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ABSTRACT

In this article, we study the generalized Hilbert matrix operator Γ_μ acting on the Bergman spaces A^p of the unit disc for $1 \leq p < \infty$. In particular, we characterize the measures μ for which the operator Γ_μ is bounded, determine the exact value of the norm for $p \geq 4$, and provide norm estimates for the other values of p. Additionally, we observe an unexpected behavior in the case $p = 2$. Finally, we characterize the measures μ for which Γ_μ is compact by calculating its exact essential norm.

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1. Introduction

The classical Hilbert matrix

$$H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

stands among the most studied matrices, acting as an operator on sequence spaces and on spaces of analytic functions of the unit disc $\mathbb{D}$, of the complex plane $\mathbb{C}$. When viewed as an operator on spaces of analytic functions, such as Hardy or Bergman spaces, the Hilbert matrix becomes a key tool in exploring the relationship between function theory and operator theory. It serves as a fundamental example in the study of Hankel matrices, that is, matrices whose entries h_{nk} depend only on the sum of the indices $n+k$ (see [29] for more details). This connection between the algebraic structure of Hankel matrices and the analytic properties of functions provides valuable insights, making the Hilbert matrix a central object of study in both operator theory and complex analysis.

More specifically, the action of the matrix H as an operator on spaces of analytic functions was initially explored in the context of Hardy spaces H^p in [8], and later in [10]. It was established that its operator norm is precisely equal to the quantity $\frac{\pi}{\sin(\frac{\pi}{p})}$, for $1 < p < \infty$, while it remains unbounded in the limit cases $p = 1$ and $p = \infty$. Subsequently, the study of the operator focused on the Bergman spaces A^p, consisting of the analytic functions in $L^p(\mathbb{D})$. In [7], Diamantopoulos showed that H is bounded on A^p if and only if $p > 2$, and provided a sharp upper bound for its norm when $p \geq 4$. A precise lower bound for $\|H\|_{A^p}$ was obtained in [10] for every $p > 2$, by utilizing appropriate test functions, thereby determining the exact value of the norm for $p \geq 4$. In [4], the authors derived a sharp upper bound in the range $2 < p < 4$ by applying novel estimates for the beta function. Specifically, it was shown that for each $p > 2$,

$$\|H\|_{A^p} = \frac{\pi}{\sin\left(\frac{2\pi}{p}\right)} = \int_0^1 \frac{t^{2/p-1}}{(1-t)^{2/p}}\, dt. \tag{1}$$

See also [26] where a simplified proof of (1) appears. However, several questions remain open regarding the behavior of the classical Hilbert matrix operator on Bergman spaces. While it is established that H is bounded in the standard weighted Bergman spaces A^p_α if and only if $1 < \alpha + 2 < p$ [20], the precise value of the norm $\|H\|_{A^p_\alpha}$ has yet to be fully determined. The conjectured value of the norm, $\pi(\sin((2+\alpha)\pi/p))^{-1}$ [21], has been proved to be correct in [22] when $\alpha > 0$ and

$$p \geq \alpha + 2 + \sqrt{(\alpha+2)^2 - \left(\sqrt{2} - \frac{1}{2}\right)(\alpha+2)}.$$

For more recent advancements, we refer to [9], and [6], where the case of negative index α is also explored. Additionally, [2] and the references therein provide an overview of the latest developments in the study of the Hilbert matrix operator, particularly its action on various spaces of analytic functions and sequence spaces.

Over the past two decades, various generalizations of the Hilbert matrix operator have been the subject of extensive research (see, for instance, [16], [5], [14], [24], [3], and [13]). In this paper, we investigate a recently introduced generalization [1], defined as follows: Let μ be a probability Borel measure on $[0, 1)$. We then consider the infinite matrix

$$\tilde{\Gamma}_\mu = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \cdots \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \cdots \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with entries

$$\gamma_{nk} = \binom{n+k}{k} \int_0^1 t^k (1-t)^n \, d\mu(t).$$

The matrix $\tilde{\Gamma}_\mu$ is related to the classical Hausdorff matrix $\mathcal{K}_\mu$ induced by the moment sequence $\{\mu_n\}$ of the measure μ, that is, for $n = 0, 1, \ldots,$

$$\mu_n = \int_0^1 t^n \, d\mu(t).$$

In particular,

$$\mathcal{K}_\mu = \begin{pmatrix} c_{00} & 0 & 0 & \cdots \\ c_{10} & c_{11} & 0 & \cdots \\ c_{20} & c_{21} & c_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with entries c_{nk} given by

$$c_{nk} = \binom{n}{k} \int_0^1 t^k (1-t)^{n-k} \, d\mu(t), \; 0 \leq k \leq n.$$

Hausdorff matrices are essential tools in summability theory, where they are used to analyze and adjust the convergence behavior of sequences. A classic example is the

Cesàro matrix, a typical example of a Hausdorff matrix, which averages sequences and provides insights into how transforming sequences can affect their convergence (see [18, Chapter 6] for a concise overview). Beyond sequences, Hausdorff matrices have also been explored in the context of spaces of analytic functions, further demonstrating their versatility and significance in mathematical analysis (see for example [17], [15] and [23]). The matrix $\tilde{\Gamma}_\mu$ is obtained by shifting the entries of the k-th column of $\mathcal{K}_\mu$, k-places up. More precisely, with respect to the standard basis $\{e_n\}_{n\geq 0}$, the matrix $\tilde{\Gamma}_\mu$ is related to $\mathcal{K}_\mu$ through the algebraic relation

$$\tilde{\Gamma}_\mu(e_n) = S^{*n} \circ \mathcal{K}_\mu(e_n),$$

where $S^*(e_0) = 0$ and $S^*(e_n) = e_{n-1}$, for $n \geq 1$. If μ is the Lebesgue measure, then the matrix $\tilde{\Gamma}_\mu$ reduces to the classical Hilbert matrix H.

In the sequel, we concentrate on the action of the matrix $\tilde{\Gamma}_\mu$ as an operator on the Bergman spaces of the unit disc. In [1], the authors studied the action of $\tilde{\Gamma}_\mu$ in the Hardy spaces H^p for $1 \leq p < \infty$, and they characterized the measures μ for which the operator $\tilde{\Gamma}_\mu$ is bounded. The matrix $\tilde{\Gamma}_\mu$ acts on the sequence of the Taylor coefficients of the analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ as follows:

$$\tilde{\Gamma}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_k \binom{n+k}{k} \int_0^1 t^k (1-t)^n \, d\mu(t) \right) z^n.$$

We prove that $\tilde{\Gamma}_\mu(f)(z)$ has an equivalent integral representation in A^p for $1 \leq p < \infty$, i.e.

$$\Gamma_\mu(f)(z) = \int_0^1 f(\varphi_t(z)) w_t(z) \, d\mu(t) = \int_0^1 T_t(f)(z) \, d\mu(t), \tag{2}$$

where $T_t(f) = w_t \cdot f \circ \varphi_t$ is a weighted composition operator with

$$\varphi_t(z) = \frac{t}{1+(t-1)z} \quad \text{and} \quad w_t(z) = \frac{1}{1+(t-1)z}.$$

We note that

$$\varphi_t(\mathbb{D}) = D\left(\frac{1}{2-t}, \frac{1-t}{2-t} \right)$$

is the open disc centered at $1/(2-t)$ with radius $(1-t)/(2-t)$. In particular, for every $0 < t < 1$, $\varphi_t(\mathbb{D}) \subset \mathbb{D}$ with $\overline{\varphi_t(\mathbb{D})} \cap \partial\mathbb{D} = \{1\}$.

In what follows, we focus on necessary and sufficient conditions for the continuity of the operators Γ_μ on A^p with $1 \leq p < \infty$. In order to formulate our main result it will be convenient to introduce the following function:

$$\Theta_p(t) = \begin{cases} \dfrac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}}, & \text{if } 2 < p < \infty; \\ \sqrt{\int_0^1 \mu[0,t] \log\left(\frac{e}{t}\right) \, d\mu(t) + \left(\int_0^1 \frac{1}{1-t} \, d\mu(t)\right)^2}, & \text{if } p = 2; \\ \dfrac{1}{(1-t)^{2/p}}, & \text{if } 1 \le p < 2. \end{cases}$$

We use the following convention:

$$\mu[a,b] = \int_a^b d\mu(s) = \int_{[a,b)} d\mu(s), \quad 0 \le a < b \le 1.$$

Theorem 1.1. *Let $1 \le p < \infty$. The operator $\Gamma_\mu : A^p \to A^p$ is bounded, if and only if*

$$\int_0^1 \Theta_p(t) \, d\mu(t) < \infty.$$

In particular, there exist positive constants $A(p), B(p)$ depending only on p such that

$$A(p) \int_0^1 \Theta_p(t) \, d\mu(t) \le \|\Gamma_\mu\|_{A^p \to A^p} \le B(p) \int_0^1 \Theta_p(t) \, d\mu(t). \tag{3}$$

When $p > 2$ the constant $A(p)$ can be chosen equal to 1 and when $p \ge 4$ the constant $B(p)$ can also be chosen equal to 1. Hence, when $p \ge 4$

$$\|\Gamma_\mu\|_{A^p \to A^p} = \int_0^1 \frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}} \, d\mu(t).$$

Observe that even for $2 < p < 4$, the lower bound of the norm remains the optimal value for which the inequality (3) holds true for all measures μ. This is because the lower bound in (3) corresponds exactly to the norm of the classical Hilbert matrix H, as specified in (1). On the other hand, from [27, Corollary 3.2], we know that when $p \to 2^+$

$$\sup_{a \in (0,1)} \|\Gamma_{\delta_a}(1)\|_{A^p} \Big/ \int_0^1 \Theta_p(t) \, d\delta_a(t) > 1,$$

where δ_a is a Dirac point measure at $a \in (0,1)$. This implies that for specific measures and particular values of p, the constant $A(p)$ could be chosen bigger than 1.

The expression of $\Theta_p(t)$ is different when $p = 2$ and in Proposition 3.3 we show that the natural condition coming from Lemma 2.3, that is

$$\int_0^1 \frac{\sqrt{\log(e/t)}}{1-t}\, d\mu(t) < \infty, \tag{4}$$

is not necessary for the boundedness of Γ_μ on A^2.

Furthermore, we consider compactness and complete continuity. We recall that an operator T on a Banach space X is compact if, for any bounded sequence $\{x_n\}$ in X, the sequence $\{T(x_n)\}$ contains a convergent subsequence. Moreover, an operator T is completely continuous on X if for any weakly convergent sequence $\{x_n\}$ in X, the sequence $\{T(x_n)\}$ converges in norm. In general, every compact operator is completely continuous, however the converse could be false when X is non-reflexive.

In order to prove that an operator T is non-compact, it is enough to show that its essential norm $\|T\|_{e,X}$ is non-zero. We recall that

$$\|T\|_{e,X} = \inf\left\{\|T-K\|_X \text{ where } K \text{ is a compact operator in } X\right\}.$$

It is clear that $\|T\|_X \geq \|T\|_{e,X}$.

Theorem 1.2. *Let $1 < p < \infty$. If $\Gamma_\mu : A^p \to A^p$ is bounded, then*

$$\|\Gamma_\mu\|_{e,A^p} = \int_0^1 \frac{t^{2/p-1}}{(1-t)^{2/p}}\, d\mu(t).$$

It is clear from the theorem above that Γ_μ is never compact on A^p if $2 \leq p < \infty$. On the other hand, if $1 < p < 2$, Γ_μ is compact in A^p if and only if $\mu = \delta_0$, that is, if μ is the Dirac point mass at 0. Indeed, in this case,

$$\Gamma_{\delta_0}(f)(z) = f(0)\frac{1}{1-z}$$

which is clearly compact as a rank one operator.

Theorem 1.3. *Let $\Gamma_\mu : A^1 \to A^1$ be bounded. Then Γ_μ is compact if and only if $\mu = \delta_0$. Nevertheless, for every probability measure μ such that Γ_μ is bounded on A^1, Γ_μ is completely continuous.*

The rest of the paper is organized as follows: In Section 2, we recall the classical properties of the Bergman space, and we prove that $\Gamma_\mu(f)$ is a well-defined analytic function and that the action of Γ_μ coincides with that of $\tilde{\Gamma}_\mu$ on A^p. We also estimate an upper bound for $\|T_t\|_{A^p}$ when $1 \leq p < \infty$. We split the proof of Theorem 1.1 into

Sections 3.1 and 3.2. In Section 3.1, we prove that Γ_μ is bounded in A^p when $1 \le p < \infty$, with $p \neq 2$. In Section 3.2, we focus on the case $p = 2$ and also provide some conditions for boundedness of which some are sufficient and some are necessary. In Section 4, we deal with compactness and we prove Theorems 1.2 and 1.3.

Before we delve into calculations, we first clarify the notation that we will use in the following sections. With $\mathbb{T} = \partial\mathbb{D}$ we refer to the unit circle in the complex plane. Given a set M, by χ_M we denote the characteristic function associated to the set M. We use the expression $\|f\|_X$ to denote the norm of an element $f \in X$. Moreover, if T is an operator from the space X to X by $\|T\|_X$ we denote its operator norm, that is

$$\|T\|_X = \sup_{\|f\|_X=1} \|T(f)\|_X.$$

Even if the two notations coincide, this should not cause confusion in this context. Finally, by the expressions $f \lesssim g$ and $g \gtrsim f$, we mean that there exists a constant $C > 0$ such that $f \le Cg$. If both $f \lesssim g$ and $f \gtrsim g$ hold, we write $f \sim g$. We also highlight that by the capital letter C, we denote constants whose values may change every time they appear.

2. Preliminaries

First of all, we recall the properties of the Bergman spaces A^p that will be used in the rest of the paper. For $1 \le p < \infty$, the Bergman space A^p consists of all the analytic functions in the unit disc for which

$$\|f\|_{A^p} = \left(\int_{\mathbb{D}} |f(z)|^p \, dA(z) \right)^{1/p} < \infty,$$

where $dA(z) = dxdy/\pi$ is the normalized Lebesgue area measure. We recall that, if $f \in A^p$ with $1 \le p < \infty$, the growth estimates

$$|f(z)| \le \left(\frac{1}{1-|z|^2} \right)^{\frac{2}{p}} \|f\|_{A^p}, \quad z \in \mathbb{D} \tag{5}$$

and for some independent $C > 0$

$$|f'(z)| \le C \left(\frac{1}{1-|z|^2} \right)^{\frac{2}{p}+1} \|f\|_{A^p}, \quad z \in \mathbb{D} \tag{6}$$

hold, see [31, p. 755] and [28, p. 338] respectively. Moreover, if $f(z) = \sum_{n\ge0} a_n z^n$, then

$$\|f\|^2_{A^2} = \sum_{n\ge0} \frac{|a_n|^2}{n+1}, \tag{7}$$

see [12, p. 11]. We remark that the Taylor partial sums of f, that is, the polynomials

$$S_N(f)(z)=\sum_{n=0}^{N}a_nz^n$$

with $N\in\mathbb{N}$, converge in A^p-norm to

$$f(z)=\sum_{n=0}^{\infty}a_nz^n$$

when $1<p<\infty$, see [12, p. 31, Theorem 4]. For more information about the Bergman spaces we refer to the classical monographs [12] and [19].

The first thing that needs to be verified is that the integral (2) involved in the definition of $\Gamma_\mu(f)$ is a well-defined analytic function in $\mathbb{D}$. We follow the reasoning of [7].

Proposition 2.1. *For $1\le p<\infty$, let*

$$\psi_p=\int_0^1\frac{1}{(1-t)^{2/p}}\,d\mu(t).$$

If $\psi_p<\infty$, then for every $f\in A^p$, $\Gamma_\mu(f)$ is a well-defined analytic function.

Proof. We first prove that for every fixed $z\in\mathbb{D}$, $\Gamma_\mu(f)(z)$ is well-defined. Indeed, due to (5), we have

$$\begin{aligned}|\Gamma_\mu(f)(z)|&\le\int_0^1\frac{1}{|1-(1-t)z|}|f(\varphi_t(z))|\,d\mu(t)\\&\le\int_0^1\frac{1}{|1-(1-t)z|^{1-2/p}}\frac{1}{(1-t)^{2/p}}\,d\mu(t)\cdot\frac{\|f\|_{A^p}}{(1-|z|)^{2/p}}\\&\le\frac{\|f\|_{A^p}}{(1-|z|)^{2/p+1}}\int_0^1\frac{1}{(1-t)^{2/p}}\,d\mu(t).\end{aligned}$$

In order to prove that $\Gamma_\mu(f)$ is analytic in $\mathbb{D}$, we show that there exists a sequence of analytic functions which converges to $\Gamma_\mu(f)$ uniformly on every compact subset of $\mathbb{D}$. Let $\{P_n\}_n$ be a family of polynomials such that

$$\lim_{n\to\infty}\|f-P_n\|_{A^p}=0,$$

see [12, p. 30, Theorems 3 and 4]. It is clear that for every $n\in\mathbb{N}$, $\Gamma_\mu(P_n)$ is analytic in $\mathbb{D}$. Therefore, for every z in a compact set $K\subset\mathbb{D}$,

$$\begin{aligned}\left|\Gamma_\mu(f)(z)-\Gamma_\mu(P_n)(z)\right| &\leq \|f-P_n\|_{A^p}\frac{\psi_p}{(1-|z|)^{2/p+1}} \\ &\leq \|f-P_n\|_{A^p}\frac{\psi_p}{\operatorname{dist}(K,\mathbb{T})^3},\end{aligned} \tag{8}$$

which concludes the proof of the proposition. □

The fact that in A^p the operators Γ_μ and $\tilde{\Gamma}_\mu$ coincide requires some standard estimates. For the sake of completeness, we will write them down.

Proposition 2.2. *Let $1 \leq p < \infty$ and $\psi_p < \infty$, where ψ_p is defined as in Proposition 2.1. If $f \in A^p$, then*

$$\Gamma_\mu(f) = \tilde{\Gamma}_\mu(f).$$

Proof. It is clear that for every analytic polynomial

$$\Gamma_\mu(P_n) = \tilde{\Gamma}_\mu(P_n).$$

We first consider the case $1 < p < \infty$ and $f(z) = \sum_{k\geq 0} a_k z^k \in A^p$. Then for every $z \in \mathbb{D}$, by the proof of Proposition 2.1,

$$\lim_{N\to\infty} |\Gamma_\mu(f)(z) - \tilde{\Gamma}_\mu(S_N)(z)| = \lim_{N\to\infty} |\Gamma_\mu(f)(z) - \Gamma_\mu(S_N)(z)| = 0,$$

where S_N is the Taylor partial sum. Since $\Gamma_\mu(f)$ and $\tilde{\Gamma}_\mu(S_N)$ are analytic functions, the Taylor coefficients coincide, that is, for every $n \in \mathbb{N}$,

$$\widehat{\Gamma_\mu(f)}(n) = \lim_{N\to\infty} \sum_{k=0}^{N} a_k \binom{n+k}{k} \int_0^1 t^k(1-t)^n\, d\mu(t),$$

which concludes the first part of the proof.

For $p = 1$ and $f(z) = \sum_{k\geq 0} a_k z^k \in A^1$, from (5) and (6), we note that

$$|a_0| \leq \|f\|_{A^1} \text{ and } |a_1| \leq C\|f\|_{A^1}.$$

Moreover, by the Cauchy formula, for every $r \geq 1/2$ and $n \geq 2$, we note that

$$\begin{aligned}|a_n| &\leq r^{-n}\int_0^{2\pi} |f(re^{i\theta})|\frac{d\theta}{2\pi} \\ &= r^{-n}(1-r)^{-1}\left\{(1-r)\int_0^{2\pi} |f(re^{i\theta})|\frac{d\theta}{2\pi}\right\} \leq r^{-n}(1-r)^{-1}2\|f\|_{A^1}.\end{aligned}$$

In particular, by choosing $r = 1 - 1/n$, we obtain that

$$|a_n| \le 2\left(1-\frac{1}{n}\right)^{-n}(n+1)\|f\|_{A^1} \le 8(n+1)\|f\|_{A^1}.$$

Therefore, for every $z \in \mathbb{D}$,

$$\begin{aligned}
|\tilde{\Gamma}_\mu(f)(z)| &= \left|\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} a_k \binom{n+k}{k}\int_0^1 t^k(1-t)^n\,d\mu(t)\right) z^n\right| \\
&\le \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} |a_k| \binom{n+k}{k}\int_0^1 t^k(1-t)^n\,d\mu(t)\right) |z|^n \\
&\le C\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}(k+1)\binom{n+k}{k}\int_0^1 t^k(1-t)^n\,d\mu(t)\right) |z|^n \cdot \|f\|_{A^1}.
\end{aligned}$$

Thus, since

$$\sum_{n=0}^{\infty}\binom{n+k}{k}(1-t)^n|z|^n t^k = \left(\frac{t}{1-(1-t)|z|}\right)^k \frac{1}{1-(1-t)|z|} = \varphi_t(|z|)^k w_t(|z|),$$

we have that

$$\begin{aligned}
|\tilde{\Gamma}_\mu(f)(z)| &\lesssim \int_0^1 \left(\sum_{k=0}^{\infty}(k+1)\varphi_t(|z|)^k\right) w_t(|z|)\,d\mu(t)\cdot\|f\|_{A^1} \\
&= \int_0^1 \frac{1}{(1-\varphi_t(|z|))^2} w_t(|z|)\,d\mu(t)\cdot\|f\|_{A^1} \\
&\le \int_0^1 \frac{1}{(1-t)^2}\,d\mu(t)\cdot\frac{\|f\|_{A^1}}{(1-|z|)^2} \\
&\le \psi_1 \cdot \frac{\|f\|_{A^1}}{(1-|z|)^2}.
\end{aligned}$$

Therefore, if $P_n \to f$ in A^1, due to (8), for every $z \in \mathbb{D}$, we have

$$\Gamma_\mu(f)(z) = \lim_{n\to\infty}\Gamma_\mu(P_n)(z) = \lim_{n\to\infty}\tilde{\Gamma}_\mu(P_n)(z) = \tilde{\Gamma}_\mu(f)(z),$$

which concludes the proof for $p = 1$. $\square$

The weighted composition operator

$$T_t(f)(z) = \frac{1}{(t-1)z+1} f\left(\frac{t}{(t-1)z+1}\right)$$

has been intensively studied in A^p in relation to the classical Hilbert matrix operator. In specific, Diamantopoulos [7] estimated its norm when $2 < p < \infty$. We extend this result to the case $1 \le p \le 2$.

Lemma 2.3. *Let $0 < t < 1$ and $f \in A^p$, $1 \le p < \infty$. Then*

$$\|T_t(f)\|_{A^p} \le C(p)\ \Psi_p(t)\ \|f\|_{A^p} \tag{9}$$

where $C(p)$ is a positive constant depending only on p and

$$\Psi_p(t) = \begin{cases} \dfrac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}}, & \text{if } 2 < p < \infty; \\[2ex] \dfrac{\sqrt{\log(e/t)}}{1-t}, & \text{if } p = 2; \\[2ex] \dfrac{1}{(1-t)^{2/p}}, & \text{if } 1 \le p < 2. \end{cases} \tag{10}$$

Moreover, if $p \ge 4$, we have $C(p) = 1$.

Proof. We recall that in [7, Lemma 2], the constant appearing in (9) was estimated for $2 < p < \infty$ as

$$C(p) = \begin{cases} 1, & \text{if } 4 \le p < \infty; \\[2ex] \left(\frac{2^{7-p}}{9(p-2)} + 2^{4-p}\right)^{1/p}, & \text{if } 2 < p < 4. \end{cases}$$

Let us consider $1 < p \le 2$. For every $z \in \mathbb{D}$, $f(z) = f(z) - f(0) + f(0)$. Therefore

$$\|T_t(f)\|_{A^p}^p \le 2^{p-1}\left(\|T_t(f - f(0))\|_{A^p}^p + \|T_t(f(0))\|_{A^p}^p\right). \tag{11}$$

We start by estimating the second term. We note that

$$\|T_t(f(0))\|_{A^p}^p = |f(0)|^p \int_{\mathbb{D}} \frac{1}{|1-(1-t)z|^p}\, dA(z).$$

Applying Forelli-Rudin estimates, see [32, Lemma 3.10], and (5), we get that there is some constant B_p depending only on p such that

$$\|T_t\,(f(0))\,\|_{A^p}^p \leq \|f\|_{A^p}^p \cdot \begin{cases} B_p, & \text{if } 1 \leq p < 2; \\ \\ B_2 \log(e/t), & \text{if } p = 2. \end{cases}$$

For the first term in (11), a change of variables yields

$$\begin{aligned} \|T_t(f - f(0))\|_{A^p}^p = & \frac{t^{2-p}}{(1-t)^2} \int\limits_{\varphi_t(\mathbb{D})} |w|^{2p-4} \, |S^*(f)(w)|^p \; dA(w) \\ \leq & \frac{t^{2-p}}{(1-t)^2} \int\limits_{\mathbb{D}} |w|^{2p-4} \, |S^*(f)(w)|^p \; dA(w) \\ = & \frac{t^{2-p}}{(1-t)^2} \Big(\int\limits_{|w| \leq 1/2} |w|^{2p-4} \, |S^*(f)(w)|^p \; dA(w) + \\ & + \int\limits_{1/2 \leq |w| < 1} |w|^{2p-4} \, |S^*(f)(w)|^p \; dA(w) \Big) \\ = & \frac{t^{2-p}}{(1-t)^2} \, (I + II) \, , \end{aligned}$$

where S^* is the backward shift and we have used the fact that

$$|\varphi_t(z)|^4 = \frac{t^2}{(1-t)^2} |\varphi_t'(z)|^2.$$

Then, by using (5), we have

$$\begin{aligned} I \leq & \int\limits_{|w| \leq 1/2} \frac{|w|^{2p-4}}{(1-|w|^2)^2} \, dA(w) \|S^*(f)\|_{A^p}^p \\ \leq & \frac{16}{9} \int\limits_{|w| \leq 1/2} |w|^{2p-4} \, dA(w) \, \|S^*\|_{A^p}^p \, \|f\|_{A^p}^p \\ \leq & \frac{2^{6-2p}}{9} \frac{1}{p-1} \|S^*\|_{A^p}^p \|f\|_{A^p}^p \end{aligned}$$

and

$$II \leq \int\limits_{1/2 \leq |w| < 1} \left(\frac{1}{2}\right)^{2p-4} |S^*(f)(w)|^p \, dA(w) \leq 2^{4-2p} \|S^*\|_{A^p}^p \|f\|_{A^p}^p.$$

Thus

$$\|T_t(f-f(0))\|_{A^p}^p \le \frac{t^{2-p}}{(1-t)^2}\frac{2^{5-2p}}{p-1}\|S^*\|_{A^p}^p\|f\|_{A^p}^p. \tag{12}$$

In the case $p=1$, for every $z\in\mathbb{D}$, we consider

$$f(z)=f(0)+zf'(0)+z^2\cdot S^{*2}(f)(z),$$

where $S^{*2}=S^*\circ S^*$. Therefore

$$\|T_t(f)\|_{A^1} \le \|T_t(z^2\cdot S^{*2}(f))\|_{A^1}+\|T_t\left(f(0)\right)\|_{A^1}+\|T_t(z\cdot f'(0))\|_{A^1}. \tag{13}$$

We have already estimated the second term. For the third one, by using (6), we note that

$$\begin{aligned}\|T_t(zf'(0))\|_{A^1} =&|f'(0)|\cdot t\int\limits_{\mathbb{D}}\frac{1}{|1-(1-t)z|^2}\,dA(z)\\ \le& CB_2\cdot[t\log(e/t)]\cdot\|f\|_{A^1}.\end{aligned}$$

Finally, with computations similar to those done for $1<p<2$, we have that

$$\begin{aligned}\|T_t(z^2\cdot S^{*2}(f))\|_{A^1} =&\frac{1}{t}\int\limits_{\mathbb{D}}\left|\frac{t}{1-(1-t)z}\right|^3\left|S^{*2}(f)\left(\frac{t}{1-(1-t)z}\right)\right|\,dA(z)\\ =&\frac{t}{(1-t)^2}\Bigg(\int\limits_{|w|\le 1/2}|w|^{3-4}\left|S^{*2}(f)(w)\right|\,dA(w)+\\ &+\int\limits_{1/2\le|w|<1}|w|^{3-4}\left|S^{*2}(f)(w)\right|\,dA(w)\Bigg)\\ \le&\frac{5t}{(1-t)^2}\|S^*\|_{A^1}^2\|f\|_{A^1}.\end{aligned}$$

Hence, by using (13), if $p=1$, we have

$$\begin{aligned}\|T_t(f)\|_{A^1} &\le \left(B_1+CB_2t\log\left(e/t\right)+5\|S^*\|_{A^1}^2\frac{t}{(1-t)^2}\right)\|f\|_{A^1}\\ &\le C\frac{1}{(1-t)^2}\|f\|_{A^1}.\end{aligned}$$

By using (11), if $1<p<2$,

$$\begin{aligned}\|T_t(f)\|_{A^p}^p &\le 2^{p-1}\left(B_p+\frac{2^{5-2p}}{p-1}\|S^*\|_{A^p}^p\frac{t^{2-p}}{(1-t)^2}\right)\|f\|_{A^p}^p\\ &\le C\frac{1}{(1-t)^2}\|f\|_{A^p}^p.\end{aligned}$$

If $p = 2$,

$$\|T_t(f)\|_{A^2}^2 \le 2\left(B_2 \log(e/t) + \|S^*\|_{A^2}^2 \frac{1}{(1-t)^2}\right)\|f\|_{A^2}^2$$
$$\le C\frac{\log(e/t)}{(1-t)^2}\|f\|_{A^2}^2.$$

The proof of the Lemma is now complete. □

3. On the boundedness and norm of the operators

First of all, notice that Γ_μ is never contractive. Indeed, if Γ_μ is bounded in A^p, then, by (5),

$$1 = \Gamma_\mu(1)(0) \le \|\Gamma_\mu(1)\|_{A^p} \le \|\Gamma_\mu\|_{A^p}.$$

We provide the estimate from below of $\|\Gamma_\mu\|_{A^p}$ in a separate lemma, which will also be used in the computation of the essential norm of Γ_μ.

Lemma 3.1. *Let $1 \le p < \infty$. If $\Gamma_\mu : A^p \to A^p$ is bounded, then*

$$\int_0^1 \frac{t^{2/p-1}}{(1-t)^{2/p}}\, d\mu(t) \le \|\Gamma_\mu\|_{A^p}.$$

Proof. Let $f_a(z) = 1/(1-z)^a$ for $0 < a < 2/p$. We know that

$$\lim_{a\to 2/p} \|f_a\|_{A^p} = \infty.$$

We consider

$$\Gamma_\mu(f_a)(z) = \int_0^1 \frac{[1-(1-t)z]^{a-1}}{(1-t)^a}\, d\mu(t)\cdot f_a(z) = \Lambda_a(z)\cdot f_a(z),$$

where

$$\Lambda_a(z) = \int_0^1 \frac{[1-(1-t)z]^{a-1}}{(1-t)^a}\, d\mu(t).$$

Let $D(1,\varepsilon)$ be a circle of radius ε centered at 1. We have that

$$\begin{aligned}
\|\Gamma_\mu(f_a/\|f_a\|_{A^p})\|_{A^p}^p &\geq \int\limits_{\mathbb{D}\cap D(1,\varepsilon)} \frac{|f_a(z)|^p}{\|f_a\|_{A^p}^p} |\Lambda_a(z)|^p \, dA(z) \\
&\geq \inf_{z\in\mathbb{D}\cap D(1,\varepsilon)} |\Lambda_a(z)|^p \cdot \int\limits_{\mathbb{D}\cap D(1,\varepsilon)} \frac{|f_a(z)|^p}{\|f_a\|_{A^p}^p} \, dA(z) \\
&= \inf_{z\in\mathbb{D}\cap D(1,\varepsilon)} |\Lambda_a(z)|^p \cdot \left(1 - \int\limits_{\mathbb{D}\setminus D(1,\varepsilon)} \frac{|f_a(z)|^p}{\|f_a\|_{A^p}^p} \, dA(z) \right).
\end{aligned}$$

Consequently, since $f_a/\|f_a\|_{A^p}$ has unitary norm in A^p and

$$|f_a(z)| < \frac{1}{\varepsilon^a} \leq \frac{1}{\varepsilon^{2/p}} \text{ when } z \in \mathbb{D} \setminus D(1,\varepsilon),$$

we find that

$$\begin{aligned}
\|\Gamma_\mu\|_{A^p}^p &\geq \liminf_{\epsilon\to 0} \ \liminf_{a\to 2/p} \ \inf_{z\in\mathbb{D}\cap D(1,\varepsilon)} |\Lambda_a(z)|^p \cdot \left(1 - \int\limits_{\mathbb{D}\setminus D(1,\varepsilon)} \frac{|f_a(z)|^p}{\|f_a\|_{A^p}^p} \, dA(z) \right) \\
&= \liminf_{\epsilon\to 0} \ \liminf_{a\to 2/p} \ \inf_{z\in\mathbb{D}\cap D(1,\varepsilon)} |\Lambda_a(z)|^p \\
&\geq \left(\liminf_{\epsilon\to 0} \ \liminf_{a\to 2/p} \ \inf_{z\in\mathbb{D}\cap D(1,\varepsilon)} \operatorname{Re} \Lambda_a(z) \right)^p.
\end{aligned}$$

Furthermore,

$$\inf_{z\in\mathbb{D}\cap D(1,\varepsilon)} \operatorname{Re} \Lambda_a(z) \geq \int_0^1 \frac{1}{(1-t)^a} \inf_{z\in\mathbb{D}\cap D(1,\varepsilon)} \operatorname{Re}(1-(1-t)z)^{a-1} \, d\mu(t). \tag{14}$$

Let us consider $2 \leq p < \infty$. Then, since $a < 1$, we note that

$$\begin{aligned}
&\inf_{z\in\mathbb{D}\cap D(1,\varepsilon)} \operatorname{Re}(1-(1-t)z)^{a-1} \\
&\quad = \inf_{z\in\mathbb{D}\cap D(1,\varepsilon)} \frac{1}{|1-(1-t)z|^{2(1-a)}} \operatorname{Re}(1-(1-t)\overline{z})^{1-a} \geq \frac{|t|^{1-\alpha}}{|t+\varepsilon|^{2(1-a)}}.
\end{aligned}$$

On the other hand, if $1 \leq p < 2$, we consider $a > 1$ and we get that

$$\inf_{z\in\mathbb{D}\cap D(1,\varepsilon)} \operatorname{Re}(1-(1-t)z)^{a-1} \geq t^{a-1}.$$

Therefore, by taking the limits and applying Fatou's Lemma, we have that

$$\liminf_{\epsilon\to 0}\ \liminf_{a\to 2/p}\ \inf_{z\in\mathbb{D}\cap D(1,\varepsilon)} \operatorname{Re}\Lambda_a(z) \geq \int_0^1 \frac{t^{2/p-1}}{(1-t)^{2/p}}\,d\mu(t).$$

The statement of the lemma is now proved. □

3.1. The case p≠2

We are now ready for the proof of Theorem 1.1 when $p \neq 2$.

Proof of Theorem 1.1 when $p \neq 2$**.** We start with the sufficient condition. An application of the generalized Minkowski's inequality for the integrals together with Lemma 2.3 imply that for $1 \leq p < \infty$

$$\|\Gamma_\mu(f)\|_{A^p} \leq \int_0^1 \|T_t(f)\|_{A^p}\,d\mu(t) \leq B(p)\int_0^1 \Theta_p(t)\,d\mu(t)\,\|f\|_{A^p},$$

where $B(p)$ is equal to $C(p)$ from the proof of Lemma 2.3.

We prove now the necessary condition. We first consider $2 < p < \infty$. Then Lemma 3.1 implies that

$$\|\Gamma_\mu\|_{A^p} \geq \int_0^1 \Theta_p(t)\,d\mu(t)$$

from which the statement of the theorem follows.

If $1 \leq p < 2$, we consider again Lemma 3.1, but we note that

$$\begin{aligned}\int_0^1 \frac{1}{(1-t)^{2/p}}\,d\mu(t) &= \int_0^{1/2} \frac{1}{(1-t)^{2/p}}\,d\mu(t) + \int_{1/2}^1 \frac{1}{(1-t)^{2/p}}\,d\mu(t)\\ &\leq 2^{2/p}\Gamma_\mu(1)(0) + 2^{2/p-1}\int_{1/2}^1 \frac{t^{2/p-1}}{(1-t)^{2/p}}\,d\mu(t)\\ &\leq \left(1+\frac{1}{2}\right)2^{2/p}\|\Gamma_\mu\|_{A^p} = 3\cdot 2^{2/p-1}\|\Gamma_\mu\|_{A^p}.\end{aligned}$$

Therefore

$$\|\Gamma_\mu\|_{A^p} \geq \frac{1}{3\cdot 2^{2/p-1}}\int_0^1 \frac{1}{(1-t)^{2/p}}\,d\mu(t) = A(p)\int_0^1 \Theta_p(t)\,d\mu(t).$$

Theorem 1.1 is now proved for $p \neq 2$. □

3.2. The case p=2

We start this section by computing $\|\Gamma_\mu(1)\|^2_{A^2}$. This quantity will be necessary for the conclusion of the proof of Theorem 1.1. We use some ideas inspired by [16].

Lemma 3.2. *Let μ be a probability measure on $[0,1)$. Then*

$$\|\Gamma_\mu(1)\|^2_{A^2} \sim \int_0^1 \mu[0,t] \log \frac{e}{t}\, d\mu(t).$$

Proof. We note that

$$\Gamma_\mu(1)(z) = \int_0^1 \frac{1}{1-(1-t)z}\, d\mu(t) = \sum_{n\geq 0} \int_0^1 (1-t)^n\, d\mu(t)\ z^n.$$

Hence, by (7), we obtain that

$$\begin{aligned}
\|\Gamma_\mu(1)\|^2_{A^2} &= \sum_{n\geq 0} \frac{\int_0^1 (1-t)^n\, d\mu(t) \int_0^1 (1-s)^n\, d\mu(s)}{n+1} \\
&= \int_0^1 \int_0^1 \sum_{n\geq 0} \frac{[(1-t)(1-s)]^n}{n+1}\, d\mu(t)\, d\mu(s) \\
&= \int_0^1 \int_0^1 1 + \sum_{n\geq 1} \frac{\{(1-t)(1-s)\}^n}{n+1}\, d\mu(t)\, d\mu(s) \\
&\sim \int_0^1 \int_0^t \log \frac{e}{1-(1-t)(1-s)}\, d\mu(s)\, d\mu(t).
\end{aligned}$$

More precisely,

$$\frac{1}{2}\|\Gamma_\mu(1)\|^2_{A^p} \leq \int_0^1 \int_0^t \log \frac{e}{1-(1-t)(1-s)}\, d\mu(s)\, d\mu(t) \leq \|\Gamma_\mu(1)\|^2_{A^p}.$$

Consequently,

$$\|\Gamma_\mu(1)\|^2_{A^2} \geq \int_0^1 \int_0^t \log \frac{e}{1-(1-t)^2}\, d\mu(s)\, d\mu(t)$$

$$
\begin{aligned}
&= \int_0^1 \mu[0,t] \log \frac{e}{t(2-t)}\, d\mu(t) \\
&= \int_0^1 \mu[0,t] \log \frac{e}{t} \left(1 - \frac{\log(2-t)}{\log(e/t)}\right) d\mu(t) \\
&\geq (1-\log(2)) \int_0^1 \mu[0,t] \log \frac{e}{t}\, d\mu(t)
\end{aligned}
$$

and

$$
\|\Gamma_\mu(1)\|_{A^2}^2 \leq 2 \int_0^1 \int_0^t \log \frac{e}{1-(1-t)}\, d\mu(s)\, d\mu(t) = 2 \int_0^1 \mu[0,t] \log \frac{e}{t}\, d\mu(t),
$$

which proves the statement of the lemma. □

We are now ready to prove Theorem 1.1 for $p = 2$.

Proof of Theorem 1.1 when $p = 2$. We recall that in Lemma 3.1, we have already verified that

$$
\|\Gamma_\mu\|_{A^2} \geq \int_0^1 \frac{1}{1-t}\, d\mu(t).
$$

Therefore, if Γ_μ is bounded in A^2, we obtain that

$$
\begin{aligned}
&\int_0^1 \mu[0,t] \log \left(\frac{e}{t}\right) d\mu(t) + \left(\int_0^1 \frac{1}{1-t}\, d\mu(t)\right)^2 \\
&\lesssim \|\Gamma_\mu(1)\|_{A^2}^2 + \|\Gamma_\mu\|_{A^2}^2 \lesssim \|\Gamma_\mu\|_{A^2}^2.
\end{aligned}
$$

On the other hand, by using (12) and Lemma 3.2, we note that

$$
\begin{aligned}
\|\Gamma_\mu(f)\|_{A^2}^2 \leq& 2\|\Gamma_\mu(f(0))\|_{A^2}^2 + 2\|\Gamma_\mu(f - f(0))\|_{A^2}^2 \\
\leq& 2|f(0)|^2 \|\Gamma_\mu(1)\|_{A^2}^2 + 2\left(\int_0^1 \|T_t(f - f(0))\|_{A^2}\, d\mu(t)\right)^2 \\
\leq& 4\|f\|_{A^2}^2 \int_0^1 \mu[0,t] \log \left(\frac{e}{t}\right) d\mu(t) + 4\|S^*\|_{A^2}^2 \|f\|_{A^2}^2 \left(\int_0^1 \frac{1}{1-t}\, d\mu(t)\right)^2.
\end{aligned}
$$

Therefore, by fixing $B(2) = 2\max(1, \|S^*\|_{A^2})$, we have that

$$\|\Gamma_\mu(f)\|_{A^2} \leq B(2)\|f\|_{A^2} \sqrt{\int_0^1 \mu[0,t] \log\left(\frac{e}{t}\right) d\mu(t) + \left(\int_0^1 \frac{1}{1-t}\, d\mu(t)\right)^2}$$

from which the estimate from above follows. □

We highlight the fact that the first term in the definition of $\Theta_2(t)$ is fundamental. Indeed, if $\mu = \delta_0$, then Γ_{δ_0} is unbounded in A^2 but

$$\int_0^1 \frac{1}{1-t}\, d\delta_0(t) < \infty.$$

It is worth mentioning that the operator Γ_μ is bounded from the closed subspace $A^2_0 = \{f \in A^2 : f(0) = 0\}$ to A^2 if and only if

$$\int_0^1 \frac{1}{1-t}\, d\mu(t) < \infty.$$

This result is in accordance with the other values of p and it shows that it is $\|\Gamma_\mu(1)\|_{A^2}$ that has an unexpected behavior. We also note that $\Gamma_\mu(f)(0) = 0$ is seldom true, even if we choose $f \in A^2_0$.

When $p \neq 2$, an application of the generalized Minkowski's inequality is enough to provide the "correct" estimate for the norm of Γ_μ. For this reason, it is tempting to consider the condition

$$\int_0^1 \sqrt{\log(e/t)}\, d\mu(t) < \infty, \tag{15}$$

which captures the behavior of

$$\int_0^1 \frac{\sqrt{\log(e/t)}}{1-t}\, d\mu(t)$$

when t is close to 0, see Lemma 2.3, instead of

$$\sqrt{\int_0^1 \mu[0,t] \log(e/t)\, d\mu(t)} < \infty.$$

However, (15) is not a necessary condition for the boundedness of Γ_μ on A^2.

Proposition 3.3. *There exists a positive, absolutely continuous measure μ for which $\Gamma_\mu : A^2 \to A^2$ is bounded, while condition (15) is not satisfied.*

Proof. We consider the measure

$$d\mu(t) = C\left(\frac{1}{2}\frac{1}{\log\left(\frac{e}{t}\right)^{\frac{3}{2}}}\frac{1}{\log\left(\log\left(\frac{e}{t}\right)\right)}\frac{1}{t} + \frac{1}{\log\left(\frac{e}{t}\right)^{\frac{3}{2}}}\frac{1}{\log\left(\log\left(\frac{e}{t}\right)\right)^2}\frac{1}{t}\right)\chi_{[0,\frac{1}{2}]}(t)\,dt,$$

where

$$C = \log(2e)^{\frac{1}{2}}\log(\log 2e).$$

It follows that

$$\mu[0,t] = \begin{cases} C\log\left(\frac{e}{t}\right)^{-\frac{1}{2}}\frac{1}{\log(\log(\frac{e}{t}))}, & \text{if } 0 < t \le \frac{1}{2}; \\ 1, & \text{if } \frac{1}{2} < t \le 1. \end{cases}$$

We note that

$$\int_0^1 \sqrt{\log(e/t)}\,d\mu(t) > \frac{C}{2}\int_0^{\frac{1}{2}} \frac{1}{\log\left(\frac{e}{t}\right)}\frac{1}{\log\left(\log\left(\frac{e}{t}\right)\right)}\frac{1}{t}\,dt = \infty,$$

even if Γ_μ is bounded on A^2 since the condition of Theorem 1.1 is satisfied. Indeed

$$\sqrt{\int_0^1 \mu[0,t]\log\left(\frac{e}{t}\right)\,d\mu(t) + \left(\int_0^1 \frac{1}{1-t}\,d\mu(t)\right)^2}$$
$$\le \sqrt{C^2\int_0^{\frac{1}{2}} \frac{1}{t}\frac{1}{\log\left(\frac{e}{t}\right)}\frac{1}{\log\left(\log\left(\frac{e}{t}\right)\right)^2}\left(1 + \frac{1}{\log\left(\log\left(\frac{e}{t}\right)\right)}\right)\,dt + 4} < \infty. \quad \square$$

Before concluding this section, we provide another necessary condition for the boundedness of Γ_μ on A^2. To formulate it, we consider the adjoint of Γ_μ acting on the classical Dirichlet space $\mathcal{D}$, consisting of analytic functions for which

$$\int_{\mathbb{D}} |f'(z)|^2\,dA(z) < \infty.$$

Lemma 3.4. *If $\Gamma_\mu : A^2 \to A^2$ is bounded, then its adjoint operator Γ_μ^* is given by*

$$\Gamma_\mu^*(f)(z) = \int_0^1 T_t^*(f)(z)\, d\mu(t) = \int_0^1 T_{1-t}(f)(z)\, d\mu(t)$$

for every f in the Dirichlet space $\mathcal{D}$.

Proof. First of all if $f(z) = \sum_n a_n z^n$ and $g(z) = \sum_n b_n z^n$ belong to A^2, then

$$\langle f, g \rangle = \sum_n \frac{a_n \overline{b_n}}{n+1} = \sum_n a_n \overline{c_n} = \langle f, G \rangle_c$$

where $c_n = b_n/(n+1)$ and $G(z) = \sum_n c_n z^n$ belongs to the Dirichlet space $\mathcal{D}$. Since also the reverse inclusion holds, we can identify the dual of A^2 with $\mathcal{D}$ by using the Cauchy pairing.

We first assume that $f(z)$ and $G(z)$ are polynomials. By using Proposition 1 of [1] and Fubini's Theorem, we obtain that

$$\begin{aligned}
\langle \Gamma_\mu(f), G \rangle_c &= \int_{\mathbb{T}} \Gamma_\mu(f)(\zeta) \overline{G(\zeta)} \frac{|d\zeta|}{2\pi} = \int_0^1 \int_{\mathbb{T}} T_t(f)(\zeta) \overline{G(\zeta)} \frac{|d\zeta|}{2\pi}\, d\mu(t) \\
&= \int_0^1 \langle T_t(f), G \rangle_c \, d\mu(t) = \int_0^1 \langle f, T_{1-t}(G) \rangle_c \, d\mu(t) \\
&= \int_{\mathbb{T}} f(\zeta) \overline{\left(\int_0^1 T_{1-t}(G)(\zeta)\, d\mu(t) \right)} \frac{|d\zeta|}{2\pi} = \left\langle f, \Gamma_\mu^*(G) \right\rangle_c .
\end{aligned}$$

Since the partial Taylor sums are dense in A^2 and $\mathcal{D}$, we have that

$$\begin{aligned}
\langle \Gamma_\mu(f), G \rangle_c &= \lim_{n\to\infty} \langle \Gamma_\mu(S_n(f)), S_n(G) \rangle_c \\
&= \lim_{n\to\infty} \left\langle S_n(f), \Gamma_\mu^*(S_n(G)) \right\rangle_c \\
&= \left\langle f, \Gamma_\mu^*(G) \right\rangle_c ,
\end{aligned}$$

which concludes the proof. $\square$

Proposition 3.5. *If $\Gamma_\mu : A^2 \to A^2$ is bounded, then*

$$\int_0^1 \left(\log \frac{e}{t} \right)^{a/2} d\mu(t) < \infty,$$

for every $0 < a < 1$.

Proof. We note that $f_a(z) = \log\left(e/(1-z)\right)^{a/2} \in \mathcal{D}$. Therefore, according to Lemma 3.4,

$$\begin{aligned}
\|\Gamma_\mu\|_{A^2} \sim \|\Gamma_\mu^*\|_{\mathcal{D}} &\geq \frac{1}{\|f_a\|_{\mathcal{D}}} \|\Gamma_\mu^*(f_a)\|_{\mathcal{D}} \\
&\geq \frac{1}{\|f_a\|_{\mathcal{D}}} |\Gamma_\mu^*(f_a)(0)| \\
&= \frac{1}{\|f_a\|_{\mathcal{D}}} \int_0^1 \left(\log \frac{e}{t}\right)^{a/2} \, d\mu(t),
\end{aligned}$$

from which the statement follows. □

We point out that, by monotone convergence, the condition

$$\lim_{a\to 1} \int_0^1 \left(\log \frac{e}{t}\right)^{a/2} \, d\mu(t) < \infty$$

is exactly (15). Even if we can say that for every $a < 1$ this integral is finite, we do not have a uniform bound, since the quantity $1/\|f_a\|_{\mathcal{D}}$ tends to zero.

4. Essential norm, compactness and complete continuity

In Lemma 3.1, we have already estimated the essential norm of Γ_μ from below. Indeed, let X be a reflexive space. If $\{w_n\} \subset X$ is a unitary weakly null sequence, then

$$\begin{aligned}
\|T\|_{e,X} = \inf_K \|T - K\|_X &\geq \inf_K \lim_{n\to\infty} \|T(w_n) - K(w_n)\|_X \\
&\geq \inf_K \lim_{n\to\infty} \left|\|T(w_n)\|_X - \|K(w_n)\|_X\right| = \lim_{n\to\infty} \|T(w_n)\|_X.
\end{aligned} \tag{16}$$

For the estimate from above of $\|\Gamma_\mu\|_{e,A^p}$, we use Lemma 3.2 of [24] (see also [25]), which we state below.

Lemma 4.1. *Let* $1 < p < \infty$. *There exists a sequence of compact operators* $\{L_n\}_n$ *such that*

$$\limsup_n \|I - L_n\|_{A^p} \leq 1.$$

Moreover, for every $0 < R < 1$, *we have*

$$\limsup_{n\to\infty} \sup_{\|f\|_{A^p}=1} \sup_{|z|\leq R} |(I - L_n)(f)(z)| = 0.$$

We are now ready to compute the essential norm $\|\Gamma_\mu\|_{e,A^p}$, $1 < p < \infty$.

Proof of Theorem 1.2. By Lemma 3.1 and (16), we know that

$$\|\Gamma_\mu\|_{e,A^p} \geq \lim_{a\to 2/p} \|\Gamma_\mu(f_a/\|f_a\|_{A^p})\|_{A^p} = \int_0^1 \frac{t^{2/p-1}}{(1-t)^{2/p}}\, d\mu(t)\ .$$

To complete the proof of the theorem, we only need to estimate the essential norm of Γ_μ from above.

Let $D_{R,t} = \varphi_t(\mathbb{D}) \cap \overline{D(0,R)}$ and $D^c_{R,t} = \varphi_t(\mathbb{D}) \setminus D(0,R)$, where $0 < R < 1$. Then,

$$\begin{aligned}\|\Gamma_\mu(f)\|_{A^p} &\leq \|\Gamma_\mu(f(0))\|_{A^p} + \|\Gamma_\mu(f - f(0))\|_{A^p} \leq |f(0)|\|\Gamma_\mu(1)\|_{A^p} \\ &\quad + \int_0^1 \frac{t^{2/p-1}}{(1-t)^{2/p}} \left(\int_{\varphi_t(\mathbb{D})} |w|^{p-4}|(f-f(0))(w)|^p\, dA(w)\right)^{1/p} d\mu(t) \\ &= I + II.\end{aligned}$$

We write $f(z) - f(0) = zg(z)$. For the second quantity, we have

$$\begin{aligned}II \leq \int_0^1 \frac{t^{2/p-1}}{(1-t)^{2/p}} \Bigg[&\left(\int_{D_{R,t}} |w|^{p-4}|wg(w)|^p\, dA(w)\right)^{1/p} \\ &+ \left(\int_{D^c_{R,t}} |w|^{p-4}|wg(w)|^p\, dA(w)\right)^{1/p} \Bigg] d\mu(t),\end{aligned}$$

hence

$$\begin{aligned}II &\leq \int_0^1 \frac{t^{2/p-1}}{(1-t)^{2/p}} \Bigg[\sup_{|w|\leq R} |g(w)| \left(\int_{\mathbb{D}} |w|^{2p-4}\, dA(w)\right)^{1/p} \\ &\qquad + \sup_{w\in D^c_{R,t}} |w|^{1-4/p} \cdot \|f - f(0)\|_{A^p} \Bigg] d\mu(t) \\ &\leq \int_0^1 \frac{t^{2/p-1}}{(1-t)^{2/p}} \Bigg[\sup_{|w|\leq R} |g(w)| C_p + \sup_{w\in D^c_{R,t}} |w|^{1-4/p} \cdot \|f\|_{A^p} \\ &\qquad + \sup_{w\in D^c_{R,t}} |w|^{1-4/p} \cdot |f(0)| \Bigg] d\mu(t).\end{aligned}$$

Let $\{L_n\}_n$ be the sequence of compact operators as described in Lemma 4.1. Then

$$\begin{aligned}\|\Gamma_\mu\|_{e,A^p} &\leq \limsup_{n\to\infty} \sup_{\|f\|_{A^p}=1} \|(\Gamma_\mu - \Gamma_\mu L_n)(f)\|_{A^p} \\ &= \limsup_{n\to\infty} \sup_{\|f\|_{A^p}=1} \|\Gamma_\mu(I-L_n)(f)\|_{A^p}\end{aligned}$$

and the last expression is smaller than

$$\begin{aligned}&\leq \limsup_{n\to\infty} \sup_{\|f\|_{A^p}=1} |(I-L_n)(f)(0)| \left(\|\Gamma_\mu(1)\|_{A^p} + \max\{1, R^{1-4/p}\} \int_0^1 \frac{t^{2/p-1}}{(1-t)^{2/p}}\, d\mu(t)\right) \\ &\quad + \limsup_{n\to\infty} \sup_{\|f\|_{A^p}=1} \sup_{|w|=R} |S^* \circ (I-L_n)(f)(w)| C_p \int_0^1 \frac{t^{2/p-1}}{(1-t)^{2/p}}\, d\mu(t) \\ &\quad + \limsup_{n\to\infty} \sup_{\|f\|_{A^p}=1} \|(I-L_n)f\|_{A^p} \max\{1, R^{1-4/p}\} \int_0^1 \frac{t^{2/p-1}}{(1-t)^{2/p}}\, d\mu(t).\end{aligned}$$

The first term tends to zero because of 4.1. Moreover

$$\sup_{|w|=R} |S^* \circ (I-L_n)(f)(w)| \leq \frac{1}{R}\left(\sup_{|w|\leq R} |(I-L_n)(f)(w)| + |(I-L_n)(f)(0)|\right).$$

Therefore, using once more Lemma 4.1,

$$\begin{aligned}&\limsup_{n\to\infty} \sup_{\|f\|_{A^p}=1} \frac{1}{R} \sup_{|w|\leq R} |(I-L_n)(f)(w)| \\ &\qquad + \limsup_{n\to\infty} \sup_{\|f\|_{A^p}=1} \frac{1}{R} |(I-L_n)(f)(0)| = 0.\end{aligned}$$

Consequently, by the boundedness of Γ_μ and Theorem 1.1, we have

$$\|\Gamma_\mu\|_e \leq \limsup_{n\to\infty} \sup_{\|f\|_{A^p}=1} \|(I-L_n)f\|_{A^p} \max\{1, R^{1-4/p}\} \int_0^1 \frac{t^{2/p-1}}{(1-t)^{2/p}}\, d\mu(t),$$

and letting $R \to 1$, we obtain the desired upper estimate. $\square$

Next, we move to the case $p = 1$. We need the following preliminary lemma.

Lemma 4.2. *Let $\{f_n\} \subset A^1$ be a weakly null convergent sequence. Then, for every fixed $0 \leq t < 1$, $T_t(f_n)$ is strongly null convergent.*

Proof. Since $\{f_n\}$ is weakly null, f_n converge to zero on every compact subset of $\mathbb{D}$. Moreover, $\varphi_t(\mathbb{D}) \subset \mathbb{D}$ touches $\mathbb{T}$ only at 1. Consequently, for every $0 \leq t < 1$, $f_n \circ \varphi_t$ converge in measure to zero and, due to [11, p. 295], $\{f_n \circ \varphi_t\}$ is also strongly null. The Lemma is proved since the multiplication by w_t does not change the behavior of $f_n \circ \varphi_t$. $\square$

Proof of Theorem 1.3. Let $f_a(z) = 1/(1-z)^a$ as in Lemma 3.1. We consider the bounded sequence

$$\left\{ \frac{f_a}{\|f_a\|_{A^1}} \right\}_{1<a<2},$$

which is converging to zero uniformly on compact subsets of $\mathbb{D}$. Then, if Γ_μ were compact, [30, Lemma 3.7] would imply that $\Gamma_\mu(f_a/\|f_a\|_{A^1})$ would tend to zero. However, in Lemma 3.1, we have verified that

$$\lim_{a\to 2} \|\Gamma_\mu(f_a/\|f_a\|_{A_1})\|_{A^1} \geq \int_0^1 \frac{t}{(1-t)^2}\, d\mu(t).$$

Consequently, if $\mu \neq \delta_0$, then $\Gamma_\mu(f_a/\|f_a\|_{A^1})$ is not strongly converging to zero and thus Γ_μ cannot be compact. On the other hand, if $\mu = \delta_0$, then Γ_μ is compact since it is a rank 1 operator.

In order to show that Γ_μ is completely continuous, we use Lemma 4.2 which states that T_t is completely continuous for every $t \in [0,1)$. Consider now a sequence of functions $\{f_n\}$ which is weakly null in A^1. Then, by the complete continuity of T_t we have that $\lim_n \|T_t f_n\|_{A^1} = 0$, for all $0 \leq t < 1$. Furthermore, using Lemma 2.3, we have

$$\|T_t f_n\|_{A^1} \leq \sup_n \|f_n\|_{A^1} \|T_t\|_{A^1} \leq C(1) \sup_n \|f_n\|_{A^1} \frac{1}{(1-t)^2}.$$

By applying the dominated convergence theorem together with Theorem 1.1, we conclude that

$$\limsup_n \|\Gamma_\mu f_n\|_{A^1} \leq \limsup_n \int_0^1 \|T_t f_n\|_{A^1}\, d\mu(t) = 0. \quad \square$$

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Data availability

No data was used for the research described in the article.

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