A short note on Schiffer's conjecture for a class of centrally symmetric convex domains in R^2

A SHORT NOTE ON SCHIFFER'S CONJECTURE FOR A CLASS OF CENTRALLY SYMMETRIC CONVEX DOMAINS IN $\mathbb{R}^2$

By

SUGATA MONDAL

Abstract. Let Ω be a bounded centrally symmetric domain in $\mathbb{R}^2$ with analytic boundary $\partial\Omega$ and center c. Let $\tau = \tau(\Omega)$ be the number of points p on $\partial\Omega$ such that the normal line to $\partial\Omega$ at p passes through c. We show that if $\tau < 8$ then Ω satisfies Schiffer's conjecture.

1 Introduction

Let Ω be a simply-connected bounded domain in the plane with Lipschitz boundary. Assume that there exists a C^2 function $u : \Omega \to \mathbb{R}$ that satisfies

$$\Delta u = \mu \cdot u, \quad \partial_\nu u\big|_{\partial\Omega} \equiv 0 \quad \text{and} \quad u\big|_{\partial\Omega} \equiv 1 \tag{1}$$

for some $\mu \neq 0$, where ∂_ν denotes the unit outward normal vector field along $\partial\Omega$. Schiffer's conjecture says that Ω is a disc [Y]. Observe that, because Euclidean discs have infinitely many radial Neumann eigenfunctions, there are infinitely many solutions to (1) when Ω is a Euclidean disc. Interest in this conjecture, for bounded connected domains in the plane with analytic boundary, partially, comes from its connection to a well-studied problem in integral geometry known as the Pompieu problem. A plane domain Ω is said to have the Pompieu property if for any non-zero continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ there exists a rigid motion σ of $\mathbb{R}^2$ such that

$$\int_{\sigma(\Omega)} f \neq 0.$$

It is well known that the unit disc does not have the Pompieu property (see the discussion below). The Pompieu problem is to determine all simply connected domains with the Pompieu property.

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In [Will76], it was proved that a bounded domain Ω in the plane does not have the Pompieu property if and only if there exists a solution to (1) on Ω. In [Will81], Williams proved a free boundary result concerning (1). As a consequence of it, in $\mathbb{R}^2$ one obtains that a bounded, simply-connected open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$ has the Pompeiu property unless $\partial\Omega$ is analytic. Hence, to prove that discs are the only bounded simply-connected plane domains with Lipschitz boundary that does not have the Pompieu property, it suffices to prove Schiffer's conjecture for domains in the plane with analytic boundary.

Perhaps not entirely unexpectedly, Schiffer's conjecture, in some form, shows up in the asymptotic properties of nodal sets of eigenfunctions [TZ09]. For the rest of the paper we will be concerned with simply-connected domains in the plane with analytic boundary, unless otherwise stated.

Although the literature on the Pompieu problem is fairly rich (see [GF91], [GF91'], [GF93], [GF94], [E93], [E93'], [E94], [D]), Schiffer's conjecture has not been studied that extensively on its own. In [Ber80] Bernstein showed that if there are infinitely many solutions to (1), then Ω must be a disc. In [BY82] it was proved that if the μ in (1) is equal to $\mu_2(\Omega)$—the second Neumann eigenvalue of Ω—then Ω is a disc. In [Av86] it was shown that if Ω is convex and if μ in (1) is at most $\mu_7(\Omega)$—the seventh Neumann eigenvalue of Ω—then Ω is a disc. Recently, Deng [De12] has obtained results similar to the last one where, among other things, he was able to drop the convexity assumption and replace the seventh Neumann eigenvalue of Ω by some larger eigenvalue.

Other recent developments around Schiffer's conjecture include [L07], [NSY20] and [KM20]. In this short note we extend an observation of Deng [De12] to prove Schiffer's conjecture for a class of centrally symmetric domains. The main result of this paper is the following theorem.

Theorem 1. *Let Ω be a centrally symmetric domain with analytic boundary $\partial\Omega$ and center c. Let $\tau = \tau(\Omega)$ be the number of points p on $\partial\Omega$ such that the normal line to $\partial\Omega$ at p passes through c. We show that if $\tau < 8$ then Ω satisfies Schiffer's conjecture.*

Remark 2. A few remarks are in order.

(1) Schiffer's conjecture for ellipses follows from [BST73, Theorem 5.1] via its connection with the Pompieu property for ellipses.

(2) It can be easily checked that for Ω an ellipse, the number $\tau = 4$. Hence, Theorem 1 gives an alternate proof that any ellipse satisfies Schiffer's conjecture. Because the condition in Theorem 1 is geometric and 'open', it follows that any controlled deformation of any given ellipse (that satisfies

the assumptions in Theorem 1) also satisfies Schiffer's conjecture. This is particularly interesting because, for most of the results on Schiffer's conjecture and the Pompeiu problem (see [GF91], [GF91'], [GF93], [GF94], [E93], [E93'], [E94]) the conditions required are algebraic and 'closed' in nature. Of course, a geometric condition that ensures $\tau < 8$ would be interesting.

(3) According to [D-L15], for any centrally symmetric convex (c-s-c) domain Ω in the plane with smooth boundary, and for any $p \in \Omega$, the average number of points on $\partial\Omega$ where the normal line to $\partial\Omega$ passes through p is at most 8. It would be interesting to see if one can work with a general point in Ω rather than the center and conclude the conjecture for all c-s-c domains from the result of [D-L15].

(4) It is not very difficult to show the existence of c-s-c domains in the plane with $\tau > 8$. To see this, first observe that for the disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, we have $\tau(D) = +\infty$. Hence, for any $n \geq 2$, one can carefully modify D to produce c-s-c domains Ω with $\tau(\Omega) = 2n$.
For example, a smoothened regular $2n$-gon P has $\tau(P) = 4n$. It is perhaps helpful to note that for any point $q \in \partial\Omega$ if the normal line to $\partial\Omega$ passes through c then q is a local extremum of the function $f : \partial\Omega \to \mathbb{R}$ given by

$$f(p) = d(c, p),$$

where $d(x, y)$ denotes the Euclidean distance between the points $x, y \in \mathbb{R}^2$.

Sketch of the proof of Theorem 1. Let Ω be a simply-connected domain in the plane with analytic boundary and let u be a solution of (1). In §2.3 we obtain certain integral inequalities that u satisfies. When Ω is centrally symmetric, using these identities, we deduce that the normal derivative along $\partial\Omega$, of a particular rotational (see §2.1) derivative of u, must vanish at at least eight points (Theorem 4).

In §3 we study the local behavior of u on $\partial\Omega$. We first obtain a local expression of u at a point on $\partial\Omega$ (Proposition 6). Then, we use this expression to give a geometric description of the points where the normal derivative of a rotational derivative of u may vanish. The proof of Theorem 1 is deduced from this geometric description.

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2 Counting nodal critical points of derivatives of u after Jian Deng

2.1 Killing fields. A Killing field on $\mathbb{R}^2$ is a vector field whose infinitesimal generators are isometries of $\mathbb{R}^2$. It is well known that any Killing field on $\mathbb{R}^2$ is either a constant vector field or a rotational vector field. A constant vector field L is a vector field on $\mathbb{R}^2$ such that there exists $a, b \in \mathbb{R}$ with either $a \neq 0$ or $b \neq 0$ such that

$$L = a \cdot \partial_x + b \cdot \partial_y.$$

Here x, y denote the cartesian coordinates of $\mathbb{R}^2$. A rotational vector field R_p with center p, on the other hand, is a vector field such that

$$R_p = \partial_\theta$$

where (r, θ) are the polar coordinates on $\mathbb{R}^2$ with center p. One property of a Killing field X, that we will be using in this paper, is that it commutes with the Laplacian

$$\Delta \cdot X = X \cdot \Delta.$$

This property implies that if $v : \mathbb{R}^2 \to \mathbb{R}$ is a C^3 function such that $\Delta v = \lambda \cdot v$ on a domain Ω, then $\Delta X v = \lambda \cdot X v$ on Ω.

2.2 Boundary behavior of solutions of (1). Let Ω be a simply connected domain with analytic boundary $\partial\Omega$. We denote the anti-clockwise oriented unit tangent and the unit outward normal vector fields along $\partial\Omega$ by ∂_τ and ∂_ν respectively.

Let v be a Neumann eigenfunction on Ω. This means that there exists a $\lambda > 0$ such that $\Delta v = \lambda \cdot v$ and v satisfies the Neumann boundary condition $\partial_\nu v = 0$ along $\partial\Omega$. It is well known that v can be extended analytically to a neighborhood $N(\Omega)$ of Ω [TZ09]. From here onwards we shall always think of v as a function on $N(\Omega)$. By the analyticity, the equality $\Delta v = \lambda \cdot v$ still holds on all of $N(\Omega)$.

For a function $f : \Omega \to \mathbb{R}$, the set $f^{-1}(0)$ of zeros of f is called the **nodal set** of f and we denote it by $\mathcal{Z}(f)$. If f is C^1, we call a point $p \in \Omega$ a **critical point** of f if the gradient of f vanishes at p. Equivalently, p is a critical point of f if all partial derivatives of f at p vanish.

Lemma 3. *Let u be a solution of* (1) *on Ω. Then each point on $\partial\Omega$ is a critical point of u. In particular, $Xu|_{\partial\Omega} \equiv 0$ for any vector field X.*

Proof. Because $\partial_\tau(p)$ and $\partial_\nu(p)$ are linearly independent and because $\partial_\nu u|_{\partial\Omega} \equiv 0$, it suffices to show that $\partial_\tau u(p) = 0$ for each $p \in \partial\Omega$. This follows from the fact that $u \equiv 1$ on $\partial\Omega$. □

Let s denote the arc length parametrization of $\partial\Omega$. Let the parametric equation of $\partial\Omega$ be given by $z(s) = (x(s), y(s))$. In the complex notation $z(s) = x(s) + i \cdot y(s)$ the derivative

$$\frac{dz}{ds} = e^{i\cdot\theta(s)},$$

where $\theta(s)$ is the angle that the tangent to $\partial\Omega$ at the point $z(s)$ makes with the x axis.

2.2.1 Boundary values of derivatives of u. Most of the results in this subsection are reformulations of some results from §2 of [De12]. We give the details here for the sake of completeness.

The main goal here is to find expressions for u_{xx}, u_{xy} and u_{yy} along $\partial\Omega$, in terms of the angle function $\theta(s)$ defined above. For this, we first recall that the unit tangent vector to $\partial\Omega$ at $z(s)$ is given by

$$\partial_\tau = \cos\theta(s) \cdot \frac{\partial}{\partial x} + \sin\theta(s) \cdot \frac{\partial}{\partial y}.$$

Also, after possibly reflecting Ω along the x-axis, the unit outward normal vector to $\partial\Omega$ at $z(s)$ is given by

$$\partial_\nu = \sin\theta(s) \cdot \frac{\partial}{\partial x} - \cos\theta(s) \cdot \frac{\partial}{\partial y}.$$

Using Lemma 3, we get

$$0 = \partial_\tau u_x = \cos\theta(s) \cdot \frac{\partial^2 u}{\partial x^2} + \sin\theta(s) \cdot \frac{\partial^2 u}{\partial x \partial y} \tag{2}$$

and

$$0 = \partial_\tau u_y = \cos\theta(s) \cdot \frac{\partial^2 u}{\partial x \partial y} + \sin\theta(s) \cdot \frac{\partial^2 u}{\partial y^2}. \tag{3}$$

Since

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\mu \cdot u,$$

a straightforward computation shows that

$$u_{xx}|_{\partial\Omega} = \frac{1}{2} \cdot (1 - \cos(2\theta(s))) \cdot \mu \cdot u|_{\partial\Omega}. \tag{4}$$

Similarly, we obtain other identities:

$$u_{xy}|_{\partial\Omega} = -\frac{1}{2} \cdot \sin(2\theta(s)) \cdot \mu \cdot u|_{\partial\Omega} \quad \text{and} \quad u_{yy}|_{\partial\Omega} = \frac{1}{2} \cdot (1 + \cos(2\theta(s))) \cdot \mu \cdot u|_{\partial\Omega}.$$

2.3 Counting the nodal critical point on the boundary. Let X be a Killing field on $\mathbb{R}^2$. Hence, from §2.1, either $X = c_1 \cdot \partial_x + c_2 \cdot \partial_y$ for some constants c_1, c_2 (with either $c_1 \neq 0$ or $c_2 \neq 0$), or $X = R_p$ for some $p \in \mathbb{R}^2$. By Lemma 3, $\partial\Omega \subset \mathcal{Z}(Xu)$. Because X is a Killing field, it follows that $\Delta Xu = \mu \cdot Xu$. Hence, by Green's formula [Cha]

$$0 = \int_{\partial\Omega} Xu \cdot \frac{\partial u_{xx}}{\partial n} ds = \int_{\partial\Omega} \frac{\partial Xu}{\partial n} \cdot u_{xx} ds. \tag{5}$$

Now, by (4), we get our first equality

$$\int_{\partial\Omega} \frac{\partial Xu}{\partial n} \cdot \frac{1}{2} \cdot (1 - \cos(2\theta(s))) \cdot \mu \cdot u|_{\partial\Omega} ds = 0.$$

Since $u|_{\partial\Omega}$ is a constant, we get

$$\int_{\partial\Omega} \frac{\partial Xu}{\partial n} \cdot \frac{1}{2} \cdot (1 - \cos(2\theta(s))) ds = 0. \tag{6}$$

Similarly, working with u_{xy} and u_{yy} we get the identities

$$\int_{\partial\Omega} \frac{\partial Xu}{\partial n} \cdot \sin(2\theta(s)) ds = 0 = \int_{\partial\Omega} \frac{\partial Xu}{\partial n} \cdot (1 + \cos(2\theta(s))) ds. \tag{7}$$

Combining these equations we get the identities

$$\begin{aligned} &\int_{\partial\Omega} \frac{\partial Xu}{\partial n} ds = 0, \quad \int_{\partial\Omega} \frac{\partial Xu}{\partial n} \cdot \sin(2\theta(s)) ds = 0 \quad \text{and} \\ &\int_{\partial\Omega} \frac{\partial Xu}{\partial n} \cdot \cos(2\theta(s)) ds = 0. \end{aligned} \tag{8}$$

Now we use these identities to obtain a lower bound on the number of nodal critical points of $R_c u$ on $\partial\Omega$. Observe that, because $R_c u \equiv 0$ on $\partial\Omega$, the tangential derivative $\partial_\tau R_c u(p) = 0$ for every $p \in \partial\Omega$. Hence, every point $p \in \partial\Omega$, where $\partial_\nu R_c u(p) = 0$, is a nodal critical point of R_c. Because $R_c u$ satisfies $\Delta u = \mu \cdot u$ in $N(\Omega)$, the nodal critical points of R_c are isolated by [Chn76]. In particular, there are only finitely many points on $\partial\Omega$ where $\partial_\nu R_c$ vanishes.

3 Centrally symmetric convex domains

Let Ω be a centrally symmetric convex domain.

Theorem 4 (Deng)**.** *Let Ω be a centrally symmetric domain with center c. Let u be a solution of* (1)*. Then there are at least eight points on $\partial\Omega$ where $\partial_\nu R_c$ vanishes.*

Proof. We first claim that u is centrally symmetric with respect to the central symmetry $\imath$ of Ω. To see this, we consider the function $v = (u + \imath(u))/2$. Clearly v satisfies (1). If v is not a multiple of u then we may consider a linear combination w, of u and v, such that w satisfies (1) and $w|_{\partial\Omega} \equiv 0$. In particular, each point of $\partial\Omega$ is a nodal critical point of w. By [Chn76], w must be identically zero.

Hence we may assume that v is a constant multiple of u. This implies that $\imath(u)$ is a constant multiple of u. Because $\imath$ is an isometry we may conclude that $\imath(u) = \pm u$. If $\imath(u) = -u$ then u would vanish at some point on $\partial\Omega$. Since $u|_{\partial\Omega} \equiv 1$, we conclude that $\imath(u) = u$.

A straightforward computation now shows that $R_c u$ is also symmetric with respect to $\imath$. Because $\sin(m\theta)$, $\cos(m\theta)$ are antisymmetric with respect to $\imath$ for m odd, we get

$$\int_{\partial\Omega} \partial_\nu R_c \cdot \sin(m \cdot \theta) ds = 0 = \int_{\partial\Omega} \partial_\nu R_c \cdot \cos(m \cdot \theta) ds, \tag{9}$$

for any odd integer m, Combining with (8) we obtain that (9) holds for $m = 0, 1, 2, 3$. Hence $\partial_\nu R_c|_{\partial\Omega}$ is orthogonal to $1, \cos(\theta), \sin(\theta), \cos(2\theta), \sin(2\theta), \cos(3\theta), \sin(3\theta)$. It follows from Strum–Liouville theory [Ar04] that $\partial_\nu R_c|_{\partial\Omega}$ has at least eight zeros. $\square$

4 Nodal sets of rotational derivatives

Let Ω be a simply-connected domain in the plane with analytic boundary $\partial\Omega$. Let v be a non-constant Neumann eigenfunction of Ω with eigenvalue μ. We recall that this means that

$$\Delta v = \mu \cdot v, \quad \text{and} \quad \partial_\nu v \equiv 0 \quad \text{along } \partial\Omega.$$

Here, as before, ∂_ν denotes the unit outward normal vector field along $\partial\Omega$. Recall that, because $\partial\Omega$ is analytic, we have a neighborhood $N(\Omega)$ where v extends analytically and satisfies $\Delta v = \mu \cdot v$ on $N(\Omega)$ [TZ09].

Now we consider rotational derivatives $R_q v$ of v. In particular, we would like to understand the nodal set $\mathcal{Z}(R_q v)$ for different choices of q as a graph on $N(\Omega)$. Because R_q is a Killing field, we have $\Delta R_q v = \mu \cdot R_q v$ on all of $N(\Omega)$. Hence, by [Chn76], $\mathcal{Z}(R_q v)$ is a locally finite graph. Therefore, by considering a smaller neighborhood $N'(\Omega)$, if necessary, we assume that $\mathcal{Z}(R_q v) \cap N'(\Omega)$ is a finite graph. From now onwards, by $\mathcal{Z}(R_q v)$ we would mean $\mathcal{Z}(R_q v) \cap N'(\Omega)$.

For $c \in \partial\Omega$ we consider a coordinate system on the plane such that c is the center of this coordinate system, the x-axis is tangent to $\partial\Omega$ at c and near c, Ω lies in the upper half-plane. We refer to such a coordinate as **adapted** to c and Ω.

Lemma 5. *Fix a coordinate adapted to c and* Ω. *Assume that, in a neighborhood of c,* v *has the expression*

$$v(x,y) = v_{0,0} + v_{0,2} \cdot y^2 + p(x,y)$$

where $v_{0,0}$ *and* $v_{0,2}$ *are constants and p is an analytic function of vanishing order at least three.*[1] *If* $v_{0,2} \neq 0$ *and q does not lie on the normal to* $\partial\Omega$ *at c, then* $\mathcal{Z}(R_q v)$ *consists of a smooth arc containing c in a neighborhood of c.*

Proof. Let $q = (x_q, y_q)$ in the adapted coordinates such that q does not lie on the normal to $\partial\Omega$ at c. Hence, $x_q \neq 0$. Now,

$$R_q(x,y) = (x - x_q) \cdot v_y(x,y) - (y - y_q) \cdot v_x(x,y).$$

Using the expression for v we have

$$R_q v(x,y) = -2v_{0,2} \cdot x_q \cdot y + f(x,y),$$

where f is an analytic function of vanishing order at least two. Because $\nabla R_q v$ does not vanish at c, the claim follows. □

Our next result is a bit more general. Let w satisfy $\Delta w = \lambda \cdot w$ on a domain U for some $\lambda \neq 0$. Let crit(w) denote the set of critical points of w. For a C^1 curve γ and $a \in \gamma$, let $N_\gamma(a)$ denote the normal line to γ at a. Finally, for a C^2 curve γ we denote the curvature and the centre of curvature of γ at a point $a \in \gamma$ by $\kappa_\gamma(a)$ and $\xi_\gamma(a)$ respectively.

Proposition 6. *Let* $C \subset U$ *be an analytic arc that is neither a subset of a circle nor a subset of a straight line. Assume that* $C \subset \mathrm{crit}(w)$. *Then the following holds:*

1. *if* $q \notin N_C(p)$, *p is a vertex of* $\mathcal{Z}(R_q w)$ *of degree two, i.e., then* $\mathcal{Z}(R_q w)$ *consists of a smooth arc containing p in a neighborhood of p,*
2. *if* $q \in N_C(p)$ *and* $q \neq \xi_C(p)$, *then p is a vertex of* $\mathcal{Z}(R_q w)$ *of degree four,*
3. *if* $q = \xi_C(p)$, *then p is a vertex of* $\mathcal{Z}(R_q w)$ *of degree at least six.*

Proof. Because w satisfies $\Delta w = \lambda \cdot w$ it follows that w is real analytic. We now consider an expression for w in a neighborhood of $q \in C$ as follows:

$$w(x,y) = w_{0,0} + w_{1,0} \cdot x + w_{0,1} \cdot y + w_{2,0} \cdot x^2 + w_{1,1} \cdot xy + w_{0,2} \cdot y^2 + O(3)$$

Here the coordinate system is adapted to C and p and $w_{i,j}$ are constants.

[1] This means that $Lp(0,0) = 0$ for any second order differential operator L.

Since $C \subset \mathrm{crit}(w)$, it follows that $w(t) = w_{0,0}$ for each $t \in C$. Since C is not a single point, by [Chn76], it follows that $w_{0,0} \neq 0$. Because p is a critical point of w, we have $w_{1,0} = 0 = w_{0,1}$. Moreover, as $C \subset \mathrm{crit}(w)$ we have $C \subset \mathcal{Z}(\partial_x w)$. In particular, $w_{2,0} = 0$. Using ∂_y instead of ∂_x in the above argument, we may further conclude that $w_{1,1} = 0$. Finally, since $\Delta w = \lambda \cdot w$, and the linear term in the above expression of w vanishes identically, it follows that the degree three polynomial in the expression for w is harmonic. In sum

$$w(x, y) = w_{0,0} + w_{0,2} \cdot y^2 + w_{3,0} \cdot (x^3 - 3xy^2) + w_{0,3} \cdot (y^3 - 3x^2 y) + O(4).$$

Because $w_{0,0} \neq 0$ and $\Delta w = \lambda \cdot w$, it follows that $w_{0,2} \neq 0$. Now consider a point $q \in N_C(p)$. In the adapted coordinates $q = (0, y_q)$. A straightforward computation shows that

$$R_q w(x, y) = 2 \cdot w_{0,2} \cdot xy + y_q \cdot (3 \cdot w_{3,0} \cdot (x^2 - y^2) - 6 \cdot w_{0,3} \cdot xy) + O(3).$$

Since $y_q \neq 0$, if $w_{3,0} \neq 0$ then $\mathcal{Z}(R_q w)$ would have two sub-arcs that pass through p and both of these arcs would intersect the x axis transversally. This is impossible because $C \subset \mathcal{Z}(R_q w)$. Hence, $w_{3,0} = 0$ and we have the following expression

$$w(x, y) = w_{0,0} + w_{0,2} \cdot y^2 + w_{0,3} \cdot (y^3 - 3x^2 y) + O(4). \tag{10}$$

The first part of the proposition now follows from Lemma 5. For the rest of the claims we begin with the following.

Lemma 7. $w_{0,3} = 0$ *if and only if* $\kappa_C(p) = 0$.

Proof. Let $y = f_p(x)$ be the equation of C near p, where f_p is an analytic function of x. Since the coordinate system is adapted to C and p we have $f_p(0) = 0 = f_p'(0)$. Now, using the Weierstrass preparation theorem we get

$$w(x, y) = w(0, 0) + (y - f_p(x))^2 \cdot g_p(x, y) \tag{11}$$

where $g_p(x, y)$ is an analytic function of x, y.

Now we compare the expressions (10) and (11) for w. Since $f_p(0) = 0$, we get $g_p(0, 0) = w_{0,2}$. Since $w_{0,0} \neq 0$ and $\Delta w = \lambda \cdot w$, we have $w_{0,2} \neq 0$. Hence, $g_p(0, 0) \neq 0$. Since $f_p'(0) = 0$, comparing the coefficient of $x^2 y$ in (10) and (11) we get

$$f_p''(0) \cdot w_{0,2} = 3 w_{0,3}.$$

Since $f_p''(0) = \kappa_C(p)$, the claim follows. □

Remark 8. It follows from the proof that $\kappa_C(p) = \frac{3w_{0,3}}{w_{0,2}}$. Hence $\xi_C(p) = (0, \frac{w_{0,2}}{3w_{0,3}})$ in the chosen coordinates.

First assume that $\rho_C(p) \neq 0$, and hence, $\xi_C(p)$ is finite. In particular, w has the following expression near p:

$$w(x, y) = w_{0,0} + w_{0,2} \cdot y^2 + w_{0,3} \cdot (y^3 - 3x^2 y) + O(4),$$

where each of the three quantities $w_{0,0}$, $w_{0,2}$ and $w_{0,3}$ is non-zero. Consider $q \in N_C(p)$. As our coordinates are adapted to C and p, we have $q = (0, y_q)$. Therefore,

$$R_q w(x, y) = (2w_{0,2} - 6y_q \cdot w_{0,3}) \cdot xy + O(3).$$

If $q \neq \xi_C(p)$ then, by Remark 8, $y_q \neq 1/3 \cdot (w_{0,2}/w_{0,3})$. Hence there are exactly two arcs in $\mathcal{Z}(R_q w)$ crossing each other at p. This proves that p is a degree four vertex of $\mathcal{Z}(R_q w)$, giving the second claim.

Finally, if $q = \xi_C(p)$ then $y_q = 1/3 \cdot (w_{0,2}/w_{0,3})$. Therefore, the order of vanishing of $R_q w$ at p is at least three. This means that there are at least three arcs in $\mathcal{Z}(R_q w)$ crossing each other at p. Hence, the degree of p as a vertex of $\mathcal{Z}(R_q w)$ is at least six. □

5 Proof of the main theorem

Let Ω be a centrally symmetric convex domain with analytic boundary $\partial\Omega$ and with center c. Let u be a solution of (1). We consider $R_c u$—the rotational derivative of u with respect to the center c of Ω. Our goal is to determine the points p on the boundary $\partial\Omega$ such that the normal derivative $\partial_\nu R_c u(p) = 0$. Because $R_c u$ vanishes along $\partial\Omega$ we obtain that $\partial_\tau R_c u|_{\partial\Omega} = 0$. Therefore, if $\partial_\nu R_c u(p) = 0$ for some point $p \in \partial\Omega$, then p must be a nodal critical point of $R_c u$.

Because $R_c u$ is a Laplace eigenfunction and $\partial\Omega \subset \mathcal{Z}(R_c u)$, it follows from [Chn76] that as a graph, the degree of p as a vertex of $\mathcal{Z}(R_c u)$ is at least four. By Proposition 6 this means that c lies on the normal to $\partial\Omega$ at p. In particular, because $\tau(\Omega)$, the number of points q on $\partial\Omega$ such that the normal line to $\partial\Omega$ at q passes through c is < 8, we conclude that the number of points $p \in \partial\Omega$, where $\partial_\nu R_c u(p) = 0$, is at most 7. This contradicts Theorem 4.

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Sugata Mondal
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF READING
READING RG6 6AX, UNITED KINGDOM
email: s.mondal@reading.ac.uk

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