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Compactness and related properties of weighted composition operators on weighted BMOA spaces

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A R T I C L E I N F O

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A B S T R A C T

It is shown that a large class of properties coincide for weighted composition operators on a large class of weighted VMOA spaces, including the ones with logarithmic weights and the ones with standard weights $(1-|z|)^{-c}$, $0 \leq c < \frac{1}{2}$. Some of these properties are compactness, weak compactness, complete continuity and strict singularity. A function-theoretic characterization for these properties is also given. Similar results are also proved for many weighted composition operators on similarly weighted BMOA spaces. The main results extend the theorems given in Laitila et al. (2023) [16], and new test functions that are suitable for the weighted setting are developed.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions $\mathbb{D} \to \mathbb{C}$. The linear space $\mathcal{H}(\mathbb{D})$ is a Fréchet space when equipped with the metrizable topology τ_0, induced by convergence on compact sub-

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sets. If $\phi, \psi \in \mathcal{H}(\mathbb{D})$ with $\phi\colon \mathbb{D} \to \mathbb{D}$, then a weighted composition operator is defined as $\psi C_\phi\colon \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})\colon f \mapsto \psi f \circ \phi$, which is the product of a multiplication operator $f \mapsto \psi f$ and a composition operator $C_\phi\colon f \mapsto f \circ \phi$. The main aim of the paper is to obtain a complete characterization of compactness of ψC_ϕ acting on the space VMOA_v in terms of conditions on ψ and ϕ, and to prove that compactness is equivalent to many other operator-theoretic properties for ψC_ϕ on VMOA_v. Concerning the operator ψC_ϕ on BMOA_v, a similar result is obtained under some extra assumptions. One sufficient additional condition for the results involving compactness to hold, is that the composition operator $f \mapsto f \circ \phi$ is bounded on BMOA_v and $\mathrm{BMOA}_v \not\subset H^\infty$.

In the literature there are many results on the equivalence of weak compactness and compactness for (weighted) composition operators. For example in [7] Eklund, Galindo, Lindström and Nieminen proved that weakly compact weighted composition operators from Bloch type spaces into a wide class of Banach spaces of analytic functions on the open unit disk are always compact. The harder well-known problem of whether every weakly compact composition operator on the space BMOA (and VMOA) is compact was solved by Laitila, Nieminen, Saksman and Tylli [14]. Very recently, Laitila, Lindström and Norrbo [16] extended the result to weighted composition operators on BMOA and simplified the function-theoretic characterization of compactness given in [13]. This is now generalized to a large class of weighted BMOA and VMOA spaces, contained in BMOA. In addition to the immediate changes the weight v presents, the known proof of the invariance of p in $\mathrm{BMOA}_{v,p}$ demands some constraints on v. As a consequence, some new estimates are developed (see Proposition 2.3) and the previous function-theoretic characterization is changed in a nontrivial way.

For the standard weights $(1-|z|)^{-c},\ 0 \le c < \frac{1}{2}$, boundedness and compactness of composition operators was characterized in [29] by Xiao and Xu. The logarithmic BMOA-space appears naturally in the study of Toeplitz and Hankel operators on BMOA and H^1 (see for example [20] and the references therein), and hence, some of the obtained results on the space BMOA_v may prove useful outside the study of weighted composition operators.

Next, some relevant vector spaces are introduced. Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle and $dm(e^{it}) = \frac{dt}{2\pi},\ t \in [0, 2\pi[$ be the normalized rotationally invariant Haar measure on $\mathbb{T}$. For $0 < p < \infty$, the Hardy space, H^p, is the linear space of functions $f \in \mathcal{H}(\mathbb{D})$ that satisfy

$$\|f\|_{H^p}^p := \sup_{r \in [0,1[} \int_{\mathbb{T}} |f(rw)|^p \, dm(w) < \infty$$

and

$$H^\infty := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)| < \infty \right\}.$$

For any function $f \in H^1$, the nontangential limit $\lim_{z\to w} f(z)$ exists for almost every $w \in \mathbb{T}$ (see e.g. [5, Section 2 until Theorem 2.2]). A useful composition operator which will be used several times is $T_c\colon f \mapsto [z \mapsto f(cz)],\ c \in \overline{\mathbb{D}}$, which is a rotational shift of argument when $c \in \mathbb{T}$ and a dilation when $c \in [0,1]$. The symbol $\hat{\phi}$ will always represent an analytic automorphism of $\mathbb{D}$, that is, a function of the form $z \mapsto b(a-z)/(1-\overline{a}z)$, where $b \in \mathbb{T}$ and $a \in \mathbb{D}$. The set of these functions is denoted by Aut. Given $0 < p < \infty$ and a weight v, that is, an integrable function $\mathbb{D} \to]0,\infty[$, the space $\mathrm{BMOA}_{v,p}$ consists of all functions $f \in H^p$ such that

$$\|f\|_{*,v,p} := \sup_{\hat{\phi}\in \mathrm{Aut}} v(\hat{\phi}(0)) \left\| f \circ \hat{\phi} - f(\hat{\phi}(0)) \right\|_{H^p} < \infty.$$

The subscript $*$ stands for semi-norm and as a superscript $*$ means the dual space. As in the classical case, the subspace $\mathrm{VMOA}_{v,p} \subset \mathrm{BMOA}_{v,p}$ is defined as

$$\mathrm{VMOA}_{v,p} := \left\{ f \in \mathrm{BMOA}_{v,p} : \lim_{\substack{\left|\hat{\phi}(0)\right|\to 1 \\ \hat{\phi}\in \mathrm{Aut}}} v(\hat{\phi}(0)) \left\| f \circ \hat{\phi} - f(\hat{\phi}(0)) \right\|_{H^p} = 0 \right\}.$$

For $a, z \in \mathbb{D}$, let $\sigma_a(z) := (a-z)/(1-\overline{a}z)$, and note that $\sigma_a(\sigma_a(z)) = z$. Since

$$b\frac{a-z}{1-\overline{a}z} = \frac{ba-bz}{1-\overline{ba}bz},\ b \in \mathbb{T}$$

and $\|T_c f\|_{H^p} = \|f\|_{H^p}\,,\ c \in \mathbb{T}$, it follows that

$$\|f\|_{*,v,p} = \sup_{a\in\mathbb{D}} v(a) \|f \circ \sigma_a - f(a)\|_{H^p} = \sup_{a\in\mathbb{D}} v(a) \left(\int_{\mathbb{T}} |f(w) - f(a)|^p P_a(w)\, dm(w) \right)^{\frac{1}{p}},$$

where $P_a(z) := (1-|a|^2)/|1-\overline{a}z|^2$ is the Poisson kernel. The following well-known identities will be used throughout the paper without any explicit mentioning:

$$1 - |\sigma_a(z)|^2 = \frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}z|^2} = (1-|z|^2)\,|\sigma_a'(z)| = (1-|z|^2)P_a(z),\ a, z \in \mathbb{D}.$$

Before we proceed, we introduce some elementary notations. The space bounded linear operators on a Banach space X is denoted by $\mathcal{L}(X)$. The evaluation maps, $\mathcal{H}(\mathbb{D}) \ni f \mapsto f(z)$, are denoted by $\delta_z, z \in \mathbb{D}$. For two quantities $A, B \geq 0$ the notation $A \lesssim B$ or $B \gtrsim A$ means that there exists a constant $C > 0$ such that $A \leq CB$. The constant will quite often depend on an exponent $1 \leq p, q < \infty$ (John-Nirenberg related) or the weight v (or the related function g). Dependencies will be mentioned by a subscript. Moreover, $A \asymp B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold, and in this case we say that A is equivalent to B. The function χ_A will appear inside integrals and is a function of all relevant integration variables. The subscript A can be a set, in which case the function takes the value 1 if the integration variables are inside the set and 0 else. The subscript A can also be a logic expression, which works as an abbreviation for the set of points satisfying the expression. Henceforth, $\mathrm{BMOA}_v := \mathrm{BMOA}_{v,2}$ and for $v = 1$, $\mathrm{BMOA} := \mathrm{BMOA}_1$. For convenience, we also define

$$\gamma(f, a, p) := \|f \circ \sigma_a - f(a)\|_{H^p}, \quad a \in \mathbb{D},\, 1 \leq p < \infty.$$

A real-valued, nonnegative function f is said to be almost increasing if there is a constant $C \geq 1$ such that $f(y) \leq Cf(x)$ whenever $y \leq x$. If $C = 1$, then the word increasing is used. For a set $M \subset \mathbb{C}$ and a number $c \in \mathbb{C}$, the notation $cM := Mc := \{cx \in \mathbb{C} : x \in M\}$ is used. When a function f defined on M is considered on a subdomain $M' \subset M$, the restriction of f to M' is denoted by $f|_{M'}$. The notation $\mathbb{C}_{\Re \geq c} := \{z \in \mathbb{C} : \Re z \geq c\}$, $c \in \mathbb{R}$, denotes a right half plane and $\mathcal{H}(M)$ is the linear space of functions analytic on some domain containing M.

In this paragraph we assume that the weight $v \colon \mathbb{D} \to]0, \infty[$ is radial (rotationally invariant), that is, $v(z) = v(|z|),\ z \in \mathbb{D}$. In the classical case, where $v \equiv 1$, the fact that Aut is a group with respect to composition yields $\|f \circ \hat{\phi}\|_{*,v,p} = \|f\|_{*,v,p}$. For more general weights, it holds that $\|f \circ \hat{\phi}\|_{*,v,p} = \|f\|_{*,v,p}$ for every $\hat{\phi} \in \mathrm{Aut}$, if and only if v is constant. In general, for every $\hat{\phi} \in \mathrm{Aut}$ it holds that

$$\left\| f \circ \hat{\phi} \right\|_{*,v,p} = \|f\|_{*,v\circ\hat{\phi}^{-1},p}\,.$$

If the weight v is almost increasing, equivalent to a radial weight and satisfies $v(b) \lesssim v(\frac{b-a}{1-a})v(a),\ 0 \leq a \leq b < 1$, it still holds that

$$f \in \mathrm{BMOA}_{v,p} \implies f \circ \hat{\phi} \in \mathrm{BMOA}_{v,p}$$

(see Lemma 3.6). In general, it is not evident that this is true (see Conjecture 2 in Section 7).

A useful tool is the Hardy-Stein estimates (see for example [30, Theorem 4.22]), from which it follows that for $0 < p < \infty$

$$\|f \circ \sigma_a - f(a)\|_{H^p}^p \asymp \int_{\mathbb{D}} |f'(z)|^2 \,|f(z) - f(a)|^{p-2} \,\frac{(1 - |z|^2)(1 - |a|^2)}{|1 - \overline{a}z|^2}\, dA(z) \quad f \in H^p. \tag{1.1}$$

The case $p = 2$ is also a consequence of the well-known Littlewood-Paley identity.

If the weight satisfies

$$\sup_{a\in\mathbb{D}} v(a)(1 - |a|)^{\frac{1}{p}-\epsilon} < \infty$$

for some $\epsilon > 0$ and $p \geq 2$, we have by (1.1) and [30, Theorem 1.12] that all bounded functions $f \in \mathcal{H}(\mathbb{D})$ with $|f'(z)|^2 \lesssim (1-|z|)^{-(1+p\epsilon)}$ belong to $\mathrm{BMOA}_{v,p}$. By inclusion, the same functions $f \in \mathrm{BMOA}_{v,q}, 1 \leq q < 2$.

The main function-theoretic characterization of compactness concerns the following two functions, α and β. For $\psi, \phi \in H^1$ and $\phi \colon \mathbb{D} \to \mathbb{D},\ a \in \mathbb{D}$, we define

$$\alpha(\psi, \phi, a) := \frac{v(a)}{v(\phi(a))} |\psi(a)| \, \|\phi_a\|_{H^2} ,$$

where $\phi_a := \sigma_{\phi(a)} \circ \phi \circ \sigma_a$. The function α is sufficient for a characterization of boundedness and compactness for composition operators, but for weighted composition operators, a complementary function

$$\beta(\psi, \phi, a) := \left\|\delta_{\phi(a)}\right\|_{(\mathrm{BMOA}_v)^*} v(a)\gamma(\psi, a, 1)$$

is also needed. It follows that for $\sup_{a\in\mathbb{D}} \beta(\psi, \phi, a) < \infty$ to hold, it is necessary that $\psi \in \mathrm{BMOA}_{v,1}$.

1.1. Main results

Let $g \in \mathcal{H}(\mathbb{C}_{\Re \geq \frac{1}{2}})$ be such that $g|_{[\frac{1}{2},\infty[}$ is (strictly) positive and almost increasing. Assume also that

(G1) There exists $\epsilon_0 > 0$ such that $\sup_{0<x<1} x\, g(\frac{1}{x})^{2+\epsilon_0} < \infty$,
(G2) $g(1/b) \lesssim g(a/b)g(1/a)$ for $0 < b \leq a < 2$,
(G3) $|g(z)| \gtrsim g(|z|),\ z \in \mathbb{C}_{\Re \geq \frac{1}{2}}$,

and let $v(z) \asymp g(\frac{1}{1-|z|})$.

If v is such a weight, also called admissible weight, we have the following two theorems.

Theorem 1.1.

$$\psi C_\phi \in \mathcal{L}(\mathrm{BMOA}_v) \iff \sup_{a\in\mathbb{D}} (\alpha(\psi, \phi, a) + \beta(\psi, \phi, b)) < \infty$$

and

$$\begin{aligned} \psi C_\phi \in \mathcal{L}(\mathrm{VMOA}_v) &\iff \sup_{a\in\mathbb{D}} (\alpha(\psi, \phi, a) + \beta(\psi, \phi, a)) < \infty \textit{ and } \psi, \psi\phi \in \mathrm{VMOA}_v \\ &\iff \sup_{a\in\mathbb{D}} (\alpha(\psi, \phi, a) + \beta(\psi, \phi, a)) < \infty, \psi \in \mathrm{VMOA}_v \textit{ and} \\ &\qquad \lim_{|a|\to 1} v(a)\psi(a)\gamma(\phi, a, 2) = 0. \end{aligned}$$

More specifically, for $X = \mathrm{BMOA}_v$ or $X = \mathrm{VMOA}_v$ and $\psi C_\phi \colon X \to X$, it holds that

$$\|\psi C_\phi\|_{\mathcal{L}(X)} \asymp_{v,g} |\psi(0)| \left\|\delta_{|\phi(0)|}\right\|_{X^*} + \sup_{a\in\mathbb{D}} \alpha(\psi,\phi,a) + \sup_{a\in\mathbb{D}} \beta(\psi,\phi,a),$$

where the evaluation map satisfies

$$\|\delta_z\|_{X^*} \asymp_{v,g} 1 + \int_0^{|z|} \frac{dt}{(1-t)v(t)}.$$

Theorem 1.2. *Let* $X = \mathrm{BMOA}_v$ *or* $X = \mathrm{VMOA}_v$*. For* $\psi C_\phi \in \mathcal{L}(X)$ *with at least one of the following properties:*

- $C_\phi \in \mathcal{L}(\mathrm{BMOA}_v)$*, and* $\mathrm{BMOA}_v \not\subset H^\infty$ *or* $\psi \in \mathrm{VMOA}_v$*,*
- $\psi C_\phi|_{\mathrm{VMOA}_v} \in \mathcal{L}(\mathrm{VMOA}_v)$*,*

the following are equivalent:

(1) ψC_ϕ *is compact,*
(2) ψC_ϕ *is weakly compact,*
(3) ψC_ϕ *is completely continuous,*
(4) ψC_ϕ *does not fix a copy of* c_0 *(*c_0*-singular),*
(5) ψC_ϕ *is unconditionally converging,*
(6) ψC_ϕ *is strictly singular,*
(7) ψC_ϕ *is finitely strictly singular,*
(8) $\limsup_{|\phi(a)|\to 1}(\alpha(\psi,\phi,a) + \beta(\psi,\phi,a)) = 0,$
(9) $\limsup_{|\phi(a)|\to 1}\left(\left\|\psi C_\phi f_a^{(\alpha)}\right\|_{\mathrm{BMOA}_v} + \beta(\psi,\phi,a)\right) = 0,$

where

$$f_a^{(\alpha)} : z \mapsto \frac{\sigma_{\phi(a)}(z) - \phi(a)}{v(\phi(a))}.$$

In the case $\psi C_\phi|_{\mathrm{VMOA}_v} \in \mathcal{L}(\mathrm{VMOA}_v)$*, the following are equivalent:*

(2) ψC_ϕ *is weakly compact,*
(10) $\psi C_\phi(\mathrm{BMOA}_v) \subset \mathrm{VMOA}_v$ *(here* ψC_ϕ *is interpreted as an operator on* $\mathcal{H}(\mathbb{D})$*),*
(11) $\forall \epsilon > 0\ \exists N > 0 : \|\psi C_\phi f\|_{\mathrm{BMOA}_v} \le N \|f\|_{H^2} + \epsilon \|f\|_{\mathrm{BMOA}_v}, \quad f \in \mathrm{VMOA}_v.$

This yields a complete characterization of the properties listed for ψC_ϕ on the space VMOA_v, but not on BMOA_v. Note that for $v \equiv 1$ the function β in this work is smaller than the one used in [13]. Since $\left\|\delta_{\phi(a)}\right\|_{X^*}$, where $X = \mathrm{BMOA}_v$ or $X = \mathrm{VMOA}_v$, is not necessarily equivalent to a radial function, it is unclear if the reverse relation holds, see Conjecture 5.

Concerning the multiplication operator, S. Ye characterized, in [25], boundedness and compactness of the multiplication operators on BMOA_v and VMOA_v with $v(z) = \ln(2/(1-|z|^2))$. For more general weights, S. Janson described the multipliers in [12, Theorem 2] using a different proof. The following result can essentially be compared to Theorem 2 by Janson:

Corollary 1.3. *Let $X = \mathrm{BMOA}$ or $X = \mathrm{VMOA}$. The multiplication operator $M_\psi \colon f \mapsto \psi f$ is bounded on X_v if and only if*

$$\psi \in H^\infty \cap \mathrm{BMOA}_w,$$

where

$$w(a) = v(a)\left(1 + \int_0^{|a|} \frac{dt}{(1-t)v(t)}\right) \asymp v(a) \left\|\delta_a\right\|_{(X_v)^*}, \quad a \in \mathbb{D}.$$

Especially, if $\sup_{a\in\mathbb{D}} \int_0^{|a|} \frac{dt}{(1-t)v(t)} < \infty$, or equivalently $X_v \subset H^\infty$, then $X_v = X_w$ is an algebra.

Moreover, the operator M_ψ is compact (or satisfies any of the other equivalent properties (2)–(7)) on X_v if and only if $\psi \equiv 0$.

Although the statement about compactness does not follow immediately from Theorem 1.2, it holds that $C_\phi \in \mathcal{L}(\mathrm{BMOA}_v)$ and $\limsup_{|\phi(a)|\to 1}(\alpha(\psi, \phi, a) > 0$ unless $\psi \equiv 0$, so the same proof as for the theorem applies in this case.

Concerning the composition operator, we have the following generalization of the part of the work [29] by J. Xiao and W. Xu that concerns boundedness and compactness of C_ϕ on the Analytic Lipschitz spaces, BMOA_v, where $v(a) = (1-|a|^2)^{-c}$, $c \in]0, 1/2[$, $a \in \mathbb{D}$.

Corollary 1.4. *Let $X = \mathrm{BMOA}$ or $X = \mathrm{VMOA}$ and let v be an admissible weight. The composition operator C_ϕ is bounded on X_v if and only if*

$$\sup_{a\in\mathbb{D}} \frac{v(a)}{v(\phi(a))} \left\|\phi_a\right\|_{H^2} < \infty,$$

and $C_\phi \in \mathcal{L}(X_v)$ is a compact operator if and only if

$$\limsup_{|\phi(a)|\to 1} \frac{v(a)}{v(\phi(a))} \left\|\phi_a\right\|_{H^2} = 0.$$

Furthermore, it is necessary that $\phi \in X_v$ for C_ϕ to be bounded.

Some examples of admissible weights are $v(z) = (1-|z|)^{-c}$, that is, $g(z) = z^c$, $0 \le c < 1/2$ and

$$v(z) = (\ln(\frac{e}{1-|z|}))^c, \text{ that is, } g(z) = (\ln(ez))^c, \ c > 0,$$

where the branch cuts are chosen appropriately (e.g. along the negative real axis). To see that condition (G2) holds for $g(z) = (\ln(ez))^c, \ c > 0$, we substitute $a = e^{-A}$ and $b = e^{-B}$, and use the fact that

$$\frac{1+B}{(1+A)(B-A+1)} \leq \frac{1}{(1+A)} + \frac{1}{(B-A+1)} \leq 1 + \frac{1}{\ln \frac{e}{2}} \qquad \Big(\ln \frac{1}{2} < A \leq B < \infty\Big).$$

In view of Theorem 1.1 and the evaluation map, the logarithmic weights mentioned above yield a space $\mathrm{BMOA}_v \subset H^\infty$ if and only if $c > 1$. Concerning the standard weights, $\mathrm{BMOA}_v \subset H^\infty$ for all $c > 0$. Moreover, if v is an admissible weight and $u \colon \mathbb{D} \to [a, b]$ for some $0 < a < b < \infty$, then vu is an admissible weight. This is clear from the fact that the norms $\|\cdot\|_{\mathrm{BMOA}_v}$ and $\|\cdot\|_{\mathrm{BMOA}_{vu}}$ are equivalent. An example of a non-radial admissible weight is $z \mapsto |2+z|\,(1-|z|)^{-1/4}$. Another useful property of the set of admissible weights is given in the paragraph where (6.1) appears, that is, the product of two admissible weights is admissible if it satisfies the growth restriction (G1).

The article is structured as follows: Section 2 contains some more definitions and some preliminary results. The most important new result in this section is Proposition 2.3, which in addition to [6, Proposition 2.6] ($\mathrm{BMOA}_{v,p} = \mathrm{BMOA}_{v,1}$ for suitable $v, 1 < p < \infty$), contains some more precise estimates, which are, for example, used to prove that the given function-theoretic condition is sufficient for $\psi C_\phi \in \mathcal{L}(\mathrm{VMOA}_v)$ to be compact (Theorem 5.12). Another important tool is the denseness of polynomials in VMOA_v, which is given in Proposition 2.7. In Theorem 2.15 it is shown that VMOA_v^{**} is isometrically isomorphic to BMOA_v. The section ends with a brief discussion regarding the demands for the weight v to be admissible. The two following sections contain some preparatory results, of which a few might be of interest on their own. Section 3 contains three important results: Lemmas 3.1 and 3.4 concern the main function-theoretic characterization for boundedness and compactness, and they are related to the functions α and β respectively. The third important result is Corollary 3.5, which is an estimate for the evaluation map. In Section 4, the test functions are developed and proofs of important properties for these functions are given.

Section 5 contains, in contrast to Section 2, the main parts of the main theorems, whose proofs make heavy use of the fact that the operator is a weighted composition operator and it acts on VMOA_v (or BMOA_v), where v is admissible. The function-theoretic characterization for boundedness is proved followed by the remaining implications (the ones that are not of a more general type) to conclude that Theorem 1.2 holds. Section 6 contains some examples of symbols ψ and ϕ making ψC_ϕ bounded or even compact. Using the obtained results, the proofs for the three main results are completed and summarized in Section 7. Section 7 also contains some conjectures.

2. Preliminaries

This section contains some more definitions and preliminary results. We begin the section by showing that the polynomials in $\mathcal{H}(\mathbb{D})$ are often dense in a proper subspace of $\mathrm{BMOA}_{v,p}$. The section ends with a brief discussion concerning the conditions (G1) and (G2).

Proposition 2.1. *Let $1 < p < \infty$. If $\sup_{a\in\mathbb{D}} v(a)(1-|a|)^{\frac{1}{p}} < \infty$, then the polynomials belong to $\mathrm{BMOA}_{v,p}$. If $\lim_{a\to 1} v(a)(1-|a|)^{\frac{1}{p}} = 0$, then $\mathrm{VMOA}_{v,p} \subset \mathrm{BMOA}_{v,p}$ is a closed subspace containing the analytic polynomials. If $p = 1$ the same statements are valid when $v(a)(1-|a|)^{\frac{1}{p}}$ is replaced by $v(a)(1-|a|)\ln\frac{e}{1-|a|}$.*

Proof. Let $Q(z) = \sum_{k=0}^{n} c_k z^k$ be a polynomial on the disk. Now

$$Q(z) - Q(a) = \sum_{k=0}^{n} c_k z^k - \sum_{k=0}^{n} c_k a^k = \sum_{k=1}^{n} c_k (z^k - a^k) = (z-a)\sum_{k=1}^{n} c_k \sum_{j=0}^{k-1} z^j a^{k-1-j}.$$

There exists a constant $C(Q,p)$ only dependent of the polynomial Q and $1 \le p < \infty$ such that for $1 < p < \infty$,

$$\gamma(Q,a,p)^p = \int_{\mathbb{T}} |(z-a)|^p \left|\sum_{k=1}^{n} c_k \sum_{j=0}^{k-1} z^j a^{k-1-j}\right|^p P_a(z)\,dm(z) \le C(Q,p)(1-|a|^2).$$

Assuming $\lim_{a\to 1} v(a)(1-|a|)^{\frac{1}{p}} = 0$, this yields

$$v(a)\gamma(Q,a,p) \le C'(Q,p)v(a)(1-|a|)^{\frac{1}{p}} \overset{|a|\to 1}{\longrightarrow} 0,$$

that is, $Q \in \mathrm{VMOA}_{v,p}$. If $(f_n) \subset \mathrm{VMOA}_{v,p}$ is a sequence, converging with respect to $\|\cdot\|_{\mathrm{BMOA}_{v,p}}$ to an analytic function f, and $\epsilon > 0$, then for n large enough

$$v(a)\gamma(f,a,p) \le v(a)\gamma(f_n,a,p) + v(a)\gamma(f_n - f,a,p) \le v(a)\gamma(f_n,a,p) + \epsilon.$$

Letting first $|a| \to 1$, then $\epsilon \to 0$, we obtain

$$\lim_{|a|\to 1} v(a)\gamma(f,a,p) = 0.$$

Hence, $\mathrm{VMOA}_{v,p}$ is a closed subspace of $\mathrm{BMOA}_{v,p}$, containing the polynomials.

Finally, if $p = 1$,

$$\gamma(Q,a,p) = (1-|a|^2)\int_0^{2\pi} J(e^{it})\left|1 - ae^{-it}\right|^{p-1}\frac{dt}{2\pi} \lesssim C(Q,p)(1-|a|^2)\ln\frac{e}{1-|a|},$$

and the statement follows similarly to the proof above. □

The first part of Lemma 2.12 yields that BMOA_v is a Banach space and since VMOA_v is closed, it is also a Banach space.

Proposition 2.1 shows some similarities between the weighted spaces BMOA_v and VMOA_v, and their unweighted variants. The following Proposition shows some contrast between the spaces, which renders some classical approaches ineffective. The function $z \mapsto z^n$ is one common tool in characterizing compactness in the unweighted setting, and can be found in, for example, [15,26,28] (and in some form also in [13, (3.13)]).

Proposition 2.2. *Let $1 < p < \infty$ and assume $\lim_{|a|\to 1} v(a)(1-|a|)^{\frac{1}{p}} = 0$. The family F consisting of $f_n\colon z \mapsto z^n, n \in \mathbb{N}$ belongs to $\mathrm{VMOA}_{v,p}$, and $\sup_n \|f_n\|_{*,v,p} < \infty$ if and only if v is bounded.*

Proof. The fact that $F \subset \mathrm{VMOA}_{v,p}$ follows from Proposition 2.1 and the trivial estimate $\|f_n\|_{\mathrm{BMOA}_{1,p}} \le 3\|f_n\|_\infty = 3$ proves one of the implications in the remaining statement. Now, let $t_n = 1 - n^{-\frac{1}{2}}, n \in \mathbb{N}$. By (1.1) and the fact that $\int_{\mathbb{T}} P_b\, dm = 1$ for every $b \in \mathbb{D}$, we have

$$\|f_n \circ \sigma_a - f_n(a)\|_{H^2}^2 \asymp \int_{\mathbb{D}} n^2 |z|^{2(n-1)} (1 - |\sigma_a(z)|^2)\, dA(z)$$

$$\ge \int_{t_n}^{1} n^2 x^{n-1} \frac{(1-x)(1-|a|^2)}{1-|a|^2 t_n}\, dx.$$

Moreover, by the well-known asymptotics for the classical beta function, we have

$$n^{-2} \overset{n\to\infty}{\sim} \int_{t_n}^{1} x^{n-1}(1-x)\, dx + O\big(2^{-\sqrt{n}}\big).$$

This yields that

$$\limsup_{n\to\infty} \|f_n \circ \sigma_a - f_n(a)\|_{H^2}^2 \gtrsim 1.$$

Finally,

$$\sup_n \|f_n\|_{*,v,p} \ge \sup_{a\in\mathbb{D}} \limsup_{n\to\infty} v(a)\, \|f_n \circ \sigma_a - f_n(a)\|_{H^2} \gtrsim \sup_{a\in\mathbb{D}} v(a). \quad \square$$

2.1. Consequences of John-Nirenberg's result

We begin by introducing the spaces $\mathrm{BMO}_{v,p}$ and $\mathrm{VMO}_{v,p}$. (Note that the following representations are not standard and these spaces will only be used in this section.) We define

$$\|f\|_{*,\mathrm{BMO}_{v,p}} := \sup_{\substack{I\subset\mathbb{T}\\ I \text{ arc}}} v(1-m(I))\eta(f,I,p),$$

where

$$\eta(f,I,p) := \left(\int_I |f(z)-m_I(f)|^p \,\frac{dm(z)}{m(I)}\right)^{\frac{1}{p}},$$

and $m_I(f) := \int_I f(w)\frac{dm(w)}{m(I)}$ is the mean of the function on the arc $I \subset \mathbb{T}$. Now

$$\mathrm{BMO}_{v,p} := \left\{ f \in H^p : \|f\|_{*,\mathrm{BMO}_{v,p}} < \infty \right\}$$

and

$$\mathrm{VMO}_{v,p} := \left\{ f \in \mathrm{BMO}_{v,p} : \lim_{\substack{m(I)\to 0\\ I \text{ arc}}} v(1-m(I))\eta(f,I,p) = 0 \right\}.$$

The following well-known results can be found in, for example, [1] (see also [8,10]). If $v \asymp 1$, then $\mathrm{BMOA}_{v,p} = \mathrm{BMOA}_v$, $0 < p < \infty$ and for $1 \leq p < \infty$, any of the semi-norms $\|f\|_{*,v,p}$ is comparable with any of $\|f\|_{*,\mathrm{BMO}_{v,q}}$, $1 \leq q < \infty$. The independency of the parameter p is, in the work of Baernstein ([1]), proved in the conformally invariant setting, BMOA, making use of the group structure of Aut. This yields a better result than the classical approach carried out in [8] and [10], which proves independency in the classical BMO setting and then apply the result that for a fixed p, $\|f\|_{*,\mathrm{BMO}_{v,p}} \asymp \|f\|_{*,v,p}$ when $v \asymp 1$. It is, however, no surprise that stronger results can exist in the analytic setting compared to the measurable setting due to additional structure. When v is almost increasing, we have the following results, pointed out in [6] by Dyakonov. (The weights $\varphi(t)$ in [6] and [27] are comparable to $v(1-t)^{-1}$ for $t \in]0,1[$.) The following proposition contains [6, Proposition 2.6], but also some new crucial estimates for the proofs of the main results.

Proposition 2.3. *Let $1 \leq q \leq p < \infty$ and $v\colon \mathbb{D} \to]0,\infty[$ be radial. Assume $v|_{[0,1[}$ is almost increasing and there is an $\epsilon_0 > 0$ such that $x \mapsto v|_{[0,1[}(1-x)x^{\frac{1}{p}-\epsilon_0}$ is almost increasing. Then*

$$\mathrm{BMOA}_{v,p} = \mathrm{BMO}_{v,q} \quad \text{and} \quad \|f\|_{*,\mathrm{BMO}_{v,q}} \asymp_{v,p,q,\epsilon_0} \|f\|_{*,v,p}.$$

Moreover, for $R \in]0,1[$

$$\sup_{m(I)\leq R} v(1-m(I))\eta(f,I,p) \asymp_{v,p,q} \sup_{m(I)\leq R} v(1-m(I))\eta(f,I,q) \lesssim \sup_{|a|\geq 1-R} v(a)\gamma(f,a,q) \tag{2.1}$$

and for any $R_{\mathrm{BMO}} \in]0,1[$ *and* $R_A \in]0, R_{\mathrm{BMO}}/2]$, *we have*

$$\sup_{|a|\geq 1-R_A} v(a)\gamma(f,a,q) \lesssim_{v,q,\epsilon_0} \sup_{m(I)\leq R_{\mathrm{BMO}}} v(1-m(I))\eta(f,I,q) + \|f\|_{*,\mathrm{BMO}_{v,q}} \left(\frac{2R_A}{R_{\mathrm{BMO}}}\right)^{\epsilon_0}. \tag{2.2}$$

Hence,

$$\mathrm{VMOA}_{v,p} = \mathrm{VMO}_{v,q} \quad \textit{and} \quad \limsup_{|a|\to 1} v(a)\gamma(f,a,p) \asymp_{v,p,q,\epsilon_0} \limsup_{m(I)\to 0} v(1-m(I))\eta(f,I,q). \tag{2.3}$$

Note that if $x \mapsto v|_{[0,1[}(1-x)x^{\frac{1}{p}-\epsilon_0}$ almost increasing for some $p = p_0$, it is almost increasing for any $0 < p \leq p_0$.

The proof is split into a few results and most of them can be found, in some form, in [1,8] and [10]. The following proposition follows immediately from the proof in the classical unweighted BMO setting (see e.g. [10, p. 73]).

Proposition 2.4. *For* $1 \leq p < \infty$ *and* $\inf_{x\in]0,1[} v(x) > 0$ *and all* $R > 0$, *it holds that*

$$\sup_{\substack{m(I)\leq R \\ I \text{ arc}}} v(1-m(I))\eta(f,I,p) \leq 2 \sup_{|a|\geq 1-R} v(a)\gamma(f,a,p).$$

Furthermore,

$$\|f\|_{*,v,p} \gtrsim \|f\|_{*,\mathrm{BMO}_{v,p}}.$$

For the proof of [10, Theorem 3.1], a dyadic decomposition of $\mathbb{T}$ is used. To be able to summarize the approximations made from the dyadic decomposition in the classical fashion, the weight needs to satisfy an extra condition stated in [6, Proposition 2.6], the fact that $(v(1-t))^{-1}$ need to be of upper type less than $1/p$. In this work, the comparable property found in [27] of almost increasing/decreasing is used. Inspired by [10, Proof of Theorem 3.1], we have the following proposition, which includes a new, suitable estimate for this work.

Proposition 2.5. *Let* $1 \leq p < \infty$ *and* $v\colon \mathbb{D} \to]0,\infty[$ *be radial. Assume there is an* $\epsilon_0 > 0$ *such that* $x \mapsto v|_{[0,1[}(1-x)x^{\frac{1}{p}-\epsilon_0}$ *is almost increasing. Then*

$$\|f\|_{*,v,p} \lesssim_{v,p,\epsilon_0} \|f\|_{*,\mathrm{BMO}_{v,p}}$$

and for any $R_{\mathrm{BMO}} \in]0,1[$ *and* $a \in \mathbb{D}$ *with* $|a| \geq 1 - R_{\mathrm{BMO}}/2$, *we have*

$$v(a)\gamma(f,a,p) \lesssim_{v,p,\epsilon_0} \sup_{m(I)\leq R_{\mathrm{BMO}}} v(1-m(I))\eta(f,I,p) + \|f\|_{*,\mathrm{BMO}_{v,p}} \left(\frac{2(1-|a|)}{R_{\mathrm{BMO}}}\right)^{\epsilon_0}. \tag{2.4}$$

Proof. Let $a \in \mathbb{D}$ and define $J_k,\ k = 0, 1, \ldots, N$ to be the arc with center $a/|a|$ and $m(J_k) = 2^k(1 - |a|)$, where N is the number such that $m(J_N) < 1 \leq 2m(J_N)$. We also put $J_{N+1} := \mathbb{T}$. The relation between a and N is given by

$$N = N(a) = \max\left\{n \in \mathbb{Z} : n < \ln \frac{1}{1 - |a|} \frac{1}{\ln 2}\right\}.$$

We have $\mathbb{T} \setminus J_0 = \bigcup_{k=1}^{N+1}(J_k \setminus J_{k-1})$ and hence, with the aid of Minkowski's inequality and a variant of [10, Lemma 3.2], we have

$$\begin{aligned}
&\left(\int_{\mathbb{T}} |f(z) - f(a)|^p \, P_a(z) \, dm(z)\right)^{\frac{1}{p}} \\
&\lesssim \left(\frac{1}{m(J_0)} \int_{J_0} |f(z) - m_{J_0}(f)|^p \ dm(z)\right)^{\frac{1}{p}} \\
&\quad + \sum_{k=1}^{N+1} \left(\int_{J_k \setminus J_{k-1}} |f(z) - m_{J_0}(f)|^p \inf_{w \in J_{k-1}} P_a(w) \, dm(z)\right)^{\frac{1}{p}}.
\end{aligned}$$

For the integrals in the second term, we apply Minkowski's inequality and [10, Lemma 3.4] to obtain

$$\begin{aligned}
&\left(\int_{J_k \setminus J_{k-1}} |f(z) - m_{J_0}(f)|^p \inf_{w \in J_{k-1}} P_a(w) \, dm(z)\right)^{\frac{1}{p}} \\
&\qquad \lesssim_p \frac{1}{2^{\frac{k}{p}}} \eta(f, J_k, p) + \frac{1}{2^{\frac{k}{p}}} \left|m_{J_k}(f) - m_{J_0}(f)\right|.
\end{aligned}$$

Furthermore, a variant of [10, Lemma 3.3] gives

$$|m_{J_k}(f) - m_{J_0}(f)| \lesssim \sum_{j=1}^{k} \eta(f, J_j, 1)$$

yielding

$$\begin{aligned}
\sum_{k=1}^{N+1} \left(\int_{J_k \setminus J_{k-1}} |f(z) - m_{J_0}(f)| \inf_{w \in J_{k-1}} P_a(w) \, dm(z)\right)^{\frac{1}{p}} &\lesssim_p \sum_{k=1}^{N+1} \frac{1}{2^{\frac{k}{p}}} \sum_{j=1}^{k} \eta(f, J_j, 1) \\
&\lesssim_p \sum_{j=1}^{N+1} \frac{1}{2^{\frac{j}{p}}} \eta(f, J_j, p).
\end{aligned}$$

Since $m(J_j) = 2^j(1-|a|)$ and

$$v(a)m(J_0)^{\frac{1}{p}-\epsilon_0} = v(1-m(J_0))m(J_0)^{\frac{1}{p}-\epsilon_0} \lesssim_v v(1-m(J_j))m(J_j)^{\frac{1}{p}-\epsilon_0} \tag{2.5}$$

for every $j = 1, \ldots, N+1$ by assumption, we have now obtained

$$v(a)\left(\int_{\mathbb{T}} |f(z)-f(a)|^p P_a(z)\,dm(z)\right)^{\frac{1}{p}} \lesssim_{v,p} \sum_{j=0}^{N+1} \frac{1}{2^{\epsilon_0 j}} v(1-m(J_j))\eta(f,J_j,p).$$

Using $v(1-m(J_j))\eta(f,J_j,p) \leq \|f\|_{*,\mathrm{BMO}_{v,p}}$, the first statement, $\|f\|_{*,v,p} \lesssim_{v,p,\epsilon_0} \|f\|_{*,\mathrm{BMO}_{v,p}}$, follows. For the second, fix $R_{\mathrm{BMO}} \in]0,1[$. For $a \in \mathbb{D}$ with $|a| \geq 1-R_{\mathrm{BMO}}/2$, let $N_I = N_I(a) \in [1,N]$ be the integer such that $m(J_{N_I(a)}) \in]R_{\mathrm{BMO}}/2, R_{\mathrm{BMO}}]$. Furthermore, using (2.5), we have

$$v(a)\sum_{j=N_I}^{N+1} \frac{1}{2^{\frac{j}{p}}}\eta(f,J_j,p) \lesssim_v \sum_{j=N_I}^{N+1} \frac{1}{2^{\epsilon_0 j}} v(1-m(J_j))\eta(f,J_j,p) \lesssim_{\epsilon_0} \|f\|_{*,\mathrm{BMO}_{v,p}} 2^{-\epsilon_0 N_I}.$$

For $|a| \geq 1-R_{\mathrm{BMO}}/2$, we have now obtained

$$\begin{aligned} v(a)\gamma(f,a,p) &= v(a)\left(\int_{\mathbb{T}} |f(z)-f(a)|^p P_a(z)\,dm(z)\right)^{\frac{1}{p}} \\ &\lesssim_{v,p} \sum_{j=0}^{N_I} \frac{1}{2^{\epsilon_0 j}} v(1-m(J_j))\eta(f,J_j,p) + \sum_{j=N_I}^{N+1} \frac{1}{2^{\epsilon_0 j}} v(1-m(J_j))\eta(f,J_j,p) \\ &\lesssim_{\epsilon_0} \sup_{m(I)\leq R_{\mathrm{BMO}}} v(1-m(I))\eta(f,I,p) + \|f\|_{*,\mathrm{BMO}_{v,p}} 2^{-\epsilon_0 N_I}. \end{aligned}$$

Using $2^{N_I}(1-|a|) = m(J_{N_I(a)}) \geq \frac{R_{\mathrm{BMO}}}{2}$, we can conclude that for any $R_{\mathrm{BMO}} \in]0,1[$ and $a \in \mathbb{D}$ with $|a| \geq 1-R_{\mathrm{BMO}}/2$, we have

$$\begin{aligned} v(a)\gamma(f,a,p) \lesssim_{v,p,\epsilon_0} &\sup_{m(I)\leq R_{\mathrm{BMO}}} v(1-m(I))\eta(f,I,p) \\ &+ \|f\|_{*,\mathrm{BMO}_{v,p}} \left(\frac{R_{\mathrm{BMO}}}{2(1-|a|)}\right)^{-\epsilon_0}. \quad \square \end{aligned}$$

Finally, the crucial ingredient for independency of p is a John-Nirenberg type result. We have the following, inspired by [27]:

Lemma 2.6 *(John-Nirenberg). Let $0 < R < 1 < M$ and $f \in L^1(\mathbb{T})$ with $0 < \|f\|_{*,\mathrm{BMO}_{1,1}} < \infty$. For any arc $I \subset \mathbb{T}$ with $m(I) \leq R$ and $\lambda > 0$, we have*

$$m(\{w \in I : |f(w)-m_I(f)| > \lambda\}) \leq m(I)\sqrt{M}e^{-\frac{\lambda}{\sup_{m(I)\leq R}\eta(f,I,1)}\frac{\ln M}{2M}}.$$

The proof of Lemma 2.6 is the same as in [10, Theorem 4.1] (see also [8, Theorem 2.1]). Instead of considering the function $f/\|f\|_{*,\mathrm{BMO}_{1,1}}$ as done in the references, one should fix $0 < R < 1$ and change the supremum in the denominator to only include arcs $I \subset \mathbb{T}$ with $m(I) \leq R$.

We are now ready to prove Proposition 2.3:

Proof of Proposition 2.3. We begin by proving

$$\|f\|_{*,\mathrm{BMO}_{v,p}} \lesssim_{v,p} \|f\|_{*,\mathrm{BMO}_{v,1}}, \tag{2.6}$$

which together with Proposition 2.5 and Proposition 2.4 implies the equivalence of norms and that $\mathrm{BMOA}_{v,p} = \mathrm{BMOA}_{v,1}$ given the assumptions hold. Let $f \in L^1(\mathbb{T})$ with $0 < \|f\|_{*,\mathrm{BMO}_{v,1}} < \infty$. Fix $I \subset \mathbb{T}$ and put $R = m(I)$. Using Lemma 2.6 for the first inequality and the fact that v is almost increasing, we get

$$\begin{aligned} v(1-R)^p \eta(f,I,p)^p &= \frac{v(1-R)^p}{m(I)} \int_0^\infty m(\{w \in I : |f(w) - m_I(f)| > \lambda\})\, d\lambda^p \\ &\lesssim_p v(1-R)^p \int_0^\infty e^{-\frac{\lambda}{\sup_{m(I)\leq R} \eta(f,I,1)}}\, d\lambda^p \\ &\lesssim_p v(1-R)^p \left(\sup_{m(I)\leq R} \eta(f,I,1) \right)^p \\ &\lesssim_v \sup_{m(I)\leq R} v(1-m(I))^p \eta(f,I,1)^p \leq \left(\|f\|_{*,\mathrm{BMO}_{v,1}} \right)^p, \end{aligned}$$

and hence,

$$\|f\|_{*,\mathrm{BMO}_{v,p}} \lesssim_{v,p} \|f\|_{*,\mathrm{BMO}_{v,1}}.$$

Moving on, given $R_{\mathrm{BMO}} \in]0,1[$, we do the same calculations for any arc with $m(I) \leq R_{\mathrm{BMO}}$ to obtain

$$v(1-R)\eta(f,I,p) \lesssim_{v,p} \sup_{m(I)\leq R} v(1-m(I))\eta(f,I,1) \leq \sup_{m(I)\leq R_{\mathrm{BMO}}} v(1-m(I))\eta(f,I,1),$$

yielding

$$\sup_{m(I)\leq R_{\mathrm{BMO}}} v(1-m(I))\eta(f,I,p) \asymp_{v,p} \sup_{m(I)\leq R_{\mathrm{BMO}}} v(1-m(I))\eta(f,I,1).$$

Combining this with Proposition 2.4 proves (2.1). Concerning (2.2), given any $R_{\mathrm{BMO}} \in]0,1[$, pick $R_A \in]0, R_{\mathrm{BMO}}/2]$ and apply $\sup_{|a|\geq 1-R_A}$ to both sides of (2.4) in Proposition 2.5 and we are done. Finally, (2.3) follows immediately from (2.1) and (2.2). □

The following proposition is a generalization of [10, Theorem 2.1].

Proposition 2.7. *Let* $1 \le p < \infty$, $v\colon \mathbb{D} \to]0,\infty[$ *be radial and put* $q = \max\{2,p\}$. *Assume* $v|_{[0,1[}$ *is almost increasing and there is an* $\epsilon_0 > 0$ *such that* $x \mapsto v|_{[0,1[}(1-x)x^{\frac{1}{q}-\epsilon_0}$ *is almost increasing. Then, for a given* $f \in \mathcal{H}(\mathbb{D})$, *the following are equivalent:*

- $f \in \mathrm{VMOA}_{v,p}$
- $\lim_{c\to 1} \|T_c f - f\|_{\mathrm{BMO}_{v,p}} = 0$,
- f *belongs to the* $\mathrm{BMOA}_{v,p}$*-closure of analytic polynomials,*

where the limit is taken arbitrary inside $\overline{\mathbb{D}}$.

Proof. First, for all $c \in \mathbb{D}$ and $f \in \mathrm{BMOA}_{v,p}$, the functions $T_c f \in \mathrm{BMOA}_{v,p}$ (see the remark right after (1.1)). Furthermore, $(T_c f - f)(0) = 0$. Let $1 \le p < \infty$, $q = \max\{2,p\}$ and assume v satisfies the assumptions and $\epsilon_0 > 0$ is the ϵ_0 given in the statement. Then $x \mapsto v|_{[0,1[}(1-x)x^{\frac{1}{q}-\epsilon_0}$ is almost increasing and bounded. By Proposition 2.3, we have $\mathrm{VMOA}_{v,p} = \mathrm{VMOA}_v = \mathrm{VMO}_{v,p} = \mathrm{VMO}_{v,2}$ with equivalent norms. It is, therefore, sufficient to prove that the three properties are equivalent for $p = 2$. To this end, assume $f \in \mathrm{VMOA}_v$. If $c \in \mathbb{T}$, it follows from the proof of [10, Theorem 2.1] that we can fix $0 < R$ small such that

$$\begin{aligned}\sup_{m(I)\le R} v(1-m(I))\eta(f - T_c f, I, 2) &\le \sup_{m(I)\le R} v(1-m(I))\eta(f, I, 2)\\ &\quad + \sup_{m(I)\le R} v(1-m(I))\eta(T_c f, I, 2)\\ &= 2\sup_{m(I)\le R} v(1-m(I))\eta(f, I, 2) < \epsilon,\end{aligned}$$

because $f \in \mathrm{VMO}_{v,2}$. For any I such that $m(I) \ge R$, we can choose c, independent of I, close enough to 1 so that $\|T_c f - f\|_{H^2} < \frac{R^{\frac{1}{2}}\epsilon}{v(1-R)}$. It follows that

$$\sup_{m(I)\ge R} v(1-m(I))\eta(T_c f - f, I, 2) \lesssim_v \frac{v(1-R)}{R^{\frac{1}{2}}}\|T_c f - f\|_{H^2} < \epsilon,$$

and hence,

$$\|T_c f - f\|_{\mathrm{BMO}_{v,2}} \lesssim_v \epsilon$$

when $c \in \mathbb{T}$ is close to 1. Now, if $c = rw_0 \in \mathbb{D}, 0 < r < 1, w_0 \in \mathbb{T}$, we first note that since $f \in H^1$

$$\int_{\mathbb{T}} P_{cz} f\, dm = f(cz) = (T_c f)(z),\ c \in \mathbb{D}, z \in \overline{\mathbb{D}}.$$

The reasoning used in [10, Theorem 2.1] combined with Minkowski's inequality gives us that for every $\delta > 0$

$$v(1-m(I))\eta(T_c f - f, I, 2) \lesssim \sup_{|\arg w|<\delta} \|T_{w_0 w} f - f\|_{*,\mathrm{BMO}_{v,2}} + \int\limits_{|\arg w|\geq\delta} P_r(w)\, \|f\|_{*,\mathrm{BMO}_{v,2}}\, dm(w).$$

Choosing $\delta > 0$ small enough and w_0 close to 1, the first term is less than ϵ. By choosing r close to 1, the second term is less than ϵ and we have proved that $f \in \mathrm{VMOA}_{v,p}$ implies $\lim_{c\to 1} \|T_c f - f\|_{\mathrm{BMO}_{v,2}} = 0$, where the limit is taken arbitrary inside $\overline{\mathbb{D}}$. Assuming $\lim_{c\to 1} \|T_c f - f\|_{\mathrm{BMO}_{v,2}} = 0$ an application of Proposition 2.3 yields

$$\lim_{c\to 1} \|T_c f - f\|_{\mathrm{BMOA}_v} = 0.$$

By choosing $c \in [0, 1[$ close enough to 1, we have $\|T_c f - f\|_{\mathrm{BMOA}_v} < \epsilon$. Since $T_c f \in \mathcal{H}(\overline{\mathbb{D}})$, the function $(T_c f)' \in \mathcal{H}(\overline{\mathbb{D}})$ can be approximated uniformly in $\mathbb{D}$ by analytic polynomials, and hence, the derivative of an analytic polynomial, say $\sup_{z\in\mathbb{D}} |p_0'(z) - (T_c f)'(z)| < \epsilon$. Now, using formula (1.1), we have

$$\|p_0 - (T_c f)\|^2_{\mathrm{BMOA}_v} \leq \epsilon^2 (1 + \sup_{a\in\mathbb{D}} v(a)(1-|a|^2))$$

and we can conclude that f belongs to the BMOA_v-closure of analytic polynomials. Finally, any function in the BMOA_v-closure of the polynomials belongs to VMOA_v according to Proposition 2.1. $\square$

For the rest of the paper, it will be assumed $v|_{[0,1[}$ is almost increasing and (strictly) positive. Using Proposition 2.4, another interesting fact about $\mathrm{BMOA}_{v,1}$ is the following result by Spanne, [27, p. 594].

Proposition 2.8. *If $v|_{[0,1[}$ is increasing and $\int_0^1 \frac{dt}{t\,v(1-t)}$ is finite, then for every $f \in \mathrm{BMOA}_{v,1}$, there exists a constant C_f (depending on f) and $\theta_f > 0$ such that for $r < \theta_f$, it holds that*

$$\operatorname*{ess\,sup}_{|t_1-t_2|<r} \left|f(e^{it_1}) - f(e^{it_2})\right| \leq C_f \int\limits_0^r \frac{dt}{t\,v(1-t)}.$$

For analytic functions, the statement above is not evidently as close to an if and only if statement (compare with [27, p. 594]). However, under some additional constraints on v, which will be present for the main results in this paper, the function defined in Lemma 3.4 could be used to prove a counterpart to Proposition 2.8 (see Corollary 3.5). Recall that $\mathrm{BMOA}_{v,1}$ only consists of constants if $v(a) \gtrsim (1-|a|)^{-c}$ for any $c > 1$ (see e.g. [9, Theorem 1.2]).

2.2. Some general Banach space theory

Let X be a Banach space and $T \in \mathcal{L}(X)$.

- A series $\sum_n x_n \subset X$ is weakly unconditionally Cauchy (wuC) if $\sum_n l(x_n)$ is unconditionally convergent, equivalently absolutely convergent, for all $l \in X^*$.
- If for every infinite dimensional subspace $M \subset X$ the operator $T|_M : M \to T(M)$ is not an isomorphism, then the operator is said to be *strictly singular* (also called Kato operator, [23, 1.9.2]).
- If for $\epsilon > 0$, there exists $N_\epsilon \geq 1$ such that for every subspace $M \subset X$ with dimension greater than N_ϵ, there is $x \in \partial B_M$ such that $\|Tx\|_X \leq \epsilon$, then the operator is said to be *finitely strictly singular* (this notion is used e.g. in [17]).
- Let M be a Banach space. The operator T *fixes a copy of* M if there exists a closed subspace $Y \subset X$ such that $Y \simeq M$ (isomorphic) and $T|_Y$ is an isomorphism onto its image $T(Y) \subset X$.
- Let $M \subset X$ be a subspace. The operator T is M-singular if it does not fix a copy of M.
- The operator T is *unconditionally converging* if it maps wuC- series to unconditionally convergent series.
- The operator T is said to be *completely continuous* if it maps weakly convergent sequences to norm convergent sequences.

For a normed space X, the closed unit ball is given by $B_X := \{f \in X : \|f\|_X \leq 1\}$. The following lemma is found in [24, C. II. Theorem 8.4'].

Lemma 2.9. *Let X be a Banach space. If $T \in \mathcal{L}(X)$ is not unconditionally converging, then it fixes a copy of c_0.*

The first statement in the lemma below is found e.g. in [23, 1.11] and the other follows more or less from the definitions.

Lemma 2.10. *Let X be a Banach space. If $T \in \mathcal{L}(X)$ is weakly compact or completely continuous, then it does not fix a copy of c_0. Moreover, if T is compact, it is both weakly compact and completely continuous.*

By the bounded inverse theorem, we have the following:

Proposition 2.11. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then the following are equivalent:*

- *T is strictly singular.*
- *For every $\epsilon > 0$ and every infinite dimensional subspace $M \subset X$, there is $x \in \partial B_M$ such that $\|Tx\|_X \leq \epsilon$.*

For more information see, for example, [4] and [19]. Inspired by [18] and [14, Proposition 6], we have

Lemma 2.12. *Let Y be a Banach space on which, for every $a \in \mathbb{D}$, $U(\cdot, a)\colon Y \to [0, \infty[$ is a complete norm yielding the evaluation maps bounded. Then the norm $\|f\| := \sup_{a\in\mathbb{D}} U(f, a)$ renders $X \subset Y$ a Banach space for some subspace $X \subset Y$.*

Moreover, if $(f_n) \subset X$ is a sequence with $\|f_n\| \asymp 1$, $\lim_{|a|\to 1} U(f_n, a) = 0$ for all n, and for all $0 < R < 1$, it holds that $\lim_{n\to\infty} \sup_{|a|\le R} U(f_n, a) = 0$. Then, there is a subsequence (f_{n_k}) equivalent to the standard basis for c_0, and hence, the identity $X \to X$ fixes a copy of c_0.

Proof. Let

$$X := \{f \in Y : \|f\| < \infty\}.$$

Clearly $\|\cdot\|_X := \|\cdot\|$ is a norm on X. Let f_n be a Cauchy sequence in X with respect to $\|\cdot\|_X$. Since $\|f\|_X \ge U(f, a)$ for all $f \in X$ and $(Y, U(\cdot, a))$ is complete, there is a limit g in Y. Since the evaluation maps are bounded, the limit, g, is independent of $a \in \mathbb{D}$. For all $a \in \mathbb{D}$, we have

$$U(g - f_n, a) = \lim_{m\to\infty} U(f_m - f_n, a) \le \lim_{m\to\infty} \|f_m - f_n\|_X\,,$$

and since the right-hand side is independent of $a \in \mathbb{D}$, we have proved that $\lim_{n\to\infty} \|f_n - g\|_X = 0$ and $\|g\|_X \le \lim_{n\to\infty} \|f_n\|_X < \infty$ since (f_n) is Cauchy in X.

For the second statement, applying the standard sliding hump technique to the assumptions (see for example [14, Proof of Proposition 6]) yields that there exists an increasing sequence $(r_k) \subset [0, 1[$ and a subsequence $(f_{n_k}) \subset (f_n)$ such that

$$\sup_{|a|\le r_k} U(f_{n_k}, a) \le 2^{-k}, \qquad \text{for all k,} \tag{2.7}$$

$$\sup_{|a|> r_{k+1}} U(f_{n_k}, a) \le 2^{-k}, \qquad \text{for all k.} \tag{2.8}$$

The sequences (r_k) and (f_{n_k}) are obtained, by first choosing e.g. $r_1 = \frac{1}{2}$ and then an element f_{n_1} such that (2.7) is satisfied, which is possible due to $\lim_{n\to\infty} \sup_{|a|\le R} U(f_n, a) = 0$. After that, we apply the fact that $\lim_{|a|\to 1} U(f_{n_1}, a) = 0$ to obtain r_2 via (2.8) and so on.

Now, let $(t_k) \in \ell^\infty$. For every $a \in \mathbb{D}$, there exists exactly one $k_a \in \{0, 1, 2, \ldots\}$ such that

$$a \in A(k_a) = \begin{cases}]r_{k_a}, r_{k_a+1}], & k_a > 0 \text{ and} \\ [0, \frac{1}{2}], & k_a = 0. \end{cases}$$

On the one hand, condition (2.7) tells us that for a fixed $a \in \mathbb{D}$, it holds that $U(f_{n_k}, a) \leq 2^{-k}$, whenever, $k > k_a$. On the other hand, condition (2.8) tells us that for a fixed $a \in \mathbb{D}$, it holds that $U(f_{n_k}, a) \leq 2^{-k}$, whenever, $k < k_a$. We can now conclude that for every K

$$\begin{aligned}\left\|\sum_{k=1}^{K} t_k f_{n_k}\right\|_X &\leq \|(t_k)\|_\infty \sum_{k=1}^{K} \|f_{n_k}\|_X = \|(t_k)\|_\infty \sup_{a\in\mathbb{D}} \left(\sum_{k>k_a} + \sum_{k<k_a} + \sum_{k=k_a}\right) U(f_{n_k}, a) \\ &\leq \|(t_k)\|_\infty \,(\sum_k 2^{-k} + \left\|f_{n_{k_a}}\right\|_X) = \|(t_k)\|_\infty \,(1 + \sup_n \|f_n\|_X).\end{aligned}$$

This is a characterization of $(\sum_k f_{n_k})$ being a weakly unconditionally Cauchy series, which is equivalent to $\sum_k t_k f_{n_k}$ converging for every $(t_k) \in c_0$. The sequence f_{n_k} is therefore, a basis, but not necessary Schauder. By the Bessaga-Pełczyński selection principle, we can extract a subsequence $(g_k) \subset (f_{n_k})$, which is basic, and since wuC-property and $\|f_{n_k}\|_X \gtrsim 1$ are inherited to (series of) subsequences, we have finally obtained a sequence $(g_k) \subset (f_n)$, which is equivalent to the standard basis of c_0. $\square$

The following lemma is well known.

Lemma 2.13. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then $T^{**}\colon X^{**} \to X^{**}$ is weak*-weak* continuous and $\iota^{-1}T^{**}\iota = T$ on X, where $\iota\colon X \to X^{**}$ is the canonical embedding.*

Lemma 2.14. *Let $1 \leq p < \infty$ and $v\colon \mathbb{D} \to]0,\infty[$ be a radial function. Then*

$$\sup_{c\in\mathbb{D}} \|T_c f\|_{\mathrm{BMOA}_{v,p}} \leq \|f\|_{\mathrm{BMOA}_{v,p}}.$$

Proof. Let $c = rw_0, w_0 \in \mathbb{T}, r \in [0,1[$. Using the fact that $f(cz) = \int_{\mathbb{T}} (T_{w_0 z} f)(w) P_r(w)\, dm(w)$, an application of Minkowski's inequality gives us

$$\begin{aligned}\|T_c f\|_{*,v,p} &= \sup_{a\in\mathbb{D}} v(a) \left(\int_{\mathbb{T}} |f(cz) - f(ca)|^p \, P_a(z)\, dm(z)\right)^{\frac{1}{p}} \\ &\leq \sup_{a\in\mathbb{D}} v(a) \int_{\mathbb{T}} \left(\int_{\mathbb{T}} |(T_{w_0 z} f)(w) - (T_{w_0 a} f)(w)|^p \, P_a(z)\, dm(z)\right)^{\frac{1}{p}} P_r(w)\, dm(w) \\ &\leq \sup_{a,w\in\mathbb{D}} v(aw_0 w) \left(\int_{\mathbb{T}} |f(z) - f(aw_0 w)|^p \, P_{aw_0 w}(z)\, dm(z)\right)^{\frac{1}{p}} \\ &= \|f\|_{*,v,p}.\end{aligned}$$

For the last inequality, we have approximated the integrand of the outer intergral by its supremum, followed by a variable substitution $z \mapsto z\overline{w_0 w}$, and the fact that $v(a) = v(aw), w \in \mathbb{T}$. $\square$

We are now ready to prove the following result, which does mainly rely on fundamental properties of weighted composition operators on Banach spaces of analytic functions and a certain duality relation.

Theorem 2.15. *Let v be an admissible weight. The spaces VMOA_v^{**} and BMOA_v are isomorphic. Moreover, if ϕ and ψ are such that $\psi C_\phi \in \mathcal{L}(\mathrm{VMOA}_v)$, then the domain of ψC_ϕ can be extended to BMOA_v and the extension $\psi C_\phi \colon \mathrm{BMOA}_v \to \mathrm{BMOA}_v$ is weakly compact if and only if $\psi C_\phi|_{\mathrm{VMOA}_v}$ is weakly compact if and only if $\psi C_\phi(\mathrm{BMOA}_v) \subset \mathrm{VMOA}_v$.*

Proof. By Lemma 2.14, we obtain

$$\sup_{c\in\mathbb{D}} \|T_c f\|_{\mathrm{BMOA}_v} \le \|f\|_{\mathrm{BMOA}_v}, f \in \mathrm{BMOA}_v.$$

Since $\lim_{r\to 1^-} \|T_r f - f\|_{H^2} = 0$ [5, Theorem 2.6] and $T_r f \in \mathrm{VMOA}_v$ for every $0 < r < 1$ according to the remark right after (1.1), we can use [21, Theorem 2.2] with $X = Y = H^2/\mathbb{C}$ and

$$\mathcal{L} = \{L_{\hat\phi} \colon f \mapsto v(a)(f \circ \sigma_a - f(a)) : \hat\phi \in \mathrm{Aut}\}.$$

The result is that $(\mathrm{VMOA}_v')^{**}$ and BMOA_v' are isometrically isomorphic, where

$$\mathrm{VMOA}_v' = \{f \in \mathrm{VMOA}_v : f(0) = 0\} \text{ and } \mathrm{BMOA}_v' = \{f \in \mathrm{BMOA}_v : f(0) = 0\}.$$

Now it follows that

$$\mathrm{VMOA}_v^{**} = (\mathrm{VMOA}_v' \oplus_1 \mathbb{C})^{**} \cong (\mathrm{VMOA}_v')^{**} \oplus_1 \mathbb{C} \cong \mathrm{BMOA}_v' \oplus_1 \mathbb{C} = \mathrm{BMOA}_v,$$

where $\cong$ stays for isometrically isomorphic and $X \oplus_1 Y$ means the direct sum is equipped with the norm $f \mapsto \|(\|f\|_X, \|f\|_Y)\|_{\ell^1}$.

Theorem 2.2 in [21] also yields that the corresponding isometric isomorphism is an extension of the canonical mapping $\iota \colon \mathrm{VMOA}_v \to (\mathrm{VMOA}_v)^{**}$. Hereafter, let ι be the extension. By the Banach-Alaoglu theorem $(B_{\mathrm{VMOA}_v^{**}}, w^*)$ is compact, and since $\delta_z \in \mathrm{VMOA}_v^*$, we have that $\iota \colon (B_{\mathrm{BMOA}_v}, \tau_0) \to (B_{\mathrm{VMOA}_v^{**}}, w^*)$ is a homeomorphism. Applying Lemma 2.13 to the weighted composition operator $\psi C_\phi \in \mathcal{L}(\mathrm{VMOA}_v)$, we get that $\iota^{-1}\psi C_\phi^{**}\iota|_{\mathrm{VMOA}_v} = \psi C_\phi$ and that $\iota^{-1}T^{**}\iota|_{B_{\mathrm{BMOA}_v}}$ is $\tau_0 - \tau_0$ continuous. Since VMOA_v is τ_0 dense in BMOA_v and $\iota^{-1}\psi C_\phi^{**}\iota|_{\mathrm{VMOA}_v} = \psi C_\phi$, we obtain $\iota^{-1}T^{**}\iota = \psi C_\phi \in \mathcal{L}(\mathrm{BMOA}_v)$.

Finally, Gantmacher's theorem yields the equivalences. $\square$

Another equivalent statement of $\psi C_\phi \in \mathcal{L}(\mathrm{VMOA}_v)$ being weakly compact is the following [22, Theorem 3.2]: for every $\epsilon > 0$ there exists $N > 0$ such that

$$\|\psi C_\phi f\|_{\mathrm{BMOA}_v} \leq N \|f\|_{H^2} + \epsilon \|f\|_{\mathrm{BMOA}_v}, \quad f \in \mathrm{VMOA}_v.$$

See [22, Corollary 3.3] for a similar result concerning the operators induced by the same symbols acting on BMOA_v.

A final remark is that, assuming Theorem 1.2 holds for ($\psi C_\phi \in \mathcal{L}(\mathrm{VMOA}_v)$), Theorem 2.15 allows us to immediately extend it to operators ($\psi C_\phi \in \mathcal{L}(\mathrm{BMOA}_v)$) satisfying the extra assumption $\psi C_\phi|_{\mathrm{VMOA}_v} \in \mathcal{L}(\mathrm{VMOA}_v)$. In this case, we can add $\psi C_\phi(\mathrm{BMOA}_v) \subset \mathrm{VMOA}_v$ to the list of characterizations of compactness given in Theorem 1.2. It is worth noticing that there are weighted composition operators in $\mathcal{L}(\mathrm{BMOA}_v)$ that are not an extension of operators $\psi C_\phi \in \mathcal{L}(\mathrm{VMOA}_v)$, in other words, there exists $\psi C_\phi \in \mathcal{L}(\mathrm{BMOA})$ such that $\psi C_\phi|_{\mathrm{VMOA}_v} \notin \mathcal{L}(\mathrm{VMOA})$. An example of an operator $\psi C_\phi \in \mathcal{L}(\mathrm{BMOA})$, which is compact due to Theorem 1.2, but for which $\psi C_\phi|_{\mathrm{VMOA}_v} \notin \mathcal{L}(\mathrm{VMOA})$ is obtained using $\psi = h \in \mathrm{BMOA}_v$ (see Lemma 3.4 and its proof) and $\phi\colon z \mapsto z/2$.

2.3. Some comments about the conditions concerning admissible weights

The functions $g(z) = z^c$, $c \geq 1/2$ satisfy all assumptions except the global growth restriction (G1). The functions

$$z \mapsto (e+z)^{ce^{\cos(\ln(\ln(e+z)))}},$$

where $0 < c < \frac{1}{2e}$ belong to $\mathcal{H}(\mathbb{C}_{\Re \geq \frac{1}{2}})$ and satisfy (G1) (it's easy to see that its growth along the positive real line is bounded by $z \mapsto z^{ce}$), but not the curvature restriction (G2). The intuition for this function follows from the fact that it can be written as an analytic staircase function $F(z) = z + \cos z$ composed with an analytic magnifier $\exp\exp$ from the left and its inverse $\ln\ln$ from the right to create the effect of a rapidly growing size of the steps as z tends toward infinity along the positive real axis. Finally, the input is translated for the function to have the right domain, and compressed from its linear asymptotic mean growth to be dominated by a suitable root-type growth for condition (G1) to hold. We have,

$$(e+z)^{ce^{\cos(\ln(\ln(e+z)))}} = \left(\exp\left(\exp(F(\ln(\ln(e+z))))\right)\right)^c.$$

To see that the function does not satisfy (G2), consider $0 < b = a^2 \leq a \leq 1$ in condition (G2). For condition (G2) to hold, it is necessary that (substituting $a \mapsto \frac{1}{a}$)

$$\sup_{a \in]1,\infty[} \frac{(e+a^2)^{ce^{\cos(\ln(\ln(e+a^2)))}}}{\left(((e+a)^2)^{ce^{\cos(\ln(\ln(e+a)))}}\right)^2} < \infty.$$

Moreover,

$$\frac{(e+a^2)^{ce^{\cos(\ln(\ln(e+a^2)))}}}{\left(((e+a)^2)^{ce^{\cos(\ln(\ln(e+a)))}}\right)^2} \approx \left((e+a)^{2c}\right)^{e^{\cos(\ln(\ln(e+a^2)))}-e^{\cos(\ln(\ln(e+a)))}}.$$

For the above to explode as $a \to \infty$ along some sequence, it is sufficient to prove that

$$\limsup_{a\to 0} \cos(\ln(\ln(e+a^2))) - \cos(\ln(\ln(e+a))) > 0.$$

This can be seen from

$$\cos(\ln(\ln(e+a^2))) = \cos(\ln(\ln(e+a)) + \ln\frac{\ln(e+a^2)}{\ln(e+a)}) \overset{a\to\infty}{\sim} \cos(\ln 2 + \ln(\ln(e+a))).$$

3. Some important lemmas

Lemma 3.1. *Let $g\colon [1,\infty[\to]0,\infty[$ be an almost increasing function. We have*

$$\sup_{x,y\in]0,1[} \frac{\left(g\frac{1}{1-y}\right)}{\left(g\frac{1}{1-x}\right)}\left(\frac{(1-y^2)(1-x^2)}{(1-xy)^2}\right) \asymp \sup_{t,x\in]0,1]} \frac{t\, g\left(\frac{1}{tx}\right)}{g\left(\frac{1}{x}\right)}.$$

One sufficient condition for the quantity above to be finite is that $x \mapsto x\, g(\frac{1}{x})$ is almost increasing.

Proof. Some elementary calculations yield

$$\sup_{x,y\in]0,1[} \frac{\left(g\frac{1}{1-y}\right)}{\left(g\frac{1}{1-x}\right)}\left(\frac{(1-y^2)(1-x^2)}{(1-xy)^2}\right) \asymp \sup_{x,y\in]0,1[} \frac{g\left(\frac{1}{y}\right)}{g\left(\frac{1}{x}\right)}\frac{xy}{(x+y)^2}.$$

Put $y = tx$ to obtain

$$\begin{aligned}
\sup_{x,y\in]0,1[} \frac{g\left(\frac{1}{y}\right)}{g\left(\frac{1}{x}\right)}\left(\frac{xy}{(x+y)^2}\right) &= \sup_{x,t\in]0,\infty[} \chi_{0<x<1}\chi_{0<t<\frac{1}{x}} \frac{g\left(\frac{1}{tx}\right)}{g\left(\frac{1}{x}\right)}\left(\frac{x^2 t}{x^2(1+t)^2}\right)\\
&= \sup_{x,t\in]0,\infty[} \chi_{0<x<\min\{1,\frac{1}{t}\}} \frac{g\left(\frac{1}{tx}\right)}{g\left(\frac{1}{x}\right)}\left(\frac{t}{(1+t)^2}\right)\\
&= \max\left\{\sup_{t\in]0,1]}\sup_{x\in]0,1[} \frac{g\left(\frac{1}{tx}\right)}{g\left(\frac{1}{x}\right)}\left(\frac{t}{(1+t)^2}\right), \sup_{t\in]1,\infty[}\sup_{x\in]0,\infty[} \chi_{0<x<\frac{1}{t}} \frac{g\left(\frac{1}{tx}\right)}{g\left(\frac{1}{x}\right)}\left(\frac{t}{(1+t)^2}\right)\right\}.
\end{aligned}$$

Since g is almost increasing, we have

$$\sup_{t\in]1,\infty[}\sup_{x\in]0,\infty[}\chi_{0<x<\frac{1}{t}}\frac{g\left(\frac{1}{tx}\right)}{g\left(\frac{1}{x}\right)}\left(\frac{t}{(1+t)^2}\right)\asymp_g 1.$$

Moreover,

$$\sup_{t\in]0,1]}\sup_{x\in]0,1[}\frac{g\left(\frac{1}{tx}\right)}{g\left(\frac{1}{x}\right)}\left(\frac{t}{(1+t)^2}\right)\asymp\sup_{t,x\in]0,1]}\frac{t\,g\left(\frac{1}{tx}\right)}{g\left(\frac{1}{x}\right)}. \quad \square$$

The weighted Bloch space will only appear in this section and it is here defined as follows:

$$\mathcal{B}_v=\{f\in\mathcal{H}(\mathbb{D}):\|f\|_{\mathcal{B}_v}:=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)v(z)\,|f'(z)|<\infty\}.$$

The following lemma is a trivial generalization of [10, Corollary 5.2] (see also [10, Lemma 5.1]).

Lemma 3.2. *It holds that*

$$\mathrm{BMOA}_{v,1}\subset\mathcal{B}_v$$

and

$$\|f\|_{\mathcal{B}_v}\le\|f\|_{\mathrm{BMOA}_{v,1}},\ f\in H^1(\mathbb{D}).$$

Proof. For $f\in H^1$ we have

$$|f'(0)|\le\|f\|_{H^1}.$$

Applying the above to $f\circ\sigma_a-f(a)$ yields

$$|f'(a)|\,(1-|a|^2)\le\gamma(f,a,1).$$

Multiply both sides by $v(a)$ and take the supremum over $a\in\mathbb{D}$ to obtain the result (after adding $|f(0)|$ to both sides). $\square$

The following Lemma provides a relation between the main condition (G1) and the weaker condition (G1') used in Lemma 3.4.

Lemma 3.3. *For an almost increasing function $g:[1,\infty[\to]0,\infty[$ and $p>q>0$, we have*

$$\sup_{0<x<1}x\,g(\frac{1}{x})^p<\infty\ \Rightarrow\ g\in L^q(]1,\infty[,d\arctan)\ \Rightarrow\ \sup_{0<x<1}x\,g(\frac{1}{x})^q<\infty.$$

Proof. The first implication follows from

$$\int_1^\infty \frac{g(x)^q\,dx}{1+x^2} = \int_0^1 g(\frac{1}{x})^q \frac{dx}{1+x^2} = \int_0^1 x^{-\frac{q}{p}} \left(xg(\frac{1}{x})^p\right)^{\frac{q}{p}} \frac{dx}{1+x^2}.$$

To prove the second implication, it is proved that for any almost increasing function f with $\sup_{t\in]1,\infty[} t^{-1}f(t)=\infty$, we have $f\notin L^1(]1,\infty[,d\arctan)$. Since $\limsup_{t\to\infty} t^{-1}f(t) = \infty$, there is an increasing sequence (t_n) tending to infinity such that $nt_n \le f(t_n)$ for all $n\ge 2$. Furthermore, let $N\colon \mathbb{N}\to\mathbb{N}$ be an increasing function such that $N(k)\le k$ for all k, $\lim_k N(k)=\infty$ and $\lim_k \frac{t_{N(k)}}{t_{k+1}}=0$. Now, for large k (we can assume $t_1\ge 1$),

$$\begin{aligned}\int_1^\infty \frac{f(t)}{1+t^2}\,dt &\ge \sum_n \int_{t_n}^{t_{n+1}} \frac{f(t)}{1+t^2}\,dt \gtrsim_f \sum_n \int_{t_n}^{t_{n+1}} \frac{f(t_n)}{t^2}\,dt \ge \sum_{n=N(k)}^{k} nt_n\left(\frac{1}{t_n}-\frac{1}{t_{n+1}}\right)\\ &\ge \sum_{n=N(k)}^{k} \frac{n}{t_{n+1}}(t_{n+1}-t_n) \ge \frac{N(k)}{t_{k+1}}\sum_{n=N(k)}^{k}(t_{n+1}-t_n) = N(k)\left(1-\frac{t_{N(k)}}{t_{k+1}}\right)\end{aligned}$$

and the statement follows from letting $k\to\infty$. $\square$

The function h in the following Lemma is the starting point for constructing the test-function associated with $\beta(\psi,\phi,a)$ (see Subsection 4.2).

Lemma 3.4. *Let $g\in\mathcal{H}(\mathbb{C}_{\Re\ge\frac{1}{2}})$ such that $g|_{[\frac{1}{2},\infty[}$ is (strictly) positive and almost increasing. Then, for $v(z)\asymp g(\frac{1}{1-|z|})$, all functions $f\in \mathrm{BMOA}_{v,1}$ satisfy*

$$|f(z)-f(0)| \lesssim_{v,g} \|f\|_{1,*}\, h(|z|), \tag{3.1}$$

where

$$h\colon z\mapsto \int_0^z \frac{dt}{(1-t)g(\frac{1}{1-t})}.$$

The constant is 1 if $v(z)\ge g(\frac{1}{1-|z|})$. Assume also that

(G1') $g\in L^2(]1,\infty[,d\arctan)$,
(G2) $g(1/b)\lesssim g(a/b)g(1/a)$ for $0<b\le a<2$ and
(G3) $|g(z)|\gtrsim g(|z|),\ z\in\mathbb{C}_{\Re\ge\frac{1}{2}}$.

Then $h_c\in\mathrm{BMOA}_v,\ c\in\mathbb{D}$ with a uniform bound for the norm with respect to c, where $h_c(z):=h(cz)$. Moreover, for a fixed $c\in\mathbb{D}$ it holds that $\gamma(h_c,a,2)^2\lesssim_{c,g} 1-|a|,\ a\in\mathbb{D}$, implying $h_c\in\mathrm{VMOA}_v$.

Proof. First, let $f \in \mathrm{BMOA}_{v,1}$ and $z \in \mathbb{D}$. By definition we have

$$|f(z) - f(0)| = \left| \int_0^z f'(t)\,dt \right| = \left| z \int_0^1 f'(tz)\,dt \right| \le |z| \int_0^1 \frac{v(zt)(1 - t^2 |z|^2)\,|f'(tz)|}{v(zt)(1 - t^2 |z|^2)}\,dt$$
$$\le |z| \int_0^1 \frac{\|f\|_{1,*}}{v(zt)(1 - t^2 |z|^2)}\,dt \lesssim_{v,g} \|f\|_{1,*}\, h(|z|),$$

where Lemma 3.2 gives the second last inequality.

Next, we prove that $\|h_c\|_{\mathrm{BMOA}_v} \lesssim \|h\|_{\mathrm{BMOA}_v} < \infty$, where $c \in \mathbb{D}$. First, put

$$C := \sup_{0<x<1} x\, g(\frac{1}{x})^2 \text{ and } M := \sup_{0<b\le a<2} \frac{g(1/b)}{g(a/b)g(1/a)}.$$

The constant C is finite due to (G1') and Lemma 3.3, and (G2) yields that M is finite. For $0 < y \le x < 2$ we have

$$\frac{x\, g(\frac{1}{x})^2}{y\, g(\frac{1}{y})^2} \overset{(G2)}{\ge} \frac{1}{M \frac{y}{x}\, g(\frac{x}{y})^2} \ge \frac{1}{MC} > 0. \tag{3.2}$$

It follows that for all $z, c \in \mathbb{D}$, we have

$$|1 - cz|\, g(\frac{1}{|1 - cz|})^2 \ge \frac{1}{MC}(1 - |z|)g(\frac{1}{1 - |z|})^2$$

so that

$$|h'(cz)|^2 = \left(|1 - cz|^2 \left| g(\frac{1}{1 - cz}) \right|^2 \right)^{-1} \overset{(G3)}{\lesssim_g} \frac{MC \left((1 - |z|)g(\frac{1}{1-|z|})^2 \right)^{-1}}{|1 - cz|}. \tag{3.3}$$

Moreover, for $c, z \in \mathbb{D} \setminus \{0\}$,

$$\frac{|1 - (c/|c|)z|}{|1 - cz|} \le \frac{|1 - cz|}{|1 - cz|} + \frac{\left| cz(1 - \frac{1}{|c|}) \right|}{|1 - cz|} \le 1 + \frac{|z(|c| - 1)|}{|1 - cz|} \le 2.$$

For $a \in \mathbb{D}$ and $c \in \mathbb{D} \setminus \{0\}$, (1.1) together with the above estimates yield

$$\gamma(h_c, a, 2)^2 \asymp |c|^2 \int_{\mathbb{D}} |h'(cz)|^2 (1 - |\sigma_a(z)|^2)\, dA(z)$$
$$\overset{(3.3)}{\lesssim_g} \int_{\mathbb{D}} g\left(\frac{1}{1 - |z|} \right)^{-2} \frac{(1 - |a|)}{|1 - (c/|c|)z|\,|1 - \overline{a}z|^2}\, dA(z)$$

$$\leq \int_0^1 \int_0^{2\pi} g\left(\frac{1}{1-r}\right)^{-2} \frac{(1-|a|)}{|1-re^{it}|\,|1-|a|\,re^{it}|^2}\,dt\,dr,$$

where the last inequality is due to Hardy-Littlewood's inequality on rearrangements (see e.g. [11, Theorem 378, p. 278]). If $|a| \leq \frac{1}{2}$,

$$v(a)^2\gamma(h_c, a, 2)^2 \lesssim_{v,g} g(1)^{-2} < \infty.$$

Henceforth, we assume $|a| > \frac{1}{2}$. It is evident that

$$\sup_{a\in\mathbb{D}} v(a)^2 \int_0^{\frac{1}{2}} \int_0^{2\pi} g\left(\frac{1}{1-r}\right)^{-2} \frac{(1-|a|)}{|1-re^{it}|\,|1-|a|\,re^{it}|^2}\,dt\,dr \lesssim_g \sup_{a\in\mathbb{D}} v(a)^2(1-|a|)$$

and

$$\sup_{a\in\mathbb{D}} v(a)^2 \int_0^1 \int_{\frac{1}{2}}^{2\pi-\frac{1}{2}} g\left(\frac{1}{1-r}\right)^{-2} \frac{(1-|a|)}{|1-re^{it}|\,|1-|a|\,re^{it}|^2}\,dt\,dr \lesssim \sup_{|a|\in\mathbb{D}} v(a)^2(1-|a|).$$

By (G1') together with Lemma 3.3 $\sup_{a\in\mathbb{D}} v(a)^2(1-|a|) < \infty$. The symmetry of the integrand with respect to t yields, it is sufficient to prove that

$$\sup_{\frac{1}{2}<|a|<1} v(a)^2 \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} g\left(\frac{1}{1-r}\right)^{-2} \frac{(1-|a|)}{|1-re^{it}|\,|1-|a|\,re^{it}|^2}\,dt\,dr < \infty$$

in order to establish $h_c \in \mathrm{BMOA}_v$ with uniformly bounded norm with respect to $c \in \overline{\mathbb{D}}$. To this end, since $\frac{1}{2} < r < 1$, using the estimate $\cos t \leq 1 - t^2/3$ when $0 < t < 1/2$, we have

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{dt}{|1-re^{it}|\,|1-|a|\,re^{it}|^2} &= \int_0^{\frac{1}{2}} \frac{dt}{(1+r^2-2r\cos t)^{\frac{1}{2}}(1+|a|^2 r^2 - 2|a|\,r\cos t)} \\ &\lesssim \int_0^{\frac{1}{2}} \frac{dt}{((1-r)^2+t^2)^{\frac{1}{2}}\left((1-|a|+|a|\,(1-r))^2+(\sqrt{|a|}t)^2\right)},\end{aligned}$$

and we can therefore conclude that

$$\int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} g\left(\frac{1}{1-r}\right)^{-2} \frac{1}{|1-re^{it}|\,|1-|a|\,re^{it}|^2}\,dt\,dr$$

$$\lesssim \int_0^{\frac{1}{2}} g\left(\frac{1}{r}\right)^{-2} \int_0^{\frac{1}{2}} \frac{dt}{(r^2+t^2)^{\frac{1}{2}}\left((1-|a|+|a|\,r)^2+(\sqrt{|a|}t)^2\right)}\,dr$$

$$\leq \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} g\left(\frac{1}{r}\right)^{-2} \frac{1}{(r^2+t^2)^{\frac{1}{2}}\left((1-|a|)^2+|a|^2\,(r^2+t^2)\right)}\,dt\,dr.$$

Now consider the two integrals as a representation of an area integral over a square in $\mathbb{R}^2$. We can get a larger integration area by considering a quarter circle in the first quadrant with radius 1. Putting $r^2+t^2=R^2$ and $r=R\cos\theta$, we obtain

$$\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} g\left(\frac{1}{r}\right)^{-2} \frac{dt\,dr}{(r^2+t^2)^{\frac{1}{2}}\left((1-|a|)^2+|a|^2\,(r^2+t^2)\right)}$$

$$=\frac{1}{\pi}\int_0^1\int_0^{\frac{\pi}{2}} g\left(\frac{1}{R\cos\theta}\right)^{-2}\frac{d\theta R\,dR}{R\left((1-|a|)^2+|a|^2\,R^2\right)} \lesssim_g \int_0^1 g\left(\frac{1}{R}\right)^{-2}\frac{dR}{(1-|a|)^2+R^2},$$

where it has been used that g is almost increasing, $\cos\theta\leq 1$ and $|a|\geq 1/2$.

It remains to prove that

$$\sup_{\frac{1}{2}<|a|<1} g\left(\frac{1}{1-|a|}\right)^2\int_0^1 g\left(\frac{1}{R}\right)^{-2}\frac{(1-|a|)\,dR}{(1-|a|)^2+R^2} = \sup_{0<b<\frac{1}{2}}\int_0^1 \frac{g\left(\frac{1}{b}\right)^2}{g\left(\frac{1}{R}\right)^2}\frac{b\,dR}{b^2+R^2} \tag{3.4}$$

is finite. Now since g is almost increasing on $[2,\infty[$, we have

$$b\geq R \;\Rightarrow\; g\left(\frac{1}{b}\right)^2 \lesssim_g g\left(\frac{1}{R}\right)^2$$

which yields

$$\int_0^b \frac{g\left(\frac{1}{b}\right)^2}{g\left(\frac{1}{R}\right)^2}\frac{b\,dR}{b^2+R^2} \leq \int_0^b \frac{b\,dR}{b^2+R^2} \leq \int_0^1 \frac{dR}{1+R^2} = \frac{\pi}{4}.$$

Finally, since

$$R>b \;\Rightarrow\; g\left(\frac{1}{b}\right) \lesssim_g g\left(\frac{R}{b}\right) g\left(\frac{1}{R}\right),$$

we have

$$\int_b^1 \frac{g\left(\frac{1}{b}\right)^2}{g\left(\frac{1}{R}\right)^2}\frac{b\,dR}{b^2+R^2} \lesssim \int_b^1 g\left(\frac{R}{b}\right)^2 \frac{b\,dR}{b^2+R^2} \le \int_1^{\frac{1}{b}} \frac{g(R)^2\,dR}{1+R^2} \le \|g\|^2_{L^2(]1,\infty[,d\arctan)} < \infty.$$

The remaining statement, that $h_c \in \mathrm{VMOA}_v,\ c \in \mathbb{D}$, follows from

$$\begin{aligned}\gamma(h_c,a,2)^2 &\asymp |c|^2 \int_{\mathbb{D}} |h'(cz)|^2 (1-|\sigma_a(z)|^2)\,dA(z)\\ &\lesssim (1-|a|)(\sup_{z\in\mathbb{D}}|h'(cz)|^2)\int_{\mathbb{D}} \frac{(1-|z|^2)}{|1-\overline{a}z|^2}\,dA(z)\\ &= (1-|a|)(\sup_{z\in\mathbb{D}}|h'(cz)|^2)\int_0^1 \frac{(1-r^2)}{1-|\overline{a}r|^2}\,dr^2 \le (1-|a|)(\sup_{z\in\mathbb{D}}|h'(cz)|^2),\end{aligned}$$

where the last equality is true, because $\int_{\mathbb{T}} P_b\,dm = 1$ for every $b \in \mathbb{D}$. We already concluded $\sup_{a\in\mathbb{D}} v(a)^2(1-|a|) < \infty$ so we are done. □

In combination with Lemma 3.2, we have the following corollary, telling us that the growth restriction of functions as approaching $\mathbb{T}$ is the same for functions in $\mathcal{B}_v$ and BMOA_v.

Corollary 3.5. *Under the assumptions of Lemma 3.4 we have, for $X = \mathrm{BMOA}_v$, $X = \mathrm{VMOA}_v$ or $X = \mathcal{B}_v$,*

$$\|\delta_z\|_{X^*} \asymp_{v,g} (1+h(|z|)) = 1 + \int_0^{|z|} \frac{dt}{(1-t)g(\frac{1}{1-t})} \asymp 1 + \int_0^{|z|} \frac{dt}{(1-t)v(t)} \lesssim \ln\frac{e}{1-|z|}.$$

Assuming (G2) in Lemma 3.4, we have the following result

Lemma 3.6. *Let $g : [1,\infty[\to]0,\infty[$ be almost increasing and assume $g(1/b) \lesssim g(a/b)g(1/a)$ for $0 < b \le a \le 1$. Then, for $a \in \mathbb{D}$,*

$$g\left(\frac{1}{1-|\sigma_a(z)|}\right) \lesssim_g g\left(\frac{1+|a|}{1-|a|}\right) g\left(\frac{1}{1-|z|}\right).$$

Proof. It holds that

$$g\left(\frac{1}{1-|\sigma_a(z)|}\right) \lesssim_g g\left(\frac{1-|z|}{1-|\sigma_a(z)|}\right) g\left(\frac{1}{1-|z|}\right).$$

To finish the proof, we apply a result for automorphisms of the disk (see e.g. [3, p. 48]) and the fact that g is almost increasing. □

The next result is an estimate for comparison of evaluation maps at z and $\phi(z)$ for an analytic self-map ϕ.

Corollary 3.7. *Let $g : [1,\infty[\to]0,\infty[$ be almost increasing and assume $g(1/b) \lesssim g(a/b)g(1/a)$ for $0 < b \le a \le 1$. If $v(z) \asymp g(\frac{1}{1-|z|})$ and $\phi\colon \mathbb{D} \to \mathbb{D}$ is an analytic self-map, we have*

$$\left\|\delta_{\phi(z)}\right\|_{\mathrm{BMOA}^*_{v,p}} \lesssim_{v,g} \left(\max\left\{v\left(\frac{2\,|\phi(0)|}{1+|\phi(0)|}\right), 1\right\} + h(|\phi(0)|)\right) \|\delta_z\|_{\mathrm{BMOA}^*_{v,p}}.$$

Proof. Let $z, a \in \mathbb{D}$. We have for $1 \le p < \infty$,

$$\begin{aligned}\|f\circ\sigma_a\|_{*,p} &= \sup_{\hat\phi\in\mathrm{Aut}} v((\sigma_a\circ\sigma_a\circ\hat\phi)(0)) \left\|f\circ\sigma_a\circ\hat\phi - (f\circ\sigma_a\circ\hat\phi)(0)\right\|_{H^p} \\ &= \sup_{\hat\phi\in\mathrm{Aut}} v((\sigma_a\circ\hat\phi)(0)) \left\|f\circ\hat\phi - (f\circ\hat\phi)(0)\right\|_{H^p} \le \left(\sup_{\hat\phi\in\mathrm{Aut}} \frac{v(\sigma_a(\hat\phi(0)))}{v(\hat\phi(0))}\right) \|f\|_{*,p}.\end{aligned}$$

Applying Lemma 3.6 and the fact that $\frac{1+|a|}{1-|a|} = \frac{1}{1-\frac{2|a|}{1+|a|}}$, it follows that

$$\|f\circ\sigma_a\|_{*,p} \lesssim_{v,g} v\left(\frac{2\,|a|}{1+|a|}\right)\|f\|_{*,p}, \quad a\in\mathbb{D},\ p\in[1,\infty[.$$

Estimate (3.1) in Lemma 3.4 yields the last inequality below:

$$\begin{aligned}|(f\circ\sigma_a)(z)| &\le (\|f\circ\sigma_a\|_{*,p} + |f(a)|)\,\|\delta_z\|_{\mathrm{BMOA}^*_{v,p}} \\ &\lesssim_{v,g} \left(\|f\|_{*,p}\, v\left(\frac{2\,|a|}{1+|a|}\right) + |f(0)| + |f(a)-f(0)|\right)\|\delta_z\|_{\mathrm{BMOA}^*_{v,p}} \\ &\lesssim_{v,g} \left(\|f\|_{\mathrm{BMOA}_{v,p}} \max\{v\left(\frac{2\,|a|}{1+|a|}\right), 1\} + \|f\|_{\mathrm{BMOA}_{v,p}}\, h(|a|)\right)\|\delta_z\|_{\mathrm{BMOA}^*_{v,p}}.\end{aligned} \tag{3.5}$$

Let ϕ be an analytic self-map of $\mathbb{D}$. Then $\sigma_{\phi(0)}\circ\phi$ is an analytic self-map of $\mathbb{D}$ with zero as a fixed point. It follows from the Schwarz Lemma that $\left|\sigma_{\phi(0)}\circ\phi(z)\right| \le |z|$ and by the maximum modulus principle, it follows that $\|\delta_{(\sigma_{\phi(0)}\circ\phi)(z)}\|_{\mathrm{BMOA}^*_{v,p}} \lesssim_{v,g} \|\delta_z\|_{\mathrm{BMOA}^*_{v,p}}$. From (3.5) we now obtain

$$\begin{aligned}\left|\delta_{\phi(z)}f\right| &= |(f\circ\phi)(z)| = \left|(f\circ\sigma_{\phi(0)})(\sigma_{\phi(0)}(\phi(z)))\right| \\ &\lesssim_{v,g} \|f\|_{\mathrm{BMOA}_{v,p}} \left(\max\{v\left(\frac{2\,|\phi(0)|}{1+|\phi(0)|}\right), 1\} + h(|\phi(0)|)\right)\left\|\delta_{(\sigma_{\phi(0)}\circ\phi)(z)}\right\|_{\mathrm{BMOA}^*_{v,p}} \\ &\lesssim_{v,g} \|f\|_{\mathrm{BMOA}_{v,p}} \left(\max\{v\left(\frac{2\,|\phi(0)|}{1+|\phi(0)|}\right), 1\} + h(|\phi(0)|)\right)\|\delta_z\|_{\mathrm{BMOA}^*_{v,p}}\end{aligned}$$

and the result is proved. □

4. The functions α and β

It is time to introduce the test functions of unit norm $g_a^{(\alpha)}$ and $g_a^{(\beta)}$, for $a \in \mathbb{D}$, which are a key ingredient in the proof of the two main results: Theorems 1.1 and 1.2. The conclusions in this section are made under suitable assumptions (see beginning of Section 5).

4.1. The function α

For $\phi : \mathbb{D} \to \mathbb{D}$, $a \in \mathbb{D}$ and $\psi \in \mathrm{BMOA}_v$ recall that

$$\alpha(\psi, \phi, a) = \frac{v(a)}{v(\phi(a))} |\psi(a)| \, \|\phi_a\|_{H^2},$$

where $\phi_a = \sigma_{\phi(a)} \circ \phi \circ \sigma_a$.

For $a \in \mathbb{D}$ define

$$f_a^{(\alpha)} : z \mapsto \frac{\sigma_{\phi(a)}(z) - \phi(a)}{v(\phi(a))}, \; z \in \mathbb{D}$$

and

$$g_a^{(\alpha)} : z \mapsto \frac{f_a^{(\alpha)}(z)}{\left\| f_a^{(\alpha)} \right\|_{\mathrm{BMOA}_v}}, \; z \in \mathbb{D}.$$

The important properties for these functions are that $g_a^{(\alpha)} \in \mathrm{VMOA}_v, \lim_{|\phi(a)| \to 1} \|g_a^{(\alpha)}\|_{H^2} = 0$ and their relation with the function α,

$$\alpha(\psi, \phi, a) \lesssim v(a) \left\| \psi(a)(g_a^{(\alpha)} \circ \phi \circ \sigma_a - g_a^{(\alpha)}(\phi(a))) \right\|_{H^2}, \; a \in \mathbb{D},$$

which will be used to prove that boundedness of ψC_ϕ implies boundedness of α and β. Their relation with the function α is also crucial when proving that $\limsup_{|a| \to 1} \alpha(\psi, \phi, a) > 0$ ensures ψC_ϕ is not c_0-singular.

Using $f_a^{(2)}(z) = \sigma_{\phi(a)}(z) - \phi(a)$, we have

$$\begin{aligned}
\left| f_a^{(2)} \circ \sigma_b(z) - f_a^{(2)}(b) \right|^2 &= \left| \sigma_{\phi(a)} \circ \sigma_b(z) - \sigma_{\phi(a)}(b) \right|^2 \\
&= \left| \frac{\phi(a)\overline{b} - 1}{1 - \overline{\phi(a)}b} \sigma_{\frac{b - \phi(a)}{1 - \phi(a)\overline{b}}}(z) - \frac{\phi(a)\overline{b} - 1}{1 - \overline{\phi(a)}b} \frac{b - \phi(a)}{1 - \phi(a)\overline{b}} \right|^2 \\
&= \left| \sigma_{\frac{b - \phi(a)}{1 - \phi(a)\overline{b}}}(z) - \frac{b - \phi(a)}{1 - \phi(a)\overline{b}} \right|^2 = \left| \frac{z \left(\left| \frac{b - \phi(a)}{1 - \phi(a)\overline{b}} \right|^2 - 1 \right)}{1 - z \overline{\frac{b - \phi(a)}{1 - \phi(a)\overline{b}}}} \right|^2.
\end{aligned}$$

Since $\int_{\mathbb{T}} P_c \, dm = 1$ for every $c \in \mathbb{D}$, we have

$$\begin{aligned} v(b)^2\gamma(f_a^{(\alpha)}, b, 2)^2 &= \frac{v(b)^2}{v(\phi(a))^2}\left(1 - \left|\frac{b-\phi(a)}{1-\phi(a)\bar{b}}\right|^2\right) \\ &\leq \frac{v(b)^2}{v(\phi(a))^2}\left(\frac{(1-|b|^2)(1-|\phi(a)|^2)}{\left|1-|\phi(a)|\,|b|\right|^2}\right), \end{aligned} \tag{4.1}$$

and hence, by Lemma 3.1

$$\sup_{b\in\mathbb{D}} \gamma(f_a^{(\alpha)}, b, 2)^2 v(b)^2 \leq \sup_{x,y\in[0,1[} \frac{v(x)^2}{v(y)^2}\left(\frac{(1-y^2)(1-x^2)}{(1-xy)^2}\right) < \infty \tag{4.2}$$

proving $f_a^{(\alpha)} \in \mathrm{BMOA}_v$ for all $a \in \mathbb{D}$ and $\sup_{a\in\mathbb{D}} \|f_a^{(\alpha)}\|_{\mathrm{BMOA}_v} < \infty$. From (4.1) it also follows that $f_a^{(\alpha)} \in \mathrm{VMOA}_v$, and therefore also $g_a^{(\alpha)} \in \mathrm{VMOA}_v$. Finally, from the equality in (4.1) it follows that $1 \leq \|f_a^{(\alpha)}\|_{\mathrm{BMOA}_v}$ and using $b = 0$ in (4.1) now leads to $\lim_{|\phi(a)|\to 1} \|g_a^{(\alpha)}\|_{H^2} = 0$.

Next, we proceed to the function β, which should not be confused with what is usually referred to as the β-function.

4.2. The function β

For $\phi\colon \mathbb{D} \to \mathbb{D},\ a \in \mathbb{D}$ and $\psi \in \mathrm{BMOA}_v$, recall that

$$\beta(\psi, \phi, a) = \left\|\delta_{\phi(a)}\right\|_{(\mathrm{BMOA}_v)^*} v(a)\gamma(\psi, a, 1).$$

Corollary 3.5 yields that

$$\beta(\psi, \phi, a) \asymp_{v,g} \left(1 + \int\limits_0^1 \frac{|\phi(a)|\, dt}{(1-t\,|\phi(a)|)g(\frac{1}{1-t|\phi(a)|})}\right) v(a)\gamma(\psi, a, 1).$$

For $a \in \mathbb{D}$ define

$$f_a^{(\beta)}\colon z \mapsto \int\limits_0^{z\overline{\phi(a)}} \frac{dt}{(1-t)g(\frac{1}{1-t})} = \int\limits_0^{z} \frac{\overline{\phi(a)}\, dt}{(1-t\overline{\phi(a)})g(\frac{1}{1-t\overline{\phi(a)}})},\ z \in \mathbb{D}$$

and

$$g_a^{(\beta)}\colon z \mapsto \frac{(1+f_a^{(\beta)})^2}{\left\|(1+f_a^{(\beta)})^2\right\|_{\mathrm{BMOA}_v}},\ z \in \mathbb{D}.$$

The important properties for the functions $g_a^{(\beta)}$ are that $g_a^{(\beta)} \in \mathrm{VMOA}_v$ and their relation with the function β,

$$\beta(\psi,\phi,a) \lesssim_{v,g} v(a)\left|g_a^{(\beta)}(\phi(a))\right|\gamma(\psi,a,1),\ a\in\mathbb{D}, \tag{4.3}$$

which will be used in a similar fashion to the α-case. Moreover, if $\mathrm{BMOA}_v \not\subset H^\infty$, we will need $\lim_{|\phi(a)|\to 1}\|g_a^{(\beta)}\|_{H^1} = 0$ to hold.

Hereafter, $X = \mathrm{VMOA}_v$ or $X = \mathrm{BMOA}_v$. From the remark right after (1.1), it is clear that $g_a^{(\beta)} \in \mathrm{VMOA}_v$, because $(f_a^{(\beta)})^2 = S_{\overline{\phi(a)}}h^2$ (dilation of analytic function), where $|\phi(a)| < 1$. Moreover, since g is almost increasing

$$(1-t\,|\phi(a)|)g(\frac{1}{1-t\,|\phi(a)|}) \gtrsim_g (1-t\,|\phi(a)|^2)g(\frac{1}{1-t\,|\phi(a)|^2}),$$

and hence,

$$\begin{aligned}1+\int_0^1 \frac{|\phi(a)|\ dt}{(1-t\,|\phi(a)|)g(\frac{1}{1-t|\phi(a)|})} &\lesssim_g 1+\int_0^1 \frac{|\phi(a)|\ dt}{(1-t\,|\phi(a)|^2)g(\frac{1}{1-t|\phi(a)|^2})}\\ &\lesssim 1+\int_0^1 \frac{|\phi(a)|^2\ dt}{(1-t\,|\phi(a)|^2)g(\frac{1}{1-t|\phi(a)|^2})},\end{aligned} \tag{4.4}$$

from which an application of Corollary 3.5 gives us

$$\left\|(1+f_a^{(\beta)})^2\right\|_X \gtrsim_{v,g} \frac{(1+f_a^{(\beta)}(\phi(a)))^2}{\left\|\delta_{\phi(a)}\right\|_{X^*}} \gtrsim_{v,g} \left\|\delta_{\phi(a)}\right\|_{X^*}. \tag{4.5}$$

The inequality in equation (3.3) in Lemma 3.4 followed by (4.4), yields

$$1+\sup_{z\in\mathbb{D}}\left|f_a^{(\beta)}(z)\right| \lesssim_g 1+\int_0^1 \frac{|\phi(a)|\ dt}{(1-t\,|\phi(a)|)g(\frac{1}{1-t|\phi(a)|})} \lesssim_{v,g} \left\|\delta_{|\phi(a)|^2}\right\|.$$

Similarly to the proof of Lemma 3.4, for every $a,b\in\mathbb{D}$ we have

$$\begin{aligned}v(a)^2\gamma((1+f_a^{(\beta)})^2,b,2)^2 &\asymp v(a)^2\,|\phi(a)|^2\int_{\mathbb{D}}\left|1+f_a^{(\beta)}(z)\right|^2\left|h'(\overline{\phi(a)}z)\right|^2(1-|\sigma_b(z)|^2)\,dA(z)\\ &\lesssim_{v,g} \left\|\delta_{|\phi(a)|^2}\right\|_{X^*}^2 v(a)^2\int_{\mathbb{D}} g\left(\frac{1}{1-|z|}\right)^{-2}\frac{(1-|b|)}{|1-z|\,\left|1-\bar{b}z\right|^2}\,dA(z)\\ &\lesssim_{v,g} \left\|\delta_{|\phi(a)|^2}\right\|_{X^*}^2 \le \left\|\delta_{|\phi(a)|}\right\|_{X^*}^2.\end{aligned}$$

Combining with (4.5), we can conclude that

$$\left\|(1+f_a^{(\beta)})^2\right\|_X \asymp_{v,g} \left\|\delta_{\phi(a)}\right\|_{X^*} \asymp_{v,g} \left\|\delta_{\phi(a)^2}\right\|_{X^*}. \tag{4.6}$$

As a consequence, (4.3) holds. Finally, concerning $\lim_{|\phi(a)|\to 1} \|g_a^{(\beta)}\|_{H^1} = 0$, we can assume $\lim_{\phi(a)\to 1} \left\|\delta_{\phi(a)}\right\|_{X^*} = \infty$, so it is sufficient to prove that $\sup_{a\in\mathbb{D}} \left\|f_a^{(\beta)}\right\|_{H^2} < \infty$. By the Littlewood-Paley identity followed by the inequality in equation (3.3), we have for $a \in \mathbb{D}$,

$$\begin{aligned}\left\|f_a^{(\beta)}\right\|_{H^2}^2 &\asymp |\phi(a)|^2 \int\limits_{\mathbb{D}} \left|(1-z\overline{\phi(a)})g(\frac{1}{1-z\overline{\phi(a)}})\right|^{-2} (1-|z|^2)\, dA(z) \\ &\lesssim_g \int\limits_{\mathbb{D}} \frac{g(\frac{1}{1-\left|z\overline{\phi(a)}\right|})^{-2}}{\left|1-z\overline{\phi(a)}\right|}\, dA(z) \lesssim_g \sup_{a\in\mathbb{D}} \int\limits_{\mathbb{D}} \frac{g(1)^{-2}}{\left|1-z\overline{\phi(a)}\right|}\, dA(z) < \infty\end{aligned}$$

and we are done.

The reason for using the parameter 1 in the factor $\gamma(\psi, a, 1)$ of the function β (compared to 2, which is used in e.g. [13]) is due to the restrictive connection between the parameters p, q and v for the statements in Proposition 2.3 to hold, which is present via (G1). On the one hand, if $\gamma(\psi, a, 2)$ is used in the definition of β, the application of Hölder's inequality in e.g. (5.1) (see also proof of Lemma 5.4) would create quantities involving the H^4-norm. In this case, to be able to apply Proposition 2.3, we need a more restrictive condition instead of (G1). On the other hand, it is essential to be able to connect the $\gamma(\psi, a, 1)$ factor in β with $\gamma(\psi, a, 2)$, for example, to obtain (5.15). This is solved by the extensive Proposition 2.3.

5. Weighted composition operators on BMOA_v and VMOA_v

Recall that an admissible weight v, is a function satisfying:
There is a $g \in \mathcal{H}(\mathbb{C}_{\Re\geq\frac{1}{2}})$ such that $g|_{[\frac{1}{2},\infty[}$ is (strictly) positive and almost increasing. Assume also that

(G1) There exists $\epsilon_0 > 0$ such that $\sup_{0<x<1} x\, g(\frac{1}{x})^{2+\epsilon_0} < \infty$,
(G2) $g(1/b) \lesssim g(a/b)g(1/a)$ for $0 < b \leq a < 2$,
(G3) $|g(z)| \gtrsim g(|z|),\ z \in \mathbb{C}_{\Re\geq\frac{1}{2}}$,

such that $v(z) \asymp g(\frac{1}{1-|z|})$.

With these assumptions, $v|_{[0,1[}$ is almost increasing, and a version of equation (3.2) in Lemma 3.4 shows that

$$x \mapsto v|_{[0,1[}(1-x)x^{\frac{1}{p}-\epsilon}$$

is almost increasing for some $\epsilon > 0$. Moreover, since v is equivalent to a radial function Proposition 2.3 yields that $\mathrm{BMOA}_v = \mathrm{BMOA}_{v,1}$ with equivalent norms. Moreover, due to Lemma 3.3, the assumptions of Lemmas 3.1 and 3.4 are satisfied and the useful properties obtained in Sections 3 and 4 hold. The results below, up to Corollary 5.6, are inspired by [13, Section 2 and Proposition 4.1].

Boundedness of ψC_ϕ on BMOA_v and VMOA_v is characterized in Theorem 5.5 and Corollary 5.6 respectively. Another standard but important type of result is Theorem 5.7.

Concerning the VMOA_v-case, some Carleson measure theory (Proposition 5.11) is sufficient, due to the nature of VMOA_v, to prove that the function-theoretic condition implies that ψC_ϕ is the uniform limit of a sequence of compact operators (ψC_ϕ composed with dilation operators), see Theorem 5.12. A candidate for a sufficient condition for $\psi C_\phi \in \mathcal{L}(\mathrm{BMOA}_v)$ to be compact is presented in Theorem 5.10. Although the structure of the space makes the Carleson measure approach less fruitful, the unit ball B_{BMOA_v} is compact with respect τ_0. This allows another, classical, approach to be carried out, namely, to prove that ψC_ϕ maps τ_0-null sequences to norm-null sequences. Due to some complications involving the weight v, no characterization is achieved in the general case (see also Conjecture 3).

In all of the proofs to these results, Proposition 2.3 has a crucial part.

Lemma 5.1 *([13, Proposition 2.1]). There is a constant $C \geq 1$ such that*

$$\|f \circ u\|_{H^2} \leq C \|f\|_{H^2} \|u\|_{H^2}$$

for all $f \in H^2$ and analytic self-maps u of $\mathbb{D}$ such that $f(0) = u(0) = 0$.

Lemma 5.2. *For $f \in \mathrm{BMOA}_v$ and $a \in \mathbb{D}$, we have*

$$v(a)\gamma(\psi C_\phi f, a, 1) \leq \|f\|_{\mathrm{BMOA}_v} \bigg(\alpha(\psi, \phi, a) + v(a) \|\psi \circ \sigma_a - \psi(a)\|_{H^2} \frac{\|\phi_a\|_2}{v(\phi(a))} \\ + \beta(\psi, \phi, a) \bigg).$$

Proof. For a fixed $a \in \mathbb{D}$ and $f \in \mathrm{BMOA}_v$ we have

$$\begin{aligned} \gamma(\psi C_\phi f, a, 1) &= \|\psi \circ \sigma_a f \circ \phi \circ \sigma_a - \psi \circ \sigma_a(0) f \circ \phi \circ \sigma_a(0)\|_{H^1} \\ &\leq |\psi(a)| \|f \circ \phi \circ \sigma_a - f(\phi(a))\|_{H^1} \\ &\quad + \|(\psi \circ \sigma_a - \psi(a))(f \circ \phi \circ \sigma_a - f(\phi(a)))\|_{H^1} \\ &\quad + |f(\phi(a))| \|\psi \circ \sigma_a - \psi(a)\|_{H^1}. \end{aligned}$$

For the first term, we have

$$\|f\circ\phi\circ\sigma_a - f(\phi(a))\|_{H^1} \le \|f\circ\phi\circ\sigma_a - f(\phi(a))\|_{H^2} = \left\|f\circ\sigma_{\phi(a)}\circ\phi_a - f(\phi(a))\right\|_{H^2}$$
$$\overset{\text{Lemma 5.1}}{\le} \gamma(f,\phi(a),2)\,\|\phi_a\|_{H^2} \le \frac{\|f\|_*\,\|\phi_a\|_{H^2}}{v(\phi(a))}.$$

For the middle term, we apply Hölder's inequality and Lemma 5.1 to obtain

$$\begin{aligned}
&\|(\psi\circ\sigma_a - \psi(a))(f\circ\phi\circ\sigma_a - f(\phi(a)))\|_{H^1}\\
&\quad\le \|\psi\circ\sigma_a - \psi(a)\|_{H^2}\,\|f\circ\phi\circ\sigma_a - f(\phi(a))\|_{H^2}\\
&\quad\le \|\psi\circ\sigma_a - \psi(a)\|_{H^2}\,\frac{\|f\|_*\,\|\phi_a\|_{H^2}}{v(\phi(a))}
\end{aligned}\tag{5.1}$$

For the last term, we have

$$|f(\phi(a))|\,\|\psi\circ\sigma_a - \psi(a)\|_1 \le \left\|\delta_{\phi(a)}\right\|_{(\mathrm{BMOA}_v)^*}\,\|f\|_{\mathrm{BMOA}_v}\,\|\psi\circ\sigma_a - \psi(a)\|_{H^1}$$

and the statement follows. □

Lemma 5.3. *Let $g\colon [1,\infty[\to]0,\infty[$ be almost increasing, $x\mapsto x\,g(\frac{1}{x})$ be almost increasing on $]0,1]$ and $v(z)\asymp g(\frac{1}{1-|z|})$. Let $X=\mathrm{BMOA}_v$ or $X=\mathrm{VMOA}_v$. If $\psi C_\phi\in\mathcal{L}(X)$, then*

$$\alpha(\psi,\phi,a) \lesssim_{v,g} \left\|\psi C_\phi g_a^{(\alpha)}\right\|_{\mathrm{BMOA}_v} + \frac{v(a)\,\|(\psi\circ\sigma_a - \psi(a))\|_{H^2}}{v(\phi(a))} \lesssim_v \|\psi C_\phi\|_{\mathcal{L}(X)}.$$

Proof. Invoking theory from Section 4 and using

$$v(\phi(a))(f_a^{(\alpha)}\circ\phi\circ\sigma_a - f_a^{(\alpha)}(\phi(a))) = \sigma_{\phi(a)}\circ\phi\circ\sigma_a = \phi_a,$$

we can conclude that

$$\begin{aligned}
\alpha(\psi,\phi,a) &= \frac{v(a)}{v(\phi(a))}\,|\psi(a)|\,\|\phi_a\|_{H^2} = v(a)\left\|\psi(a)(f_a^{(\alpha)}\circ\phi\circ\sigma_a - f_a^{(\alpha)}(\phi(a)))\right\|_{H^2}\\
&\le v(a)\left\|\psi\circ\sigma_a f_a^{(\alpha)}\circ\phi\circ\sigma_a - \psi(a)f_a^{(\alpha)}(\phi(a))\right\|_{H^2}\\
&\quad + v(a)\left\|(\psi\circ\sigma_a - \psi(a))f_a^{(\alpha)}\circ\phi\circ\sigma_a\right\|_{H^2}.
\end{aligned}\tag{5.2}$$

For the first term, Lemma 3.1 yields:

$$\begin{aligned}
v(a)\left\|\psi\circ\sigma_a f_a^{(\alpha)}\circ\phi\circ\sigma_a - \psi(a)f_a^{(\alpha)}(\phi(a))\right\|_{H^2} &\le \left\|\psi C_\phi f_a^{(\alpha)}\right\|_{\mathrm{BMOA}_v}\\
&\le \|\psi C_\phi\|_{\mathcal{L}(X)}\left\|f_a^{(\alpha)}\right\|_{\mathrm{BMOA}_v}\\
&\lesssim_{v,g} \|\psi C_\phi\|_{\mathcal{L}(X)}.
\end{aligned}$$

For the second term, we have

$$\begin{aligned} v(a)\left\|(\psi\circ\sigma_a-\psi(a))f_a^{(\alpha)}\circ\phi\circ\sigma_a\right\|_{H^2} &\le \left\|f_a^{(\alpha)}\circ\phi\circ\sigma_a\right\|_\infty v(a)\left\|(\psi\circ\sigma_a-\psi(a))\right\|_{H^2} \\ &\le 2\frac{v(a)\left\|(\psi\circ\sigma_a-\psi(a))\right\|_{H^2}}{v(\phi(a))} \le 2\frac{\|\psi\|_*}{v(\phi(a))} \lesssim_v \|\psi C_\phi 1\|_* \le \|\psi C_\phi\|_{\mathcal{L}(X)}. \quad \square \end{aligned}$$

Lemma 5.4. *Let v be admissible and assume $\psi C_\phi \in \mathcal{L}(X)$, where $X = \mathrm{BMOA}_v$ or $X = \mathrm{VMOA}_v$. Then*

$$\begin{aligned} \beta(\psi,\phi,a) &\lesssim_{v,g} \left\|\psi C_\phi g_a^{(\beta)}\right\|_{*,1} + \alpha(\psi,\phi,a) + \frac{v(a)\left\|\psi\circ\sigma_a-\psi(a)\right\|_{H^2}\|\phi_a\|_{H^2}}{v(\phi(a))} \\ &\lesssim_{v,g} \|\psi C_\phi\|_{\mathcal{L}(X)}. \end{aligned}$$

Proof. Theory from Section 4, Hölder's inequality and Lemma 5.1 yield

$$\begin{aligned} \beta(\psi,\phi,a) &\lesssim_{v,g} v(a)\left\|g_a^{(\beta)}(\phi(a))(\psi\circ\sigma_a-\psi(a))\right\|_{H^1} \\ &= v(a)\left\|(\psi\circ\sigma_a)g_a^{(\beta)}\circ\phi\circ\sigma_a-\psi(a)g_a^{(\beta)}(\phi(a))\right\|_{H^1} \\ &\quad + v(a)\left\|(\psi\circ\sigma_a-\psi(a))(g_a^{(\beta)}(\phi(a))-g_a^{(\beta)}\circ\phi\circ\sigma_a)\right\|_{H^1} \\ &\quad + v(a)\left\|\psi(a)(g_a^{(\beta)}(\phi(a))-g_a^{(\beta)}\circ\phi\circ\sigma_a)\right\|_{H^1} \\ &\le \left\|\psi C_\phi g_a^{(\beta)}\right\|_{*,1} \\ &\quad + \left(\frac{v(a)\left\|\psi\circ\sigma_a-\psi(a)\right\|_{H^2}\|\phi_a\|_{H^2}}{v(\phi(a))} + \alpha(\psi,\phi,a)\right)\left\|g_a^{(\beta)}\right\|_{\mathrm{BMOA}_v} \\ &\lesssim_{v,g} \|\psi C_\phi\|_{\mathcal{L}(X)} + \frac{\|\psi C_\phi 1\|_{\mathrm{BMOA}_v}}{v(0)} + \alpha(\psi,\phi,a). \end{aligned}$$

Lemma 5.3 gives the statement. $\square$

Theorem 5.5 *(Boundedness). For an admissible weight v, we have*

$$\psi C_\phi \in \mathcal{L}(\mathrm{BMOA}_v) \iff \sup_{a\in\mathbb{D}}(\alpha(\psi,\phi,a)+\beta(\psi,\phi,a)) < \infty.$$

More specifically,

$$\|\psi C_\phi\|_{\mathcal{L}(\mathrm{BMOA}_v)} \asymp_{v,g} |\psi(0)|\left(1+h(|\phi(0)|)\right) + \sup_{a\in\mathbb{D}}\alpha(\psi,\phi,a) + \sup_{a\in\mathbb{D}}\beta(\psi,\phi,a),$$

where

$$h\colon z \mapsto \int_0^z \frac{dt}{(1-t)g(\frac{1}{1-t})}.$$

Proof. Since $\|\psi\|_* \lesssim_{v,g} \|\psi\|_{*,1} \leq \sup_{a\in\mathbb{D}} \beta(\psi,\phi,a)$, Proposition 2.3 and Lemma 5.2 yield

$$\begin{aligned}\sup_{\|f\|_{\mathrm{BMOA}_v}\leq 1} \sup_{a\in\mathbb{D}} v(a)\gamma(\psi C_\phi f, a, 2) &\lesssim_{v,g} \sup_{a\in\mathbb{D}} \left(\alpha(\psi,\phi,a) + \|\psi\|_* \frac{\|\phi_a\|_2}{v(\phi(a))} + \beta(\psi,\phi,a)\right) \\ &\lesssim_{v,g} \sup_{a\in\mathbb{D}} \alpha(\psi,\phi,a) + \sup_{a\in\mathbb{D}} \beta(\psi,\phi,a).\end{aligned}$$

Moreover,

$$\sup_{\|f\|_{\mathrm{BMOA}_v}\leq 1} |(\psi C_\phi f)(0)| \leq |\psi(0)| \left\|\delta_{\phi(0)}\right\|_{\mathrm{BMOA}_v^*} \lesssim_{v,g} |\psi(0)| \left(1 + h(|\phi(0)|)\right)$$

and we can conclude that

$$\|\psi C_\phi\|_{\mathcal{L}(\mathrm{BMOA}_v)} \lesssim_{v,g} |\psi(0)| \left(1 + h(|\phi(0)|)\right) + \sup_{a\in\mathbb{D}} \alpha(\psi,\phi,a) + \sup_{a\in\mathbb{D}} \beta(\psi,\phi,a).$$

For the lower estimate, we have

$$\begin{aligned}|\psi(0)| \left(1 + h(|\phi(0)|)\right) &\lesssim_{v,g} |\psi(0)| + |\psi(0)|\, g_0^{(\beta)}(\phi(0)) \leq \|\psi C_\phi 1\|_{\mathrm{BMOA}_v} + (\psi C_\phi g_0^{(\beta)})(0) \\ &\lesssim_{v,g} \|\psi C_\phi\|_{\mathcal{L}(\mathrm{BMOA}_v)},\end{aligned}$$

after which Lemmas 5.3 and 5.4 yield the lower bound for $\|\psi C_\phi\|_{\mathcal{L}(\mathrm{BMOA}_v)}$. $\square$

Notice that the condition given in Proposition 2.3 is sufficient to prove that

$$\psi C_\phi \in \mathcal{L}(\mathrm{BMOA}_v) \iff \sup_{a\in\mathbb{D}} (\alpha(\psi,\phi,a) + \sup_{a\in\mathbb{D}} \beta(\psi,\phi,a)) < \infty.$$

The following corollary can be compared to [13, Proposition 4.1] with a slightly different proof.

Corollary 5.6. *For an admissible weight v, the following statements are equivalent:*

- $\psi C_\phi \in \mathcal{L}(\mathrm{VMOA}_v)$
- $\sup_{a\in\mathbb{D}}(\alpha(\psi,\phi,a) + \beta(\psi,\phi,a)) < \infty$ *and* $\psi, \psi\phi \in \mathrm{VMOA}_v$
- $\sup_{a\in\mathbb{D}}(\alpha(\psi,\phi,a) + \beta(\psi,\phi,a)) < \infty$, $\psi \in \mathrm{VMOA}_v$ *and* $\lim_{|a|\to 1} v(a)\psi(a)\gamma(\phi,a,2) = 0$.

More specifically, if $\psi C_\phi \colon \mathrm{VMOA}_v \to \mathrm{VMOA}_v$, *then*

$$\|\psi C_\phi\|_{\mathcal{L}(\mathrm{VMOA}_v)} \asymp_{v,g} |\psi(0)| \left(1 + h(|\phi(0)|)\right) + \sup_{a\in\mathbb{D}} \alpha(\psi,\phi,a) + \sup_{a\in\mathbb{D}} \beta(\psi,\phi,a).$$

Proof. Assume first $\psi C_\phi \in \mathcal{L}(\mathrm{VMOA}_v)$. According to Proposition 2.1 the maps $z \mapsto 1$ and $z \mapsto z$ belong to VMOA_v and by assumption, we can conclude $\psi, \psi\phi \in \mathrm{VMOA}_v$. By Lemmas 5.3 and 5.4, we have $\sup_{a\in\mathbb{D}}(\alpha(\psi,\phi,a)+\beta(\psi,\phi,a)) < \infty$. On the other hand, assume $\psi, \psi\phi \in \mathrm{VMOA}_v$ and $\sup_{a\in\mathbb{D}}(\alpha(\psi,\phi,a)+\beta(\psi,\phi,a)) < \infty$. Theorem 5.5 yields $\psi C_\phi \colon \mathrm{VMOA}_v \to \mathrm{BMOA}_v$ is bounded. All that is left to prove is that the codomain is VMOA_v. Proposition 2.7 yields it is sufficient to prove that any analytic polynomial is mapped into VMOA_v. To this end, let f be an analytic polynomial. We have,

$$\gamma(\psi C_\phi f, a, 2) \le \|(\psi\circ\sigma_a - \psi(a)) f\circ\phi\circ\sigma_a\|_{H^2} + \|\psi(a)(f\circ\phi\circ\sigma_a - f(\phi(a)))\|_{H^2}\,.$$

Since f is bounded, for the first term we have

$$\|(\psi\circ\sigma_a - \psi(a)) f\circ\phi\circ\sigma_a\|_{H^2} \le \|f\|_\infty \gamma(\psi, a, 2),$$

and since $\psi \in \mathrm{VMOA}_v$

$$\lim_{|a|\to 1} v(a)\,\|(\psi\circ\sigma_a - \psi(a)) f\circ\phi\circ\sigma_a\|_{H^2} = 0.$$

For the second term, using Lemma 5.1,

$$v(a)\,\|\psi(a)(f\circ\phi\circ\sigma_a - f(\phi(a)))\|_{H^2} \le \alpha(\psi,\phi,a) v(\phi(a))\gamma(f,\phi(a),2).$$

It is now sufficient to prove that for every sequence $(a_n)\subset\mathbb{D}$ with $\lim_n |a_n| = 1$ there is a subsequence (a_{n_k}) such that

$$\lim_k \alpha(\psi,\phi,a_{n_k}) v(\phi(a_{n_k}))\left\|f\circ\sigma_{\phi(a_{n_k})} - f(\phi(a_{n_k}))\right\|_{H^2} = 0. \tag{5.3}$$

Since $\phi(a_n)$ is bounded, there is always a subsequence, either entirely in a compact subset of $\mathbb{D}$ or that converges to a point on $\mathbb{T}$. If $|\phi(a_{n_k})| \to 1$ as $k\to\infty$, (5.3) follows from $\sup_{a\in\mathbb{D}} \alpha(\psi,\phi,a) < \infty$ and $f \in \mathrm{VMOA}_v$. If $(\phi(a_{n_k}))$ is contained in a compact subset of $\mathbb{D}$, we note that

$$\|\phi_a\|_{H^2} \asymp \|\phi\circ\sigma_a - \phi(a)\|_{H^2}\,.$$

To conclude the proof, it is sufficient to prove that

$$\lim_{|a|\to 1} v(a)\psi(a)\,\|\phi\circ\sigma_a - \phi(a)\|_{H^2} = 0.$$

This follows from the fact that

$$\begin{aligned}\psi(a)\,\|\phi\circ\sigma_a - \phi(a)\|_{H^2} &\le \|\psi\circ\sigma_a\phi\circ\sigma_a - \psi(a)\phi(a)\|_{H^2} + \|(\psi\circ\sigma_a - \psi(a))\phi\circ\sigma_a\|_{H^2}\\ &\le \gamma(\psi\phi, a, 2) + \gamma(\psi, a, 2)\end{aligned}$$

and $\psi, \psi\phi \in \mathrm{VMOA}_v$. $\square$

Concerning the proof above, the case where the denseness of polynomials is used is when $\mathrm{VMOA}_v \not\subset H^\infty$, that is, when $\int_0^1 \frac{dt}{t\,v(1-t)}$ is infinite (see Proposition 2.8).

5.1. Compactness and related properties for ψC_ϕ on BMOA_v and VMOA_v

The test functions that are used in the following theorem can be found in Section 4.

Theorem 5.7. *Let $\psi C_\phi \in \mathcal{L}(\mathrm{VMOA}_v)$ and v be an admissible weight (see beginning of Section 5). If $\limsup_{|\phi(a)|\to 1}(\alpha(\psi,\phi,a)+\beta(\psi,\phi,a)) > 0$, then ψC_ϕ fixes a copy of c_0. Moreover, in this case ψC_ϕ is not unconditionally converging. Furthermore, the same statements hold for $\psi C_\phi \in \mathcal{L}(\mathrm{BMOA}_v)$ if at least one of the following holds:*

- $\mathrm{BMOA}_v \not\subset H^\infty$,
- $\psi \in \mathrm{VMOA}_v$.

Proof. Let $(a_n) \subset \mathbb{D}$ be a sequence such that $\lim_n |\phi(a_n)| \to 1$ and at least one of the following holds: $\lim_n \alpha(\psi,\phi,a_n) > 0$ or $\lim_n \beta(\psi,\phi,a_n) > 0$. If $\lim_n \alpha(\psi,\phi,a_n) > 0$, then Lemma 5.3 yields

$$\lim_n \left\| \psi C_\phi g_{a_n}^{(\alpha)} \right\|_{\mathrm{BMOA}_v} > 0.$$

We can, therefore, by going to a subsequence if necessary, assume

$$\inf_n \left\| \psi C_\phi g_{a_n}^{(\alpha)} \right\|_{\mathrm{BMOA}_v} > 0.$$

Now, one can apply Lemma 2.12, first for $(g_{a_n}^{(\alpha)})$ (recall $\|g_{a_n}^{(\alpha)}\|_{\mathrm{BMOA}_v} = 1$ for all n) and then for $(\psi C_\phi g_{a_{n_k}}^{(\alpha)})$ to obtain the statement of fixing a copy of c_0, where (n_k) is the sequence of indices obtained after the first application of Lemma 2.12.

Since $(\sum_k g_{a_{n'(k)}}^{(\alpha)})$ is *wuC*, where $(n'(k))$ is the sequence of indices obtained after the two applications of the Lemma, it is sufficient to prove that $(\sum_k \psi C_\phi g_{a_{n'(k)}}^{(\alpha)})$ is not unconditionally convergent, but since the terms are bounded from below in norm it can't converge in norm and we are done.

Similarly, if $\lim_n \alpha(\psi,\phi,a_n) = 0$ and $\lim_n \beta(\psi,\phi,a_n) > 0$, Lemma 5.4 yields

$$\inf_n \left\| \psi C_\phi g_{a_n}^{(\beta)} \right\|_{\mathrm{BMOA}_v} > 0.$$

If $\mathrm{BMOA}_v \not\subset H^\infty$, we can apply Lemma 2.12 and the statement follows in the same manner as above. Else $\sup_{z\in\mathbb{D}} \|\delta_z\|_{X^*} < \infty$, where $X = \mathrm{VMOA}_v$ or $X = \mathrm{BMOA}_v$. If $\psi C_\phi|_{\mathrm{VMOA}_v} \in \mathcal{L}(\mathrm{VMOA}_v)$, it follows from Corollary 5.6 that $\psi \in \mathrm{VMOA}_v$, in which case $\lim_n \beta(\psi,\phi,a_n) > 0$ is impossible and we are done. □

The following proof is a standard procedure.

Lemma 5.8. *For any weight v yielding the evaluation maps $f \mapsto f(z),\ f \in \mathrm{BMOA}_v, z \in \mathbb{D}$ bounded, the norm-closed unit ball B_{BMOA_v} is τ_0-compact.*

Proof. Since the evaluation maps are bounded, the unit ball is bounded with respect to τ_0 by the Banach-Steinhaus theorem. By Fatou's lemma, we have for any τ_0-convergent sequence $(f_n) \subset B_{\mathrm{BMOA}_v}$ with an analytic function f as limit (defined on $\mathbb{D}$),

$$\|f\|_{\mathrm{BMOA}_v} \leq \liminf_n \|f_n\|_{\mathrm{BMOA}_v} \leq 1.$$

This shows that B_{BMOA_v} is τ_0-closed in $\mathcal{H}(\mathbb{D})$, because $(\mathcal{H}(\mathbb{D}), \tau_0)$ is a Fréchet space. By Montel's theorem B_{BMOA_v} is τ_0-compact. $\square$

Lemma 5.9. *Given that v is admissible, for any analytic self-map $\phi\colon \mathbb{D} \to \mathbb{D}$, we have*

$$\inf_{z\in\mathbb{D}} \frac{v(z)}{v(\phi(z))} \gtrsim_{v,g} v\left(\frac{2\,|\phi(0)|}{1+|\phi(0)|}\right)^{-1} > 0.$$

Moreover, if $\psi \in \mathrm{BMOA}_v$, then

$$\psi C_\phi \in \mathcal{L}(\mathrm{BMOA}_v) \implies \psi C_\phi \in \mathcal{L}(\mathrm{BMOA}),$$
$$\alpha(\psi,\phi,a) \gtrsim_{v,g,\phi} \alpha(\psi,\phi,a)_{\mathrm{BMOA}} \text{ and } \beta(\psi,\phi,a) \gtrsim_{v,g,\phi} \beta(\psi,\phi,a)_{\mathrm{BMOA}},$$

where the BMOA *subscript stands for that $v \equiv 1$ is used.*

Notice that $\beta(\psi,\phi,a)_{\mathrm{BMOA}}$ is not the same as in [13] and [16].

Proof. Applying [3, Corollary 2.40], we have

$$1-|\phi(z)| \geq (1-|z|)\frac{1-|\phi(0)|}{1+|\phi(0)|},$$

which gives (using (G2) with $b = (1-|z|)\frac{1-|\phi(0)|}{1+|\phi(0)|},\ a = 1-|z|$)

$$g\left(\frac{1}{1-|\phi(z)|}\right) \leq g\left(\frac{1}{1-|z|}\frac{1+|\phi(0)|}{1-|\phi(0)|}\right) \lesssim_g g\left(\frac{1}{1-|z|}\right) g\left(\frac{1+|\phi(0)|}{1-|\phi(0)|}\right).$$

Since $v(z) \asymp g\left(\frac{1}{1-|z|}\right)$ the first statement follows. Now, we obtain

$$\alpha(\psi,\phi,a) = \frac{v(a)}{v(\phi(a))}\,|\psi(a)|\,\|\phi_a\|_{H^2} \gtrsim_{v,g,\phi} |\psi(a)|\,\|\phi_a\|_{H^2} = \alpha(\psi,\phi,a)_{\mathrm{BMOA}} \quad \forall a \in \mathbb{D}.$$

For the function β, we apply Corollary 3.5 to obtain

$$\left\|\delta_{\phi(a)}\right\|_{(\mathrm{BMOA}_v)^*} \gtrsim_{v,g} 1 + \int\limits_0^{|\phi(a)|} \frac{dt}{(1-t)} \frac{1}{g(\frac{1}{1-|\phi(a)|})} \gtrsim_{v,g} \frac{\ln \frac{e}{1-|\phi(a)|}}{v(|\phi(a)|)} \gtrsim_{v,g} \frac{\left\|\delta_{\phi(a)}\right\|_{\mathrm{BMOA}^*}}{v(a)v\left(\frac{2|\phi(0)|}{1+|\phi(0)|}\right)},$$

and hence,

$$\left\|\delta_{\phi(a)}\right\|_{\mathrm{BMOA}_v^*} v(a)\gamma(\psi, a, 1) \gtrsim_{v,g,\phi} \left\|\delta_{\phi(a)}\right\|_{\mathrm{BMOA}^*} \gamma(\psi, a, 1).$$

Finally, apply Theorem 5.5 twice, first for the weighted case, then for the unweighted case to obtain $\psi C_\phi \in \mathcal{L}(\mathrm{BMOA}_v) \implies \psi C_\phi \in \mathcal{L}(\mathrm{BMOA})$. □

Theorem 5.10 *(Sufficiency for compactness). Assume v is admissible (see beginning of Section 5) and that $\psi C_\phi \in \mathcal{L}(\mathrm{BMOA}_v)$. If in addition, at least one of the following holds:*

(1) $C_\phi \in \mathcal{L}(\mathrm{BMOA}_v)$ or
(2) $\psi C_\phi|_{\mathrm{VMOA}_v} \in \mathcal{L}(\mathrm{VMOA}_v)$,

then $\lim_{|\phi(a)|\to 1}(\alpha(\psi, \phi, a) + \beta(\psi, \phi, a)) = 0$ is sufficient to ensure ψC_ϕ is compact on BMOA_v.

Proof. If v is bounded, the result follows from [13] and [16] (notice that (1) is trivially true), so we can assume v is unbounded. The major part of the proof is similar to the second part of [13, Proof of Theorem 3.1]. Let (f_n) be a bounded sequence in BMOA_v, which converges to 0 with respect to τ_0 (converges uniformly on compact subsets of $\mathbb{D}$). By Corollary 3.5 and Lemma 5.8, it follows that the unit ball B_{BMOA_v} is τ_0-compact. Since (f_n) is contained in a τ_0-compact set, it is sufficient to prove that $\lim_n \|\psi C_\phi f_n\|_{\mathrm{BMOA}_v} = 0$ to obtain ψC_ϕ is compact.

Similarly to [13, (3.17)], we have for any $r \in]0,1[$

$$\begin{aligned}\|\psi C_\phi f_n\|_{\mathrm{BMOA}_v} \lesssim |\psi(0) f_n(\phi(0))| &+ \sup_{|\phi(a)|>r} v(a)\gamma(\psi C_\phi f_n, a, 1) \\ &+ \sup_{|\phi(a)|\le r} v(a)\gamma(\psi C_\phi f_n, a, 1).\end{aligned} \tag{5.4}$$

The first term converges to zero, because $f_n \to 0$ as $n \to \infty$ w.r.t. τ_0. For the second term, Lemma 5.2 gives us

$$\begin{aligned}&\sup_{|\phi(a)|>r} v(a)\gamma(\psi C_\phi f_n, a, 1) \\ &\le \|f_n\|_{\mathrm{BMOA}_v} \sup_{|\phi(a)|>r} \left(\alpha(\psi, \phi, a) + \|\psi\|_* \frac{\|\phi_a\|_{H^2}}{v(\phi(a))} + \beta(\psi, \phi, a)\right),\end{aligned}$$

which yields

$$\lim_{r\to 1}\sup_{|\phi(a)|>r} v(a)\gamma(\psi C_\phi f_n, a, 1) = 0$$

by assumption.

Therefore, for a given $\epsilon > 0$, we can choose $r \in]0,1[$ large enough to ensure that

$$\sup_n \sup_{|\phi(a)|>r} v(a)\gamma(\psi C_\phi f_n, a, 1) < \epsilon.$$

It remains to prove that for $r \in]0,1[$ arbitrarily close to 1,

$$\lim_{n\to\infty}\sup_{|\phi(a)|\le r} v(a)\gamma(\psi C_\phi f_n, a, 1) \lesssim \epsilon.$$

To this end, for the third term in (5.4), we have

$$\begin{aligned}\sup_{|\phi(a)|\le r} v(a)\gamma(\psi C_\phi f_n, a, 1) \le& \sup_{|\phi(a)|\le r} v(a)\left\|\psi\circ\sigma_a(f_n\circ\phi\circ\sigma_a - f_n(\phi(a)))\right\|_{H^1}\\ &+ \sup_{|\phi(a)|\le r} v(a)\left|f_n(\phi(a))\right|\left\|\psi\circ\sigma_a - \psi(a)\right\|_{H^1}.\end{aligned} \tag{5.5}$$

For the second term above (in (5.5)), we have

$$\sup_{|\phi(a)|\le r} v(a)\left|f_n(\phi(a))\right|\left\|\psi\circ\sigma_a - \psi(a)\right\|_{H^1} \le \sup_{z\in r\overline{\mathbb{D}}}\left|f_n(z)\right|\|\psi\|_{\mathrm{BMOA}_v} \overset{n\to\infty}{\longrightarrow} 0,$$

since $\lim_n f_n = 0$ w.r.t. τ_0. Now, for $a \in \mathbb{D}$ and $t \in [0,1[$ we define

$$F_{n,a} := f_n\circ\phi\circ\sigma_a - f_n(\phi(a)) = f_n\circ\sigma_{\phi(a)}\circ\phi_a - f_n(\phi(a))$$

and

$$E = E(\phi, a, t) := \{w \in \mathbb{T} : |\phi_a(w)| > t\}.$$

Following Laitila, let $\frac{1}{2} < t < 1$ and note that by [13, (3.19)]

$$F_{n,a}(z) \le 2\left|\phi_a(z)\right|\sup_{|w|\le t}\left|f_n\circ\sigma_{\phi(a)}(w) - f_n(\phi(a))\right|, \quad z \in \overline{\mathbb{D}} \text{ with } \phi_a(z) \in t\overline{\mathbb{D}}.$$

It follows that for $a \in \mathbb{D}$

$$\left\|\chi_{\mathbb{T}\setminus E}\psi\circ\sigma_a F_{n,a}\right\|_{H^1} \lesssim \frac{\|\psi\|_* + v(\phi(a))\alpha(\psi,\phi,a)}{v(a)}\sup_{|w|\le t}\left|f_n\circ\sigma_{\phi(a)}(w) - f_n(\phi(a))\right|,$$

and because $\psi C_\phi \in \mathcal{L}(\mathrm{BMOA}_v)$ and

$$\left|\sigma_{\phi(a)}(w)\right|^2 = 1 - \frac{(1-|\phi(a)|^2)(1-|w|^2)}{\left|1-\overline{\phi(a)}w\right|^2} \le 1 - \frac{(1-r^2)(1-t^2)}{4} < 1,$$

for $|w| \leq t$ and $|\phi(a)| \leq r$, an application of Theorem 5.5 yields

$$\lim_n \sup_{|\phi(a)|\leq r} v(a) \left\|\chi_{\mathbb{T}\setminus E}\psi \circ \sigma_a F_{n,a}\right\|_{H^1} = 0,$$

using $\lim_n f_n = 0$ w.r.t. τ_0.

In view of (5.5), all that remains to show is that we can, for $0 < r < 1$ arbitrarily close to 1, choose $0 < t < 1$ near 1 to ensure that

$$\lim_n \sup_{|\phi(a)|\leq r} v(a) \left\|\chi_E \psi \circ \sigma_a (f_n \circ \phi \circ \sigma_a - f_n(\phi(a)))\right\|_{H^1} < \epsilon.$$

Applying Hölders inequality twice, we have

$$\begin{aligned}\left\|\chi_E \psi \circ \sigma_a (f_n \circ \phi \circ \sigma_a - f_n(\phi(a)))\right\|_{H^1} &\leq \left(\left\|\chi_E \psi \circ \sigma_a\right\|_{H^1}\right)^{\frac{1}{2}} \left(\left\|\chi_E \psi \circ \sigma_a F_{n,a}^2\right\|_{H^1}\right)^{\frac{1}{2}} \\ &\leq \left(\left\|\chi_E \psi \circ \sigma_a\right\|_{H^1}\right)^{\frac{1}{2}} \left(\left\|F_{n,a}\right\|_{H^2}\right)^{\frac{1}{2}} \left(\left\|\psi \circ \sigma_a F_{n,a}\right\|_{H^2}\right)^{\frac{1}{2}}.\end{aligned}$$

For the last factor, we have (note that $\|\psi\|_* = \|\psi C_\phi 1\|_* \leq \|\psi C_\phi\|_{\mathcal{L}(\mathrm{BMOA}_v)}$)

$$\begin{aligned}\left\|\psi \circ \sigma_a F_{n,a}\right\|_{H^2} &\leq \left\|\psi \circ \sigma_a f_n \circ \phi \circ \sigma_a - \psi(a) f_n(\phi(a))\right\|_{H^2} + |f_n(\phi(a))| \left\|\psi \circ \sigma_a - \psi(a)\right\|_{H^2} \\ &\leq 2\frac{\|\psi C_\phi\|_{\mathcal{L}(\mathrm{BMOA}_v)}}{v(a)} \sup_n \|f_n\|_{\mathrm{BMOA}_v} \left\|\delta_{\phi(a)}\right\|_{(\mathrm{BMOA}_v)^*}.\end{aligned}$$

Next, we prove that there is a positive number $M = M(v, \psi, \phi, (f_n), r, \epsilon)$ such that

$$\sup_{|\phi(a)|\leq r} v(a) \left\|\chi_E \psi \circ \sigma_a\right\|_{H^1} \left\|F_{n,a}\right\|_{H^2} \lesssim_{v,\psi,\phi,(f_n)} \epsilon + M \sup_{|\phi(a)|\leq r} \left\|\chi_E \psi \circ \sigma_a\right\|_{H^1}. \tag{5.6}$$

Assuming $C_\phi \in \mathcal{L}(\mathrm{BMOA}_v)$, Lemma 5.1 and Theorem 5.5 implies

$$\left\|F_{n,a}\right\|_{H^2} \leq \left\|f_n \circ \sigma_{\phi(a)} - f_n(\phi(a))\right\|_{H^2} \left\|\phi_a\right\|_{H^2} \leq \frac{\|\phi_a\|_{H^2}}{v(\phi(a))} \sup_n \|f_n\|_{\mathrm{BMOA}_v} \lesssim_{v,\phi,(f_n)} \frac{1}{v(a)} \tag{5.7}$$

and (5.6) follows.

Assuming $\psi C_\phi|_{\mathrm{VMOA}_v} \in \mathcal{L}(\mathrm{VMOA}_v)$, the fact that $\frac{1}{2} < t < 1$ gives us

$$\begin{aligned}v(a) \left\|\chi_E \psi \circ \sigma_a\right\|_{H^1} &\lesssim v(a) \left\|\psi \circ \sigma_a \phi_a\right\|_{H^1} \\ &\leq v(a) \left\|(\psi \circ \sigma_a - \psi(a))\right\|_{H^2} + v(a)\psi(a) \left\|\phi_a\right\|_{H^2}\end{aligned}$$

and by Corollary 5.6, using

$$\sup_{|\phi(a)|\leq r} \left\|\phi_a\right\|_{H^2} \lesssim_r \sup_{|\phi(a)|\leq r} \left\|\phi \circ \sigma_a - \phi(a)\right\|_{H^2},$$

we have

$$\sup_{|a|\geq s} v(a)\left\|\chi_E\psi\circ\sigma_a\right\|_{H^1} < \epsilon$$

for some $s = s(v,\psi,\phi,r,\epsilon) \in]0,1[$. Using the first two inequalities in (5.7), we can conclude that $\sup_{a\in\mathbb{D}}\sup_n \|F_{n,a}\|_{H^2} < \infty$, and hence,

$$\begin{aligned}\sup_{|\phi(a)|\leq r} v(a)\left\|\chi_E\psi\circ\sigma_a\right\|_{H^1}\left\|F_{n,a}\right\|_{H^2} &\lesssim_{v,\psi,\phi,(f_n)} \epsilon + \sup_{\substack{|a|\leq s\\ |\phi(a)|\leq r}} v(a)\left\|\chi_E\psi\circ\sigma_a\right\|_{H^1}\\ &\lesssim_v \epsilon + v(s)\sup_{|\phi(a)|\leq r}\left\|\chi_E\psi\circ\sigma_a\right\|_{H^1}.\end{aligned}$$

Summing up, we have proved that

$$\left\|\chi_E\psi\circ\sigma_a(f_n\circ\phi\circ\sigma_a - f_n(\phi(a)))\right\|_{H^1} \lesssim_* \epsilon + M\sup_{|\phi(a)|\leq r}\left\|\chi_E\psi\circ\sigma_a\right\|_{H^1},$$

where $\lesssim_*$ depend on v,ψ,ϕ and (f_n), and $M = M(v,\psi,\phi,(f_n),r,\epsilon)$.

The final part of the proof is to obtain

$$\lim_{t\to 1}\sup_{|\phi(a)|\leq r}\left\|\chi_E\psi\circ\sigma_a\right\|_{H^2} = 0.$$

Following the proof of [16, Proof of Theorem 2.1], we have that if $\psi C_\phi \in \mathcal{L}(\mathrm{BMOA})$ and

$$\lim_{|\phi(a)|\to 1}\left\|(\psi\circ\sigma_a)\phi_a\right\|_{H^2} = 0,$$

then

$$\lim_{t\to 1}\sup_{|\phi(a)|\leq r}\left\|\chi_E\psi\circ\sigma_a\right\|_{H^2} = 0 \quad \forall r\in]0,1[.$$

By assumption $\psi C_\phi \in \mathcal{L}(\mathrm{BMOA}_v)$, and by Lemma 5.9 and Theorem 5.5 (see also [13, Theorem 2.1], where a slightly different definition is used for β) the functions ψ and ϕ in ψC_ϕ give rise to a bounded operator $\psi C_\phi \in \mathcal{L}(\mathrm{BMOA})$ via the function-theoretic characterization. Furthermore,

$$\alpha(\psi,\phi,a) \geq \frac{v(a)\left\|(\psi\circ\sigma_a)\phi_a\right\|_{H^2}}{v(\phi(a))} - \frac{\|\psi\|_{\mathrm{BMOA}_v}}{v(\phi(a))}, \quad a\in\mathbb{D}$$

and with the aid of Lemma 5.9,

$$\alpha(\psi,\phi,a) + \frac{\|\psi\|_{\mathrm{BMOA}_v}}{v(\phi(a))} \gtrsim_{v,g,\phi} \left\|(\psi\circ\sigma_a)\phi_a\right\|_{H^2}.$$

Using the fact that v is unbounded, we conclude

$$\lim_{|\phi(a)|\to 1} \|(\psi \circ \sigma_a)\phi_a\|_{H^2} = 0. \quad \square$$

By turning via Carleson measure theory, it is proved that $\psi C_\phi \in \mathcal{L}(\mathrm{VMOA}_v)$ is compact given

$$\lim_{|a|\to 1} (\alpha(\psi, \phi, a) + \beta(\psi, \phi, a)) = 0.$$

We will use the following variant of Carleson sets with center z, where $\frac{1}{2} < |z| < 1$:

$$S(z) := \{w \in \mathbb{D} : 0 < 1 - |w| < 2(1 - |z|) \text{ and } |\arg w - \arg z| < 2\pi(1 - |z|)\}. \tag{5.8}$$

This is not the standard definition, but these scaled, open sets serves the same purpose as the classical ones when working with the measure μ_f in Proposition 5.11. It is clear that the longest euclidean distance from the center of the Carleson set is the distance to one of the corners away from the origin. Based on this, some elementary calculations yield

$$\inf_{w\in S(z)} |\sigma_z'(w)| \gtrsim \frac{1}{1 - |z|}, \quad \frac{1}{2} < |z| < 1. \tag{5.9}$$

The following proposition is a straightforward generalization of [13, Lemma 4.6] and [8, Lemma 3.3].

Proposition 5.11. *Let v be admissible and*

$$d\mu_f(z) := |f'(z)|^2 (1 - |z|^2)\, dA(z).$$

We have

$$\|f\|_{*,v}^2 \asymp \sup_{|z|\in]\frac{1}{2},1[} \frac{v(z)^2}{1 - |z|} \mu_f(S(z)).$$

Moreover,

$$\|f\|_{\mathrm{BMOA}_v} \lesssim_v \frac{1 + \sup_{z\in]0,r[} v(z)}{(1 - r)^{\frac{5}{2}}} \sup_{|z|\le r} |f(z)| + \sup_{|a|\ge r} v(a)\gamma(f, a, 2). \tag{5.10}$$

Proof. Let $z \in \mathbb{D}$ with $|z| > \frac{1}{2}$. On the one hand, we have by (5.9) and (1.1)

$$\begin{aligned} \frac{v(z)^2}{1 - |z|} \mu_f(S(z)) &\lesssim v(z)^2 \int\limits_{S(z)} |\sigma_z'(w)|\, |f'(w)|^2 (1 - |w|^2)\, dA(w) \\ &\lesssim v(z)^2 \gamma(f, z, 2)^2. \end{aligned} \tag{5.11}$$

On the other hand, if we put

$$E_n := E_n(z) := \left\{ w \in \mathbb{D} : \left| w - \frac{z}{|z|} \right| < 2^n(1-|z|) \right\},$$

and

$$N_z := \max\{n \in \mathbb{N} : 2^n(1-|z|) < 1\}$$

we obtain for $n = 1, 2, \ldots, N_z$

$$S_n := S\Big(\frac{z}{|z|}\Big(1 - 2^{n-1}(1-|z|)\Big)\Big) \supset E_n, \tag{5.12}$$

and therefore,

$$v\Big(1 - 2^{n-1}(1-|z|)\Big)^2 \mu_f(E_n) \le \Big(\sup_{|\zeta| \in]\frac{1}{2},1[} \frac{v(\zeta)^2}{1-|\zeta|} \mu_f(S(\zeta)) \Big) 2^{n-1}(1-|z|).$$

We have $\sup_{w \in \mathbb{D}} |\sigma_z'(w)| \lesssim (1-|z|)^{-1}$ and for $n \ge 2$ and $|z| \in]\frac{1}{2}, 1[$, we have

$$\sup_{w \in E_n \setminus E_{n-1}} |\sigma_z'(w)| \lesssim \frac{2^{-2n}}{1-|z|} \quad \text{and} \quad \sup_{w \in \mathbb{D} \setminus E_{N_z}} |\sigma_z'(w)| \lesssim 1 - |z|\,.$$

Combining the above estimate with (1.1) after which the path of integration, $\mathbb{D}$, is partitioned into $E_1, E_n \setminus E_{n-1}, n = 2, \ldots, N_z$ and $\mathbb{D} \setminus E_{N_z}$ yield

$$\begin{aligned} v(z)^2 \gamma(f, z, 2)^2 &\lesssim \frac{v(z)^2}{1-|z|} \mu_f(E_1) + \sum_{n=2}^{N_z} v(z)^2 \frac{2^{-2n}}{1-|z|} \mu_f(E_n \setminus E_{n-1}) \\ &\quad + v(z)^2(1-|z|)\mu_f(\mathbb{D} \setminus E_{N_z}) \\ &\overset{(5.12)}{\lesssim} \Big(\sup_{|\zeta| \in]\frac{1}{2},1[} \frac{v(\zeta)^2}{1-|\zeta|} \mu_f(S(\zeta)) \Big) \Bigg(1 + \sum_{n=2}^{N_z} \frac{v(z)^2 2^{-n}}{v\Big(1 - 2^{n-1}(1-|z|)\Big)^2} \Bigg) \\ &\quad + v(z)^2(1-|z|)\mu_f(\mathbb{D}). \end{aligned}$$

Clearly, for $z \in \mathbb{D}$ with $|z| > 1/2$,

$$v(z)^2(1-|z|)\mu_f(\mathbb{D}) \lesssim \Big(\sup_{|\zeta| \in]\frac{1}{2},1[} \frac{v(\zeta)^2}{1-|\zeta|} \mu_f(S(\zeta)) \Big)$$

and using (G2), we have

$$\frac{v(z)^2 2^{-n}}{v\Big(1 - 2^{n-1}(1-|z|)\Big)^2} \lesssim_{v,g} g\big(2^n\big)^2 2^{-n}, \quad n \ge 2.$$

To finish the proof of

$$\|f\|_{*,v}^2 \asymp_{v,g} \sup_{|z|\in]\frac{1}{2},1[} \frac{v(z)^2}{1-|z|}\mu_f(S(z)),$$

we note that

$$\sum_{n=3}^{\infty} g\left(2^n\right)^2 2^{-n} \lesssim_g \int_1^{\infty} \frac{g(e^x)^2}{e^x}\,dx \asymp \int_e^{\infty} g(x)^2\,d\arctan(x),$$

where the substitution $x \mapsto \ln x$ and $\frac{dx}{x^2} \asymp d\arctan(x)$ have been applied. Lemma 3.3 and (G1) yield that $\int_e^\infty g^2\,d\arctan$ is finite.

To prove the last statement, let $r \in]\frac{1}{2},1[$ and consider

$$\|f\|_{\mathrm{BMOA}_v} \lesssim |f(0)| + \sqrt{\sup_{|z|\in]r,1[} \frac{v(z)^2}{1-|z|}\mu_f(S(z))} + \sqrt{\sup_{|z|\in]\frac{1}{2},r]} \frac{v(z)^2}{1-|z|}\mu_f(S(z))}.$$

Equation (5.11) proves that

$$\sup_{|z|\in]r,1[} \frac{v(z)^2}{1-|z|}\mu_f(S(z)) \lesssim \sup_{|z|\in]r,1[} v(z)^2\gamma(f,z,2)^2.$$

For a given $z \in]\frac{1}{2},r[$, $S(z)\setminus\overline{r\mathbb{D}}$ can be covered by a $N = N(r,z) = \inf\{n \in \mathbb{N} : \frac{1-|z|}{1-r} \leq n\} + 1$ number of Carleson sets, $(S(z_j))_{j=1}^N$, where $|z_j| = r$ for all j. It follows that for some $j_0 \in [1,N]$,

$$\begin{aligned}
\frac{v(z)^2}{1-|z|}\mu_f(S(z)) &\leq \frac{v(z)^2}{1-|z|}\sum_{j=1}^{N}\mu_f(S(z_j)) + \frac{v(z)^2}{1-|z|}\mu_f(\overline{r\mathbb{D}}) \\
&\leq \frac{v(z)^2}{1-|z|}N\mu_f(S(z_{j_0})) + \frac{\sup_{z\in]0,r[} v(z)^2}{1-r}\int_{r\mathbb{D}} \left|f'(w)\right|^2 (1-|w|^2)\,dA(w) \\
&\leq \frac{v(z)^2(1-r)}{v(r)^2 1-|z|}N\frac{v(r)^2}{1-r}\mu_f(S(z_{j_0})) + \sup_{|z|\leq r}\left|f'(z)\right|^2 \frac{\sup_{z\in]0,r[} v(z)^2}{1-r} \\
&\lesssim_v \sup_{|z|\in]r,1[} \frac{v(z)^2}{1-|z|}\mu_f(S(z)) + \sup_{|z|\leq r} \frac{\left|f(z)\right|^2}{(1-r)^4}\frac{\sup_{z\in]0,r[} v(z)^2}{1-r},
\end{aligned}$$

where the Cauchy formula yields the comparison estimate between f' and f. □

Theorem 5.12 *(Sufficiency for compactness on* VMOA_v*). Assuming v is admissible (see beginning of Section 5), if $\psi C_\phi \in \mathcal{L}(\mathrm{VMOA}_v)$ and $\lim_{|\phi(a)|\to 1}(\alpha(\psi,\phi,a)+\beta(\psi,\phi,a)) = 0$, then ψC_ϕ is compact.*

Proof. If v is bounded, the result is proved in [13, Theorem 4.3], hence, we assume v is unbounded. First, we prove that if $\psi C_\phi \in \mathcal{L}(\mathrm{VMOA}_v)$, then

$$\lim_{|\phi(a)|\to 1}(\alpha(\psi,\phi,a)+\beta(\psi,\phi,a))=0 \implies \lim_{|a|\to 1}(\alpha(\psi,\phi,a)+\beta(\psi,\phi,a))=0. \tag{5.13}$$

Indeed, if (a_n) is a sequence with $\lim_n |a_n| \to 1$, we extract an arbitrary subsequence, also called (a_n). If there is a subsequence $(a_{n_k}) \subset (a_n)$ with $\lim_k |\phi(a_{n_k})| = 1$, we can conclude that

$$\lim_k(\alpha(\psi,\phi,a_{n_k})+\beta(\psi,\phi,a_{n_k}))=0$$

for such a subsequence. If this is not the case, $|\phi(a_n)| \in \overline{R\mathbb{D}}$ for some $R<1$, and hence, by Corollary 5.6

$$\lim_n \alpha(\psi,\phi,a_n) \lesssim_{v,\phi,(a_n)} \lim_n v(a_n)\psi(a_n)\left\|\phi(a_n)-\phi\circ\sigma_{a_n}\right\|_{H^2}=0$$

and

$$\lim_n \beta(\psi,\phi,a_n) \lesssim_{v,\phi,(a_n)} \lim_n v(a_n)\left\|\psi\circ\sigma_{a_n}-\psi(a_n)\right\|_{H^2}=0.$$

This proves that for an arbitrary sequence (a_n) with $\lim_n |a_n|=1$, every subsequence of $(\alpha(\psi,\phi,a_n)+\beta(\psi,\phi,a_n))_n$ has a convergent subsequence, with limit zero, which means it converges to zero as $n\to\infty$.

We are now ready to prove that ψC_ϕ is the uniform limit (in operator norm) of the compact operators $\psi C_\phi K_n, n\in\mathbb{N}$, where $K_n := T_{\frac{n}{n+1}}, f\mapsto [z\mapsto f(\frac{n}{n+1}z)], f\in \mathrm{VMOA}_v$. The fact that $K_n \in \mathcal{L}(\mathrm{VMOA}_v)$ is compact follows from the following: Let (f_k) be a bounded sequence in VMOA_v and f_0 be the limit w.r.t. τ_0 (convergence on compact subsets) of some subsequence $(f_{k'})$. Applying (1.1), followed by and the Cauchy formula to estimate the f' by f yield

$$\|K_n f_{k'} - K_n f_0\|_{\mathrm{BMOA}_v} \lesssim_{v,n} \|K_{n+1}(f_{k'}-f_0)\|_\infty \overset{k'\to\infty}{\longrightarrow} 0.$$

Although f_0 might not be in VMOA_v, the function $K_n f_0 \in \mathrm{VMOA}_v$ by the remark after (1.1).

Continuing, for $n\in\mathbb{N}$, we have by Proposition 5.11, for every $r\in]\frac{1}{2},1[$,

$$\begin{aligned}&\sup_{f\in B_{\mathrm{VMOA}_v}} \|(\psi C_\phi - \psi C_\phi K_n)f\|_{\mathrm{BMOA}_v}\\ &\lesssim_v \sup_{f\in B_{\mathrm{VMOA}_v}} \frac{1+\sup_{z\in]0,r[} v(z)}{(1-r)^{\frac{5}{2}}} \sup_{|z|\le r} |(\psi C_\phi - \psi C_\phi K_n)f(z)| \\ &\quad + \sup_{f\in B_{\mathrm{VMOA}_v}} \sup_{|a|\ge r} v(a)\gamma((\psi C_\phi - \psi C_\phi K_n)f,a,2).\end{aligned} \tag{5.14}$$

Furthermore, as in the proof of [13, (4.13)] for $r \in]\frac{1}{2}, 1[$ an application of Cauchy's integral formula yields

$$\lim_{n} \sup_{f \in B_{\mathrm{VMOA}_v}} \sup_{|z| \le r} |(\psi C_\phi - \psi C_\phi K_n) f(z)| = 0.$$

An application of Proposition 2.3 yields that for any $R_{\mathrm{BMO}} \in]0,1[$, $R_A \in]0, R_{\mathrm{BMO}}/2[$ and $f \in \mathrm{VMOA}_v$, we have

$$\sup_{|a| \ge 1 - R_A} v(a)\gamma((\psi C_\phi - \psi C_\phi K_n) f, a, 2) \lesssim_{v,\epsilon_0} \sup_{|a| \ge 1 - R_{\mathrm{BMO}}} v(a)\gamma((\psi C_\phi - \psi C_\phi K_n) f, a, 1) \\ + \|(\psi C_\phi - \psi C_\phi K_n) f\|_{*,v,2} \left(\frac{2R_A}{R_{\mathrm{BMO}}} \right)^{\epsilon_0}.$$

Combining this estimate with Lemma 5.2 and the fact that $\sup_n \|K_n\|_{\mathcal{L}(\mathrm{VMOA}_v)} < \infty$ yield

$$\sup_{f \in B_{\mathrm{VMOA}_v}} \sup_{|a| \ge 1 - R_A} v(a)\gamma(\psi C_\phi (I - K_n) f, a, 2) \\ \lesssim_{v,\epsilon_0} \sup_{|a| \ge 1 - R_{\mathrm{BMO}}} \left(\alpha(\psi, \phi, a) + v(a) \|\psi \circ \sigma_a - \psi(a)\|_{H^2} \frac{\|\phi_a\|_2}{v(\phi(a))} + \beta(\psi, \phi, a) \right) \\ + \|\psi C_\phi\|_{\mathcal{L}(\mathrm{VMOA}_v)} \left(\frac{2R_A}{R_{\mathrm{BMO}}} \right)^{\epsilon_0}.$$

Using $r = 1 - R_A$ in (5.14) gives us

$$\limsup_{n} \sup_{f \in B_{\mathrm{VMOA}_v}} \|(\psi C_\phi - \psi C_\phi K_n) f\|_{\mathrm{BMOA}_v} \\ \lesssim_{v,\epsilon_0} \sup_{|a| \ge 1 - R_{\mathrm{BMO}}} \left(\alpha(\psi, \phi, a) + v(a) \|\psi \circ \sigma_a - \psi(a)\|_{H^2} + \beta(\psi, \phi, a) \right) \\ + \|\psi C_\phi\|_{\mathcal{L}(\mathrm{VMOA}_v)} \left(\frac{2R_A}{R_{\mathrm{BMO}}} \right)^{\epsilon_0}.$$

Since $\psi \in \mathrm{VMOA}_v$ by Corollary 5.6, letting $R_A \to 0$ followed by $R_{\mathrm{BMO}} \to 0$ yield

$$\limsup_{n} \|\psi C_\phi - \psi C_\phi K_n\|_{\mathcal{L}(\mathrm{VMOA}_v)} \lesssim_{v,\epsilon_0} \limsup_{|a| \to 1} (\alpha(\psi, \phi, a) + \beta(\psi, \phi, a)), \tag{5.15}$$

where the right-hand side, considering (5.13), is zero by assumption. □

Similarly to [13, Theorem 4.3], using Lemmas 5.3 and 5.4, we have the following Corollary:

Corollary 5.13. *Assuming v is admissible, if $\psi C_\phi \in \mathcal{L}(\mathrm{VMOA}_v)$, then the essential norm, is given by*

$$\begin{aligned}\|\psi C_\phi\|_{e,\mathcal{L}(\mathrm{VMOA}_v)} &:= \inf_{\substack{K\in\mathcal{L}(\mathrm{VMOA}_v)\\ K\ compact}} \|\psi C_\phi - K\|_{\mathcal{L}(\mathrm{VMOA}_v)} \\ &\asymp \limsup_{|a|\to 1}(\alpha(\psi,\phi,a)+\beta(\psi,\phi,a)) \\ &\asymp \limsup_{|\phi(a)|\to 1}(\alpha(\psi,\phi,a)+\beta(\psi,\phi,a)).\end{aligned}$$

We end this section with showing that for many admissible weights, the function h defined in Lemma 3.4 is not in VMOA_v. Therefore, VMOA_v is a proper subspace of BMOA_v.

Proposition 5.14. *If v is admissible and g satisfies the reverse inequality (G3), that is, $|g(z)| \lesssim g(|z|),\ z \in \mathbb{C}_{\Re\geq\frac{1}{2}}$, then $h \in \mathrm{BMOA}_v \setminus \mathrm{VMOA}_v$, where*

$$h\colon z \mapsto \int_0^z \frac{dt}{(1-t)g(\frac{1}{1-t})}.$$

Proof. We begin by proving that $h \notin \mathrm{VMOA}_v$. Let $a \in]\frac{1}{2}, 1[$ and $S(a)$ be the Carleson set defined in (5.8) and put $S_a = S(a) \cap a\overline{\mathbb{D}}$. For $z \in S_a$, it holds that $1-|z| \geq 1-a$, $|1-z| \lesssim 1-a$, and from (5.9), it follows that $|1-\overline{a}z|^2 \lesssim (1-a)^2$. By the Littlewood-Paley identity, (1.1) with $p = 2$ and the assumption $|g(z)| \lesssim g(|z|),\ z \in \mathbb{C}_{\Re\geq\frac{1}{2}}$, we have

$$\gamma(h,a,2) \gtrsim \int_{S_a} \frac{(1-|z|^2)(1-|a|^2)}{|1-z|^2\left|g(\frac{1}{1-z})\right|^2|1-\overline{a}z|^2}\,dA(z) \gtrsim_g \frac{\int_{S_a} dA(z)}{(1-a)^2 g(\frac{1}{1-a})^2} \asymp g(\frac{1}{1-a})^{-2}.$$

We can conclude that

$$\lim_{a\to 1} v(a)^2\gamma(h,a,2) \gtrsim_g 1,$$

proving $h \notin \mathrm{VMOA}_v$. By Lemma 3.4 $h \in \mathrm{BMOA}_v$. □

The condition $|g(z)| \lesssim g(|z|),\ z \in \mathbb{C}_{\Re\geq\frac{1}{2}}$ is trivially fulfilled for the standard weights, $g(z) = z^c, 0 \leq c < 1/2$. The condition is also fulfilled for $g(z) = (\ln(ez))^c, c > 0$. Indeed, for $z \in \mathbb{C}_{\Re\geq\frac{1}{2}}$ we have $(\ln(e|z|)) \geq \ln(e/2)$, and hence,

$$(\ln(e|z|))^2 \leq |\ln(ez)|^2 \leq (\ln(e|z|))^2 + (\pi/2)^2 \lesssim (\ln(e|z|))^2.$$

6. Examples

This section contains some practical examples of symbols ψ and ϕ making ψC_ϕ bounded and compact on BMOA_v and VMOA_v, where v is an admissible weight (see beginning of Section 5). Before we proceed, we have the following useful lemma.

Lemma 6.1. *Let v, be admissible. The weight*

$$w(a) = v(a)\left(1 + \int_0^{|a|} \frac{dt}{(1-t)v(t)}\right), \quad a \in \mathbb{D}$$

satisfies $\sup_{0<x<1} x\, w(1-x)^{2+\epsilon_0} < \infty$, where $\epsilon_0 > 0$ is given in (G1).

Proof. For $a \in \mathbb{D}$, (G2) gives us

$$w(a) \asymp_{v,g} v(a) + \int_0^{|a|} \frac{v(|a|)\, dt}{(1-t)v(t)} \lesssim_{v,g} v(a) + \int_0^{|a|} v\left(\frac{|a|-t}{1-t}\right) \frac{dt}{(1-t)} = v(|a|) + \int_0^{|a|} v(t) \frac{dt}{(1-t)},$$

where the substitution $t \mapsto (|a|-t)/(1-t)$ was used to obtain the last equality. Furthermore, by (G1) we have

$$\int_0^{|a|} v(t) \frac{dt}{(1-t)} \lesssim_{v,g,\epsilon_0} \int_0^{|a|} (1-t)^{-\left(1+\frac{1}{2+\epsilon_0}\right)}\, dt \lesssim_{\epsilon_0} (1-|a|)^{-\frac{1}{2+\epsilon_0}}$$

yielding

$$\sup_{a\in\mathbb{D}} (1-|a|) w(a)^{2+\epsilon_0} \lesssim_{v,g,\epsilon_0} \sup_{a\in\mathbb{D}} (1-|a|) v(a)^{2+\epsilon_0} + 1$$

and we are done. $\square$

Via the function-theoretic characterization of boundedness and compactness of ψC_ϕ, it is clear that if ψ and ϕ makes ψC_ϕ act in a bounded (compact) manner on VMOA_v, then ψC_ϕ will act in a bounded (compact) manner on BMOA_v too. By Proposition 2.1, VMOA_v contains all analytic polynomials for any weight, v, satisfying $\lim_{a\to 1} v(a)^2(1-|a|) = 0$. Therefore, for polynomial symbols ψ and ϕ for which $\sup_{a\in\mathbb{D}} \alpha(\psi,\phi,a) < \infty$, ψC_ϕ acts boundedly on VMOA_v (and BMOA_v). More is true, let $\psi = q_1$ and $\phi = q_2$ be two fractions of polynomials, where the denominators have no zeros in $\overline{\mathbb{D}}$. By (1.1), it follows that $\|q_j\|_{\mathrm{BMOA}_v} \asymp_{q_j} \|p_j\|_{\mathrm{BMOA}_v}$ for some polynomials p_j, $j = 1, 2$ for any admissible weight v. It also follows that $q_1, q_2, q_1 q_2 \in \mathrm{VMOA}_v$, and by Corollaries 3.5, 3.7 and Lemma 6.1, $\lim_{|a|\to 1} \beta(q_1, q_2, a) = 0$. All that remains for boundedness and compactness, is to prove that $\sup_{a\in\mathbb{D}} \alpha(q_1, q_2, a) < \infty$ and $\lim_{|q_2(a)|\to 1} \alpha(q_1, q_2, a) = 0$ respectively. Note that $\lim_{|q_2(a)|\to 1} \alpha(q_1, q_2, a) = 0$ grants boundedness. This follows from $q_2 \in \mathrm{VMOA}_v$, $q_1 \in H^\infty$ and

$$\sup_{|q_2(a)|\le R} \|(q_2)_a\|_{H^2} \asymp_R \|q_2(a) - q_2 \circ \sigma_a\|_{H^2}, \quad 0 < R < 1.$$

It is worth noting that if $|q_2(\eta)| = 1$ for some $\eta \in \mathbb{T}$, the continuity of q_2 ensures there is a disk $D := \eta(c + (1-c)\mathbb{D})$, $0 < c < 1$ such that $\lim^*_{|q_2(a)|\to 1} \alpha(q_1, q_2, a) = 0$, where the limit is taken outside the disk D.

Turning our attention to the multiplication operator $M_\psi \colon f \mapsto \psi f$, Corollary 1.3 yields it is bounded but not compact on $X = \mathrm{BMOA}_v$ or $X = \mathrm{VMOA}_v$ (v admissible), for any $\psi \in H^\infty \cap \mathrm{BMOA}_w \setminus \{0\}$, where $w(a) = v(a) \|\delta_a\|_{(X)^*}, a \in \mathbb{D}$. It is worth noting that the product of admissible weights is admissible if and only if (G1) is satisfied for the product. Moreover, by Corollary 3.5 the admissible weight

$$w_1(a) := v(a) \ln \frac{e}{1-|a|}, a \in \mathbb{D} \tag{6.1}$$

dominates w yielding $\mathrm{BMOA}_{w_1} \subset \mathrm{BMOA}_w$.

Another interesting fact is the following result concerning the composition operator C_ϕ. Let v be an admissible weight. Assume $\psi \in X$, where $X = \mathrm{BMOA}_v$ or $X = \mathrm{VMOA}_v$ and ϕ is a self-map of $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$. If $\psi C_\phi \colon X \to X$ is a bounded (compact) operator, then for every $n \in \mathbb{N}$, $\psi C_{\phi^n} : X \to X$ is bounded (compact). To this end, since v is almost increasing, (G2) yields

$$v(a^n) \lesssim_v v(a) \lesssim_{v,g} v\left(1 - \frac{1-|a|}{1-|a|^n}\right) v(a^n) \lesssim_v v(1 - \frac{1}{n}) v(a^n), \quad a \in \mathbb{D}, n \in \mathbb{N}. \tag{6.2}$$

Furthermore, put $c_0 := \epsilon_0/(2(1+\epsilon_0))$. By Lemma 6.1 and Corollary 3.5, we have

$$\limsup_{|a|\to 1} (1-|a|)^{1-c_0} v(a)^2 \|\delta_a\|^2_{X^*} = 0. \tag{6.3}$$

We will establish

$$\|(\phi^n)_a\|^2_{H^2} \lesssim_n (1-|a|)^{1-c_0} + \|\phi_a\|^2_{H^2}, \tag{6.4}$$

and after multiplying both sides of (6.4) with $v(a)^2\psi(a)^2/v(\phi(a))^2$ (6.2) yields

$$\alpha(\psi, \phi^n, a)^2 \lesssim_{n,v,g} \frac{(1-|a|)^{1-c_0} v(a)^2 \psi(a)^2}{v(\phi(a))^2} + \alpha(\psi, \phi, a)^2.$$

In view of Theorems 1.1 and 1.2, the statement concerning boundedness follows from taking the limit $|a| \to 1$ together with (6.3), and compactness follows from considering $|\phi(a)| \to 1$.

To prove (6.4), let $n \in \mathbb{N}$ and $r \in]0, \frac{1}{2}[$ be small enough so that for $a \in \mathbb{D}$ with $|a| \geq 1 - r$, $|\phi(z) - \phi(a)| < \frac{1}{2n^2}$ for every

$$z \in J(a) := \left\{ w \in \mathbb{T} : \left| w - \frac{a}{|a|} \right| < 2(1 - |a|)^{c_0/2} \right\}.$$

Furthermore, $\sup_{z \in \mathbb{T} \setminus J(a)} P_a(z) \lesssim (1 - |a|)^{1-c_0}$. It follows that for $|a| \geq 1 - r$ and $z \in J(a)$, we have

$$\left| \sum_{k=0}^{n-1} \overline{\phi(a)}^k \phi(z)^k - \sum_{k=0}^{n-1} \overline{\phi(a)}^k \phi(a)^k \right| \leq \sum_{k=0}^{n-1} \left| \phi^k(z) - \phi(a)^k \right| \leq n^2 \left| \phi(z) - \phi(a) \right| < \frac{1}{2}, \quad (6.5)$$

and hence,

$$\begin{aligned} \|(\phi^n)_a\|_{H^2}^2 &= \int_{\mathbb{T}} \left| \frac{\phi(a)^n - \phi^n}{1 - \overline{\phi(a)}^n \phi^n} \right|^2 P_a \, dm \\ &\lesssim (1 - |a|)^{1-c_0} + \int_{J(a)} \left| \frac{\sum_{k=0}^{n-1} \phi(a)^k \phi^{n-k-1}}{\sum_{k=0}^{n-1} \overline{\phi(a)}^k \phi^k} \right|^2 \left| \frac{\phi(a) - \phi}{1 - \overline{\phi(a)}\phi} \right|^2 P_a \, dm \\ &\overset{(6.5)}{\leq} (1 - |a|)^{1-c_0} + \int_{J(a)} \left| \frac{n}{\sum_{k=0}^{n-1} |\phi(a)|^{2k} - \frac{1}{2}} \right|^2 \left| \frac{\phi(a) - \phi}{1 - \overline{\phi(a)}\phi} \right|^2 P_a \, dm \\ &\leq (1 - |a|)^{1-c_0} + 4n^2 \int_{\mathbb{T}} \left| \frac{\phi(a) - \phi}{1 - \overline{\phi(a)}\phi} \right|^2 P_a \, dm \leq (1 - |a|)^{1-c_0} + 4n^2 \|\phi_a\|_{H^2}^2 . \end{aligned} \quad (6.6)$$

The following partial complement to (6.4) can be proved using similar calculations to (6.6): If we only consider sequences (a_j), with $\inf_j |\phi(a_j)| > 0$, then

$$\left\| \phi_{a_j} \right\|_{H^2}^2 \lesssim_{n, (\phi(a_j))} (1 - |a_j|)^{1-c_0} + \left\| (\phi^n)_{a_j} \right\|_{H^2}^2 . \quad (6.7)$$

If $\phi(\mathbb{D}) \subset b\overline{\mathbb{D}}$ for some $0 < b < 1$, then by Corollary 3.5

$$\alpha(\psi, \phi, a) \asymp_{b,v} |\psi(a)| \, v(a) \, \|\phi \circ \sigma_a - \phi(a)\|_{H^2} \lesssim_{\psi,v} \ln \frac{e}{1 - |a|} v(a) \, \|\phi \circ \sigma_a - \phi(a)\|_{H^2}$$

and

$$\beta(\psi, \phi, a) \asymp_b v(a) \, \|\psi \circ \sigma_a - \psi(a)\|_{H^1} .$$

It follows that if $\phi \in \mathrm{VMOA}_{w_1} \cap H^\infty$, then ϕ can be scaled to grant that the operator $\psi C_\phi \colon X \to X$ is bounded (compact) if $\psi \in X$, where $X = \mathrm{BMOA}_v$ or $X = \mathrm{VMOA}_v$.

Since w_1 is admissible (see (6.1)), the remark right after (1.1) yields many examples $\phi \in \mathrm{VMOA}_{w_1} \cap H^\infty$, e.g. a dilation of an analytic function.

Recall that a Blaschke product is a function of the form

$$z \mapsto z^m \prod_n \frac{|b_n|}{b_n} \frac{b_n - z}{1 - \overline{b_n} z},$$

where $m = 0, 1, \ldots$ and $(b_n) \subset \mathbb{D} \cup \{1\} \setminus \{0\}$ is a sequence satisfying $\sum_n (1 - |b_n|) < \infty$ ([5, p. 20]). An interesting fact is that an infinite Blaschke product ($b_n \neq 1$ for infinitely many n) as the symbol ϕ gives rise to a bounded composition operator if and only if $v \asymp 1$. When $v \asymp 1$, the statement follows from $\alpha(1, \phi, a) \asymp \|\phi_a\|_{H^2} \leq 1$. When v is unbounded, let $(a_n) \subset \mathbb{D}$ be the zeros of ϕ. Since for every $a \in \mathbb{D}$, the functions $\sigma_{\phi(a)}$ and σ_a are automorphisms on $\mathbb{T}$ and ϕ has a modulus 1 a.e., it follows that $\|\phi_a\| = 1$ for all $a \in \mathbb{D}$. Moreover, $v(a_n)/v(\phi(a_n)) = v(a_n)/v(0) \to \infty$ as $n \to \infty$, due to $\lim_n |a_n| = 1$, resulting in $\sup_{a \in \mathbb{D}} \alpha(1, \phi, a) = \infty$.

Next, we consider the polynomial $\phi(z) = \frac{1+z}{2}, z \in \mathbb{D}$. To see that it is bounded, it is enough to prove that

$$\limsup_{|\phi(a)| \to 1} \alpha(1, \phi, a) < \infty,$$

which is the same as

$$\limsup_{a \to 1} \alpha(1, \phi, a) < \infty.$$

In [13, Example 5.1], J. Laitila showed that

$$\|\phi_a\|_{H^2}^2 = \frac{1 - |a|^2}{2(1 - \Re a)}.$$

Since $|\phi(a)| \geq |a|$ for $a \in 1/2 + (1/2)\overline{\mathbb{D}}$, we have for these $a \in \mathbb{D}$, $v(a)/v(\phi(a)) \lesssim 1$. It remains to examine the tangential limits $a \to 1$, that lie outside this disk. These are $a \in \mathbb{D}$ near 1 for which $\Re a \leq |a|^2$. Using this and (G2), we have

$$1 - |\phi(a)|^2 = 1 - \frac{1}{4}(1 + |a|^2 + 2\Re a) \leq \frac{3}{4}(1 - \Re a)$$

and

$$\begin{aligned} \frac{v(a)}{v(\phi(a))} &\asymp_{v,g} \frac{v(a^2)}{v(\phi(a)^2)} \lesssim_{v,g} v\left(1 - \frac{1 - |a|^2}{1 - |\phi(a)|^2}\right) \lesssim_v v\left(1 - \frac{3}{4} \frac{1 - |a|^2}{1 - |\phi(a)|^2}\right) \\ &\lesssim_v v\left(1 - \frac{1 - |a|^2}{1 - \Re a}\right). \end{aligned}$$

It now follows from (G1) that

$$\sup_{a\in\mathbb{D}} \alpha(1,\phi,a) = \sup_{a\in\mathbb{D}} \frac{v(a)}{v(\phi(a))} \|\phi_a\|_{H^2} < \infty \tag{6.8}$$

proving $C_\phi\colon \mathrm{VMOA} \to \mathrm{VMOA}$ is bounded. Considering the limit $a \to 1$ along the real line together with Lemma 5.9, it is clear that C_ϕ is not compact.

Let $\phi(z) = \frac{1+z}{2}, z \in \mathbb{D}$ and $\psi(z) = 1 - z, z \in \mathbb{D}$, which makes $M_\psi \in \mathcal{L}(\mathrm{VMOA}_v)$ in accordance with Proposition 2.1 and the discussion above. In [13, Example 5.1], J. Laitila proved that ψC_ϕ acts as a compact operator $\mathrm{VMOA} \to \mathrm{VMOA}$ although neither M_ψ nor C_ϕ is compact. This example works in the same manner on VMOA_v. All that remains to prove for this statement is that ψC_ϕ is compact on VMOA_v. From the first paragraph after the proof to Lemma 6.1, it is sufficient to prove that

$$\lim_{a\to 1} \alpha(\psi,\phi,a) = 0.$$

This is true due to (6.8) and

$$\alpha(\psi,\phi,a) = |1-a|\, \alpha(1,\phi,a).$$

7. Proof of main results

Proof of Theorem 1.1. Theorem 5.5, Corollary 5.6 and Corollary 3.5 yield the statements. □

Lemma 7.1. *Let $X = \mathrm{BMOA}_v$ or $X = \mathrm{VMOA}_v$, where v is an admissible weight (see beginning of Section 5). It holds for $\psi C_\phi \in \mathcal{L}(X)$ that*

$$\psi C_\phi \text{ is compact} \quad \Longrightarrow \quad \limsup_{|\phi(a)|\to 1} \left\| \psi C_\phi f_a^{(\alpha)} \right\|_{\mathrm{BMOA}_v} = 0 \quad \Longrightarrow \quad \limsup_{|\phi(a)|\to 1} \alpha(\psi,\phi,a) = 0,$$

where

$$f_a^{(\alpha)}\colon z \mapsto \frac{\sigma_{\phi(a)}(z) - \phi(a)}{v(\phi(a))}.$$

Proof. Note that $C := \sup_{a\in\mathbb{D}} \|f_a^{(\alpha)}\|_{\mathrm{BMOA}_v} < \infty$ by (4.2) and $\psi C_\phi|_{CB_X}$ is always $\tau_0 - \tau_0$ continuous. If $X = \mathrm{BMOA}_v$, Lemma 5.8 yields (CB_X, τ_0) is compact. Now, [2, Lemma 3.3] proves the first implication, noting that for every $0 < R < 1$

$$\sup_{|z|\le R} \left| f_a^{(\alpha)}(z) \right| \lesssim \frac{R(1-|\phi(a)|^2)}{1-|\phi(a)|\,R},$$

and hence, $f_a^{(\alpha)} \to 0$ w.r.t. τ_0 as $|\phi(a)| \to 1$. On closer inspection, the use of B_X being compact w.r.t. τ_0 is redundant for the implication used above. Indeed, if ψC_ϕ is compact,

$K := \overline{\psi C_\phi(CB_X)}^{\|\cdot\|_{\mathrm{BMOA}_v}}$ is compact and by a standard argument, the norm topology induced by $\|\cdot\|_{\mathrm{BMOA}_v}$ restricted to K is equivalent to the topology on K induced by τ_0. Therefore, the first implication also holds for $X = \mathrm{VMOA}_v$.

Concerning the second implication, the boundedness of ψC_ϕ ensures $\sup_{a\in\mathbb{D}} \beta(\psi, \phi, a) < \infty$ and $\psi \in \mathrm{BMOA}_v$. If v is unbounded, Lemma 5.3 gives the second implication. If v is bounded, Lemma 5.3 combined with [13, (3.10)] gives the implication. □

Proof of Theorem 1.2. In the following, Theorem (*) means either Theorem 5.10 or Theorem 5.12 depending on if ψC_ϕ is operating on BMOA_v or VMOA_v. The statements are proved as follows:

$$(8) \overset{\text{Theorem } (*)}{\Longrightarrow} (1) \overset{\text{Lemma } 2.10}{\Longrightarrow} \begin{smallmatrix}(2)\\ \text{and}\\ (3)\end{smallmatrix} \Longrightarrow \begin{smallmatrix}(2)\\ \text{or}\\ (3)\end{smallmatrix} \overset{\text{Lemma } 2.10}{\Longrightarrow} (4) \overset{\text{Theorem } 5.7}{\Longrightarrow} (8),$$

and

$$(4) \overset{\text{Lemma } 2.9}{\Longrightarrow} (5) \overset{\text{Theorem } 5.7}{\Longrightarrow} (4)$$

followed by

$$(1) \overset{[17,\text{ pp. } 120\text{-}121]}{\Longrightarrow} (7) \overset{\text{Def.}}{\Longrightarrow} (6) \overset{\text{Def.}}{\Longrightarrow} (4).$$

Lemma 7.1 connects (9) with the rest of the equivalent statements in the first list.

Theorem 2.15 together with the rest of Subsection 2.2 proves the equivalence of (2), (10) and (11). □

The only implications in the proof that do not hold for an arbitrary bounded operator on a Banach space $T \in \mathcal{L}(X)$ are the ones involving (8), (9). The implication (5) $\Rightarrow$ (4) holds in general, which can be seen using the standard basis of c_0 as a wuC series, which does not converge unconditionally.

To finish the section, some open problems and conjectures are gathered.

Conjecture 1. The condition $x \mapsto v(1-x)x^{\frac{1}{p}-\epsilon}$ is almost increasing for some $\epsilon > 0$ can be replaced by the milder assumption $\inf_{x\in]0,1[} v(x) > 0$ in Proposition 2.3.

The problematic part is Proposition 2.5. An initial idea is to apply Baernstein's approach. However, instead of Aut being a group with respect to composition, a straightforward approach would demand that

$$\{\hat{\phi} \in \mathrm{Aut} : \left|\hat{\phi}(0)\right| \geq R\}$$

is a group, which is not true (it is not closed under composition). The binary nature of integrating over an interval (integration over $\mathbb{T}$ with the binary function χ_I) allows

Lemma 2.6 and the very beginning of the proof of Proposition 2.3 regardless of a strictly positive weight v. In the conformal setting (integration over $\mathbb{T}$ with the Poisson kernel P_a as a weight) an unbounded weight v complicate matters.

Conjecture 2. There exists an increasing, radial (not admissible) weight $v\colon \mathbb{D} \to]0,\infty[$, $\hat{\phi} \in \mathrm{Aut}$ and $f \in \mathrm{BMOA}_v$ such that $f \circ \hat{\phi} \notin \mathrm{BMOA}_v$.

A direct approach yields that this is the same as proving that there exists $\hat{\phi} \in \mathrm{Aut}$ and $f \in \mathrm{BMOA}_v$ such that

$$\sup_{a\in\mathbb{D}} \frac{v(\hat{\phi}(a))}{v(a)} v(a)\gamma(f,a,2) = \infty.$$

Examining the quotient $\frac{v(\hat{\phi}(a))}{v(a)}$, we have the following: Using $v(z) = g(\frac{1}{1-|z|^2})$, for some increasing unbounded function $g\colon [1,\infty[\to]0,\infty[$, the automorphisms $\hat{\phi}_c := \sigma_{-\sqrt{1-c}}$, $0 < c < 1$ yield

$$g\left(\frac{1}{1-\left|\hat{\phi}_c(a)\right|^2}\right) = g\left(\frac{\left|1+\sqrt{1-c}a\right|^2}{c(1-|a|^2)}\right) \geq g\left(\frac{1}{c}\frac{1}{1-|a|^2}\right),\ a \in]0,1[. \tag{7.1}$$

For any increasing unbounded function $G\colon [1,\infty[\to [1,\infty[$ of any growth rate, we can create a continuous g, dominated by G, as follows: Let $x_0 = 1$ and $g(x) = 1$ for all $x \in [x_0, x_1]$, where $x_1 > x_0$ is a point where $G(x_1) \geq 1$. On $[x_1, x_1+1]$ we define g to increase to the value $G(x_1+1)$. Define g to be constant for all $x \in [x_1+1, x_2]$, where $x_2 > x_1+1$ is such that $G(x_2)/G(x_1+1) \geq 2$. Continuing this process we obtain a sequence (x_n), with the property that $G(x_{n+1})/G(x_n+1) \geq n+1$ for all n and $([x_n, x_n+1[)_n \cup ([x_n+1, x_{n+1}[)_n$ is a partition of $[1,\infty[$, where g is increasing on the first family of intervals and constant on the latter. Moreover, as $x \geq 1$, the function g is increasing, unbounded and satisfies

$$\frac{g(2x_n)}{g(x_n)} = \frac{g(2x_n)}{g(x_{n-1}+1)} \geq \frac{g(1+x_n)}{g(x_{n-1}+1)} \geq \frac{G(x_n)}{G(x_{n-1}+1)} \geq n$$

for all n. In combination with (7.1), it follows that such a weight, $a \mapsto \frac{v(\hat{\phi}_{1/2}(a))}{v(a)}$ is unbounded on $[0,1[$. The remaining part would be to find a function f such that $\limsup_{|a|\to 1} v(a)\gamma(f,a,2) > 0$ if such exists. It would also be interesting if one could find an analytic function g satisfying the desired properties. Notice that if v increases too fast, then the space consists only of constant functions, in which case composition with an automorphism will not change the function at all. Recall that if $v(a) \gtrsim (1-|a|)^{-(1+\epsilon)}$ for any $\epsilon > 0$, then BMOA_v consists of constant functions.

Conjecture 3. We can drop the assumption in Theorem 1.2 that at least one of the following is true:

(1) $C_\phi \in \mathcal{L}(\mathrm{BMOA}_v)$, and $\mathrm{BMOA}_v \not\subset H^\infty$ or $\psi \in \mathrm{VMOA}_v$,
(2) $\psi C_\phi|_{\mathrm{VMOA}_v} \in \mathcal{L}(\mathrm{VMOA}_v)$.

This extra assumption is only needed in Theorem 5.10 and Theorem 5.7. Being able to remove it would yield a complete characterization of e.g. compactness of $\psi C_\phi \in \mathcal{L}(\mathrm{BMOA}_v)$ for admissible weights.

Conjecture 4. Given an admissible weight v (see beginning of Section 5), all analytic polynomial symbols $\phi\colon \mathbb{D} \to \mathbb{D}$ make C_ϕ bounded.

In section 6, a few examples of polynomial symbols rendering C_ϕ bounded are given.

Conjecture 5. Given an admissible weight v and $X = \mathrm{BMOA}_v$ or $X = \mathrm{VMOA}_v$ such that X is not contained in H^∞, it holds that

$$\sup_{a\in\mathbb{D}} \left\|\delta_{\phi(a)}\right\|_{X^*} \gamma(\psi, a, 2) \lesssim_{v,g,\psi,\phi} \sup_{a\in\mathbb{D}} \left\|\delta_{\phi(a)}\right\|_{X^*} \gamma(\psi, a, 1) \text{ and}$$

and

$$\limsup_{|\phi(a)|\to 1} \left\|\delta_{\phi(a)}\right\|_{X^*} \gamma(\psi, a, 2) \lesssim_{v,g,\psi,\phi} \limsup_{|\phi(a)|\to 1} \left\|\delta_{\phi(a)}\right\|_{X^*} \gamma(\psi, a, 1).$$

If $X \subset H^\infty$ Proposition 2.3 yields the statements above.

Declaration of competing interest

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No data was used for the research described in the article.

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