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Sequential Monitoring for Changes in GARCH(1,1) Models Without Assuming Stationarity

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ABSTRACT

In this article, we develop two families of sequential monitoring procedure to (timely) detect changes in the parameters of a GARCH(1,1) model. Our statistics can be applied irrespective of whether the historical sample is stationary or not, and indeed without previous knowledge of the regime of the observations before and after the break. In particular, we construct our detectors as the CUSUM process of the quasi-Fisher scores of the log likelihood function. To ensure timely detection, we then construct our boundary function (exceeding which would indicate a break) by including a weighting sequence which is designed to shorten the detection delay in the presence of a changepoint. We consider two types of weights: a lighter set of weights, which ensures timely detection in the presence of changes occurring "early, but not too early" after the end of the historical sample; and a heavier set of weights, called "Rényi weights" which is designed to ensure timely detection in the presence of changepoints occurring very early in the monitoring horizon. In both cases, we derive the limiting distribution of the detection delays, indicating the expected delay for each set of weights. Our methodologies can be applied for a general analysis of changepoints in GARCH(1,1) sequences; however, they can also be applied to detect changes from stationarity to explosivity or vice versa, thus allowing to check for "volatility bubbles", upon applying tests for stationarity before and after the identified break. Our theoretical results are validated via a comprehensive set of simulations, and an empirical application to daily returns of individual stocks.

JEL Classification: Primary 62M10, 91B84, secondary 60G10, 62F12

1 | Introduction

In recent years, developing tools for the (ex-ante or ex-post) detection of the onset or the collapse of a bubble in financial markets has been one of the most active research areas in financial econometrics; we refer the reader, in particular, to the seminal articles on ex-post detection by Phillips et al. (2011), and Phillips et al. (2015a) and also to Skrobotov (2023) for a review. As far as ex-ante detection – that is, finding the onset or collapse of a bubble in real time, as new data come in – is concerned, this important issue has also been studied in numerous recent contributions. Although a comprehensive literature review goes beyond the scope of this article, we refer to the articles by Homm and Breitung (2012), Phillips et al. (2015b), and Whitehouse et al. (2023), *inter alia*. In particular, the article by Whitehouse et al. (2023) also contains a comprehensive literature review of in-sample and online bubble detection methods. A common trait to the vast majority of the existing literature is its reliance on a linear specification, usually an AutoRegressive (AR) model, to capture regime changes in the dynamics of log prices. Whilst such a modelling choice can be justified from the theoretical point of view and it is analytically tractable (see, e.g., Phillips and Yu 2011, and Aue and Horváth 2007), a major issue is that using an AR model is fraught with difficulties when monitoring for changes from an explosive towards a stationary regime. Several promising solutions have been proposed such as the reverse regression approach by Phillips and Shi (2018), Horváth and Trapani (2023). propose a different model, based on a random coefficient autoregressive specification – viz. $y_i = (\rho + \epsilon_{1,i}) y_{i-1} + \epsilon_{2,i}$ – where inference is always standard normal irrespective of stationarity or the lack thereof (Aue and Horváth 2011), and develop a family of sequential monitoring procedures based on the weighted CUSUM process, to check whether the coefficient ρ changes over time. Such a testing set-up also encompasses both the case of a switch from a stationary to an explosive regime (thus indicating the start of a bubble phenomenon), and a change from an explosive to a stationary regime (thus signalling the collapse of a bubble).

The theory developed by Horváth and Trapani (2023) paves the way to a more general research question. Namely, developing sequential monitoring techniques which are robust to both the initial regime (i.e., which can be used irrespective of whether the observations in the training/historical sample are stationary or explosive) and the type of change which occurs after a changepoint (i.e., which can detect changes from stationarity to another stationary regime, or to an explosive regime; or from an explosive regime to another explosive regime or a stationary one). From a technical viewpoint, this question is nontrivial for at least three reasons. First, proposing a changepoint detection methodology whose asymptotics is the same irrespective of stationarity or explosivity is not easy per se, because the partial sum processes which constitute the building blocks of, e.g., CUSUM-based statistics require completely different approximations depending on whether the observations are stationary or not. Second, in order to ensure timely changepoint detection, weighted versions of the CUSUM process need to be considered, with different sets of weights ensuring optimal detection delays for different changepoint locations within the monitoring horizon. Third, it is important to offer, to the applied user, a set of results on the limiting distribution of the detection delays, so as to gauge the expected detection delay depending on the location of the changepoint and the weighing scheme used.

Hence, in this article, we investigate, with an emphasis on completeness, the issue of sequential detection for changes in the parameters a GARCH(1,1) sequence[1]

$$y_i = \sigma_i \epsilon_i \quad \text{and} \quad \sigma_i^2 = \omega + \alpha y_{i-1}^2 + \beta \sigma_{i-1}^2, \quad 1 \le i < \infty \tag{1}$$

In particular, we develop two families of detectors based on the weighted CUSUM process of the quasi-Fisher scores associated with (1): one with "mild" weights, designed to detect changes that may occur "not too early" after the start of the monitoring period; and one with heavy weights, designed instead to detect changes occurring "very early" after the start of the monitoring period. The latter is based on applying to the CUSUM process a set of (heavy) weights, resulting in a family of test statistics known as *Rényi statistics* (see Horváth et al., (2020a), for in-sample tests, and Ghezzi et al. 2024, for sequential monitoring). Our methodologies can, in principle, be applied to detect any type of change in the vector (ω, α, β); however, given that our main interest is in detecting changes between regimes (e.g., from stationarity to nonstationarity, or vice versa) and that ω is not identified under nonstationarity, we focus on monitoring for changes only in the sub-vector (α, β). Detecting shifts in the behaviour of the (conditional) volatility process σ_i is important in general; as Hillebrand (2005) notes, when neglecting a break inference is biased in finite samples, and the sum of the estimated autoregressive parameters α and β converges to one. Furthermore, changes (and, occasionally, explosions) in the volatility of time series are often observed in practice (see e.g., Bloom 2007, and Jurado et al. 2015). Our methodologies can detect all types of changes, including those which involve a change from a stationary to a non-stationary regime, or vice versa a change from an explosive to a stationary regime. Finding the start or the end of an explosive regime in σ_i is of practical relevance because, as Richter et al. (2023) put it, one "often sees sudden, integrated, or mildly explosive behaviour in the second moment of the process which bounces back after a while" (p. 468). Changes between stationarity and explosivity in (1) can be interpreted as *volatility bubbles*, i.e., events in which the second moment of the data (rather than the data, e.g., prices, themselves) experiences periods of exhuberance. The link between a volatility phenomenon and a "proper" bubble has not been fully explored yet (see Jurado et al. 2015), and, empirically, explosive regimes in volatility can be ascribed to various sources in addition to bubbles (Sornette et al. 2018). Nevertheless, as Jarrow and Kwok (2023) put it, "price bubbles result from excess speculative trading decoupled from the asset's fundamentals (dividends and liquidation value), which increases the asset's price volatility to extreme levels" (p. 478). Hence, analysing the regimes of the volatility of financial variables can contribute to a better understanding of bubbles.

Based on the discussion above, in this article we propose a battery of tests for the sequential monitoring of the volatility of financial variables, which – being able to also capture changes between stationarity and explosivity – complements the existing tests for bubbles based on the conditional mean. Specifically, we make at least three contributions. First, we study online detection of changes by proposing a family of test statistics which can be used irrespective, and with no previous knowledge, of the stationarity or not of the observations, thus lending themselves to being used also to detect changes between stationarity and explosivity and vice versa, which – to the best of our knowledge – is a novel result in the literature, and which complements the ex-post detection statistics studied in Richter et al. (2023). Second, we develop the full-blown theory for Rényi statistics in the context of sequential monitoring of a GARCH(1,1) model. Third, we derive the limiting distribution of detection delays for all our monitoring schemes, including those based on Rényi statistics; this is an entirely novel result, which complements the results by Horváth et al. (2020a).

The remainder of the article is organised as follows. We discuss our workhorse model and the main assumptions, as well as the test statistics, in Section 2. The theory is reported in Section 3: in particular, the asymptotics under the null is in Section 3.1, and the full-blown asymptotics of the detection delay in the presence of a changepoint is in Section 3.2. We validate our theory through a comprehensive set of simulations (Section 4), and an empirical application to daily returns of individual stocks (Section 5). Section 6 concludes.

Notation. We define the Euclidean norm of a vector a as $\|a\|$. We use: "a.s" for "almost sure(ly)"; "$\rightarrow$" for the ordinary limit; "$\xrightarrow{D}$" for convergence in distribution; "$\xrightarrow{P}$" for convergence in probability; "$\stackrel{D}{=}$" for equality in distribution. Positive, finite constants are denoted as $c_0, c_1, \ldots$ and their value may change from line to line. Other, relevant notation is introduced later on in the article.

2 | Model, Assumptions and Hypothesis Testing

2.1 | Model, Assumptions and Hypotheses of Interest

The time dependent GARCH(1,1) sequence is defined by the recursion

$$y_i = \sigma_i \epsilon_i \quad \text{and} \quad \sigma_i^2 = \omega_i + \alpha_i y_{i-1}^2 + \beta_i \sigma_{i-1}^2, \quad 1 \le i < \infty \tag{2}$$

where σ_0^2, y_0^2 are initial values, and α_i, β_i and ω_i are positive parameters.

Whilst the hypothesis testing framework is spelt out below, our monitoring schemes are all based on the maintained assumption that we have m observations which form a stable period (this is also known as the *non-contamination assumption*in Chu et al. 1996), viz.

$$(\omega_1, \alpha_1, \beta_1) = (\omega_2, \alpha_2, \beta_2) = \ldots = (\omega_m, \alpha_m, \beta_m) \tag{3}$$

The value of m is, *de facto*, the only tuning parameter of our procedures (and it is, in general, an important tuning parameter in all applications of online changepoint detection). As our theory shows, it is required that $m \to \infty$ but, on the other hand, larger values of m correspond to larger detection delays – as such, the choice of m is an empirical matter which reflects the usual trade-off between size and power.

We denote the value of the common parameter in (3) as $\boldsymbol{\theta}_0 = (\alpha_0, \beta_0, \omega_0)^\top$. We now review the conditions for the stationarity of y_i (see e.g., Nelson 1991, Bougerol and Picard 1992, Francq and Zakoïan 2012 and Horváth and Wang 2024):

1. if $E \log\left(\alpha_0 \epsilon_0^2 + \beta_0\right) < 0$, σ_i converges almost surely exponentially fast to a unique, strictly stationary and ergodic solution $\left\{\bar{\sigma}_i, -\infty < i < \infty\right\}$ for all ϵ_0 and σ_0;
2. if $E \log\left(\alpha_0 \epsilon_0^2 + \beta_0\right) > 0$, then σ_i is nonstationary with $\sigma_i \xrightarrow{a.s.} \infty$ exponentially fast;
3. if $E \log\left(\alpha_0 \epsilon_0^2 + \beta_0\right) = 0$, then σ_i is nonstationary Horváth and Wang (2024); show that $\exp\left(-i^\zeta\right)\sigma_i \xrightarrow{P} 0$, for all $\zeta < 1/2$.[2]

Note also that, since $\log\left(\alpha_0 \epsilon_0^2 + \beta_0\right) = \log\left(\alpha_0 \epsilon_0^2 + \beta_0\right) I\left\{\alpha_0 \epsilon_0^2 \le 1\right\} + \left(\alpha_0 \epsilon_0^2 + \beta_0\right) I\left\{\alpha_0 \epsilon_0^2 > 1\right\}$

$$\log \beta_0 \le \log\left(\alpha_0 \epsilon_0^2 + \beta_0\right) I\left\{\alpha_0 \epsilon_0^2 \le 1\right\} \le \log\left(1 + \beta_0\right)$$

$$\log \beta_0 \le \log\left(\alpha_0 \epsilon_0^2 + \beta_0\right) I\left\{\alpha_0 \epsilon_0^2 > 1\right\} \le \alpha_0 \epsilon_0^2 + \beta_0$$

which, under our assumptions, entails that it always holds that $E\left|\log\left(\alpha_0 \epsilon_0^2 + \beta_0\right)\right| < \infty$.

We will develop several monitoring schemes for the null hypothesis that the parameter $\boldsymbol{\theta}_0$ undergoes no changes after the training period $1 \le i \le m$, i.e.,

$$H_0 : \left(\omega_{m+k}, \alpha_{m+k}, \beta_{m+k}\right) = \left(\omega_0, \alpha_0, \beta_0\right), \text{for all } k \ge 1 \tag{4}$$

Under the alternative, we assume that there is a change at time $m + k^*$; whilst this would correspond to having $\left(\omega_{m+k^*-j}, \alpha_{m+k^*-j}, \beta_{m+k^*-j}\right) \neq \left(\omega_{m+k^*+j}, \alpha_{m+k^*+j}, \beta_{m+k^*+j}\right)$ for all $j \ge 0$, it is well known that the ω_i's cannot be identified in explosive, nonstationary regimes (Francq and Zakoïan 2012). Hence, we will test for

$$\begin{aligned} H_A : (\alpha_m, \beta_m) &= (\alpha_{m+1}, \beta_{m+1}) = (\alpha_{m+2}, \beta_{m+2}) \\ &= \ldots = (\alpha_{m+k^*-1}, \beta_{m+k^*-1}) \neq (\alpha_{m+k^*}, \beta_{m+k^*}) \\ &= (\alpha_{m+k^*+1}, \beta_{m+k^*+1}) = \ldots \end{aligned} \tag{5}$$

i.e., for the possible presence of changes in α_i and β_i only. Note that these are anyway the parameters of interest, since the stationarity (or lack thereof) of y_i is not affected by ω_i.

We require the following assumptions on $\boldsymbol{\theta}_0$, and on the innovations $\{\epsilon_i, -\infty < i < \infty\}$.

Assumption 1. It holds that: $\alpha_0 > 0$, $\beta_0 > 0$ and $\omega_0 > 0$.

Assumption 2. It holds that: *(i)* $\{\epsilon_i, -\infty < i < \infty\}$ are independent and identically distributed random variables; *(ii)* ϵ_0^2 is nondegenerate and *(iii)* $E\epsilon_0 = 0$, $E\epsilon_0^2 = 1$, and $E|\epsilon_0|^\kappa < \infty$ with some $\kappa > 4$.

Assumptions 1 and 2 are standard. In particular, it is worth noting that, in Assumption 1, there is no requirement as to the stationarity properties of $\left\{y_i, 1 \le i \le m\right\}$: the observations in the training sample can belong to a stationary or explosive volatility regime. Conversely, Assumption 1 excludes from our analysis the case where the parameters are on the boundary of the parameter space. In principle, it would be possible to consider the case where the parameters sit at the boundary: this case has been explored in several contributions, starting from Chernoff (1954). Applications to the GARCH family have been considered by Iglesias and Linton (2007), Francq and Zakoian (2007), Francq and Zakoïan (2009), and Pedersen (2017), *inter alia*. The main conclusion of all these articles is that, when the parameters lie on the boundary, limiting distributions are no longer Gaussian, but "reflected Gaussian" (more precisely, they are given by the projection of a Gaussian multivariate distribution onto a convex cone). All our results below are based on the properties of Gaussian processes, and therefore the case where a parameter is on the boundary requires a separate treatment, which we leave for future study.

2.2 | Estimation and Monitoring Schemes

As stated in the Introduction, the main purpose of our analysis is to offer a detection scheme which finds changes in the parameters of a GARCH(1,1) model as soon as possible after the training period. In this section, we propose several detectors, all based on the CUSUM process of the quasi-Fisher scores.

As is typical, we estimate the parameter θ_0, using the training sample, by Quasi Maximum Likelihood (QML). The log likelihood function (ignoring a negative scaling factor) is

$$\overline{\ell}_i(\theta) = \log \overline{\sigma}_i^2(\theta) + \frac{y_i^2}{\overline{\sigma}_i^2(\theta)}, \qquad 1 \le i \le m$$

where $\theta = (\omega, \alpha, \beta)^\top$, $\omega > 0, \alpha > 0$ and $\beta > 0$; the random functions $\overline{\sigma}_i^2(\theta)$

$$\overline{\sigma}_i^2(\theta) = \omega + \alpha y_{i-1}^2 + \beta \overline{\sigma}_{i-1}^2(\theta), \qquad 1 \le i \le m$$

and we denote the initial values from which the recursions start as y_0 and $\overline{\sigma}_0^2$, respectively. The QML estimator computed from the historical sample is denoted as $\hat{\theta}_m$, with

$$\hat{\theta}_m = \text{argmin}\left\{ \sum_{i=1}^m \overline{\ell}_i(\theta) : \ \theta \in \Theta \right\}$$

where the (compact) space $\mathbf{\Theta}$ is defined as $\mathbf{\Theta} = \left\{ \theta : \underline{\omega} \le \omega \le \overline{\omega}, \underline{\alpha} \le \alpha \le \overline{\alpha}, \underline{\beta} \le \beta \le \overline{\beta} \right\}, 0 < \underline{\omega}, \overline{\omega}, \underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta} < \infty$. The compactness requirement is standard in this theory (indeed, Jordan 2003 considers the case for ARCH models, but we are not aware of similar results in the context of GARCH models). However, whilst we do need to exclude the cases where parameters are on the boundary of the parameter space, we do not require any upper bounds on $\overline{\alpha}$ or $\overline{\beta}$, which is compatible with the idea that we explicitly allow for the GARCH process to be explosive at times (thus allowing for $\overline{\alpha} > 1$ and/or $\overline{\beta} > 1$).

Starting with the initial values y_m and σ_m^2, we define the random functions $\overline{\sigma}_{m+k}^2(\theta)$ based on the observations after the historical sample by the recursion

$$\overline{\sigma}_{m+k}^2(\theta) = \omega + \alpha y_{m+k-1}^2 + \beta \overline{\sigma}_{m+k-1}^2(\theta), \qquad k \ge 1$$

with the log likelihood function (again ignoring a negative scaling factor) given by

$$\overline{\ell}_{m+k}(\theta) = \log \overline{\sigma}_{m+k}^2(\theta) + \frac{y_{m+k}^2}{\overline{\sigma}_{m+k}^2(\theta)}, \qquad k \ge 1$$

Hence, we define the CUSUM process of the quasi-Fisher scores as

$$\mathscr{r}_{m,k}(\theta) = \sum_{i=m+1}^{m+k} \left(\frac{\partial \overline{\ell}_i(\theta)}{\partial \alpha}, \frac{\partial \overline{\ell}_i(\theta)}{\partial \beta} \right)^\top, \ k \ge 1 \tag{6}$$

Heuristically, under the null of no change, the scores have zero mean. Hence, the partial sum process $\mathscr{r}_{m,k}(\theta)$ calculated at $\hat{\theta}_m$ should also fluctuate around zero with increasing variance. Conversely, in the presence of a break (at, say, k^*), $\hat{\theta}_m$ is a biased estimator for the "new" parameter θ_{m+k^*}; thus, $\mathscr{r}_{m,k}(\theta)$, calculated at $\hat{\theta}_m$, should have a drift term. In the light of these heuristic considerations, we propose the following *detector*

$$\mathcal{D}_m(k) = \mathscr{r}_{m,k}^\top(\hat{\theta}_m)\hat{\mathbf{D}}_m^{-1}\mathscr{r}_{m,k}(\hat{\theta}_m) \tag{7}$$

where

$$\hat{\mathbf{D}}_m = \frac{1}{m}\sum_{i=1}^m \left(\frac{\partial \overline{\ell}_i(\hat{\theta}_m)}{\partial \alpha}, \frac{\partial \overline{\ell}_i(\hat{\theta}_m)}{\partial \beta} \right)^\top \left(\frac{\partial \overline{\ell}_i(\hat{\theta}_m)}{\partial \alpha}, \frac{\partial \overline{\ell}_i(\hat{\theta}_m)}{\partial \beta} \right)$$

Based on (7), a break is flagged as soon as the detector $\mathcal{D}_m(k)$ exceeds a threshold. We call such a threshold the *boundary function*. Similarly to Chu et al. (1996), Horváth et al. (2004), Horváth et al. (2022), Homm and Breitung (2012), we use the boundary function, designed for a *closed-ended procedure* – i.e., for a procedure which terminates at a certain time, say $\mathscr{n}$, if there is no change

$$g_m(k) = \mathscr{c}\mathscr{n}(k/\mathscr{n})^\eta, \qquad \text{with} \ \ 0 \le \eta < 1 \tag{8}$$

On account of (7) and (8), a changepoint is found at a stopping time τ_m defined as

$$\tau_m = \begin{cases} \min\left\{k :\in [1, 2, \dots, \mathscr{n}-1], \quad \mathcal{D}_m(k) \ge g_m(k)\right\} \\ \mathscr{n}, \ \text{if} \ \ \mathcal{D}_m(k) < g_m(k) \quad \text{for all} \ \ 1 \le k \le \mathscr{n}-1 \end{cases} \tag{9}$$

The (user-chosen) parameter η in (8) determines the timeliness of changepoint detection of our sequential monitoring procedure. Aue and Horváth et al. (2004) and Aue et al. (2008) show that, as η approaches 1, changepoints are detected with a smaller and smaller delay depending on their location. On the other hand, different values of η work better for different changepoint locations, as also pointed out in a recent contribution by Kirch and Stoehr (2022b). In particular, values of $0 \le \eta < 1$ are able to offer short detection delays for breaks that do not occur "too early" after m.

To detect earlier changes Ghezzi et al. (2024), suggest using Rényi type statistics, with

$$\overline{\tau}_m = \begin{cases} \min\left\{k :\in [r, 2, \dots, \mathscr{n}-1], : \quad \mathcal{D}_m(k) \ge \overline{g}_m(k)\right\} \\ \mathscr{n}, \ \text{if} \ \ \mathcal{D}_m(k) < \overline{g}_m(k) \quad \text{for all} \ \ r \le k \le \mathscr{n}-1, \end{cases} \tag{10}$$

where r is a trimming sequence specified in Assumption 4, and

$$\overline{g}_m(k) = \mathscr{c}r(k/r)^\eta, \qquad \text{with} \quad \eta > 1 \tag{11}$$

We note that (9) and (10) exclude the case $\eta = 1$. Indeed Aue and Horváth (2004), show that, as far as stopping times under the alternative are concerned, using $\eta = 1$ would produce the shortest detection time, at least when the changepoint is located very close to the beginning of the monitoring period. However, as far as the asymptotic theory is concerned, as we discuss before presenting Theorem 2, the case $\eta = 1$ requires to be treated separately, using a different limiting theory and different boundary functions, following Horváth et al. (2007). Let

$$a(x) = (2\log x)^{1/2} \quad \text{and} \quad b_2(x) = 2\log x + \log\log x$$

We use the boundary functions

$$g_m^*(k) = k\left(\frac{\mathscr{c} + b_2(\log \mathscr{n})}{a(\log \mathscr{n})} \right)^2 \tag{12}$$

$$\overline{g}_m^*(k) = k\left(\frac{\mathscr{c} + b_2(\log(\mathscr{n}/r))}{a(\log(\mathscr{n}/r))} \right)^2 \tag{13}$$

The stopping times – denoted as τ_m^* and $\overline{\tau}_m^*$ – are defined in the same way as τ_m and $\overline{\tau}_m$ in (9) and (10), respectively, using now the boundaries $g_m^*(k)$ and $\overline{g}_m^*(k)$.

As mentioned above, we consider a closed-ended scheme, which terminates $\mathcal{n}$ periods after m. The following assumptions characterise the length of the monitoring horizon and of the trimming sequence r defined in (10). In particular, Assumption 3 is designed in order to consider only early changepoint detection; the actual choice of the length of the monitoring horizon $\mathcal{n}$ is also an empirical matter; Assumption 3 only requires that $\mathcal{n}$ should pass to infinity, but should not be "too large" (using the size of the training sample m as a benchmark) – thus suggesting that, after $\mathcal{n}$ periods, the monitoring procedure should be terminated and, if need be, restarted.

Assumption 3. It holds that $m \to \infty$ and $\mathcal{n} = \mathcal{n}(m) \to \infty$, with $\mathcal{n}/m \to 0$.

Assumption 4. It holds that, in Equation (10), $r \to \infty$ and $r/\mathcal{n} \to 0$.

3 | Asymptotics

3.1 | Asymptotics Under the Null

Let $\mathbf{W}(t) = (W_1(t), W_2(t)), t \geq 0$ be a two dimensional standard Wiener process – i.e., $\{W_1(t), t \geq 0\}$ and $\{W_2(t), t \geq 0\}$ are two independent Gaussian processes with $EW_1(t) = EW_2(t) = 0$, and covariance kernel $EW_1(t)W_1(s) = EW_2(t)W_2(s) = \min(t, s)$.

Theorem 1. *We assume that H_0 of (4) and Assumptions 1–3 hold.*

i. *If $0 \leq \eta < 1$, then we have*

$$\lim_{m\to\infty} P\{\tau_m = \mathcal{n}\} = P\left\{\sup_{0<t\leq 1} \frac{1}{t^\eta}\|\mathbf{W}(t)\|^2 \leq c\right\}$$

ii. *If in addition, Assumption 4 also holds and $\eta > 1$, then we have*

$$\lim_{m\to\infty} P\{\overline{\tau}_m = \mathcal{n}\} = P\left\{\sup_{1\leq t<\infty} \frac{1}{t^\eta}\|\mathbf{W}(t)\|^2 \leq c\right\}$$

Theorem 1 offers a rule to calculate the asymptotic critical values; for a given nominal level α, the critical value c_α is defined as

$$P\left\{\sup_{0<t\leq 1} \frac{1}{t^\eta}\|\mathbf{W}(t)\|^2 \leq c_\alpha\right\} = 1-\alpha, \text{ for all } 0 \leq \eta < 1$$

$$P\left\{\sup_{1<t<\infty} \frac{1}{t^\eta}\|\mathbf{W}(t)\|^2 \leq c_\alpha\right\} = 1-\alpha, \text{ for } \eta > 1$$

Using the scale transformation of the Wiener process, it immediately follows that $\{\mathbf{W}(t), t > 0\} \stackrel{\mathcal{D}}{=} \{t\mathbf{W}(1/t), t > 0\}$. Hence, for all $\eta > 1$

$$\sup_{1\leq t<\infty} \frac{1}{t^\eta}\|\mathbf{W}(t)\|^2 \stackrel{\mathcal{D}}{=} \sup_{0<t\leq 1} \frac{1}{t^{1-\eta}}\|\mathbf{W}(t)\|^2$$

In conclusion, we note that using $\eta < 1$ or $\eta > 1$ results in a similar limit, but the range on which the supremum is taken is "swapped" from $(0, 1)$ to $(1, \infty)$. The proofs show that the crossing of the boundary happens on different sets when we have $\eta < 1$ and $\eta > 1$.

Theorem 1 does not consider the case $\eta = 1$, which corresponds to the stopping times τ_m^* and $\overline{\tau}_m^*$ based on the boundaries defined in (12) and (13), respectively. Intuitively, Theorem 1 fails when $\eta = 1$ because – by the Law of the Iterated Logarithm – $\sup_{0<t\leq 1} t^{-1}\|\mathbf{W}(t)\|^2 = \infty$ a.s. Hence, the proof of Theorem 1 cannot be extended to the case $\eta = 1$ (in essence, there can be no convergence to a limit which does not exist): when $\eta = 1$, a different normalisation (based on $a(x)$ and $b_2(x)$), and a different limit theorem (the Darling-Erdős theorem – see Darling and Erdős 1956 – which is an Extreme Value-type theorem), need to be used. We study such a case in the following theorem.

Theorem 2. *We assume that H_0 of (4) and Assumptions 1–3 hold.*

i. *Then, for all $-\infty < c < \infty$, it holds that $\lim_{m\to\infty} P\{\tau_m^* = \mathcal{n}\} = \exp(-e^{-c})$.*

ii. *If in addition Assumption 4 also holds, then we have $\lim_{m\to\infty} P\{\overline{\tau}_m^* = \mathcal{n}\} = \exp(-e^{-c})$.*

Theorem 2 stipulates that the asymptotic critical values, for a given nominal level α, can be calculated as

$$c_\alpha = \overline{c}_\alpha = -\log\big(-\log(1-\alpha)\big) \tag{14}$$

using c_α or $\overline{c}_\alpha$ according as (12) or (13) is used.

3.2 | Asymptotics Under the Alternative

We now study the behaviour of our monitoring schemes under the alternative, focussing, in particular, on the limiting distribution of the detection delay. The limiting distribution of the detection delay when $0 \leq \eta < 1$ is in Section 3.2.1. In Section 3.2.2, we report the limiting distribution of the detection delay when using $\eta > 1$. In both cases, we require a great deal of notation; no to overshadow the main arguments, we relegate some of this to Appendix A in the Supplement.

In both cases, under the alternative H_A, of (5), the parameter $\boldsymbol{\theta}_0 = (\alpha_0, \beta_0, \omega_0)^\top$ changes to $\boldsymbol{\theta}_A = (\alpha_A, \beta_A, \omega_A)^\top$ satisfying

Assumption 5. $\alpha_A > 0, \quad \beta_A > 0, \quad \omega_0 > 0, \qquad \overline{\boldsymbol{\theta}}_0 \neq \overline{\boldsymbol{\theta}}_A$, $\overline{\boldsymbol{\theta}}_0 = (\alpha_0, \beta_0)^\top$ and $\overline{\boldsymbol{\theta}}_A = (\alpha_A, \beta_A)^\top$.

3.2.1 | Detection Delays With $0 \leq \eta < 1$

We begin by investigating the asymptotic behaviour of the stopping time τ_m defined in (9). Whilst the result in Theorem 3 is valid for all cases, we need to introduce some preliminary notation, separately, for the two cases: (1) when the sequence is *stationary* after the change and (2) when the sequence is *explosive* after the change.

We begin by introducing some preliminary notation for the former case, i.e.,

$$E\log(\alpha_A\epsilon_0^2+\beta_A)<0 \tag{15}$$

Under (15), after the change the observations are exponentially close to $\{\hat{x}_i, -\infty<i<\infty\}$, a stationary GARCH(1,1) sequence[3] given by

$$\hat{x}_i=\hat{h}_i\epsilon_i \quad \text{and} \quad \hat{h}_i^2=\omega+\alpha_A\hat{x}_{i-1}^2+\beta_A\hat{h}_{i-1}^2 \tag{16}$$

We also define the log likelihood function

$$\hat{\ell}_i(\boldsymbol{\theta})=\log\hat{h}_i^2(\boldsymbol{\theta})+\frac{\hat{x}_i^2}{\hat{h}_i^2(\boldsymbol{\theta})}$$

where $\hat{h}_i^2(\boldsymbol{\theta})=\omega+\alpha\hat{x}_{i-1}^2+\beta\hat{h}_{i-1}^2(\boldsymbol{\theta})$. Let

$$\mathscr{r}^{(1)}(\boldsymbol{\theta})=E\left(\frac{\partial\hat{\ell}_i(\boldsymbol{\theta})}{\alpha},\frac{\partial\hat{\ell}_i(\boldsymbol{\theta})}{\beta}\right)^\top$$

and define the size of the change as

$$\boldsymbol{\Delta}=\mathscr{r}^{(1)}(\boldsymbol{\theta}_0)\neq\mathbf{0} \tag{17}$$

We define the covariance matrix

$$\begin{aligned}\boldsymbol{\Sigma}_1=E&\left[\left(\frac{\partial\hat{\ell}_i(\boldsymbol{\theta}_0)}{\alpha},\frac{\partial\hat{\ell}_i(\boldsymbol{\theta}_0)}{\beta}\right)-\boldsymbol{\Delta}\right]^\top\\ &\times\left[\left(\frac{\partial\hat{\ell}_i(\boldsymbol{\theta}_0)}{\alpha},\frac{\partial\hat{\ell}_i(\boldsymbol{\theta}_0)}{\beta}\right)-\boldsymbol{\Delta}\right]\end{aligned} \tag{18}$$

and

$$t^*=\lim_{m\to\infty}k^*\Big/\left(\mathscr{n}^{1-\eta}\frac{\mathscr{c}}{A_m}\right)^{1/(2-\eta)} \tag{19}$$

where $A_m=\boldsymbol{\Delta}^\top\hat{\mathbf{D}}_m^{-1}\boldsymbol{\Delta}$. Finally (as far as preliminary notation is concerned), we define $u_{\mathscr{n}}>0$ as the unique solution of the equation

$$u_{\mathscr{n}}^2=\left(u_{\mathscr{n}}+k^*\Big/\left(\mathscr{n}^{1-\eta}\frac{\mathscr{c}}{A_m}\right)^{1/(2-\eta)}\right)^\eta \tag{20}$$

and $u^*>0$ as the solution of

$$u^*=(u^*+t^*)^{\eta/2} \tag{21}$$

It is easy to see that $u_{\mathscr{n}}\to u^*$ and that $u^*=1$, if $t^*=0$.

We now introduce the preliminary notation for the case when the observations turn into an explosive sequence after the time of change, i.e.,

$$E\log(\alpha_A\epsilon_0^2+\beta_A)\geq 0 \tag{22}$$

Jensen and Rahbek (2004) proved that

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=m+k^*+1}^{m+k^*+k}E\left(\frac{\partial\bar{\ell}_i(\boldsymbol{\theta})}{\alpha},\frac{\partial\bar{\ell}_i(\boldsymbol{\theta})}{\beta}\right)^\top=\mathscr{r}^{(2)}(\boldsymbol{\theta})$$

exists when $E\log(\alpha_A\epsilon_0^2+\beta_A)>0$; Horváth and Wang (2024) extend the validity of this result to the case $E\log(\alpha_A\epsilon_0^2+\beta_A)=0$.

Similarly to $\boldsymbol{\Delta}$ in Equation (17), we define the size of the change under H_A as $\boldsymbol{\Upsilon}=\mathscr{r}^{(2)}(\boldsymbol{\theta}_0)\neq\mathbf{0}$. Similarly to (19), we define

$$\tilde{t}^*=\lim_{m\to\infty}k^*\Big/\left(\mathscr{n}^{1-\eta}\frac{\mathscr{c}}{B_m}\right)^{1/(2-\eta)}$$

where $B_m=\boldsymbol{\Upsilon}^\top\hat{\mathbf{D}}_m^{-1}\boldsymbol{\Upsilon}$. Finally, similarly to $u_{\mathscr{n}}$ and u^* we define $\tilde{u}_{\mathscr{n}}$ and $\tilde{u}^*$ as the solutions of the equations

$$\tilde{u}_{\mathscr{n}}^2=\left(\tilde{u}_{\mathscr{n}}+k^*\Big/\left(\mathscr{n}^{1-\eta}\frac{\mathscr{c}}{B_m}\right)\right)^\eta,\ \text{and}\ \tilde{u}^*=\left(\tilde{u}^*+\tilde{t}^*\right)^{\eta/2}$$

After the change in the parameters, the gradient of the likelihood function is approximated with the sequences

$$\begin{aligned}\mathscr{v}_{m+k^*+i,1}&=\sum_{j=1}^\infty\epsilon_{m+k^*+i-j}^2\frac{1}{\beta_A}\sum_{k=1}^j\frac{\beta_A}{\alpha_A\epsilon_{m+k^*+i-k}^2+\beta_A}\\ \mathscr{v}_{m+k^*+i,2}&=\sum_{j=1}^\infty\frac{1}{\beta_A}\sum_{k=1}^j\frac{\beta_A}{\alpha_A\epsilon_{m+k^*+i-k}^2+\beta_A}\end{aligned}$$

Analogously to $\boldsymbol{\Sigma}_1$ in (18), we finally introduce

$$\boldsymbol{\Sigma}_2=E\left(1-\epsilon_{m+k^*+1}^2\right)^2E\begin{pmatrix}\mathscr{v}_{m+k^*+i,1}\\ \mathscr{v}_{m+k^*+i,2}\end{pmatrix}\left(\mathscr{v}_{m+k^*+i,1},\mathscr{v}_{m+k^*+i,2}\right) \tag{23}$$

We are now ready to present our results. To do so, we need some further notation which we report in Appendix A.1. As shown in Theorem 3, in several cases the delay τ_m-k^* converges to a standard normal random variable after being centred and rescaled; the centring and rescaling for τ_m-k^* depend on whether $t^*<\infty$ or $t^*=\infty$ ($\tilde{t}^*<\infty$ or $\tilde{t}^*=\infty$, equivalently). We define the Gaussian process $\{\boldsymbol{\Gamma}(t),t\geq 0\}$, with $E\boldsymbol{\Gamma}(t)=\mathbf{0}$ and $E\boldsymbol{\Gamma}(t)\boldsymbol{\Gamma}^\top(s)=\min(t,s)\mathbf{D}$; we let $\mathcal{N}$ denote a standard normal random variable; and we define

$$\lim_{m\to\infty}\frac{k^*}{\mathscr{n}}=\bar{u}$$

Theorem 3. *We assume that H_A of (5) and Assumptions 1–3 and 5 hold. Then, for all $0\leq\eta<1$*

i. If

$$\lim_{m\to\infty}\frac{k^*}{\mathscr{n}^{(1-\eta)/(2-\eta)}}<\infty \tag{24}$$

then

$$\frac{\tau_m-k^*-v_{1,\mathscr{n}}}{v_{2,\mathscr{n}}}\xrightarrow{D}\bar{s}_1\mathcal{N}(0,1)$$

where $v_{1,\mathscr{n}}$ and $v_{2,\mathscr{n}}$ are defined in (A.1) and (A.2), and $\bar{s}_1$ is defined in (A.7).

ii. If

$$\lim_{m\to\infty}\frac{k^*}{\mathscr{n}^{(1-\eta)/(2-\eta)}}=\infty \tag{25}$$

and $\bar{u}=0$, then

$$\frac{\tau_m-k^*-v_{3,\mathscr{n}}}{v_{4,\mathscr{n}}}\xrightarrow{D}\bar{s}_2\mathcal{N}(0,1)$$

where $v_{3,\mathscr{n}}$ and $v_{4,\mathscr{n}}$ are defined in (A.3) and (A.4), and $\bar{s}_2$ is defined in (A.8).

iii. If $0 < \bar{u} < 1$, then

$$\lim_{m\to\infty} P\left\{ \frac{\tau_m - k^*}{(k^*)^{1/2}} > x \right\} = P\left\{ \bar{u}^{1-\eta} \max\left(\mathcal{A}_1, \mathcal{A}_2(x)\right) \le c \right\}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2(x)$ are defined in (A.5) and (A.6).

To understand the practical implications of Theorem 3, note that (up to some positive and finite constant)

$$v_{1,n} \approx \left(m^{1-\eta}\right)^{1/(2-\eta)}, \qquad v_{2,n} \approx \left(m^{1-\eta}\right)^{1/(4-2\eta)} \approx v_{1,n}^{1/2}$$
$$v_{3,n} \approx \left[n^{1-\eta}(k^*)^{\eta}\right]^{1/2} \quad \text{and} \quad v_{4,n} \approx (k^*)^{1/2}$$

The case (24) corresponds to a "very early" break. In this case, Theorem 3 states that the expected delay is approximately $v_{1,n}$, i.e., that it is approximately equal to $\left(m^{1-\eta}\right)^{1/(2-\eta)}$. Clearly, as η increases, $v_{1,n}$ decreases; the dispersion around the expected delay, measured by $v_{2,n}$, also decreases, indicating that the choice of η plays a role in determining the delay in detecting (very early) changepoints and that larger values of η reduce such a delay. In the presence of an "early, but not so early" break – corresponding to case *(ii)* of the theorem, where recall that $k^* = o(n)$, the expected delay $v_{3,n}$ still decreases as η increases, as long as $k^* = n^{\gamma}$, for $\gamma > 1/(2-\eta)$, but the dispersion around the expected delay – given by the standardisation $v_{4,n}$ – does not depend on η. Finally, the case of a late(r) change is studied in part *(iii)* of the theorem: in such a case, η – and therefore the weight function in the definition of the detector – does not play any role. Part *(iii)* of the theorem also states that, in the case of a late(r) change, the delay increases as the changepoint location, k^*, increases. To better understand this, we note that our procedure is tailored to detect early changes, in essence by comparing the sample mean of k observations sequentially against the sample mean of the training sample. If k^* is large, then the sample mean of the observations after the training sample is dominated by observations which still satisfy the null hypothesis. So the procedure would need more time to cumulate a large amount of observations whose behaviour is that implied by the alternative. This corresponds to the "stylised fact" noted in this literature that it is virtually impossible to propose one rule which is optimal for any changepoint location (see e.g., Kirch and Stoehr 2022b, and Kirch and Stoehr 2022a).

3.2.2 | Detection Delays When $\eta > 1$

We now investigate the asymptotic behaviour of the stopping time $\tilde{\tau}_m$ defined in (10) – that is, when the detector is a Rényi type statistic with $\eta > 1$. In such a case, the asymptotic behaviour of the detection delay uses the same notation irrespective of whether $E\log(\alpha_A\epsilon_0^2 + \beta_A) < 0$ or ≥ 0. Here, we spell out only some of the relevant notation; further notation is in Appendix A.2. Let

$$a = \lim_{m\to\infty} \frac{k^*}{r} \in [0, \infty]$$

We define two independent normal random vectors $\mathbf{N}_1$ and $\mathbf{N}_2$ such that $E\mathbf{N}_1 = \mathbf{0}$, $E\mathbf{N}_2 = \mathbf{0}$, $E\mathbf{N}_1\mathbf{N}_1^{\top} = \mathbf{D}$ and

$$E\mathbf{N}_2\mathbf{N}_2^{\top} = \begin{cases} \mathbf{\Sigma}_1, & \text{if} \quad E\log(\alpha_A\epsilon_0^2 + \beta_A) < 0 \\ \mathbf{\Sigma}_2, & \text{if} \quad E\log(\alpha_A\epsilon_0^2 + \beta_A) \ge 0 \end{cases}$$

Theorem 4. *We assume that H_A of (5) and Assumptions 1–3 and 5 hold. Then, for all $1 < \eta < 2$.*

i. If $k^* \le r$ and $a = 0$ hold, then

$$\lim_{m\to\infty} P\{\overline{\tau}_m = r\} = 1$$

ii. If $k^* \le r$ and $a > 0$ hold, then

$$\lim_{m\to\infty} P\{\overline{\tau}_m = r\} = P\left\{(a^{1/2}\mathbf{N}_1 + a\mathbf{\Delta} + \mathbf{N}_2)^{\top}\mathbf{D}^{-1}(a^{1/2}\mathbf{N}_1 + a\mathbf{\Delta} + \mathbf{N}_2) > c\right\}$$

iii. If $k^* > r$ and $a < \infty$, then

$$\lim_{m\to\infty} P\left\{\overline{\tau}_m > k^* + xr^{1/2}\right\} = P\left\{\max\left(\mathcal{B}_1, \mathcal{B}_2(x)\right) \le c\right\}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2(x)$ are defined in (A.9) and (A.10).

iv. If $k^* > r, \bar{u} < 1$ and $a = \infty$ hold, then

$$\frac{\overline{\tau}_m - k^* - v_{5,n}}{v_{6,n}} \xrightarrow{\mathcal{D}} \overline{s}_2\mathcal{N}$$

where $v_{5,n}$ and $v_{6,n}$ are defined in (A.11) and (A.12).

Similarly to Theorem 3, Theorem 4 describes the detection delay when using Rényi type statistics depending on the location of the break; to the best of our knowledge, this is the first time such a result has ever been derived. Part *(i)* of the theorem is also derived in Ghezzi et al. (2024), and, in essence, it states that if the break occurs before the trimming sequence r in the Rényi type statistics, then it is identified straight at r – that is, as soon as the Rényi type statistics starts the monitoring. Parts *(ii)* and *(iii)*of the theorem refine and extend the results in Ghezzi et al. (2024). Finally, part *(iv)* states that, in the case of a break occurring late – or, better, much later than r – a large value of η could even be detrimental because the centring sequence $v_{5,n}$ diverges with k^*, at a faster rate as τ increases. This confirms the common wisdom (see Kirch and Stoehr 2022a and Kirch and Stoehr 2022b), and the findings in Ghezzi et al. (2024), that Rényi type statistics are designed for the fast detection of very early occurring breaks, whereas they may yield suboptimal results for later breaks.

In conclusion, we point out that all the test statistics proposed above are, essentially, nuisance free, as can be expected in a likelihood-based set-up. On the other hand, the one tuning parameter which needs setting – and which does impact on the performance of the test statistics – is the length of the training sample m. Such a choice reflects the typical trade-off between size and power: on the one hand, m is required to pass to infinity in order for the asymptotic approximations to be accurate (and therefore, practically, m should be as large as possible); on the other hand, the results in Theorems 3 and 4 indicate that the larger m, the longer the detection delay. This issue characterises all applications of sequential monitoring (see e.g., the discussion, and the references, in He et al. 2024), and a possible solution, which we recommend by way of guideline, is to decide a *minimum* m based on simulations for a given combination of η and of the length of the monitoring horizon which ensures size control, and use such a minimum m as the

TABLE 1 | Critical values.

Based on Theorem 1(*i*)				Based on Theorem 1(*ii*)			
η/α	10%	5%	1%	η/α	10%	5%	1%
$\eta = 0.0$	5.838	7.215	10.474	$\eta = 1.3$	5.609	7.024	10.235
$\eta = 0.3$	6.173	7.556	10.819	$\eta = 1.5$	5.516	6.909	10.090
$\eta = 0.5$	6.537	7.934	11.188	$\eta = 1.7$	5.436	6.822	10.014
$\eta = 0.7$	7.191	8.622	11.861	$\eta = 2.0$	5.340	6.715	9.913

size of the training sample. In the next section, we provide a set of simulations which could form the basis for the selection of m. The "non-contamination" assumption – which requires no break within the training sample – can also be tested for, using e.g., the test by Horváth and Wang (2024).

4 | Simulations

In this section, we assess the finite sample performance of our monitoring procedures via Monte Carlo simulations. We use $\lfloor \cdot \rfloor$ to denote the integer part of numbers.

According to the theory in Section 3, we can have two classes of monitoring schemes, based on $\eta \neq 1$ (covered by Theorem 1) and $\eta = 1$ (covered by Theorem 2). For the sake of brevity, here we only focus on the case $\eta \neq 1$. We consider several data generating processes (DGP). We use three lengths of the training sample $m = 500, 1000, 5000$ and two lengths of the monitoring $\mathcal{n} = 250, 500$. The sequential procedure is performed 5,000 times with independently generated samples, and the percentage of simulations for which the detector crosses the boundary functions is reported for several values of η. For the Rényi type statistic based on Theorem 1 (*ii*), we follow Horváth et al. (2021a) and set $r = \left\lfloor \sqrt{\mathcal{n}} \right\rfloor$. Guidelines on implementation are provided in Section B.1 of the Supplement. To obtain critical values, we simulate two independent standard Wiener processes $W_1(t)$ and $W_2(t)$ on a grid of 100,000 equally spaced points in the unit interval $[0, 1]$ and compute $\sup_{0<t\leq 1}\left(W_1^2(t) + W_2^2(t)\right)/t^{\eta}$ and $\sup_{0<t\leq 1}\left(W_1^2(t) + W_2^2(t)\right)/t^{1-\eta}$. We repeat this by 100,000 times and obtain the empirical 90%, 95% and 99% percentiles of the above two quantities, corresponding to the critical values at 10%, 5% and 1% levels based on Theorem 1 (*i*) and (*ii*). Critical values are shown in Table 1.

The boundary functions in Section 2 are designed for the case $m \to \infty$. However, preliminary simulations show that the empirical sizes based on those boundary functions tend to be larger than the nominal levels in finite samples, in particular for DGPs with the Student's t errors. To make our monitoring schemes more practical under small finite samples, we suggest to "tune" the boundary functions as

$$\mathfrak{g}_m(k) = c\mathcal{n}\left(1 + \frac{1}{\log(m)}\right)^2\left(1 + \frac{k}{m}\right)^2\left(\frac{k}{\mathcal{n}}\right)^{\eta}, \qquad \text{with } 0 \leq \eta < 1 \tag{26}$$

$$\bar{\mathfrak{g}}_m(k) = cr\left(1 + \frac{1}{\log(m)}\right)^2\left(1 + \frac{k}{m}\right)^2\left(\frac{k}{r}\right)^{\eta}, \qquad \text{with } \eta > 1 \tag{27}$$

for (8) and (11), respectively. The intuition underpinning the term $\left(1 + 1/\log(m)\right)^2$ is to boost the boundary function in small samples. The term $(1 + k/m)^2$ as is typically used when the monitoring horizon is "long" (see Horváth et al., (2022), Horváth et al., (2021b), and Horváth et al. 2022). To understand the term $(1 + k/m)^2$, we note that, in the case of $c > 0$, the limiting distribution is determined by

$$\max_{1\leq k\leq \mathcal{n}} \frac{1}{m}\left(\frac{k}{m+k}\right)^{-\eta}\left(1 + \frac{k}{m}\right)^{-2}\left|W_2(k) - \frac{k}{m}W_1(m)\right|^2$$
$$\stackrel{\mathcal{D}}{=} \max_{1/m\leq t\leq \mathcal{n}/m}\left(\frac{t}{1+t}\right)^{-\eta}(1+t)^{-2}\left|W_2(t) - tW_1(1)\right|^2(1 + o_P(1))$$

where $\left\{W_1(t), t \geq 0\right\}$ and $\left\{W_2(t), t \geq 0\right\}$ are two independent Wiener processes. By computing the covariance kernel, one gets that

$$\left\{W_2(t) - tW_1(1), t \geq 0\right\} \stackrel{\mathcal{D}}{=} \{(1+t)W(t/(1+t)), 0 < t\}$$

where $W(\cdot)$ is a Wiener process. We conclude

$$\max_{1\leq t\leq \mathcal{n}/m}\left(\frac{t}{1+t}\right)^{-\eta}\frac{1}{(1+t)^2}\left|W_2(t) - tW_1(1)\right|^2$$
$$\stackrel{\mathcal{D}}{=} \max_{1\leq t\leq \mathcal{n}/m}\left(\frac{t}{1+t}\right)^{\eta}\left|W\left(\frac{t}{1+t}\right)\right|^2$$

The term $(1 + k/m)^2$ at the denominator is needed to get a weighted Wiener process in the limit, i.e., a "simple" limit. If $c = 0$, the term $(1 + k/m)^2$ collapses onto 1, so it is inconsequential on the limiting distribution our monitoring scheme. Although this term is inconsequential for the asymptotic theory in our set-up, we find that it can further improve the empirical size. Both tuning terms are asymptotically negligible. and only play a role in finite samples to achieve better size control at no expense for power. The proposed tuning is tailored to DGPs with Student's t errors, rather than Gaussian errors; indeed, heavy tails are a well-known stylised fact of financial returns.

4.1 | Empirical Size Under the Null

Under the null, the realisation of GARCH(1,1) in the training period $(1 \leqslant i \leqslant m)$ and in the monitoring period $(m + 1 \leqslant i \leqslant m + \mathcal{n})$ is

$$y_i = \sigma_i\epsilon_i, \quad \text{and} \quad \sigma_i = \omega_0 + \alpha_0 y_{i-1}^2 + \beta_0\sigma_{i-1}^2, \quad 1 \leq i \leq m + \mathcal{n}$$

where ϵ_i follows a standard normal distribution or the Student's t distribution with 7 degrees of freedom. Since our

TABLE 2 | Empirical size based on Theorem 1 (i).

	$\epsilon_i \sim \mathcal{N}(0,1)$			$\epsilon_i \sim$ Student's t		
	$m = 500$	$m = 1000$	$m = 5000$	$m = 500$	$m = 1000$	$m = 5000$
			Stationary GARCH(1,1)			
			$n = 250$			
$\eta = 0.0$	6.5%	4.8%	3.5%	8.3%	8.0%	6.2%
$\eta = 0.3$	7.1%	5.5%	3.8%	10.1%	8.9%	7.5%
$\eta = 0.5$	8.5%	6.2%	4.5%	11.3%	10.8%	9.1%
$\eta = 0.7$	10.1%	8.8%	6.6%	14.5%	13.2%	10.9%
$\eta = 0.9$	13.9%	10.5%	9.9%	18.2%	14.7%	10.3%
			$n = 500$			
$\eta = 0.0$	5.2%	4.3%	2.7%	8.3%	5.4%	4.6%
$\eta = 0.3$	5.8%	4.8%	2.9%	9.3%	6.1%	5.2%
$\eta = 0.5$	6.6%	5.4%	3.7%	11.4%	8.2%	6.9%
$\eta = 0.7$	8.7%	7.4%	5.6%	14.3%	10.5%	8.8%
$\eta = 0.9$	13.9%	10.9%	8.4%	18.1%	16.1%	11.5%
			Nonstationary GARCH(1,1)			
			$n = 250$			
$\eta = 0.0$	5.4%	4.3%	3.7%	8.8%	5.2%	5.0%
$\eta = 0.3$	6.2%	5.2%	4.2%	10.7%	6.5%	5.5%
$\eta = 0.5$	7.5%	6.7%	5.1%	12.3%	8.5%	6.8%
$\eta = 0.7$	9.9%	9.4%	5.9%	14.7%	12.2%	9.1%
$\eta = 0.9$	13.6%	11.0%	11.4%	17.3%	14.3%	11.6%
			$n = 500$			
$\eta = 0.0$	2.6%	2.9%	2.9%	5.7%	4.4%	4.0%
$\eta = 0.3$	3.8%	3.2%	3.0%	7.2%	5.3%	4.1%
$\eta = 0.5$	4.5%	3.9%	3.6%	8.7%	7.7%	5.3%
$\eta = 0.7$	7.4%	6.0%	5.1%	13.0%	9.6%	8.1%
$\eta = 0.9$	12.1%	10.3%	9.6%	17.2%	13.8%	11.0%

monitoring procedure does not require the historical sample to be stationary or not, we choose the following two set of GARCH(1,1) parameters, taken from Francq and Zakoïan (2012): (i) $(\omega_0, \alpha_0, \beta_0) = (0.10, 0.18, 0.80)$, which represents the stationary case; (ii) $(\omega_0, \alpha_0, \beta_0) = (0.10, 0.30, 0.80)$, corresponding to the nonstationary case since $E \log(\alpha\epsilon_0^2 + \beta) > 0$ under errors following either the standard normal or the Student's t distributions.

Table 2 reports the empirical sizes at 5% significance level for the monitoring scheme based on Theorem 1(i) for different values of η. A noticeable feature is that a larger η results in a higher rejection rates and a smaller η is more conservative in rejection; this can be expected, since, as η approaches 1, the limit undergoes a transition from a Gaussian to a non Gaussian, Extreme-Value-type limit, which is compounded by the notoriously slow convergence towards Extreme Value distributions (see e.g., Hall 1979, for a general reference, and Gombay and Horváth 1996, for a more specific treatment). Under the Student's t errors, $\eta = 0.3$ is a good choice because the monitoring procedure has reasonably good empirical sizes when $m = 1{,}000$, and the empirical sizes for $m = 5{,}000$ are closer to the theoretical level of 5%. Under Gaussian errors, the monitoring procedure is slightly under-sized, which is mainly due to the additionally tuning we imposed in (26). For practical use, the tendency to under-reject with Gaussian errors may not necessarily be a concern, because the empirical power does not seem to be affected, as shown in Section 4.2. Lastly, the simulation results show that our monitoring schemed works reasonably well for both stationary and nonstationary GARCH(1,1) models.

Table B.1 in the Supplement contains the empirical sizes for Rényi type statistics. The rejection rates are slightly higher than the 5% nominal level. We note that, in principle, it would be possible to design a different tuning for Rényi type statistics.

4.2 | Empirical Power Under H_A

We now turn to the analysis of the empirical power. Under the alternative, the data is generated by

$$y_i = \sigma_i \epsilon_i$$

$$\sigma_i = \begin{cases} \omega_0 + \alpha_0 y_{i-1}^2 + \beta_0 \sigma_{i-1}^2, & \text{if } 1 \leq i < m + k^* \\ \omega_A + \alpha_A y_{i-1}^2 + \beta_A \sigma_{i-1}^2, & \text{if } m + k^* \leq i \leq m + n \end{cases}$$

TABLE 3 | Empirical power based on Theorem 1 *i*) for a change at $k^* = \lfloor \sqrt{n} \rfloor$.

$n = 500$	$H_{A,1}$			$H_{A,2}$		
Before $k^* = \sqrt{n}$	$\beta_0 = 0.80$			$\beta_0 = 0.80$		
After $k^* = \sqrt{n}$	$\beta_1 = 0.60$			$\beta_1 = 0.90$		
$\epsilon_i \sim \mathcal{N}(0,1)$	$m = 500$	$m = 1000$	$m = 5000$	$m = 500$	$m = 1000$	$m = 5000$
$\eta = 0.0$	96.76%	99.94%	100.00%	99.76%	100.00%	100.00%
$\eta = 0.3$	96.38%	99.94%	100.00%	99.76%	100.00%	100.00%
$\eta = 0.5$	96.06%	99.94%	100.00%	99.74%	100.00%	100.00%
$\eta = 0.7$	95.44%	99.84%	100.00%	99.72%	100.00%	100.00%
$\eta = 0.9$	93.70%	99.74%	100.00%	99.42%	100.00%	100.00%
$\epsilon_i \sim$ Student's t						
$\eta = 0.0$	80.26%	96.22%	99.92%	96.28%	99.44%	100.00%
$\eta = 0.3$	79.16%	95.80%	99.92%	96.04%	99.36%	100.00%
$\eta = 0.5$	78.20%	95.40%	99.92%	95.94%	99.34%	100.00%
$\eta = 0.7$	76.88%	94.48%	99.92%	95.76%	99.18%	99.98%
$\eta = 0.9$	71.96%	90.76%	99.48%	94.62%	98.94%	100.00%
$n = 500$	$H_{A,3}$			$H_{A,4}$		
Before $k^* = \sqrt{n}$	$\beta_0 = 0.90$			$\beta_0 = 0.90$		
After $k^* = \sqrt{n}$	$\beta_1 = 0.80$			$\beta_1 = 1.00$		
$\epsilon_i \sim \mathcal{N}(0,1)$	$m = 500$	$m = 1000$	$m = 5000$	$m = 500$	$m = 1000$	$m = 5000$
$\eta = 0.0$	100.00%	100.00%	100.00%	99.82%	100.00%	100.00%
$\eta = 0.3$	100.00%	100.00%	100.00%	99.74%	100.00%	100.00%
$\eta = 0.5$	100.00%	100.00%	100.00%	99.70%	100.00%	100.00%
$\eta = 0.7$	100.00%	100.00%	100.00%	99.60%	100.00%	100.00%
$\eta = 0.9$	99.98%	100.00%	100.00%	99.18%	100.00%	100.00%
$\epsilon_i \sim$ Student's t						
$\eta = 0.0$	99.96%	100.00%	100.00%	94.00%	98.90%	100.00%
$\eta = 0.3$	99.94%	100.00%	100.00%	93.76%	98.84%	100.00%
$\eta = 0.5$	99.94%	100.00%	100.00%	93.40%	98.70%	100.00%
$\eta = 0.7$	99.94%	100.00%	100.00%	92.98%	98.42%	99.98%
$\eta = 0.9$	99.92%	100.00%	100.00%	92.06%	98.06%	99.90%

where the parameter $\theta_0 = (\alpha_0, \beta_0, \omega_0)^\top$ changes to $\theta_A = (\alpha_A, \beta_A, \omega_A)^\top$ at time $m + k^*$. We consider two scenarios for the time of change: (a) $k^* = \lfloor \sqrt{n} \rfloor$ corresponds to a change occurring "early, but not too early" after the historical sample; (b) $k^* = \lfloor 0.5n \rfloor$ indicates a change happening much later than r.

There are many possible ways of changes under the alternative. To keep our results clean, we set $\omega_0 = \omega_A = 0.1$ and $\alpha_0 = \alpha_A = 0.18$, and concentrate on a change in β under the following four representative alternatives:

- $H_{A,1}$: $\beta_0 = 0.8, \beta_A = 0.6$, i.e., a change from a stationary to another stationary regime,
- $H_{A,2}$: $\beta_0 = 0.8, \beta_A = 0.9$, i.e., a change from a stationary to an explosive regime,
- $H_{A,3}$: $\beta_0 = 0.9, \beta_A = 0.8$, i.e., a change from an explosive to a stationary regime,
- $H_{A,4}$: $\beta_0 = 0.9, \beta_A = 1.0$, i.e., a change from an explosive to another explosive regime.

Tables 3 and 4 show the empirical power of the monitoring scheme based on Theorem 1 (i) at 5% significance level when $n = 500$ for a change at $k^* = \lfloor \sqrt{n} \rfloor$ and $k^* = \lfloor 0.5n \rfloor$, respectively.[4] There are five major observations. First, our monitoring scheme is highly effective in detecting changes under $H_{A,2}$ and $H_{A,3}$ for both early and late changes. These alternatives result in a change between a stationary regime and an explosive regime, which is relatively easy to detect. Second, the monitoring scheme exhibits high power in detecting early changes under $H_{A,1}$ and $H_{A,4}$. These alternatives represent a change within either a

TABLE 4 | Empirical power based on Theorem 3.1(i) for a change at $k^* = \lfloor 0.5n \rfloor$.

$n = 500$	$H_{A,1}$			$H_{A,2}$		
Before $k^* = 0.5n$	$\beta_0 = 0.80$			$\beta_0 = 0.80$		
After $k^* = 0.5n$	$\beta_1 = 0.60$			$\beta_1 = 0.90$		
$\epsilon_i \sim \mathcal{N}(0,1)$	$m = 500$	$m = 1000$	$m = 5000$	$m = 500$	$m = 1000$	$m = 5000$
$\eta = 0.0$	50.26%	77.88%	98.46%	82.08%	96.04%	99.90%
$\eta = 0.3$	47.84%	76.16%	98.12%	80.92%	95.74%	99.90%
$\eta = 0.5$	46.38%	74.32%	97.56%	79.80%	95.26%	99.88%
$\eta = 0.7$	43.26%	70.56%	96.60%	77.86%	94.12%	99.86%
$\eta = 0.9$	39.26%	62.12%	93.92%	73.26%	91.56%	99.74%
$\epsilon_i \sim$ Student's t						
$\eta = 0.0$	25.68%	43.10%	72.88%	72.42%	84.98%	97.02%
$\eta = 0.3$	25.02%	41.18%	70.48%	71.70%	84.30%	96.68%
$\eta = 0.5$	24.80%	39.46%	68.18%	70.62%	83.44%	96.22%
$\eta = 0.7$	25.18%	37.68%	64.14%	69.76%	81.90%	95.56%
$\eta = 0.9$	25.30%	31.70%	53.44%	66.16%	78.54%	93.40%
$n = 500$	$H_{A,3}$			$H_{A,4}$		
Before $k^* = \sqrt{n}$	$\beta_0 = 0.90$			$\beta_0 = 0.90$		
After $k^* = \sqrt{n}$	$\beta_1 = 0.80$			$\beta_1 = 1.00$		
$\epsilon_i \sim \mathcal{N}(0,1)$	$m = 500$	$m = 1000$	$m = 5000$	$m = 500$	$m = 1000$	$m = 5000$
$\eta = 0.0$	100.00%	100.00%	100.00%	76.78%	93.14%	99.90%
$\eta = 0.3$	100.00%	100.00%	100.00%	75.64%	92.36%	99.88%
$\eta = 0.5$	100.00%	100.00%	100.00%	74.40%	91.58%	99.86%
$\eta = 0.7$	100.00%	100.00%	100.00%	71.98%	89.88%	99.66%
$\eta = 0.9$	98.32%	99.84%	100.00%	66.42%	86.74%	99.40%
$\epsilon_i \sim$ Student's t						
$\eta = 0.0$	99.96%	100.00%	100.00%	64.16%	78.82%	94.30%
$\eta = 0.3$	99.94%	100.00%	100.00%	63.22%	78.14%	93.76%
$\eta = 0.5$	99.94%	100.00%	100.00%	62.38%	77.00%	93.26%
$\eta = 0.7$	99.94%	100.00%	100.00%	61.28%	75.30%	92.16%
$\eta = 0.9$	96.74%	98.96%	99.94%	59.60%	70.92%	88.54%

stationary or an explosive regime. Third, there is a deterioration in power when detecting late changes under $H_{A,1}$ and $H_{A,4}$, although satisfactory levels can be achieved by using a large(r) training sample size of $m = 5,000$. Fourth, the power is relatively lower when using the Student's t distribution errors compared to normal errors. Lastly, there is only a marginal decline observed in the power with a larger value of η.

Tables B.2 and B.3 in the Supplement provide the empirical power for the Rényi type statistics based on Theorem 1 (ii) under the same setting. When detecting early changes at $k^* = \lfloor \sqrt{n} \rfloor$, similar observations as above apply; the monitoring schemes with $\eta = 1.3$ and 1.5 proves to be effective. However, one noticeable difference is that a larger value of η is detrimental in the power. In particular, $\eta = 1.7$ and 2 suffer a remarkable loss of power under $H_{A,1}$. As far as late changes ($k^* = \lfloor 0.5n \rfloor$) are concerned, the Rényi type statistics become much less effective, as predicted by the theory. This is because Rényi type statistics are devised for the fast detection of very early changes, whilst being suboptimal for late changes.

It is also worthwhile to examine the stopping time τ_m and $\overline{\tau}_m$ to investigate the detection delays of our monitoring procedures. Figure 1 shows the boxplot of the detection delays of τ_m and $\overline{\tau}_m$ for a change at $k^* = \lfloor \sqrt{n} \rfloor$ under $H_{A,2}$ when $m = 500$, $n = 500$. For the monitoring procedure based on Theorem 1 (i), it is consistent with our theory that larger values of η reduce the detection delay. Considering the Rényi type statistics based on Theorem 1 (ii), there is only a marginal difference in using various values of η. Comparing the detection delay between the monitoring procedures based on Theorem 1 (i) and (ii), we can clearly see the merit of the Rényi type statistics for the fast detection of early changes, as evidenced by shorter detection delays.

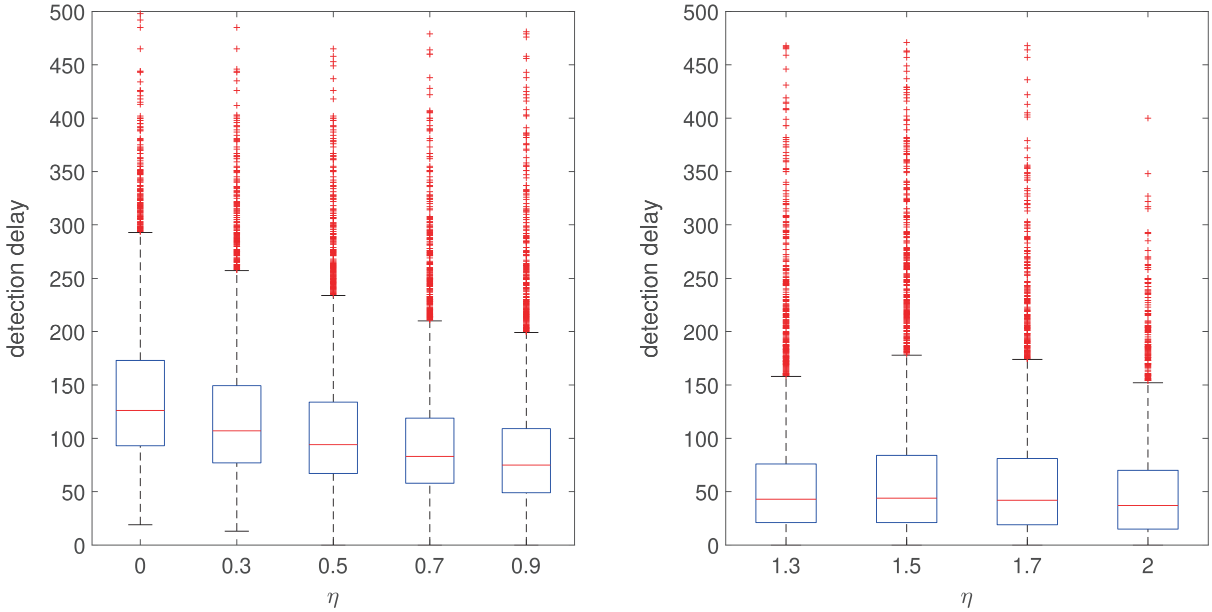

FIGURE 1 | Boxplot of detection delays when $m = 500$, $n = 500$ for a change at $k^* = \lfloor \sqrt{n} \rfloor$ under $H_{A,2}$. Left Panel: the monitoring procedure based on Theorem 1 (i); Right Panel: the monitoring procedure based on Theorem 1 (ii).

5 | Empirical Illustration

We use daily returns of individual stocks, focusing on four stocks: Apple Inc. (ticker: AAPL, Permno: 14593), Middlefield Banc Corp. (ticker: MBCN, Permno: 14932), Genetic Technologies Ltd (ticker: GENE, Permno: 90899) and NTS Realty Holdings LP (ticker: NLP, Permno: 90508); daily returns (without dividend) are downloaded from the CRSP database.[5] We consider two periods to showcase the detection for four types of changes. Depending on the specific purpose of the researcher, one can choose between the monitoring procedures based on Theorem 1 (i) and (ii). Based on our simulations, if the aim is to quickly detect very early changes, we suggest using the Rényi type statistics based on Theorem 1*(ii)*, with some tolerance for the compromise in size and power; conversely, if the purpose is to have good size control and high power, it is recommended to use the monitoring procedure based on Theorem 1 (i). In this application, our preference is to have a good balance of size and power, and the procedure based on Theorem 1 (i) (with the choice of $\eta = 0.3$) delivers a good performance with sample sizes similar to the dataset used in this section.[6] Before applying our monitoring procedure, we use the test developed by (Horváth and Wang 2024, HW(2024) henceforth) to detect changes in GARCH(1,1). A rejection indicates there is no change of (α, β) in the GARCH(1,1) during the historical sample. Further, we examine whether our historical sample is stationary or not by using the nonstationarity test developed by (Francq and Zakoïan 2012, FZ(2012) henceforth). At the end of our monitoring horizon, we use the FZ(2012) test again to check the stationarity of the samples after the change (if there is one).

5.1 | Change From a Stationary Regime

To illustrate changepoint detection from a stationary regime, we choose the training period of 2016–2019 (1,007 trading days) and the monitoring period of 2020–2021 (507 trading days). The training period is before the outbreak of COVID-19, whereas the monitoring period is in the pandemic. We apply our monitoring procedure for the stocks of AAPL and MBCN during this period. Table 5 (Columns 1 and 2) reports the results of the sequential monitoring procedure, as well as other information, including HW(2024) test, FZ(2012) test, and parameter estimates. HW(2024) test indicates that there is no parameter change during the training sample for AAPL and MBCN. The nonstationarity test of FZ(2012) indicates that they are both stationary during the training sample. Our sequential monitoring detects a change of AAPL on July 31st, 2020 and a change of MBCN on May 8th, 2020. Based on the FZ(2012) test for the sample after the change, we can conclude that AAPL experienced a change from a stationary to another stationary regime, whereas MBCN shifted from a stationary regime to a nonstationary one. Figure 2 contains returns series (upper panel) during the monitoring period and the detector versus the boundary function (lower panel).

5.2 | Change From a Nonstationary Regime

We now consider detection from a nonstationary regime, and use 2007–2010 (1,011 trading days) as the training period and 2011–2022 (505 trading days) as the monitoring period. The training period covers the global financial crisis (GFC), whereas the monitoring period follows the GFC but includes the European debt crisis. In this period, we monitor GENE and NLP. The results of the sequential monitoring procedure, alongside other Supporting Information, are displayed in Columns 3 and 4 of Table 5. Based on the HW(2024) test, we cannot reject that the return series of GENE and NLP have change in the training period. As evidenced by the nonstationarity test of FZ(2012), both stocks are in the nonstationary regime during the training period. Our sequential monitoring procedure reveals a change of GENE on

TABLE 5 | Monitoring results of the four stocks.

	AAPL	MBCN	GENE	NLP
Training Sample				
Start date	2016-01-04	2016-01-04	2007-01-03	2007-01-03
End date	2019-12-31	2019-12-31	2010-12-31	2010-12-31
Sample size	1006	1006	1006	1008
HW(2024) test				
Test stat	1.463	1.395	1.176	1.258
Rej. of HW(2024)	Not Rej.	Not Rej.	Not Rej.	Not Rej.
FZ(2012) NS test				
p-value	0.00%	0.00%	35.29%	33.81%
Stationary or not	Stationary	Stationary	Nonstationary	Nonstationary
Parameter estimates				
$\hat{\alpha}_0$	0.135	0.183	0.287	0.099
$\hat{\beta}_0$	0.745	0.579	0.816	0.916
Monitoring sample				
Start date	2020-01-02	2020-01-02	2011-01-03	2011-01-03
End date	2021-12-31	2021-12-31	2012-12-31	2012-12-31
Sample size	505	505	494	502
Our sequential monitoring				
Rejection	Rej.	Rej.	Rej.	Rej.
Time of Change	2020-07-31	2020-05-08	2011-04-27	2012-09-04
After the change				
FZ(2012) NS test				
p-value	0.00%	10.38%	0.00%	100.00%
Stationary or not	Stationary	Nonstationary	Stationary	Nonstationary
Parameter estimates				
$\hat{\alpha}_A$	0.052	0.091	0.488	0.001
$\hat{\beta}_A$	0.935	0.913	0.528	1.053

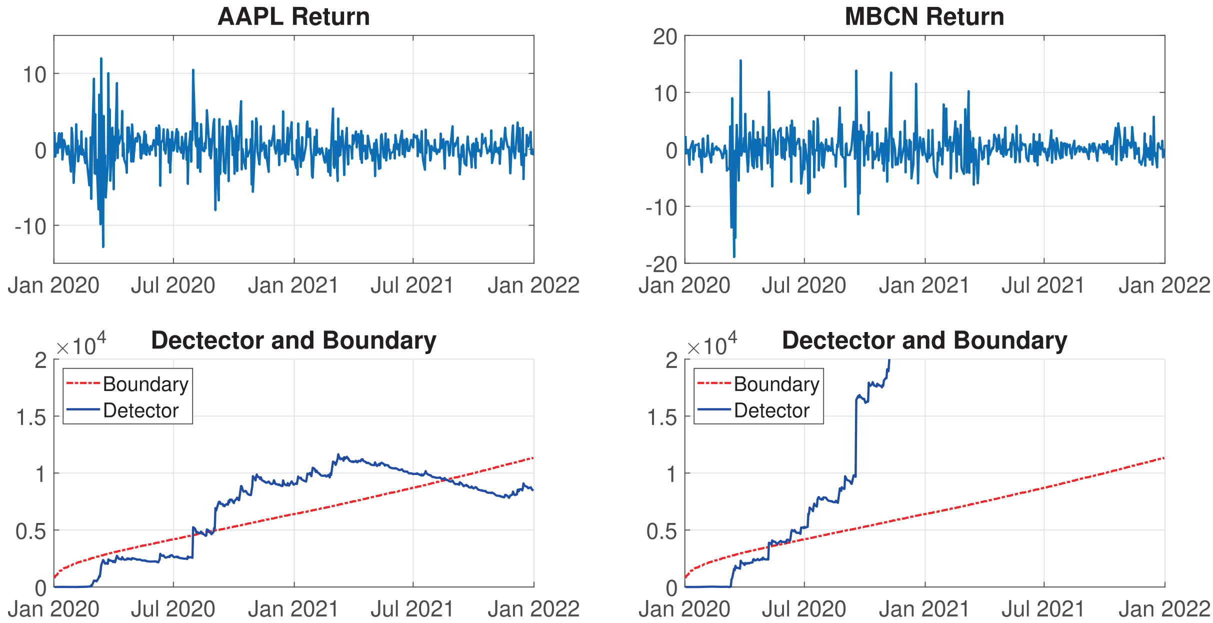

FIGURE 2 | Upper panel: The return series of AAPL (left) and MBCN (right) during the monitoring period; Lower panel: The detector $\mathcal{D}_m(k)$ versus the boundary function $\mathfrak{g}_m(k)$ of AAPL (left) and MBCN (right).

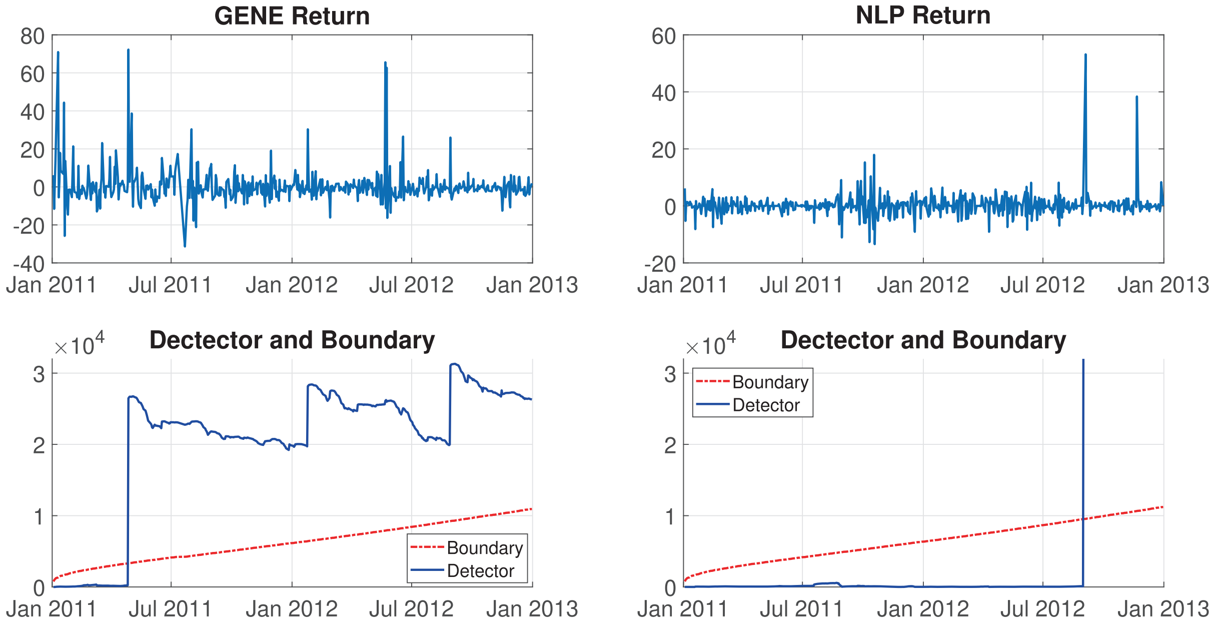

FIGURE 3 | Upper panel: The return series of GENE (left) and NLP (right) during the monitoring period; Lower panel: The detector $\mathcal{D}_m(k)$ versus the boundary function $\mathfrak{g}_m(k)$ of GENE (left) and NLP (right).

April 27th, 2011 and a change of NLP on September 4th, 2012. After applying FZ(2012) nonstationarity test on the sample after the change, it is found that the change of GENE is from a nonstationary regime to a stationary regime, whilst the change of NLP is from a nonstationary to another nonstationary regime. It is also interesting to note that GENE after the change is in a strict stationary regime, but not in a second-order stationary regime. Figure 3 shows their returns series (upper panel) during the monitoring period and the detector versus the boundary function (lower panel). The sudden jump of the detector for NLP can be attributed to a "going private" proposal made on August 31st, 2012 and kept in negotiation until a definitive merger agreement announced on December 27th, 2012.[7] This is a prolonged event, rather than an outlier, that is associated with the change in the GARCH(1,1) parameters of the stock.

6 | Conclusions and Discussions

In this article, we complement the existing literature on (ex-ante) testing for bubble phenomena by proposing a family of weighted, CUSUM-based statistics to detect changes in the parameters of a GARCH(1,1) process. Our monitoring procedure can be applied irrespective of whether, in the training sample, the observations are stationary or explosive, and it is able to detect all types of changes: (a) from a stationary to another stationary regime (which is helpful to avoid the issues concerning the consistent estimation of a GARCH(1,1) process spelt out in Hillebrand 2005); (b) from a stationary to an explosive regime (which contains information on the possible inception of a bubble); (c) from an explosive to a stationary regime (which, in the light of the previous point, could shed light on the cooling off the turbulence associated with a bubble on a financial market); and (d) from an explosive to another explosive regime (which, depending on the direction of the change – towards a more or a less explosive regime – could indicate whether exuberant volatility is heating up or cooling down). On account of their nature as *omnibus* tests, it is important to emphasise that our procedures only indicate a change in the parameters of a GARCH(1,1) model. Subsequent analysis, after a changepoint has been found, is required in order to ascertain the stationarity/explosivity of the (volatility of the) observations before and after the break. Hence, our methodologies constitute the first step of the analysis – with the (major) advantage that they can be used with no previous knowledge as to the nature of the data – and can be complemented a posteriori by the application of a test for stationarity like the one proposed in Francq and Zakoïan (2012). Building on the theory developed in this article, further statistics can be proposed which are tailored to more specific changes: for example, a detector could be based on $\hat{\alpha} + \hat{\beta}$, monitoring when this quantity exceeds 1 (thus indicating a change to explosivity).[8] An important question is also how to select the "optimal" η, which guarantees the shortest detection delay; as mentioned above, such an optimal value of η does not exist in general. However, building on the theory developed in this article, one could consider a combination of different values of η, along the same lines as Ghezzi et al. (2024). Another possible approach would be to use a rolling, as opposed to recursive, estimator.

Technically, we propose two families of statistics, both based on weighted versions of the CUSUM process of the quasi-Fisher scores: one family uses lighter weights, and it is designed to detect, optimally, changes occurring not immediately after the start of the monitoring horizon; the other family uses heavier, Rényi-type weights, which make it more sensitive to changepoints occurring immediately after the end of the training period. For both cases, we study the limiting distribution of the detection delays; to the best of our knowledge, no such results exist for the case of a GARCH(1,1) models, and no results in general exist for the case of Rényi statistics. Given the interest in the detection of bubble phenomena, and the scant amount of contributions in the context of detection of changes in the volatility, we believe that our article should be a useful addition to the toolbox of the financial econometrician.

Conflicts of Interest

The authors declare no conflicts of interest.

Endnotes

[1] A complete literature review on (the very popular) GARCH models goes beyond the scope of this article; we refer to the book by Francq and Zakoian (2019) as a state-of-the-art reference.

[2] Such divergence can be shown in probability, but a.s. divergence to infinity cannot be established.

[3] Note that the sequence y_{i-1}, per se, needs not be stationary since it starts from an initial value, whence its replacement with its stationary approximation $\hat{x}_{i-1}$.

[4] The empirical power of $n = 250$ (not reported) is marginally lower than the empirical power of $n = 500$.

[5] We choose to use the daily returns without dividend, rather than log difference of prices, to avoid the complication due to stock splits.

[6] We relegate the results using Rényi weights to Section B.4 of the Supplement.

[7] https://www.sec.gov/Archives/edgar/data/1278384/000127838412000029/ex_99-1.htm, assessed on 1 February 2025.

[8] We are grateful to an anonymous Referee for suggesting this to us.

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Supporting Information

Additional supporting information can be found online in the Supporting Information section.