(M) = 0$ and $\lambda_{\mathrm{ess}}(M) = \infty$ in the case where M is closed (that is, compact and connected without boundary) and that the spectrum of M below $\lambda_{\mathrm{ess}}(M)$ consists of a locally finite set of eigenvalues of finite multiplicity. We assume throughout that

$$\lambda_0(M) < \lambda_{\mathrm{ess}}(M) \tag{1.1}$$

and enumerate the eigenvalues of M in $[0, \lambda_{\mathrm{ess}}(M))$ according to their size and multiplicity as

$$0 \leq \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \dots, \tag{1.2}$$

Received: November 24, 2023. **Revised:** November 4, 2024. **Accepted:** November 8, 2024

where we recall that the eigenvalue $\lambda_0(M)$ has multiplicity one since its eigenfunctions do not change sign. For $\lambda \geq 0$, we denote by $N_M(\lambda)$ and $N_M(\lambda-)$ the number of $\lambda_k(M)$ in $[0,\lambda]$ and $[0,\lambda)$, respectively. More generally, for any subset $I \subseteq [0,\lambda_{ess}(M))$, we denote by $N_M(I)$ the number of $\lambda_k(M)$ in I.

Let $p\colon M' \to M$ be a finite Riemannian covering of complete and connected Riemannian manifolds. Then

$$\lambda_0(M) = \lambda_0(M') \quad \text{and} \quad \lambda_{ess}(M) = \lambda_{ess}(M'); \tag{1.3}$$

see (2.5). Since the lifts $p^*\varphi = \varphi \circ p$ of eigenfunctions φ of M are eigenfunctions of M', we always have

$$\lambda_k(M') \leq \lambda_k(M). \tag{1.4}$$

Likewise, for any subset $I \subseteq [0,\lambda_{ess}(M))$,

$$N_M(I) \leq N_{M'}(I). \tag{1.5}$$

We say that p is *I-stable* if

$$N_M(I) = N_{M'}(I). \tag{1.6}$$

With respect to this terminology, results of Magee et al. say among others that,
(1) for any orientable, convex cocompact, non-compact hyperbolic surface S with Hausdorff dimension of its limit set $\delta > 1/2$ and any $\sigma \in (3\delta/4,\delta)$, any finite Riemannian cover $p\colon S' \to S$ is asymptotically almost surely $[\delta(1-\delta),\sigma(1-\sigma)]$-stable as $|p| \to \infty$; see [20, 21]. Note that here $\lambda_0(S) = \delta(1-\delta) < 1/4 = \lambda_{ess}(S)$.
(2) for any orientable and compact hyperbolic surface S and any $\varepsilon > 0$, any finite Riemannian cover $p\colon S' \to S$ is asymptotically almost surely $[0, 3/16-\varepsilon]$-stable as $|p| \to \infty$; see [22].
(3) for any orientable and non-compact hyperbolic surface S of finite area and any $\varepsilon > 0$, any finite Riemannian cover $p\colon S' \to S$ is asymptotically almost surely $[0, 1/4-\varepsilon]$-stable as $|p| \to \infty$; see [15]. Here $\lambda_0(S) = 0 < 1/4 = \lambda_{ess}(S)$.

Clearly, I-stability means that lifting yields an isomorphism between eigenspaces of M and M' for all eigenvalues of M and M' in I. In particular:
(1) if $J \subseteq [0,\infty)$ is a further subset and $I \subseteq J$, then

$$I\text{-instability of } p \text{ implies } J\text{-instability of } p; \tag{1.7}$$

(2) if $q\colon S'' \to S'$ is a further finite Riemannian covering of complete and connected Riemannian manifolds, then

$$I\text{-instability of } p \text{ or } q \text{ implies } I\text{-instability of } p \circ q. \tag{1.8}$$

For $k > 0$, we say that p is λ_k*-stable* if

$$N_M(\lambda) = N_{M'}(\lambda), \tag{1.9}$$

where $\lambda = \lambda_k(M) < \lambda_{ess}(M)$. By definition, λ_k-instability $N_M(\lambda) < N_{M'}(\lambda)$ can occur in two ways: Either p is *strictly* λ_k*-unstable*, that is, $N_M(\lambda-) < N_{M'}(\lambda-)$, or else p is *weakly* λ_k*-unstable*, that is, $N_M(\lambda-) = N_{M'}(\lambda-)$, but the multiplicity of λ as an eigenvalue increases, $\mu(\lambda, M) < \mu(\lambda, M')$. By (1.7),

$$\lambda_k\text{-instability implies } \lambda_\ell\text{-instability, for any } 1 \leq k \leq \ell, \tag{1.10}$$

where $\lambda_\ell(M) < \lambda_{ess}(M)$. In particular, λ_1-instability implies λ_k-instability for all $k \geq 1$. For that reason, our main focus is on λ_1-stability and instability.

For an eigenfunction φ on a Riemannian manifold, the set $\mathcal{Z}(\varphi) = \{\varphi = 0\}$ is called the *nodal set of* φ and the connected components of $\{\varphi \neq 0\}$ are called *nodal domains of* φ. (In general, we use the term *domain* to indicate *connected open sets*.) One of our main arguments uses connectedness of preimages in M' of nodal sets in M.

Theorem A. If $p\colon M' \to M$ is a λ_k-stable finite Riemannian covering of complete and connected Riemannian manifolds, where $\lambda_0(M) < \lambda_k(M) < \lambda_{\mathrm{ess}}(M)$, then the preimage $p^{-1}(U)$ of any nodal domain U of any λ-eigenfunction φ on M is connected, for any $\lambda_0(M) \le \lambda \le \lambda_k(M)$. In fact, if U is any nodal domain of any λ-eigenfunction φ on M for any such λ and $j \ge 1$ denotes the number of connected components of $p^{-1}(U)$, then

$$N_{M'}(\lambda-) \ge N_M(\lambda-) + j - 1.$$

Theorem A is a special case of Theorem 4.2. An easy application of the main results of [18] and [12] and Theorem A yields the following

Corollary B. A closed manifold M carries a Riemannian metric g, such that p is strictly λ_1-unstable with respect to g, for any non-trivial finite covering $p\colon M' \to M$ of closed manifolds, where M' is endowed with the lifted Riemannian metric g'. In fact, for an appropriate choice of g,

$$N_{M'}(\lambda-) \ge |p| \quad (= N_M(\lambda-) + |p| - 1),$$

where $\lambda = \lambda_1(M, g)$ and $|p|$ denotes the degree (number of sheets) of p.

Proof. By [18, Main Theorem] and [12, Theorem 1.1], M carries a Riemannian metric g, which has a topological ball U as a nodal domain. (Note that the proof in [18] also works in the non-orientable case.) Since balls are simply connected, $p^{-1}(U)$ has $|p|$ disjoint lifts. Now the claim follows from Theorem A. ■

Corollary C. The orientable closed surface S of genus two carries a hyperbolic metric such that any non-trivial Riemannian covering $p\colon S' \to S$, that is not generated by one element, is strictly λ_1-unstable.

Here we say that a covering $p\colon M' \to M$ of connected manifolds is *generated by k elements* if, for one (or any) point $x \in M$, there are k loops in M at x such that any two points in $p^{-1}(x)$ can be connected by lifts to M' of concatenations of these loops and their inverses; see also Section 3.1. This property is independent of the choice of x.

Proof of Corollary C. By [24], there exists a hyperbolic metric on S which has a $\lambda_1(S)$-eigenfunction φ such that one of its nodal domains U is either a disc or an annulus. In the first case, the preimage of U is disconnected with $|p|$ components and the assertion is a consequence of Theorem A. In the second case, if the preimage of U is connected and $x \in U$, then any two points of $p^{-1}(x)$ can be connected by a lift of an iterate of any loop in U at x which generates $\pi_1(U, x)$; see also Lemma 3.1. This shows the assertion in the second case. ■

Theorem D. Suppose that M is complete and connected with $\lambda_1(M) < \lambda_{\mathrm{ess}}(M)$ and carries a $\lambda_1(M)$-eigenfunction φ such that its nodal set is not connected.

(1) Then there is a two-sheeted Riemannian covering of M which is strictly λ_1-unstable.
(2) If M is orientable, then M carries an n-sheeted cyclic Riemannian covering which is strictly λ_1-unstable, for any $n \ge 2$.

Theorem D is a special case of Theorem 4.6.

Let S be a complete and connected Riemannain surface. (To avoid misunderstandings: a Riemannian surface is a surface with a Riemannian metric.) Recall that S is said to be of *finite type* if it is diffeomorphic to the interior of a compact surface with (possibly empty) boundary.

For a domain U in S and a point $x \in U$, we identify Γ with $\pi_1(S, x)$ and denote the *image of* $\pi_1(U, x)$ *in* Γ by Γ_U. Corresponding assertions will be independent of the choice of x.

Theorem E. Assume that S is of finite type with $\chi(S) < 0$, and let φ be a λ-eigenfunction, where $\lambda_0(S) < \lambda < \lambda_{\mathrm{ess}}(S)$. Then φ has $\nu \geq 2$ nodal domains and at least one, U, such that $\chi(S)/\nu \leq \chi(U) \leq 1$. For any such U, Γ admits a surjective homomorphism I to $\mathbb{Z}_2^\mu$, respectively $\mathbb{Z}^\mu$ if S is orientable, where $\mu \geq -(\nu - 1)\chi(S)/\nu$, such that $\Gamma_U \subseteq \ker I$. In particular, if $\ker I \subseteq \Gamma' \subseteq \Gamma$ is a finite index subgroup, then the corresponding Riemannian covering $p\colon S' \to S$ is strictly λ-unstable. More precisely,

$$N_{S'}(\lambda-) \geq N_S(\lambda-) + |p| - 1,$$

where the number of sheets of the covering $|p| = |\Gamma'\backslash\Gamma|$.

Theorem E is proved in Section 5. It applies, for example, to non-compact hyperbolic surfaces S of finite area with $0 < \lambda < 1/4$ since, for them, $\lambda_{\mathrm{ess}}(S) = 1/4$. The number μ is determined in the proof of Theorem E.

Remark 1.11 (Weyl's law). Let $p\colon M' \to M$ be a non-trivial finite Riemannian covering of closed Riemannian manifolds. Then, by Weyl's law,

$$\lim_{\lambda\to\infty} \frac{N_M(\lambda)}{\lambda^{m/2}} = C_m \operatorname{Vol} M \quad \text{and} \quad \lim_{\lambda\to\infty} \frac{N_{M'}(\lambda)}{\lambda^{m/2}} = C_m \operatorname{Vol} M',$$

where C_m equals the volume of the ball of radius $1/2\pi$ in $\mathbb{R}^m$. Since $\operatorname{Vol} M' = |p| \operatorname{Vol} M$ and $|p| \geq 2$, we get that $N_{M'}(\lambda) > N_M(\lambda)$ for all sufficiently large λ. Therefore, stability of p can only hold in a bounded range of λ; see Corollary G for a stronger result in this direction for hyperbolic manifolds.

Theorem F. Given $\lambda_0(\tilde{M}) < \lambda < \lambda_{\mathrm{ess}}(M)$ and $\ell \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that any finite covering $p\colon M' \to M$ with $|p| \geq n$ satisfies $\lambda_\ell(M') < \lambda$.

Theorem F is a special case of the somewhat more general Theorem 6.2. It was suggested to us by a referee and improves our previous version. Theorem F generalizes work of Sunada [30, Proposition 6] and Brooks [8, Theorem 2] on so-called towers of coverings.

The assumption $\lambda_0(\tilde{M}) < \lambda_{\mathrm{ess}}(M)$ is satisfied if M is compact since the essential spectrum of closed Riemannian manifolds is empty. On the other hand, the assumption $\lambda < \lambda_{\mathrm{ess}}(M)$ is inessential and just serves as a reminder that counting beyond $\lambda_{\mathrm{ess}}(M)$ is meaningless in general.

Let $H = G/K$ be a Riemannian symmetric space of non-compact type, where K is the stabilizer of a point $x_0 \in H$. Let $G = KAN$ be an Iwasawa decomposition of G. Then $S = AN$ is a solvable Lie subgroup of G which acts simply transitively on H. Then Ax_0 is a maximal totally geodesic flat subspace of H through x_0, and Nx_0 intersects Ax_0 perpendicularly. Let h be the mean curvature of the orbit Nx_0 as a submanifold of H. Then $\lambda_0(H) = h^2/4$; see, for example, [5, Theorem 5.2].

Example 1.12. Let $H = H^k_{\mathbb{F}}$ be a hyperbolic space of dimension $m = kd$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and $d = \dim_{\mathbb{R}} \mathbb{F}$, endowed with its standard metric of maximal sectional curvature -1. Then Ax_0 is a geodesic through x_0, and Nx_0 is the horosphere through x_0 perpendicular to Ax_0. The mean curvature of horospheres in H is equal to $h = m + d - 2$.

Corollary G. Let M be a closed quotient of $H = G/K$, $\varepsilon > 0$, and $\ell \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that $\lambda_\ell(M') < h^2/4 + \varepsilon$ for any finite covering $p\colon M' \to M$ of closed quotients of H with $|p| \geq n$.

In the case of hyperbolic surfaces, $h = 1$, and Corollary G shows that beyond 1/4, spectral stability in the sense of Magee et al. cannot continue to hold. Notice also that, for surfaces, Corollary G is analogous to Buser's [9, Theorem 8.1.2], where, instead of passing to coverings, the hyperbolic metric is chosen appropriately.

An inspection of the proof of Lemma 6.1 shows that the n in Corollary G can be chosen to depend only on H, λ, and a lower bound for the injectivity radius of M.

Since $\lambda_0(\tilde{M}) = \lambda_0(M)$ if the fundamental group of M is amenable (see [7, Theorem 1] or [4, Theorem 1.2]) Theorem F has the following consequence.

Corollary H. Suppose that the fundamental group of M is amenable and that $\lambda_0(M) < \lambda_{\mathrm{ess}}(M)$. Then given any $\varepsilon > 0$ and $\ell \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $\lambda_\ell(M') < \lambda_0(M) + \varepsilon$ for any finite covering $p\colon M' \to M$ with $|p| \geq n$.

2 Setup and Preliminaries

Let M be a complete and connected Riemannian manifold of dimension m. Let Δ denote the Laplace–Beltrami operator, acting on the space of smooth functions $C^\infty(M)$ on M. Recall that Δ is *essentially self-adjoint* on $C_c^\infty(M) \subseteq L^2(M)$. Its closure will also be denoted by Δ. The spectrum of the closure, depending on the context denoted by

$$\sigma(M, \Delta) = \sigma(\Delta) = \sigma(M),$$

can be decomposed into two sets,

$$\sigma(M) = \sigma_d(M) \sqcup \sigma_{\mathrm{ess}}(M),$$

the *discrete spectrum* and the *essential spectrum*. Recall that $\sigma_d(M)$ consists of isolated eigenvalues of Δ of finite multiplicity and that $\sigma_{\mathrm{ess}}(M)$ consists of those $\lambda \in \mathbb{R}$ for which $\Delta - \lambda$ is not a Fredholm operator. By elliptic regularity theory, $\sigma(M) = \sigma_d(M)$ if M is compact. By the above characterization of the discrete spectrum, $\sigma(M) = \sigma_{\mathrm{ess}}(M)$ if M is homogeneous and non-compact.

Denote by $\lambda_0(M) \leq \lambda_{\mathrm{ess}}(M)$ the bottom of $\sigma(M)$ and $\sigma_{\mathrm{ess}}(M)$, respectively. If M is compact, then $\lambda_0(M) = 0 < \lambda_{\mathrm{ess}}(M) = \infty$. Furthermore, 0 is an eigenvalue of Δ of multiplicity one with constant functions as eigenfunctions. In general, $\lambda_0(M)$ may be positive and may belong to $\sigma_d(M)$ or we may have $\lambda_0(M) = \lambda_{\mathrm{ess}}(M)$. We shall be interested in the case where

$$\lambda_0(M) < \lambda_{\mathrm{ess}}(M). \tag{2.1}$$

Then $\lambda_0(M)$ is an eigenvalue of Δ of multiplicity one with eigenfunctions which do not change sign. Moreover, by the above, we have

$$\sigma(M) \cap [0, \lambda_{\mathrm{ess}}(M)) \subseteq \sigma_d(M). \tag{2.2}$$

We enumerate the eigenvalues of Δ in $[0, \lambda_{\mathrm{ess}}(M))$ by size,

$$\lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \cdots < \lambda_{\mathrm{ess}}(M), \tag{2.3}$$

where repetitions account for multiplicities. In general, the number of eigenvalues of Δ below $\lambda_{\mathrm{ess}}(M)$ might be infinite, even if $\lambda_{\mathrm{ess}}(M) < \infty$. On the other hand, we have the *variational characterization* of $\lambda_k(M) < \lambda_{\mathrm{ess}}(M)$ by

$$\lambda_k(M) = \inf_F \max_{0 \neq \varphi \in F} \operatorname{Ray} \varphi, \tag{2.4}$$

where F runs over all subspaces of $H^1(M)$ of dimension $k+1$ and $\operatorname{Ray} \varphi = \int |\nabla \varphi|^2 / \int \varphi^2$ denotes the *Rayleigh quotient* of $\varphi \neq 0$. The infimum is achieved by the linear span of the $\lambda_j(M)$-eigenfunctions, where $0 \leq j \leq k$.

2.1 Spectrum under finite Riemannian coverings

Unless otherwise specified, Riemannian manifolds are assumed to be complete and connected. Similarly, $p\colon M' \to M$ will denote a finite Riemannian covering of Riemannian manifolds with $|p|$ sheets and group Γ of covering transformations. Recall that Γ is transitive on the fibers of p if and only if p is normal. Since p is finite, we have

$$\lambda_0(M') = \lambda_0(M) \quad \text{and} \quad \lambda_{\text{ess}}(M') = \lambda_{\text{ess}}(M). \tag{2.5}$$

To show the first equality, recall that λ_0 is the supremum of the *positive spectrum*, that pull back and averaging are inverses to each other on the respective spaces of positive functions on M and M', and that both, pull back and averaging, are compatible with the Laplacian; see [29, Theorem 2.1]. As for the second equality, recall the well known fact that λ_{ess} is the $\limsup$ of λ_0 on the family of neighborhoods of infinity of the corresponding manifold and that the first equality does not require completeness; see, for example, [6, Proposition 4.8].

As indicated in (2.1), we assume throughout that

$$\lambda_0(M) < \lambda_{\text{ess}}(M). \tag{2.6}$$

Then $\lambda_0(M)$ is an eigenvalue of Δ on M' and M of multiplicity one with unique positive eigenfunctions φ_0' and φ_0 of respective L^2-norms equal to one.

For any functions φ' on M' and φ on M, let $p_*\varphi'$ be the function on M such that $(p_*\varphi')(x)$ is the average of the $\varphi'(y)$, $y \in p^{-1}(x)$, and $p^*\varphi = \varphi \circ p$ be the pull-back of φ to M'. Say that φ' is p-invariant if φ' is constant along the fibers of p. Obviously, this holds if and only if there is a function φ on M such that $\varphi' = p^*\varphi$ or, equivalently, if and only if $\varphi' = p^*p_*\varphi'$. Clearly, p_* and p^* preserve all the standard regularity and integrability conditions. For $\varphi' \in L^2(M')$ and $\varphi \in L^2(M)$, we have

$$\langle \varphi', p^*\varphi \rangle_{M'} = |p| \langle p_*\varphi', \varphi \rangle_M, \tag{2.7}$$

where the indices M' and M indicate the scalar products in $L^2(M')$ and $L^2(M)$, respectively. Furthermore,

$$L^2(M') = \mathbf{im}\, p^* \oplus \mathbf{ker}\, p_*. \tag{2.8}$$

For any $\lambda \geq 0$, let E_λ' and E_λ be the λ-eigenspaces of Δ on M' and M, respectively. Since lifts of λ-eigenfunctions on M are λ-eigenfunctions on M', p^*E_λ is equal to the space of p-invariant functions in E_λ'.

Proposition 2.9. For any $\lambda \geq 0$,

(1) $E_\lambda' = p^*E_\lambda \oplus \mathbf{ker}(p_*|E_\lambda')$ is an L^2-orthogonal splitting;
(2) $\sqrt{|p|}p_* \colon p^*E_\lambda \to E_\lambda$ is an orthogonal isomorphism with inverse $p^*/\sqrt{|p|}$.

Note here that λ-eigenspaces of Δ are closed subspaces of $L^2(M)$ and $L^2(M')$, even if λ belongs to the essential spectrum of M respectively M'.

3 Connectedness Under Coverings

Fix a point $x \in M$. For $x' \in p^{-1}(x)$ and a loop $c \colon [0,1] \to M$ at x, let $c_{x'}$ be the lift of c to M' starting at x'. Then $x'[c] = c_{x'}(1)$ defines a right action of $\Gamma = \pi_1(M,x)$ on the fiber $p^{-1}(x)$ of p over x, where $[c] \in \Gamma$ denotes the homotopy class of c.

Lemma 3.1. Let $U \subseteq M$ be a connected open subset containing x. Then the connected components of $p^{-1}(U)$ are in canonical one-to-one correspondence with the orbits of Γ_U on $p^{-1}(x)$, where Γ_U denotes the image of $\pi_1(U,x)$ in Γ.

Proof. Let $c \colon [0,1] \to U$ be a loop at x, $x' \in p^{-1}(x)$, and $c_{x'}$ be the lift of c starting at x'. Then $c_{x'}(1)$ is contained in $p^{-1}(U)$, and hence $x'[c]$ belongs to the component of $p^{-1}(U)$ containing x'. Conversely, if $x'' \in p^{-1}(x)$ belongs to the same component of $p^{-1}(U)$ as x', then there is a path $c' \colon [0,1] \to p^{-1}(U)$ from x' to x''. Then $c' = c_{x'}$ and, therefore, $x'' = c_{x'}(1)$, where $c = p \circ c'$ is a loop at x. ■

Fixing a point $x' \in p^{-1}(x)$, we get a canonical identification $p^{-1}(x) = \Gamma'\backslash\Gamma$, where Γ' denotes the image of $\pi_1(M', x')$ in Γ. With respect to this identification, the right action of Γ on $p^{-1}(x)$ corresponds to the right action of Γ on $\Gamma'\backslash\Gamma$.

Corollary 3.2. After the choice of a point $x' \in p^{-1}(x)$, the connected components of $p^{-1}(U)$ are in canonical one-to-one correspondence with the elements of $\Gamma'\backslash\Gamma/\Gamma_U$, the space of orbits of the right action of Γ_U on $\Gamma'\backslash\Gamma$.

When it comes to the existence of λ_k-unstable coverings, the case $\Gamma_U \subseteq \Gamma'$ is of interest. Our next result corresponds to the lifting property of covering projections.

Lemma 3.3. If $\Gamma_U \subseteq \Gamma'$, then the right action of Γ_U on $\Gamma'\backslash\Gamma$ fixes $\Gamma' e$. Hence, the action has more than one orbit unless $\Gamma' = \Gamma$. If the normal closure $N_c(\Gamma_U)$ of Γ_U in Γ is contained in Γ', then the right action of Γ_U on $\Gamma'\backslash\Gamma$ is trivial. In this case, the action has $|p| = |\Gamma'\backslash\Gamma|$ orbits.

Proof. If $g \in \Gamma_U$, then $\Gamma' eg = \Gamma' g = \Gamma'$ since $g \in \Gamma'$. Under the second assumption, if $g \in \Gamma_U$ and $h \in \Gamma$, then $hg = g'h$ for some $g' \in N_c(\Gamma_U)$ and hence $\Gamma' hg = \Gamma' g'h = \Gamma' h$ since, by assumption, $g' \in N_c(\Gamma_U) \subseteq \Gamma'$. ■

Example 3.4 (Abelian coverings). For a domain $U \subseteq M$ and a point $x \in U$, consider the Hurewicz homomorphism

$$H_x : \pi_1(M, x) \to \pi_1(M, x)/[\pi_1(M, x), \pi_1(M, x)] = H_1(M)$$

and the projection

$$H_1(M) \to H_1(M)/i_*(H_1(U)) =: A,$$

where H_1 indicates first homology groups with coefficients in $\mathbb{Z}$ and $i : U \to M$ denotes the inclusion. Under their composition, the preimage of $0 \in A$ in $\pi_1(M, x)$ equals Γ_U. Hence, the preimage Γ' of any finite index subgroup A' of A is a normal subgroup of Γ containing Γ_U such that $\Gamma'\backslash\Gamma \cong A'\backslash A$ is a finite Abelian group. A question, among others adressed in Theorem 4.6 and Theorem E, is whether A is trivial.

3.1 Minimal number of generators

Say that $\Gamma'\backslash\Gamma$ is *generated by k elements* if there is a subset G of Γ with $|G| = k$ such that $\Gamma' \cup G$ generates Γ; then the elements of G are also called *generators of $\Gamma'\backslash\Gamma$*. In the case where Γ' is a normal subgroup of Γ, this terminology coincides with the usual one for the group $\Gamma'\backslash\Gamma$.

The *minimal number of generators* of $\Gamma'\backslash\Gamma$ is denoted by $\mu(\Gamma'\backslash\Gamma)$. Note that, for any subgroups $\Gamma'' \subseteq \Gamma' \subseteq \Gamma$, we have $\mu(\Gamma'\backslash\Gamma) \le \mu(\Gamma''\backslash\Gamma)$. In particular, $\mu(\Gamma'\backslash\Gamma) \le \mu(\Gamma)$, the minimal number of generators of Γ.

Lemma 3.5. If $\mu(\Gamma'\backslash\Gamma) = k$ and $\Delta \subseteq \Gamma$ is a subgroup generated by ℓ elements, then the right action of Δ on $\Gamma'\backslash\Gamma$ has at least $k - \ell + 1$ orbits.

Proof. The claim is true for $\ell \ge k$. Suppose now, by induction, that it is true for $\ell + 1 \le k$. Then the right action of the subgroup Δ_g of Γ generated by Δ and any additional element $g \in \Gamma$ has at least $k - \ell$ orbits in $\Gamma'\backslash\Gamma$. If all of these would be orbits of Δ already, for any choice of $g \in \Gamma$, then the Δ-orbits would be invariant under Γ. However, that cannot be because Γ has only one orbit. Hence, there is a choice of a $g \in \Gamma$ which decreases the number of orbits by one. ■

Corollary 3.6. Let $p : M' \to M$ be a finite Riemannian covering of complete and connected Riemannian manifolds. Let φ be a λ_1-eigenfunction of M and U a nodal domain of φ. Then p is λ_1-unstable if $\mu(\Gamma'\backslash\Gamma) > \mu(\Gamma_U)$.

Proof. This follows immediately from Theorem A in connection with Corollary 3.2 and Lemma 3.5. ■

As for the inequality in Corollary 3.6, recall from above that we always have the upper bound $\mu(\Gamma'\backslash\Gamma) \le \mu(\Gamma)$.

Remark 3.7. In general, the calculation of the minimal number of generators of quotients $\Gamma'\backslash\Gamma$ is a difficult problem. However, in the case where Γ' is a normal subgroup of Γ such that $A = \Gamma'\backslash\Gamma$ is a finite Abelian group,

$$A = \mathbb{Z}/k_1 \times \cdots \times \mathbb{Z}/k_n,$$

then $\mu(A)$ equals the maximal number of k_i which share a common divisor.

3.2 Asymptotic estimate of minimal number of generators

Pyber [25] conjectures that almost all finite groups are nilpotent (see [25, p. 218] for the precise assertion). It is therefore interesting to get estimates on the minimal number of generators of finite nilpotent groups.

For a prime p, a finite group G is called a *p-group* if the orders of all elements of G are divisible by p. This holds if and only if the order of G is a power of p; that is, $|G| = p^\alpha$ for some positive integer α. The number of isomorphism classes $f(n)$ of groups of order $n = p^\alpha$ is given by

$$f(n) = n^{(2/27+o(1))\cdot\alpha^2}, \tag{3.8}$$

by Higman [16, Theorem 3.5] and Sims [28, Proposition 1.1]. Because any p-group is nilpotent and hence solvable, the number $f(d,n)$ of groups of order $n = p^\alpha$ with a generating set of at most d elements satisfies

$$f(d,n) \le n^{(d+1)\cdot\alpha}, \tag{3.9}$$

by Mann [23, Theorem 2]. Combining (3.8) and (3.9), we conclude that the proportion of groups of order $n = p^\alpha$ with a generating set of at most d elements tends to zero as α tends to infinity.

For a non-trivial finite group G, a Sylow p-subgroup is a non-trivial subgroup P of G such that $|P|$ is the highest power of p dividing $|G|$. If G is nilpotent, then G is the product of its p_i-Sylow subgroups P_i,

$$G = P_1 \times \cdots \times P_k, \tag{3.10}$$

where the p_i run through the primes dividing $|G|$; see [11, Theorem 3, Chapter 6]. We conclude

Theorem 3.11. Let n_i be a sequence of natural numbers such that the maximal exponent of the prime factors of n_i tends to infinity with i. Then, for any $d \ge 1$, asymptotically almost surely, every nilpotent group of order n_i has at least d generators.

Proof. Let G be a finite nilpotent group. Because each Sylow subgroup of G is normal in G, if G is generated by d elements, then each Sylow subgroup of G is generated by at most d elements. The proof now follows from the discussion of p-groups above. ■

We observe that it is necessary to assume, in Theorem 3.11, that the maximal exponent of the prime factors of n_i tends to infinity with i. This follows from a result of Guralnick [13] and Lucchini [19] that says that a finite group is generated by at most $d+1$ elements if each of its Sylow subgroups is generated by at most d elements. For more discussion on number of generators and possible applications and implications of these see Remark 5.12.

4 A Basic Argument and Applications

Let $\lambda_0(M) \le \lambda < \lambda_{\mathrm{ess}}(M)$, $\varphi \in E_\lambda$, and $(U_i)_{i\in I}$ be the family of pairwise different nodal domains of φ. For each $i \in I$, let $(U_{ij})_{j\in J_i}$ be the family of pairwise different nodal domains of $p^*\varphi$ over U_i. For each $i \in I$ and $j \in J_i$, let φ_{ij} be the function on M' which coincides with $p^*\varphi/k_{ij}$ on U_{ij}, vanishes on all other U_{ik}, and is equal to $p^*\varphi/|p|$ on the rest of M', where k_{ij} is the degree of the covering $p\colon U_{ij} \to U_i$. The set J of pairs ij

with $i \in I$ and $j \in J_i$ labels the set of all nodal domains U_{ij} of $p^*\varphi$, sorted by the nodal domains U_i of φ. For any function ψ on M, we have

$$\int_{M'} \varphi_{ij} p^* \psi = \int_M \varphi \psi. \tag{4.1}$$

Notice the similarity, and difference, between (2.7) and (4.1). The definition of the φ_{ij} is adapted to what is needed in the comparison of N_M and $N_{M'}$. Recall here that $N_M(\lambda)$, $N_{M'}(\lambda)$ and $N_M(\lambda-)$, $N_{M'}(\lambda-)$ denote the number of eigenvalues of M and M' in $[0,\lambda]$ and $[0,\lambda)$, respectively.

Theorem 4.2. For any $\lambda_0(M) \le \lambda < \lambda_{\mathrm{ess}}(M)$, there are at least $|J| - |I|$ linearly independent eigenfunctions on M' with eigenvalues in $(0,\lambda)$, which are perpendicular to $p^*(L^2(M))$. In particular,

$$N_{M'}(\lambda-) \ge N_M(\lambda-) + \sum_{i\in I}(|J_i| - 1) = N_M(\lambda-) + \sum_{i\in I} |J_i| - |I|.$$

Note that $|J_i| \ge 1$ for all i so that the summands in the middle, the *contributions* $|J_i| - 1$ *of the* U_i, are all non-negative.

Proof of Theorem A. The contribution of U to the estimate in Theorem 4.2 is $j-1$. Since the contributions of the other nodal domains are non-negative, the claim follows. ∎

Proof of Theorem 4.2. Let X be the space spanned by the φ_{ij} and $Y = p^*(\varphi^\perp)$. By (4.1), X and Y satisfy the assumptions of Lemma A.2, applied to the quadratic form Q associated to the operator $A = \Delta' - \lambda$. Namely, $Q \le 0$ on X. Furthermore, by (4.1), X and Y are perpendicular in $H = L^2(M')$. Finally, since φ is an eigenfunction of Δ, $PY \subseteq Y$ by Proposition 2.9, where P is the spectral projection of Δ' associated to $[0,\lambda)$. It remains to clarify the dimensions of X and $X \cap H_0$, where here $H_0 = E'_\lambda$.

To determine $\dim X$, suppose that $\sum_{i\in I, j\in J_i} \alpha_{ij}\varphi_{ij} = 0$. Let $i \in I$ and $j \in J_i$. Then on U_{ij}, the φ_{ik}, for $k \ne j$, vanish. Hence, on U_{ij},

$$\alpha_{ij}\varphi_{ij} = -\sum_{\substack{k\in I\setminus\{i\}\\ l\in J_k}} \alpha_{kl}\varphi_{kl}.$$

Now on U_{ij}, φ_{ij} is equal to $p^*\varphi/k_{ij}$ and each φ_{kl} on the right to $p^*\varphi/|p|$. Since $p^*\varphi \ne 0$ on U_{ij}, we get

$$\frac{\alpha_{ij}}{k_{ij}} = -\sum_{\substack{k\in I\setminus\{i\}\\ l\in J_k}} \frac{\alpha_{kl}}{|p|} =: \alpha_i.$$

Hence, on $p^{-1}(U_i)$,

$$\sum_{j\in J_i} \alpha_{ij}\varphi_{ij} = \alpha_i p^*\varphi.$$

We also get that $\alpha_{ij} = \alpha_i k_{ij}$. Since $\sum_{j\in J_i} k_{ij} = |p|$ for all $i \in I$, we infer that $\sum_{j\in J_i} \alpha_{ij} = \alpha_i |p|$. Hence, the above displayed equality also holds on the rest of M'. Therefore, the α_i satisfy the linear equation

$$\sum_{i\in I} \alpha_i = 0,$$

which has $|I| - 1$ independent solutions. In conclusion,

$$\dim X = \sum_{i\in I} |J_i| - |I| + 1.$$

To determine $\mathbf{dim}\, X \cap E'_\lambda$, note first that $X \cap E'_\lambda$ contains $p^*\varphi$. Conversely, any linear combination of the φ_{ij} is a multiple of $p^*\varphi$ on any of the nodal domains U_{kl} of $p^*\varphi$. Hence, by the unique continuation property, any smooth function in X is a multiple of $p^*\varphi$. In particular, $X \cap E'_\lambda$ consists of multiples of $p^*\varphi$. Therefore, $X \cap E'_\lambda$ has dimension one. ■

Corollary 4.3. If k of the nodal domains of φ are simply connected, then

$$N_{M'}(\lambda-) \geq N_M(\lambda-) + k(|p| - 1).$$

Proof. The preimage under p of any of the simply connected nodal domains has $|p|$ components. Hence, their contribution to the estimate in Theorem 4.2 is $k(|p| - 1)$. Since the contributions of the other nodal domains are non-negative, the claim follows. ■

In Corollary 4.4, for any nodal domain U of φ, we let $\Gamma' = p_*\pi_1(M, x')$, for any given $x \in U$ and $x' \in p^{-1}(x)$ (and Γ_U as usual). The assertions are independent of the choice of x and x'. Generalizing Corollary 4.3, we have

Corollary 4.4. If k of the nodal domains U of φ satisfy

(1) $\Gamma_U \subseteq \Gamma'$, then $N_{M'}(\lambda-) \geq N_M(\lambda-) + k$;
(2) $N_c(\Gamma_U) \subseteq \Gamma'$, then $N_{M'}(\lambda-) \geq N_M(\lambda-) + k(|p| - 1)$.

Proof. For any U as in the first assertion, $p^{-1}(U)$ has at least two, in the second $|p|$ components, by Lemma 3.3. ■

Remark 4.5. Since liftings of eigenfunctions from M to M' are eigenfunctions on M' with the same eigenvalues, an inequality $N_{M'}(\lambda-) \geq N_M(\lambda-)+C$ implies that $N_{M'}(\kappa-) \geq N_M(\kappa-)+C$ and $N_{M'}(\kappa) \geq N_M(\kappa) + C$, for any $\kappa \geq \lambda$.

Theorem 4.6. Suppose that M is complete and connected with $\lambda_1(M) < \lambda_{\mathrm{ess}}(M)$ and carries a $\lambda_1(M)$-eigenfunction φ such that its nodal set $\mathcal{Z}(\varphi)$ has at least $\mu + 1 \geq 2$ components. Let U be one of the two nodal domains of φ. Then there is a surjective homomorphism $I\colon \Gamma \to \mathbb{Z}_2^\mu$ such that $\Gamma_U \subseteq \mathbf{ker}\, I$. If M is orientable, there is a surjective homomorphism $I\colon \Gamma \to \mathbb{Z}^\mu$ such that $\Gamma_U \subseteq \mathbf{ker}\, I$. In both cases, if Γ' is a finite index subgroup of Γ containing $\mathbf{ker}\, I$, then the corresponding Riemannian covering $p\colon M' \to M$ is strictly λ_1-unstable. More precisely, with $|p| = |\Gamma' \backslash \Gamma|$ and $\lambda = \lambda_1(M)$,

$$N_{M'}(\lambda-) \geq N_M(\lambda-) + \mu(|p| - 1).$$

Proof. Suppose first that 0 is a regular value of φ. Then $\{\varphi \leq 0\}$ and $\{\varphi \geq 0\}$ are smooth domains in M with interiors $\{\varphi < 0\}$ and $\{\varphi > 0\}$, and the boundary of each of these domains equals $\{\varphi = 0\} = \mathcal{Z}(\varphi)$, with $\pm\nabla\varphi$ pointing into $\{\pm\varphi > 0\}$ along $\mathcal{Z}(\varphi)$, respectively. Since φ is a $\lambda_1(M)$-eigenfunction, it has exactly two nodal domains, that is, $\{\varphi < 0\}$ and $\{\varphi > 0\}$ are connected, hence path connected. Hence, so are $\{\varphi \leq 0\}$ and $\{\varphi \geq 0\}$. Therefore, between any two points in $\mathcal{Z}(\varphi)$, there exist paths in $\{\varphi \leq 0\}$ and $\{\varphi \geq 0\}$ joining them.

By assumption, $\mathcal{Z}(\varphi)$ has at least $\mu + 1$ connected components, $Z_1, \dots, Z_{\mu+1}$. Let $Z = Z_j$ for some $1 \leq j \leq \mu$, $Z' = Z_{\mu+1}$, and $z \in Z$ and $z' \in Z'$ be points. By what we said above, there exist paths c_- in $\{\varphi \leq 0\}$ and c_+ in $\{\varphi \geq 0\}$ between z and z'. Their union c is a closed loop in M that has intersection number one with Z. In particular, intersection with the different $Z = Z_j$ defines a non-trivial homomorphism I from Γ to $\mathbb{Z}^\mu$ if M is oriented and to $\mathbb{Z}_2^\mu$ otherwise.

Let now U be one of the nodal domains of φ. Then Γ_U is contained in $\mathbf{ker}\, I$. Hence, the normal closure $N_c(\Gamma_U)$ is contained in the normal subgroup $\mathbf{ker}\, I$ of Γ. Since $\mathbf{ker}\, I \subseteq \Gamma'$, Corollary 4.4 applies and shows the claim.

If 0 is not a regular value of φ, the above argument still applies in principle, but we need some preparation to define intersection numbers. To that end, write

$$\begin{aligned}\mathcal{Z}(\varphi) &= \{z \in \mathcal{Z}(\varphi) \mid d\varphi(z) \neq 0\} \cup \{z \in \mathcal{Z}(\varphi) \mid d\varphi(z) = 0\} \\ &= \mathcal{Z}(\varphi)_{\mathrm{reg}} \cup \mathcal{Z}(\varphi)_{\mathrm{sing}},\end{aligned}$$

the *regular* and *singular part* of $\mathcal{Z}(\varphi)$. Let Z and Z' be as above and set

$$Z_{\mathrm{reg}} = Z \cap \mathcal{Z}(\varphi)_{\mathrm{reg}}, \quad Z_{\mathrm{sing}} = Z \cap \mathcal{Z}(\varphi)_{\mathrm{sing}}.$$

Recall from the (elementary) proof of [14, Lemma 1.9] that any point in Z_{sing} is contained in an open ball B in M such that $Z_{\mathrm{sing}} \cap B$ is contained in a finite union of embedded submanifolds of dimension $\dim M - 2$. Let now $U = \{\varphi < 0\}$ and $x \in U$ as above.

Claim (1) Any loop in M at x is homotopic to a loop which does not meet Z_{sing} and meets Z_{reg} transversally in at most finitely many points.

To show (1), let c be a loop in M at x. Then $I = c^{-1}(Z_{\mathrm{sing}})$ is compact. Hence, I can be covered by finitely many consecutive intervals such that the image of each of these intervals is contained in a ball B as above. Since the codimension of the corresponding embedded submanifolds as above is two, c can be deformed consecutively to a loop at x which does not meet Z_{sing}. This shows the first assertion. The second follows from standard transversality theory, applied to the smooth hypersurface Z_{reg}.

Claim (2) For any two homotopic loops in M at x, which do not meet Z_{sing} and meet Z_{reg} transversally in at most finitely many points, the (oriented respectively non-oriented) intersection numbers with Z coincide.

To show (2), let c_1 and c_2 be two such homotopic loops. Since they do not meet Z_{sing} and intersect Z_{reg} transversally in at most finitely many points, there is an $\varepsilon > 0$, which is a regular value of φ such that $\{\varphi = \varepsilon\}$ has a component Z_ε such that the intersections of c_1 with Z and Z_ε are in one-to one correspondence to each other, and similarly for c_2. Now the intersection numbers of c_1 and c_2 with Z_ε are well defined and agree with the intersection numbers with Z_{reg}, by what we said. Since c_1 and c_2 are homotopic, their intersection numbers with Z_ε agree, hence also the ones with Z_{reg}.

Now (1) and (2) show that intersection numbers with the different $Z = Z_j$ respectively $Z_{j,\mathrm{reg}}$ define a homomorphism Γ to $\mathbb{Z}^\mu$ in the oriented case and $\mathbb{Z}_2^\mu$ otherwise. The rest of the proof is as in the regular case. ∎

Remark 4.7. Under appropriate assumptions, the conclusion of Theorem D holds with λ_k-unstable in place of λ_1-unstable. More precisely, if there is a $\lambda_k(M)$-eigenfunction φ on M such that there are components of $\mathcal{Z}(\varphi)$ together with loops in M which intersect exactly once, then the above arguments apply and show λ_k-instability. This may indicate that λ_k-instability becomes more likely, the more nodal domains $\lambda_k(M)$-eigenfunctions have.

4.1 Absolute estimate

In the above, we estimated $N_{M'}$ against N_M. That is what is behind the definition of the φ_{ij} in the beginning of the section. An easier approach leads to an estimate of $N_{M'}$ without comparing it with N_M. The point is as follows: Let $(D_i)_{i\in I}$ be a family of domains in M and, for each $i \in I$, $(D_{ij})_{j\in J_i}$ be the connected components of $p^{-1}(D_i)$. Let $\lambda \geq 0$ and φ_i be a smooth function on D_i with compact support and Rayleigh quotient $\leq \lambda$. For each $i \in I$ and $j \in J_i$, let φ_{ij} now be the function which equals $p^*\varphi_i$ on D_{ij} and vanishes outside of D_{ij}. Then the φ_{ij} are pairwise L^2-orthogonal and have Rayleigh quotient $\leq \lambda$. Hence,

$$N_{M'}(\lambda) \geq \sum_{i\in I} |J_i|. \tag{4.8}$$

Clearly, $N_M(\lambda) \geq |I|$, but that does not lead to an inequality as in Theorem 4.2.

In the following discussion, we use (4.8) only in the case of one domain in M, that is, $|I| = 1$; cf. Remark 4.13. Set

$$\sigma(M) = \inf_D \lambda_0(D), \tag{4.9}$$

where the infimum is taken over all simply connected domains $D \subset M$. By monotonicity, $\lambda_0(M) \le \sigma(D)$. Recall also that $\lambda_0(M') = \lambda_0(M)$.

In general, (4.9) poses the optimization problem of the existence of a simply connected domain D in M such that $\sigma(M) = \lambda_0(D)$ and of a Dirichlet eigenfunction φ on D for $\lambda_0(D)$.

Proposition 4.10. If $\sigma(M) < \lambda_{\mathrm{ess}}(M)$, then M' has at least $|p|$ eigenvalues in $[0, \sigma(M)]$.

The assumption $\sigma(M) < \lambda_{\mathrm{ess}}(M)$ is satisfied if M is closed since the essential spectrum of closed Riemannian manifolds is empty. Recall also that $\lambda_{\mathrm{ess}}(M') = \lambda_{\mathrm{ess}}(M)$ since p is finite.

Proof of Proposition 4.10. For any $\sigma(M) < \lambda < \lambda_{\mathrm{ess}}(M)$, let $D \subseteq M$ be a simply connected domain such that there is a $\varphi \in C_c^\infty(M)$ with support in D with Rayleigh quotient $< \lambda$. There are precisely $|p|$ lifts of D to M', and they are pairwise disjoint. For any such lift C, let $\varphi_C \in C_c^\infty(M')$ be the function with support in C such that $\varphi_C = \varphi \circ p$ on C. Then the φ_Cs and their gradients are pairwise L^2-orthogonal and have the same Rayleigh quotient as φ. ∎

For $\ell \ge 1$, let $\sigma_\ell(M) = \inf_D \lambda_0(D)$, where the infimum is taken over all domains $D \subset M$ such that the fundamental group of D is generated by at most ℓ elements. Notice that $\sigma(M) = \sigma_0(M)$.

Proposition 4.11. If the minimal number of generators of $\Gamma' \backslash \Gamma$ is k and $\sigma_\ell(M) < \lambda_{\mathrm{ess}}(M)$ for some $\ell \le k$, then M' has at least $k - \ell + 1$ eigenvalues in $[0, \sigma_\ell(M)]$.

Proof. For any $\sigma_\ell(M) < \lambda < \lambda_{\mathrm{ess}}(M)$, there is a domain D in M with $\sigma_\ell(M) \le \lambda_0(D) < \lambda$ such that the fundamental group of D is generated by at most ℓ elements. Hence, the preimage of D under p has at least $k-\ell+1$ components, by Corollary 3.2 and Lemma 3.5. Therefore, M' has at least $k-\ell+1$ eigenvalues in $[0, \lambda)$. ∎

Remark 4.12. For a Riemannian surface S, $\sigma_1(S)$ coincides with the *analytic systol of* S, introduced in [1]. Recall that S has at most $-\chi(S)$ eigenvalues in $[0, \sigma_1(S)]$, by [2, Theorem 1.5]. Here we get that the covering surface S' has at least two eigenvalues in $[0, \sigma_1(S)]$, provided that the minimal number of generators of $\Gamma' \backslash \Gamma$ is at least two.

Remark 4.13. As in Theorem 4.2, we can also consider families of pairwise disjoint domains to get a more general estimate than the one in Proposition 4.11. Namely, if $(D_i)_{i \in I}$ is a finite family of pairwise disjoint domains in M such that the fundamental group of D_i is generated by at most ℓ_i elements and such that $\lambda = \max \lambda_0(D_i) < \lambda_{\mathrm{ess}}(M)$, then M has at least $\sum_{i \in I}(k - \ell_i + 1)$ eigenvalues in $[0, \lambda)$. The point is that the different lifts of functions $\varphi_i \in C_c^\infty(D_i)$ to the different components of $p^{-1}(D_i)$ are pairwise L^2-perpendicular.

5 Coverings of Surfaces

Let S be a connected surface. Assume that S is of finite type, that is, S is diffeomorphic to the interior of a compact surface $\bar{S}$ with boundary. Equivalently, the Euler characteristic $\chi(S) > -\infty$. The connected components of $\bar{S} \setminus S$ consist of circles, called *holes* or *circles at infinity*.

Suppose that S is endowed with a complete Riemannian metric which has a square-integrable eigenfunction φ_0 at the bottom λ_0 of its spectrum. Recall that the multiplicity of λ_0 is one and that φ_0 does not change sign, hence can be chosen to be positive. If the area $|S| < \infty$, then φ_0 is constant and $\lambda_0 = 0$. Eigenfunctions of S for eigenvalues $\lambda > \lambda_0$ are perpendicular to φ_0 and hence change sign. In particular, they have at least two nodal domains. The structure of the nodal set of such an eigenfunction φ was clarified in [10, Theorem 2.5]:

Theorem 5.1 (Cheng). The nodal set $\mathcal{Z}(\varphi)$ of φ is a locally finite graph in S. Moreover, $z \in \mathcal{Z}(\varphi)$ has valence $2n$ if and only if φ vanishes to order n at z. The opening angles between the edges at z are equal to π/n. Furthermore, $\mathcal{Z}(\varphi)$ is a locally finite union of immersed circles and lines.

We will need the following topological result for the study of nodal domains.

Lemma 5.2. Let S be a connected surface of finite type and $U \subset S$ an open domain with piecewise smooth boundary. Assume that the complement U^c of U in S contains only finitely many components which are discs or annuli. Then U has finite type.

Proof. The proof rests on the fact that a surface (orientable or not) has finite type if and only if any family of simple closed curves, which are not null-homotopic, pairwise disjoint, and pairwise not freely homotopic, is finite; cf. [27,pp. 259–260]. Assuming that U is not of finite type, there is an infinite family $\mathcal{F}$ of simple closed curves in U, which satisfies these conditions with respect to U.

If c is a member of $\mathcal{F}$ and c is null-homotopic in S, then c bounds an embedded disc D in S, $c = \partial D$. Now $\partial U \cap D$ cannot contain components of ∂U which are line segments and hence consists of finitely many simple closed curves, which bound discs in D which belong to U^c. There are only finitely many such discs, by assumption. Hence, the union U' of such discs with U is of finite type if and only if U is. Therefore, we can assume from now on that the complement of U does not contain components which are discs.

Let now c_0 and c_1 be members of $\mathcal{F}$ which are freely homotopic in S. Then there is an embedded annulus A in S such that $c_0 \cup c_1 = \partial A$. Now $\partial U \cap A$ cannot contain components of ∂U which are line segments, nor can it contain closed curves which are homotopic to zero, by assumption. Hence, it consists of two boundary curves $\hat{c}_0$ and $\hat{c}_1$, such that the parts of A between c_0 and $\hat{c}_0$, respectively, c_1 and $\hat{c}_1$ belong to U and the rest, $\hat{A}$, to U^c. Now $\hat{A}$ is an annulus. Hence, there are only finitely many such, by assumption. Since S is of finite type, we arrive at a contradiction to the assumption that $\mathcal{F}$ is infinite. ■

Lemma 5.3. A λ-eigenfunction φ on S with $\lambda < \lambda_{\text{ess}}(S)$ has only finitely many nodal domains.

Proof. Let $\mathcal{N}$ be the family of different nodal domains of φ. For any $U \in \mathcal{N}$, let φ_U be the function which coincides with φ on U and vanishes otherwise. Then the φ_U are pairwise L^2-perpendicular, are in $H^1(S)$, and have Rayleigh quotient $\leq \lambda$. Since $\lambda < \lambda_{\text{ess}}(S)$, the variational characterization of eigenvalues implies that $\mathcal{N}$ is finite. ■

Corollary 5.4. For $\lambda < \lambda_{\text{ess}}(S)$, the nodal domains of λ-eigenfunctions on S are geometrically finite.

Proof. By Theorem 5.1, any nodal domains U of φ is a domain in S with piecewise smooth boundary. (If ∂U has a self-intersection at a critical point x of φ, push U a bit inside, away from x, to get ∂U embedded.) Now Lemma 5.2 and Lemma 5.3 imply the assertion. ■

5.1 On the topology of nodal domains

Assume from now on that $\chi(S) \leq 0$ and let φ be an eigenfunction of S perpendicular to φ_0.

Lemma 5.5. If none of the nodal domains of φ is a disc, then each nodal domain U of φ is a domain of finite type; in particular $\chi(U) > -\infty$. Moreover $\chi(U) < 0$ unless U is a disc or an annulus or a Möbius band.

Proof. Let U be a nodal domain of φ. If a component D of U^c is a closed disc, then a component of the open set $D \setminus \mathcal{Z}(\varphi)$ is a disc, a contradiction to the assumption. Now Lemma 5.2 implies the assertion. ■

Lemma 5.6. If φ is an eigenfunction of S perpendicular to φ_0, then φ has a nodal domain U such that $\chi(S)/2 \leq \chi(U) \leq 1$. More generally, if $\ell \geq 2$ denotes the number of nodal domains of φ, then φ has a nodal domain U such that

$$\chi(S)/\ell \leq \chi(U) \leq 1.$$

Proof. Since $\chi(S) \le 0$, the assertion holds if a nodal domain U is a disc. Assume now that no nodal domain of φ is a disc. Then all nodal domains of φ are domains of finite type, in particular with finite Euler characteristic. By [10, Theorem 2.5], the nodal set Z of φ is a graph with vertices of (even) order at least two. Therefore, $\chi(Z) \le 0$ and hence

$$\sum \chi(U) \ge \chi(Z) + \sum \chi(U) = \chi(S),$$

where the sum is over all nodal domains U of φ. Hence, there is at least one nodal domain U of φ such that $\chi(S)/\ell \le \chi(U) \le 1$. ■

Let Γ be the fundamental group of S. If S is closed, $S = \bar{S}$, the minimal number of generators of Γ is

$$\nu = \nu(\Gamma) = \nu(S) = 2 - \chi(S). \tag{5.7}$$

If S is non-compact, then Γ is a free group and the minimal number of generators of Γ is

$$\nu = \nu(\Gamma) = \nu(S) = 1 - \chi(S). \tag{5.8}$$

Lemma 5.9. Let φ be an eigenfunction of S with $\ell \ge 2$ nodal domains.

(1) If $\ell \ge \nu(S)$, at least one of the nodal domains is a disc or an annulus or a Möbius band.
(2) If $\ell < \nu(S)$, then φ has a nodal domain with minimal number of generators of its fundamental group at most $1 - \chi(S)/\ell$.

Proof. If all nodal domains of φ have negative Euler characteristic, then $\ell \le -\chi(S)$. The first claim now follows from (5.7) and (5.8). As for the second, we may assume that all nodal domains of φ have negative Euler characteristic. Hence, by Lemma 5.6, φ has a nodal domain U with $-\chi(S)/\ell \ge -\chi(U)$. But then $\nu(U) = 1 - \chi(U) \le 1 - \chi(S)/\ell$ by (5.8). ■

Corollary 5.10. There is a nodal domain U of φ such that the fundamental group of U admits a system with at most $\nu/2$ generators if S is closed and $(\nu + 1)/2$ generators otherwise, where $\nu = \nu(S)$.

Consider now a finite Riemannian covering $p\colon S' \to S$ of complete and connected Riemannian surfaces of finite type. Write $S = \Gamma\backslash\tilde{S}$ and $S' = \Gamma'\backslash\tilde{S}$, where the fundamental groups $\Gamma \supseteq \Gamma'$ of S and S' are viewed as groups of covering transformations of the universal covering surface $\tilde{S}$ of S and S'.

Proposition 5.11. If p is λ_1-stable, where $\lambda_1(S) < \lambda_{\mathrm{ess}}(S)$, then the minimal number of generators of $\Gamma'\backslash\Gamma$ is at most $\nu/2$ if S is closed and $(\nu + 1)/2$ otherwise, where $\nu = \nu(S)$.

Proof. Let φ be a $\lambda_1(S)$-eigenfunction. By Corollary 5.10, φ has a nodal domain U such that the fundamental group of U admits a system with at most $\nu/2$ respectively $(\nu + 1)/2$ generators. If the minimal number of generators for $\Gamma'\backslash\Gamma$ is strictly bigger than $\nu/2$ respectively $(\nu + 1)/2$, then the right action of $\pi_1(U)$ on $\Gamma'\backslash\Gamma$ cannot be transitive. By Corollary 3.2 together with Theorem 4.2, we get a contradiction to λ_1-stability. ■

Remark 5.12. Let n_i be a sequence of natural numbers as in Theorem 3.11. As pointed out by one of the referees, it is natural to suspect, from Theorem 3.11, that it is unlikely that, asymptotically almost surely, n_i-sheeted normal coverings of S, with nilpotent covering transformation group, are λ_1-stable. With Pyber's conjecture [25,p. 218] we may further remove the "nilpotent" assumption.

We would like to emphasise that Theorem 3.11 is not sufficient to conclude the above. The reason is clear: we need an estimate on the number of groups that arise as a quotient of the fundamental group of S that have minimal number of generators strictly more than $\nu/2$ if S is closed and strictly more than $(\nu+1)/2$ otherwise. Because we are interested in groups that arise as a quotient of the fundamental group of S, we know for a fact that the minimal number of generators of any such group is at most ν.

5.2 Intersection numbers and coverings

Let U be a nodal domain of an eigenfunction φ of S perpendicular to φ_0. Then the complement U^c of U is a surface of finite type, and we let V be a component of U^c. Then V has $k \geq 1$ boundary circles in S, that we call *doors* $d_1, \dots, d_k$, through which it is connected to U and $\ell \geq 0$ boundary circles in $\bar{S} \setminus S$, that we call *exits* $e_1, \dots, e_\ell$ *at infinity*. We draw the first kind in green, meaning that we may enter U through them, and the latter kind in red, indicating that we exit S through them eventually.

We now view V as a regular plane polyhedron P in the orientable case, respectively Q in the non-orientable case, with the colored holes in its interior and with the standard identifications of its edges, indicated by the labellings

$$\begin{aligned} &P_0 : aa^{-1}, && \text{where } g = 0, \\ &P_g : a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}, && \text{where } g \geq 1, \\ &Q_g : a_1 a_1 \cdots a_g a_g, && \text{where } g \geq 1. \end{aligned}$$

In terms of (orientable respectively non-orientable) *genus* g and numbers k and ℓ of holes, the negative of the Euler characteristic of V is

$$-\chi(V) = 2g + k + \ell - 2 \quad \text{and} \quad -\chi(V) = g + k + l - 2 \tag{5.13}$$

in the orientable (P_g) and non-orientable (Q_g) case, respectively.

If $\ell \geq 2$, we draw disjoint segments $h_2, \dots, h_\ell$ from e_1 to the other red circles $e_2, \dots, e_\ell$. We get a homomorphism

$$I = I_V : H_1(S) \to \mathbb{Z}^\mu \quad \text{respectively} \quad I = I_V : H_1(S) \to \mathbb{Z}_2^\mu, \tag{5.14}$$

the *intersection homomorphism*, where

$$\begin{aligned} \mu &= \mu(V) = 2g + k - 1 + \max\{\ell - 1, 0\} && \text{for } P_g, \\ \mu &= \mu(V) = g + k - 1 + \max\{\ell - 1, 0\} && \text{for } Q_g, \end{aligned} \tag{5.15}$$

by taking,

(1) in the oriented case, oriented intersection numbers with the circles in V coming from the edges of P labeled $a_1, b_1, \dots, a_g, b_g$, the boundary circles $d_2 \dots, d_k$, and the segments $f_2 \dots, f_\ell$;
(2) in the non-orientable case, intersection numbers modulo two with the circles in V coming from the edges of Q labeled $a_1, \dots, a_g$, the boundary circles $d_2 \dots, d_k$, and the segments $f_2 \dots, f_\ell$.

From (5.13) and (5.15), we conclude

Lemma 5.16. Irrespective of whether V is orientable or not, we have

$$\mu(V) = -\chi(V) + \begin{cases} 1 & \text{if } \ell = 0 \\ 0 & \text{if } \ell \geq 1 \end{cases} \geq -\chi(V).$$

Remark 5.17. We have $\mu = \mu(V) \geq 1$ unless (g, k, ℓ) equals $(0, 1, 0)$ or $(0, 1, 1)$, and then V is a disc with one green or an annulus with one green and one red boundary circle, respectively. There are also only a few cases with $\mu = 1$. Using the above (g, k, l) notation, if V is orientable, then it is one of the following:

(0, 2, 0) an annulus, boundary circles green; $\chi(V) = 0$;
(0, 2, 1) a pair of pants, two boundary circles green, one red; and
(0, 1, 2) a pair of pants, one boundary circle green, two red.

If V is non-orientable, it is one of the following:

(1, 1, 0) a *Möbius band*, boundary circle green; $\chi(V) = 0$.
(1, 1, 1) a projective plane with two holes, boundary circles green and red.

Lemma 5.18. Under the Hurewicz homomorphism (see Example 3.4) from Γ to $H_1(S)$ respectively $H_1(S; \mathbb{Z}_2)$, Γ_U is contained in the kernel $\ker I_V$.

Proof. By definition, homotopy classes of loops in Γ_U have representatives which are contained in U and hence have empty intersection with curves in $V \subseteq U^c$. ∎

Theorem 5.19. Let $p\colon S' \to S$ be a finite Riemannian covering of complete and connected Riemannian surfaces and φ be a λ-eigenfunction of M, where $\lambda_0(S) < \lambda < \lambda_{\mathrm{ess}}(S)$. Let k be the number of nodal domains U of φ such that, for some component V of U^c, $\ker I_V \subseteq \Gamma'$, where Γ' is the image of $\pi_1(S', x')$ in $\pi_1(S, x)$ under $p_\#$ for some $x' \in p^{-1}(x)$ (see Section 3). Then,

$$N_{S'}(\lambda-) \geq N_S(\lambda-) + k(|p| - 1).$$

Proof. By Lemma 5.18 and since $\ker I_V$ is a normal subgroup of Γ, $N_\Gamma(\Gamma_U)$ is contained in $\ker I_V$, for any nodal domain U and any component V of its complement as in the assertion. Hence, $p^{-1}(U)$ has $|p|$ components for any such U, by Lemma 3.3. Now Theorem 4.2 implies the assertion. ∎

Proof of Theorem E. By Lemma 5.6, φ has a nodal domain U such that $\chi(U) \geq \chi(S)/\nu$. Then, by Theorem 5.1,

$$\chi(U^c) = \chi(S) - \chi(U) \leq (\nu - 1)\chi(S)/\nu < 0.$$

Therefore, the components V_j of U^c with $\chi(V_j) < 0$ satisfy

$$\sum_j \chi(V_j) \leq (\nu - 1)\chi(S)/\nu. \tag{5.20}$$

For each i, we have

$$\mu(V_j) \geq -\chi(V_j),$$

by Lemma 5.16. Therefore, if μ equals the sum of the $\mu(V_j)$, then the sum

$$I = \oplus_j I_{V_j}\colon \oplus_j H_1(S) \to \oplus_j \operatorname{\mathbf{im}} I_{V_j}$$

is a homomorphism to $\mathbb{Z}^\mu$, respectively, $\mathbb{Z}_2^\mu$ as asserted, except that we compose it with the corresponding Hurewicz homomorphism to have it defined on Γ. ∎

5.3 Estimating the number of unstable coverings

Fix a base point $x \in S$, and consider finite Riemannian coverings $p\colon (S', x') \to (S, x)$ of pointed complete and connected Riemannian surfaces. Note that the isomorphism classes of such pointed coverings with n sheets correspond one-to-one with index n subgroups of $\Gamma = \pi_1(S, x)$. Denote by $a(n)$ the number of isomorphism classes of all such n-sheeted coverings and by $u(n)$ the number of isomorphism classes of λ_1-unstable ones among them.

Corollary 5.21. If S is of finite type with $\chi(S) < 0$ and $\lambda_1(S) < \lambda_{\mathrm{ess}}(S)$, then

$$u(2n) \geq \frac{a(n)}{2n-1}.$$

Proof. We want to estimate $a(n)$ against $u(2n)$. Now for any n-sheeted pointed covering $p\colon (S', x') \to (S, x)$ as above, Theorem E applies to S' in place of S and shows, that there is a λ_1-unstable twofold Riemannian covering

$$p'\colon (S'', x'') \to (S', x').$$

Then the composition $q = p \circ p'$ is $2n$-sheeted, and q is λ_1-unstable, whether p is λ_1-unstable or not. Now the number of isomorphism classes of such p versus the given q is estimated by an upper bound on the number of index n subgroups Γ' of Γ containing the given Γ'' of index $2n$. Since the index of Γ'' in Γ' is two, Γ' is generated by Γ'' together with an element $g \in \Gamma \setminus \Gamma''$. Furthermore, since the index of Γ'' in Γ is $2n$, there are at most $2n - 1$ such g modulo Γ'' (such that the subgroup Γ' generated by g and Γ'' has index n in Γ). ■

Remark 5.22. Let $\Gamma = \pi_1(S, x)$ and $\tilde{S}$ be the universal covering surface of S, endowed with the lifted metric. Let ρ be a homomorphism from Γ to the symmetric group S_n. Associated to ρ, Magee et al. consider the orbit space S' of the product action of Γ on $\tilde{S} \times \{1, \dots, n\}$, see [22,paragraph following (1.1)]. The natural projection $p\colon S' \to S$ is an n-sheeted covering, and the construction also yields a labeling of $p^{-1}(x)$ by $\{1, \dots, n\}$, a *labeled covering*. Notice that S' is connected if and only if the action of $\rho(\Gamma)$ on $\{1, \dots, n\}$ is transitive.

Clearly, isomorphism classes of labeled coverings of (S, x) are characterized by the representations ρ. Disregarding the labeling, but fixing a base point $x' \in p^{-1}(x)$ corresponds to dividing the labeling by S_{n-1}. Thus the number of all isomorphism classes of n-sheeted connected and labeled coverings equals $(n-1)!\,a(n)$ with $a(n)$ as above. Since λ_1-stability is independent of the choice of base points and labelings, we conclude that the analog of the inequality of Corollary 5.21 holds for isomorphism classes of connected labeled coverings as well.

Magee et al. point out that, as $n \to \infty$, the number of non-connected n-sheeted labeled coverings vanishes asymptotically in proportion to the connected ones.

6 High Coverings

In this section, we let $\hat{p}\colon \hat{M} \to M$ be an infinite normal covering of complete and connected Riemannian manifolds. The main example is the universal covering whenever the fundamental group of M is infinite. Another example is the homology covering, $\pi_1(\hat{M}) = [\pi_1(M), \pi_1(M)]$, whenever the first Betti number of M is positive; that is, the homology group $H_1(M) = \pi_1(M)/[\pi_1(M), \pi_1(M)]$ of M has positive rank.

Let $\hat{x} \in \hat{M}$ and $x = \hat{p}(\hat{x})$. We identify the group Γ of covering transformations of $\hat{p}$ with the quotient of $\pi_1(M, x)$ by the normal subgroup $\hat{p}_\#(\pi_1(\hat{M}, \hat{x}))$.

We are interested in the spectrum of *finite intermediate coverings*, that is, finite Riemannian coverings $p\colon M' \to M$ such that $\hat{p} = p \circ q$, where $q\colon \hat{M} \to M'$ is a Riemannian covering (of complete and connected Riemannian manifolds).

Let $p\colon M' \to M$ be a finite intermediate covering as above. For $r > 0$, we denote by $N_p(r)$ the maximal possible number of disjoint geodesic balls of radius r in M' with centers in $p^{-1}(x)$ and by $N_\Gamma(r)$ the number of classes in Γ (identified as above) which contain loops in M at x of length $< 2r$. Observe that $N_\Gamma(r) \geq 1$ for any r with equality for $r \leq$ the injectivity radius of M.

Lemma 6.1. In the above setup, we have $|p| \leq N_p(r) N_\Gamma(r)$.

Proof. If two geodesic balls of radius r with centers in $p^{-1}(x)$ intersect, then there is a geodesic of length $< 2r$ between their centers. The image of such a geodesic in M is a geodesic loop at x of length $< 2r$. Conversely, any geodesic loop at x of length $< 2r$ lifts to a geodesic of length $< 2r$ joining points in $p^{-1}(x)$, implying that the geodesic balls of radius r about the corresponding endpoints intersect. Moreover, if two such loops belong to the same class in Γ, then their lifts to M' join the same pairs of points in $p^{-1}(x)$ (even their lifts to $\hat{M}$ do).

Let x_i, $1 \le i \le N_p(r)$, be a maximal family of points in $p^{-1}(x)$ such that the geodesic balls B_i of radius r around them are disjoint. Let $y \in p^{-1}(x)$ be such that $y \neq x_i$ for any $1 \le i \le N_p(r)$. Then, the geodesic ball of radius r with center at y will intersect one of the B_i. By what we said above, each B_i intersects at most $N_\Gamma(r)$ of these balls. Hence, the total number of balls (which is the number of points in $p^{-1}(x)$, i.e., $|p|$) is at most $N_p(r)N_\Gamma(r)$. ■

Theorem 6.2. Given $\lambda_0(\hat{M}) < \lambda < \lambda_{\mathrm{ess}}(M)$ and $\ell \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that any finite intermediate covering $p\colon M' \to M$ with $|p| \ge n$ satisfies $\lambda_\ell(M') < \lambda$.

Proof. There is a smooth function φ on $\hat{M}$ with support contained in some geodesic ball $B(\hat{x}, r) \subset \hat{M}$ of radius r and with Rayleigh quotient $\mathbf{Ray}\,\varphi < \lambda$. Let $x = \hat{p}(\hat{x})$ and consider a maximal number of disjoint geodesic balls of radius r in M' with centers $x_1, \dots, x_l$ in $p^{-1}(x)$. Let $\hat{x}_1, \dots, \hat{x}_l$ be lifts of them to $\hat{M}$. Since $\hat{p}$ is normal, there exist $g_i \in \Gamma$ such that $\hat{x}_i = g_i\hat{x}$, for all $1 \le i \le l$. The pushdowns ψ_i of the $\varphi_i = \varphi \circ g_i$ to M' in the sense of [4, Section 4] have support in the geodesic balls of radius r about the x_i, and their Rayleigh quotients are $< \lambda$. Since these geodesic balls are disjoint, the ψ_i and their gradients are pairwise L^2-orthogonal. Hence, $\lambda_l(M_k) < \lambda$, by the variational characterization of eigenvalues. Now $l \ge n = |p|/N_\Gamma(r)$ by Lemma 6.1, and hence the assertion follows. ■

If Γ is amenable, then $\lambda_0(\hat{M}) = \lambda_0(M)$; see [7, Theorem 1] or [4, Theorem 1.2]. Hence, Theorem 6.2 has the following consequence (extending Corollary H).

Corollary 6.3. Suppose that Γ is amenable and that $\lambda_0(M) < \lambda_{\mathrm{ess}}(M)$. Then given any $\varepsilon > 0$ and $\ell \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that any finite intermediate covering $p\colon M' \to M$ with $|p| \ge n$ satisfies $\lambda_\ell(M') < \lambda_0(M) + \varepsilon$.

If $\hat{p}$ is the homology covering, then $\Gamma \cong H_1(M)$ is Abelian, and hence Corollary 6.3 applies to intermediate coverings of $\hat{p}$ if the first Betti number of M is positive.

Acknowledgments

We are grateful to the Max Planck Institute for Mathematics and the Hausdorff Center for Mathematics in Bonn for their support and hospitality. We would like to thank the referees for very helpful suggestions. The second named author would also like to thank C.S. Rajan and C. Drutu for helpful discussions.

Appendix A. A Remark About Spectral Theory

Let A be an unbounded operator with dense domain $\mathcal{D}$ in a Hilbert space H. Consider the polar decomposition $A = U|A|$ of A and the associated orthogonal decomposition

$$H = H_- \oplus H_0 \oplus H_+ \tag{A.1}$$

with $Ux = \pm x$ for $x \in H_\pm$ and $H_0 = \ker A = \ker U$ as in [17, Section VI.7]. Since A and $|A|$ commute with U, the decomposition of H is invariant under A and $|A|$. In particular,

$$\mathcal{D} = (\mathcal{D} \cap H_-) \oplus H_0 \oplus (\mathcal{D} \cap H_+)$$

and, similarly,

$$\mathcal{D}_Q = (\mathcal{D}_Q \cap H_-) \oplus H_0 \oplus (\mathcal{D}_Q \cap H_+)$$

for the domain $\mathcal{D}_Q \supseteq \mathcal{D}$ of the quadratic form $Q = Q(x, y) = \langle Ax, y \rangle$ in H associated to A. Since $\ker A = \ker |A| = H_0$, we have $Q < 0$ on $\mathcal{D}_Q \cap H_-$ and $Q > 0$ on $\mathcal{D}_Q \cap H_+$.

The following is a refined version of the usual variational characterization of eigenvalues of A in the case where the spectrum of A is discrete.

Lemma A.2. Let $X \subseteq \mathcal{D}_Q$ and $Y \subseteq H$ be subspaces such that $Q \le 0$ on X, $X \perp Y$, and $PY \subseteq Y$, where P denotes the orthogonal projection of H onto H_-. Then,

$$X \cap \ker P = X \cap H_0 \quad \text{and} \quad PX \perp Y.$$

In particular, if $\dim X < \infty$, then

$$\dim(H_- \ominus Y) \ge \dim PX = \dim X - \dim(X \cap H_0).$$

Proof. Write $x \in X$ as $x = x_- + x_0 + x_+$ according to (A.1), where $Px = x_-$. Now we have

$$Q(x, x) = Q(x_-, x_-) + Q(x_+, x_+) \le 0$$

and hence

$$Q(x_+, x_+) \le -Q(x_-, x_-).$$

Since $Q > 0$ on H_+, $x_- = 0$ implies that $x_+ = 0$ so that then $x = x_0 \in H_0$. This shows the first assertion. As for the second, we have

$$\langle PX, Y \rangle = \langle X, PY \rangle = 0$$

since P is orthogonal, $PY \subseteq Y$, and $X \perp Y$. ■

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International Mathematics Research Notices, 2025, 1–20
https://doi.org/10.1093/imrn/rnae259
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