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Pólya's conjecture for Dirichlet eigenvalues of annuli

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Abstract
We prove Pólya's conjecture for the eigenvalues of the Dirichlet Laplacian on annular domains. Our approach builds upon and extends the methods we previously developed for disks and balls. It combines variational bounds, estimates of Bessel phase functions, refined lattice point counting techniques and a rigorous computer-assisted analysis. As a by-product, we also derive a two-term upper bound for the Dirichlet eigenvalue counting function of the disk, improving upon Pólya's original estimate.

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35P15 (primary), 35P20, 33C10, 11P21, 65G20 (secondary)

Contents

1 | INTRODUCTION AND MAIN RESULTS

1.1 | Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and let

$$\lambda_1(\Omega) \leqslant \lambda_2(\Omega) \leqslant \dots$$

be the eigenvalues of the Dirichlet Laplacian $-\Delta_\Omega^{\mathrm{Dir}}$ in Ω, enumerated with multiplicities. Denote by

$$\mathcal{N}_\Omega^{\mathrm{Dir}}(\lambda) := \#\left\{n \in \mathbb{N} : \lambda_n(\Omega) \leqslant \lambda^2\right\}$$

the corresponding eigenvalue counting function. By Weyl's law, its asymptotics is given by

$$\mathcal{N}_\Omega^{\mathrm{Dir}}(\lambda) = \frac{\mathrm{Area}(\Omega)}{4\pi}\lambda^2 + o\left(\lambda^2\right) \qquad \text{as } \lambda \to +\infty. \tag{1.1}$$

More than 70 years ago, George Pólya [14] conjectured that the leading term of the asymptotics (1.1) is, in fact, a uniform bound on $\mathcal{N}_\Omega^{\mathrm{Dir}}(\lambda)$ for any $\Omega \subset \mathbb{R}^2$: namely that for all $\lambda > 0$,

$$\mathcal{N}_\Omega^{\mathrm{Dir}}(\lambda) < \frac{\mathrm{Area}(\Omega)}{4\pi}\lambda^2. \tag{1.2}$$

Pólya himself later proved this for domains which *tile* the plane [15]. No other planar domains for which Pólya's conjecture holds were known until very recently, when the authors of this paper proved it for disks and finite sectors, see [1]. That paper also gives a more comprehensive account of Pólya's conjecture for the Dirichlet Laplacian and its counterpart in the Neumann case, as well as its higher-dimensional generalisations.

1.2 | Main result

The goal of this article is to extend the proof of Pólya's conjecture to annuli

$$A_r = \left\{(x_1, x_2) \in \mathbb{R}^2 : r^2 < x_1^2 + x_2^2 < 1\right\},$$

with inner radius $r \in (0,1)$ and outer radius one. Note that any annulus is of this form up to rescaling. Note also that $r = 0$ corresponds to the punctured disk which has the same spectrum as the disk (see also [12, Example 2.2.23]), and therefore, is covered by the results of [1].

For brevity, we will denote the eigenvalues of an annulus and its eigenvalue counting function by

$$\lambda_{r,n} := \lambda_n(A_r), \qquad \mathcal{N}_r(\lambda) := \mathcal{N}^{\mathrm{Dir}}_{A_r}(\lambda).$$

Given that $\mathrm{Area}(A_r) = \pi(1 - r^2)$, Pólya's conjecture (1.2) for annuli states that the inequality

$$\mathcal{N}_r(\lambda) < \frac{1 - r^2}{4}\lambda^2 \tag{1.3}$$

holds for all pairs (r, λ) in the parameter space

$$\mathrm{R\Lambda} := \{(r, \lambda) : 0 < r < 1, \lambda > 0\}.$$

It will be often convenient to work with the parameter

$$\mu := r\lambda \in [0, \lambda)$$

instead of r. In terms of this parameter, the equivalent form of Pólya's conjecture for the Dirichlet Laplacian on annuli is that the inequality

$$\mathcal{N}_{\frac{\mu}{\lambda}}(\lambda) < \frac{\lambda^2 - \mu^2}{4} \tag{1.4}$$

holds for all pairs (λ, μ) in the parameter space

$$\mathrm{\Lambda M} := \{(\lambda, \mu) : 0 < \mu < \lambda\}.$$

Our main result is

Theorem 1.1. *Pólya's conjecture for the Dirichlet Laplacian holds for all planar annuli. In other words, inequality (1.3) holds for all* $(r, \lambda) \in \mathrm{R\Lambda}$*, or, equivalently, (1.4) holds for all* $(\lambda, \mu) \in \mathrm{\Lambda M}$.

Note that Theorem 1.1 gives a first example of a non-simply connected planar domain for which Pólya's conjecture has been verified.

Remark 1.2. The eigenvalues of planar annuli have also been investigated in [7] and more recently in [5]. Using number-theoretic techniques, these papers improve remainder estimates in the two-term Weyl asymptotics for annuli. Note that the two-term asymptotics implies that Pólya's conjecture holds for sufficiently large (but unspecified) values of λ. While the present paper shares with [5, 7] the method of relating the eigenvalue count to a lattice point count, our results are in a way complementary, as we focus on accurate eigenvalue bounds rather than asymptotics.

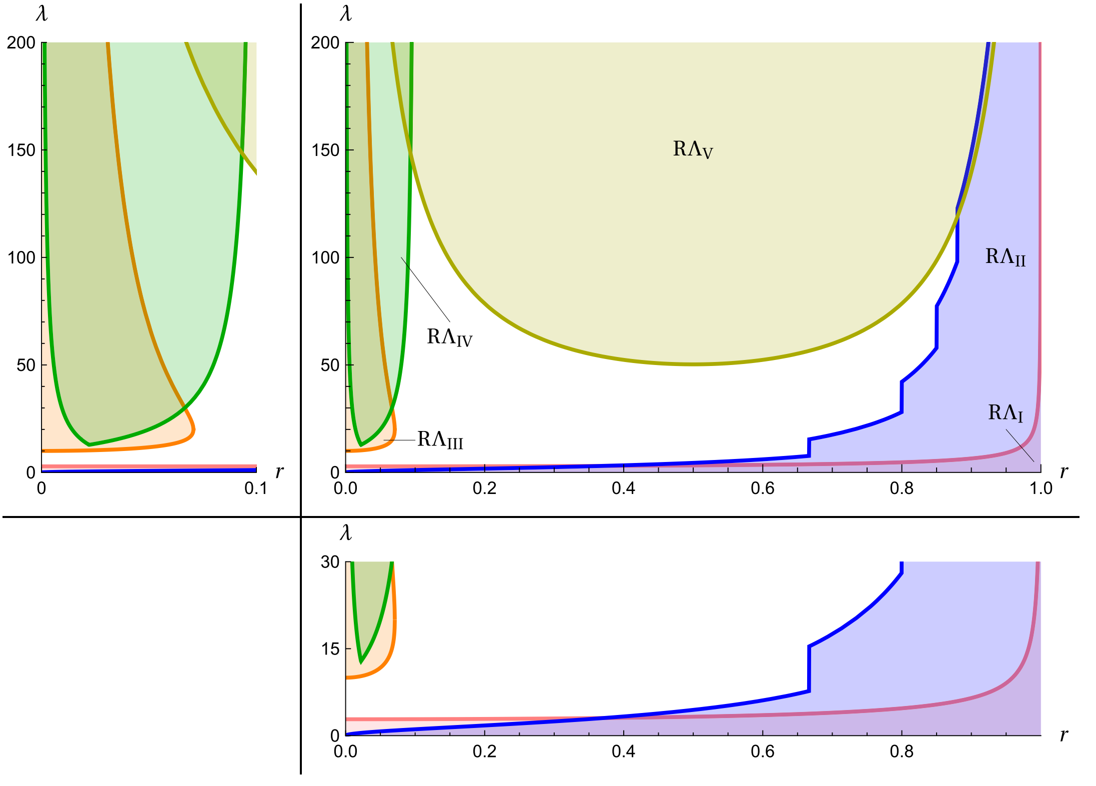

FIGURE 1 Summary of all regions in (r, λ)-plane for which Pólya's conjecture is analytically proven to be true, with two side figures zooming near the axes. The exact definitions of the regions can be found in Theorem 2.1 for $R\Lambda_{\mathrm{I}}$, Theorem 2.3 for $R\Lambda_{\mathrm{II}}$, Theorem 7.1 for $R\Lambda_{\mathrm{III}}$, Theorem 7.2 for $R\Lambda_{\mathrm{IV}}$ and Theorem 7.6 for $R\Lambda_{\mathrm{V}}$.

1.3 | Ideas of the proof

The proof of Theorem 1.1 combines analytic arguments with a rigorous computer-assisted verification. We analytically prove Theorem 1.1 in five distinct but partially intersecting regions $R\Lambda_{\aleph}$, $\aleph = \mathrm{I}, \dots, \mathrm{V}$, of $R\Lambda$. We postpone their exact definitions until later, but show their graphical representation in Figure 1.

Different approaches are used to deal with each of these regions. In the region $R\Lambda_{\mathrm{I}}$, the result follows from the Faber–Krahn and Krahn–Szego isoperimetric inequalities for the first and the second Dirichlet eigenvalues. In the region $R\Lambda_{\mathrm{II}}$, we use a comparison between the counting functions for the annuli and for certain carefully chosen flat cylinders, see Lemma 2.2. We then take advantage of the fact that the eigenvalues of these cylinders can be computed explicitly, see Theorem 2.3. Note that the arguments in the first two regions are of purely spectral-theoretic nature.

The proofs in the remaining three regions are based on results concerning the Bessel phase functions as well as lattice-counting techniques. Here, we extend the approach developed in [1] for the disk. In order to adapt it to the case of annuli, we have to overcome several new technical challenges.

First, instead of estimating the Bessel phase function itself, we now need to bound the difference between the values of the phase functions at two distinct points, which are chosen depending on the spectral parameter and the inner radius of the annulus (recall that the outer radius is fixed and is equal to one). Some of these bounds follow from the results of [3], but some are new, see §5.

Second, we have to use a more elaborate reduction to a lattice counting problem. In the region $R\Lambda_{III}$, we count points below a single curve as in [1]. In the other two regions, in order to get accurate enough bounds, we need to split the interval $[0, \lambda]$ into two (in $R\Lambda_{IV}$) and three (in $R\Lambda_V$) subintervals, respectively, and in each subinterval we count integer points under different curves, depending on which bound for the phase functions we are using (see Lemma 5.2 and Figure 6).

Third, we need to improve upon the lattice counting techniques of [1]. Recall that [1, Theorem 5.1] provided a way to estimate the number of lattice points under a graph of a non-negative decreasing convex function with a Lipschitz constant $\frac{1}{2}$. We need to extend this bound to concave functions, as well as to relax the assumption on the Lipschitz constant. We obtain several results in this direction in §3. We believe that they could be of independent interest, and for illustration we present one of them below.

The following definition is central for our analysis.

Definition 1.3. Let $a, b \in \mathbb{Z}$ with $a < b$, and let $g : [a, b] \to \mathbb{R}$. The *trapezoidal floor sum* of g on $[a, b]$ is defined as

$$\mathbf{T}(g, a, b) = \frac{1}{2}\sum_{m=a}^{b-1}\left(\lfloor g(m)\rfloor + \lfloor g(m+1)\rfloor\right) = \frac{1}{2}\lfloor g(a)\rfloor + \sum_{m=a+1}^{b-1}\lfloor g(m)\rfloor + \frac{1}{2}\lfloor g(b)\rfloor .$$

For a non-negative function g, the trapezoidal floor sum $\mathbf{T}(g, a, b)$ counts the number of points of the lattice $\mathbb{Z} \times \mathbb{N}$ lying on or under the graph of g on the interval $[a, b]$, with the exception that the lattice points with the edge abscissas a and b are counted with the weight $\frac{1}{2}$. A useful observation is that for an integer $p \in (a, b)$,

$$\mathbf{T}(g, a, b) = \mathbf{T}(g, a, p) + \mathbf{T}(g, p, b).$$

For example, we prove the following estimate on the trapezoidal floor sum for concave Lipschitz functions that is used in the region $R\Lambda_V$.

Theorem 1.4. *Let $\alpha, \beta \in \mathbb{Z}$, $\alpha < \beta$. Let g be a decreasing concave function on $[\alpha, \beta]$ which is Lipschitz with a constant $c \in (0, 1)$, and assume additionally that*

$$\lfloor g(\alpha)\rfloor = \lfloor g(p)\rfloor > \lfloor g(p+1)\rfloor$$

for some integer $p \in [\alpha, \beta)$. Then

$$\mathbf{T}(g, \alpha, \beta) \leqslant \int_\alpha^\beta g(z)\,\mathrm{d}z - \frac{1-c}{2}(\beta - p).$$

Note that without the second term on the right, the bound is an easy consequence of concavity only, see Proposition 3.1. The proof of Theorem 1.4 can be found in §3. For some related asymptotic estimates, see also [10, 11].

For decreasing convex functions, we prove a related result under the assumption that the function is Lipschitz with a constant $\frac{1}{2}$ on the whole interval, and *additionally* is Lipschitz with a constant $\frac{1}{3}$ on a subinterval, see Theorem 3.4. One consequence of this is the following improvement of Pólya's conjecture for the Dirichlet Laplacian on the disk [1].

Theorem 1.5. *Let $\mathbb{D}$ be a unit disk. Then the following two-term Pólya-type bound for the Dirichlet Laplacian holds:*

$$\mathcal{N}_{\mathbb{D}}^{\mathrm{Dir}}(\lambda) < \frac{\lambda^2}{4} - \frac{\lfloor \omega_0 \lambda \rfloor}{2}, \tag{1.5}$$

where

$$\omega_0 := \frac{\sqrt{3}}{2\pi} - \frac{1}{6} \approx 0.108998.$$

We prove Theorem 1.5 in § 7.2.

Remark 1.6. While the present paper was under review, estimate (1.5) was further improved in [6, Theorem 1.7] to

$$\mathcal{N}_{\mathbb{D}}^{\mathrm{Dir}}(\lambda) < \frac{\lambda^2}{4} - \frac{\lfloor \omega_0 \lambda \rfloor}{8} - \frac{3\lfloor \omega_1 \lambda \rfloor}{8},$$

where

$$\omega_1 := \frac{\sqrt{2+\sqrt{2}}}{2\pi} - \frac{3\sqrt{2-\sqrt{2}}}{16} \approx 0.150574.$$

The analytic results establishing Pólya's conjecture for the regions $\mathrm{R}\Lambda_{\aleph}$ can be equivalently formulated for the regions $\Lambda\mathrm{M}_{\aleph}$, $\aleph = \mathrm{I}, \dots, \mathrm{V}$, see Figure 2. In fact, such a reformulation is more convenient for the computer-assisted part of the proof of Theorem 1.1.

Set

$$\Lambda\mathrm{M}_{\text{theory}} := \bigcup_{\aleph=\mathrm{I}}^{\mathrm{V}} \Lambda\mathrm{M}_{\aleph}.$$

Importantly, the remaining region

$$\Lambda\mathrm{M} \setminus \Lambda\mathrm{M}_{\text{theory}}$$

is bounded, see Theorem 8.1, and we implement a rigorous computer-assisted algorithm verifying Pólya's conjecture there. Our method builds upon the approach of [1, Section 8]. We show that if a required inequality on the lattice count holds at a given point with some margin, then it holds in a certain explicitly described domain around this point, see Lemma 8.4. This observation, together with the technique of verified rational approximations, allows us to rigorously check Pólya's conjecture in a region $\Lambda\mathrm{M}_{\text{comp}} \supset \Lambda\mathrm{M} \setminus \Lambda\mathrm{M}_{\text{theory}}$. This completes the proof of Theorem 1.1.

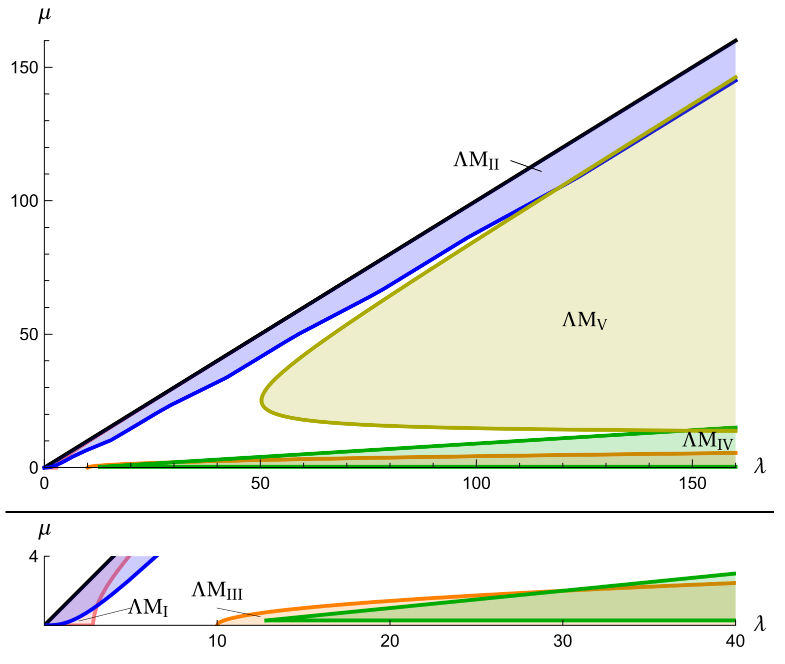

FIGURE 2 Summary of all regions in (λ, μ)-plane for which Pólya's conjecture is analytically proven to be true. The bottom figure zooms near the λ-axis and the origin.

1.4 | Plan of the paper

The paper is organised as follows. In §2, we prove Pólya's conjecture in the region $\mathrm{R}\Lambda_{\mathrm{I}}$ using isoperimetric inequalities for the first two Dirichlet eigenvalues, and in the region $\mathrm{R}\Lambda_{\mathrm{II}}$ via comparison with the spectra of flat cylinders. In §3, we present abstract results on the trapezoidal floor sums for concave and convex functions satisfying different assumptions on their Lipschitz constants. In §4, we relate the eigenvalue counting function of an annulus to the difference of the Bessel phase functions. Bounds on these differences in terms of some elementary functions and a reduction to a lattice counting problem are presented in §5. Some useful auxiliary properties of these elementary functions are collected in §6. In §7, we apply the results on the trapezoidal floor sums and the bounds on the difference of the Bessel phase functions in order to verify Pólya's conjecture in the regions $\mathrm{R}\Lambda_{\mathrm{III}}$, $\mathrm{R}\Lambda_{\mathrm{IV}}$ and $\mathrm{R}\Lambda_{\mathrm{V}}$. This section is the most technical part of the paper. Finally, in §8, we present a rigorous computer-assisted algorithm which affirms Pólya's conjecture in the remaining part of $\mathrm{R}\Lambda$.

2 | REGIONS I AND II VIA ISOPERIMETRIC INEQUALITIES AND COMPARISON WITH FLAT CYLINDERS

By the Faber–Krahn and Krahn–Szego inequalities (see, e.g. [8, 12]), we know that Pólya's conjecture is true for any domain $\Omega \subset \mathbb{R}^d$ for values of λ such that the corresponding Weyl's term satisfies $C_d|\Omega|\lambda^2 \leqslant 2$, which in the case of annulus A_r is equivalent to

$$\lambda^2 - 8 \leqslant \mu^2.$$

Therefore, we arrive at the following

Theorem 2.1. *Let*

$$\eta_{\mathrm{I}}(r) := \sqrt{\frac{8}{1-r^2}}, \qquad \zeta_{\mathrm{I}}(\lambda) := \sqrt{\lambda^2 - 8}.$$

Inequality (1.4) holds for all $(\lambda, \mu) \in \Lambda\mathrm{M}_{\mathrm{I}}$, *where*

$$\Lambda\mathrm{M}_{\mathrm{I}} := \left\{(\lambda, \mu) : 0 < \mu < \lambda \leqslant 2\sqrt{2}\right\} \cup \left\{(\lambda, \mu) : \lambda > 2\sqrt{2},\ \zeta_{\mathrm{I}}(\lambda) \leqslant \mu < \lambda\right\} \subset \Lambda\mathrm{M}.$$

Equivalently, inequality (1.3) holds for all $(r, \lambda) \in \mathrm{R}\Lambda_{\mathrm{I}}$, *where*

$$\mathrm{R}\Lambda_{\mathrm{I}} := \{(r, \lambda) : 0 < r < 1,\ \lambda \leqslant \eta_{\mathrm{I}}(r)\} \subset \mathrm{R}\Lambda.$$

We can extend the result of Theorem 2.1 by a more involved argument. In order to do so, consider first the Dirichlet Laplacian on a flat cylinder $C_h := (1-h, 1) \times \mathbb{S}^1$, with $h > 0$ (cf. [4], where some related results were obtained). An elementary separation of variables shows that the (unordered) eigenvalues of $-\Delta_{C_h}^{\mathrm{Dir}}$ are given by

$$\widetilde{\lambda}_{n,m}(C_h) = m^2 + \frac{\pi^2 n^2}{h^2}, \qquad (n, m) \in \mathbb{N} \times \mathbb{Z}.$$

The corresponding eigenvalue counting function is

$$\widetilde{\mathcal{N}}_h(\lambda) := \mathcal{N}_{C_h}(\lambda) = \#\left\{(n, m) \in \mathbb{N} \times \mathbb{Z} : m^2 + \frac{\pi^2 n^2}{h^2} \leqslant \lambda^2\right\}.$$

Recall that the eigenvalue counting function of the annulus A_r is denoted by $\mathcal{N}_r(\lambda)$. We have the following

Lemma 2.2. *Let* $0 < r < 1$, *and set* $h = h_r := \frac{1-r}{\sqrt{r}}$. *Then*

$$\mathcal{N}_r(\lambda) \leqslant \widetilde{\mathcal{N}}_h(\lambda) \qquad \textit{for all } \lambda > 0.$$

Proof. Let

$$f(\rho, \psi) = \sum_{m \in \mathbb{Z}} f_m(\rho) \mathrm{e}^{\mathrm{i}m\psi} \tag{2.1}$$

be a non-zero function from the Sobolev space $H_0^1(A_r)$, so that $f_m(r) = f_m(1) = 0$. The Rayleigh quotient of the operator $-\Delta_{A_r}^{\mathrm{Dir}}$ acting on f is

$$\mathrm{R}_r[f] = \frac{\sum\limits_{m \in \mathbb{Z}} \int\limits_r^1 \left(\rho |f_m'(\rho)|^2 + \frac{m^2}{\rho} |f_m(\rho)|^2\right) \mathrm{d}\rho}{\sum\limits_{m \in \mathbb{Z}} \int\limits_r^1 |f_m(\rho)|^2 \rho \, \mathrm{d}\rho}. \tag{2.2}$$

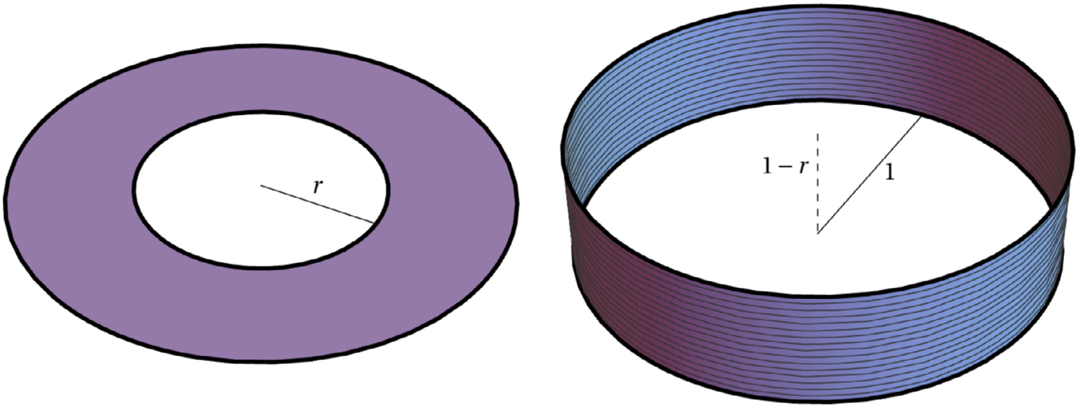

FIGURE 3 An annulus A_r and a cylinder C_{1-r}. Note that we are comparing the spectrum of the Dirichlet Laplacian in A_r with the spectrum of the Dirichlet realisation of (2.3) in C_{1-r}, which, in turn, coincides with the spectrum of the Dirichlet Laplacian in C_{h_r}.

At the same time, the eigenvalues of $-\Delta^{\mathrm{Dir}}_{C_{h_r}}$ coincide with those of the operator

$$-r\frac{\partial^2}{\partial\rho^2} - \frac{\partial^2}{\partial\psi^2} \tag{2.3}$$

acting in $H^1_0(C_{1-r})$ (with the cylindrical coordinates (ρ,ψ) coinciding with the polar coordinates in A_r), see Figure 3.

Note that f is an element of $H^1_0(C_{1-r})$ if and only if it belongs to $H^1_0(A_r)$, and the Rayleigh quotient of the operator (2.3) acting on (2.1) is

$$\widetilde{\mathrm{R}}_r[f] = \frac{\sum\limits_{m\in\mathbb{Z}} \int\limits_r^1 \left(r|f'_m(\rho)|^2 + m^2|f_m(\rho)|^2\right) \mathrm{d}\rho}{\sum\limits_{m\in\mathbb{Z}} \int\limits_r^1 |f_m(\rho)|^2 \,\mathrm{d}\rho}. \tag{2.4}$$

The comparison of the integrands in (2.4) and (2.2) immediately gives $\mathrm{R}[f] > \widetilde{\mathrm{R}}_r[f]$ for any $f \in H^1_0(A_r) \setminus \{0\}$, and the result follows from the variational principle. ☐

Theorem 2.3. *Let*

$$r_0 := 0, \quad r_1 := \frac{2}{3}, \quad r_2 := \frac{4}{5}, \quad r_3 := \frac{17}{20}, \quad r_4 := \frac{22}{25}, \quad r_5 := 1, \tag{2.5}$$

and set

$$\eta_{\mathrm{II}}(r) := \frac{(j+1)\pi\sqrt{r}}{1-r} \quad \textit{if} \quad r_j \leqslant r < r_{j+1}, \qquad j = 0,1,2,3,4, \tag{2.6}$$

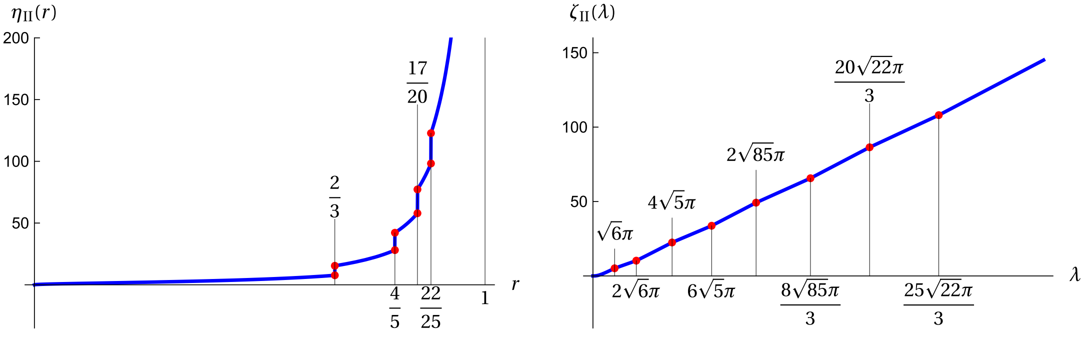

FIGURE 4 The plots of $\eta_{\mathrm{II}}(r)$ and $\zeta_{\mathrm{II}}(\lambda)$. The red dots indicate the positions of singularities in definitions (2.6) and (2.7).

$$\zeta_{\mathrm{II}}(\lambda) := \begin{cases} \lambda - \frac{i\pi}{2\lambda}\left(\sqrt{4\lambda^2 + i^2\pi^2} - i\pi\right) & \text{if} \quad i\pi\frac{\sqrt{r_{i-1}}}{1-r_{i-1}} \leqslant \lambda < i\pi\frac{\sqrt{r_i}}{1-r_i}, \qquad i = 1,2,3,4,5, \\ r_i\lambda & \text{if} \quad i\pi\frac{\sqrt{r_i}}{1-r_i} \leqslant \lambda < (i+1)\pi\frac{\sqrt{r_i}}{1-r_i}, \qquad i = 1,2,3,4, \end{cases} \tag{2.7}$$

see Figure 4.

Inequality (1.3) holds for all $(r, \lambda) \in \mathrm{R}\Lambda_{\mathrm{II}}$*, where*

$$\mathrm{R}\Lambda_{\mathrm{II}} := \{(r,\lambda) : 0 < r < 1,\ 0 < \lambda < \eta_{\mathrm{II}}(r)\}. \tag{2.8}$$

Equivalently, inequality (1.4) holds for all $(\lambda, \mu) \in \Lambda\mathrm{M}_{\mathrm{II}}$*, where*

$$\Lambda\mathrm{M}_{\mathrm{II}} = \{(\lambda,\mu) : \zeta_{\mathrm{II}}(\lambda) < \mu < \lambda\} \subset \Lambda\mathrm{M}. \tag{2.9}$$

Proof. Fix $r \in (0,1)$. With $h = h_r$, we have

$$\widetilde{\mathcal{N}}_{h_r}(\lambda) = \#\left\{(n,m) \in \mathbb{N}\times\mathbb{Z} : n^2 + \frac{(1-r)^2}{r\pi^2}m^2 \leqslant \frac{(1-r)^2}{r\pi^2}\lambda^2\right\}$$

$$= \sum_{n=1}^{\left\lfloor \frac{1-r}{\pi\sqrt{r}}\lambda \right\rfloor} \left(1 + 2\left\lfloor \sqrt{\lambda^2 - \frac{\pi^2 r}{(1-r)^2}n^2} \right\rfloor\right),$$

where we assumed the notational convention $\sum_{n=1}^{0} := 0$. By Lemma 2.2, inequality (1.3) will hold for a given r and λ if we can show that

$$\widetilde{\mathcal{N}}_{h_r}(\lambda) < \frac{1-r^2}{4}\lambda^2. \tag{2.10}$$

Assume, first of all, that $\left\lfloor \frac{1-r}{\pi\sqrt{r}}\lambda \right\rfloor = 0$, that is,

$$\lambda < \frac{\sqrt{r}\pi}{1-r}.$$

Then $\widetilde{\mathcal{N}}_{h_r}(\lambda) = 0$, and (2.10) follows immediately.

Let now $\left\lfloor \frac{1-r}{\pi\sqrt{r}}\lambda \right\rfloor = j \in \mathbb{N}$, that is,

$$j\frac{\sqrt{r}\pi}{1-r} \leqslant \lambda < (j+1)\frac{\sqrt{r}\pi}{1-r}. \tag{2.11}$$

In practice, we will only consider cases when $j \leqslant 4$, but the general scheme may be extended further. Let

$$L_n = \left\lfloor \sqrt{\lambda^2 - \frac{\pi^2 r}{(1-r)^2}n^2} \right\rfloor, \qquad n = 1, \dots, j,$$

so that

$$L_n^2 + \frac{\pi^2 r}{(1-r)^2}n^2 \leqslant \lambda^2 < (L_n+1)^2 + \frac{\pi^2 r}{(1-r)^2}n^2. \tag{2.12}$$

Then

$$\widetilde{\mathcal{N}}_{h_r}(\lambda) = j + 2L_1 + \cdots + 2L_j.$$

In order to effectively obtain some restrictions on r which guarantee that inequality (2.10) holds, we will replace it by a stronger inequality based on the lower bounds on λ^2 from (2.12):

$$\begin{aligned} \frac{1-r^2}{4}\lambda^2 - \widetilde{\mathcal{N}}_{h_r}(\lambda) &= \sum_{n=1}^{j}\left(\tau_n\frac{1-r^2}{4}\lambda^2 - 1 - 2L_n\right) \\ &\geqslant \sum_{n=1}^{j}\left(\tau_n\frac{1-r^2}{4}\left(L_n^2 + \frac{\pi^2 r}{(1-r)^2}n^2\right) - 1 - 2L_n\right) > 0, \end{aligned} \tag{2.13}$$

where

$$(\tau_1, \dots, \tau_j) = \boldsymbol{\tau} \in (0,1]^j \qquad \text{with } \tau_1 + \cdots + \tau_j = 1, \tag{2.14}$$

are some constants to be chosen later. Each summand in the right-hand side of (2.13) is minimised, over $L_n \in \mathbb{R}$, by taking $L_n = \frac{4}{(1-r^2)\tau_n}$. Substituting these values into (2.13) and multiplying the resulting inequality by $4(1-r^2)$, we obtain the inequality

$$S_j(r;\boldsymbol{\tau}) := \left(\sum_{n=1}^{j} n^2\tau_n\right)\pi^2 r(1+r)^2 - 16\left(\sum_{n=1}^{j}\frac{1}{\tau_n}\right) - 4j(1-r^2) > 0. \tag{2.15}$$

For every fixed choice of a vector $\boldsymbol{\tau}$ satisfying constraints (2.14), the cubic (in r) polynomial $S_j(r;\boldsymbol{\tau})$ is monotone increasing for $r \geqslant 0$ with $S_j(0;\boldsymbol{\tau}) < 0$. Therefore, it has a single positive root $r_j^*(\boldsymbol{\tau})$. If we can choose a vector $\boldsymbol{\tau}^+$ and a number $r^+ \in (0,1)$ such that

$$S_j\left(r^+;\boldsymbol{\tau}^+\right) > 0,$$

then $r^+ > r_j^*(\boldsymbol{\tau}^+)$, and so inequality (2.15) with $\boldsymbol{\tau} = \boldsymbol{\tau}^+$, and therefore also inequality (2.10), holds for $r \geqslant r^+$ and for λ satisfying (2.11).

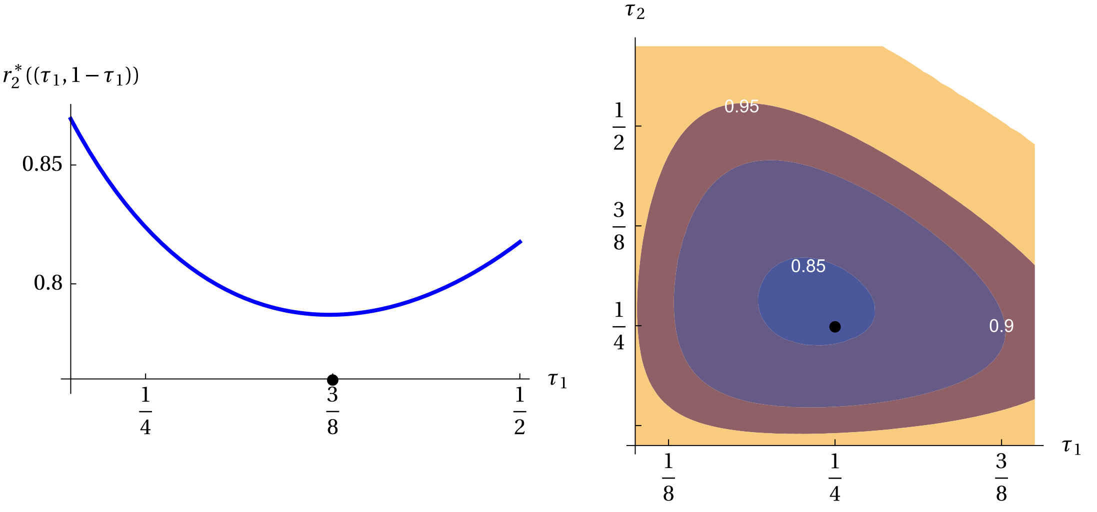

FIGURE 5 On the left, the plot of numerically computed $r_2^*((\tau_1, 1-\tau_1))$ as a function of τ_1. On the right, the numerically computed contour plot of $r_3^*((\tau_1, \tau_2, 1-\tau_1-\tau_2))$ in (τ_1, τ_2)-coordinates.

We now finish the proof of the theorem by looking at the specific cases $j \in \{1, 2, 3, 4\}$. In each case, we choose $r^+ := r_j$, which is given by (2.5).

Case $j = 1$. The only possible choice is $\boldsymbol{\tau}^+ = (1)$, and we get

$$S_1(r_1; \boldsymbol{\tau}^+) = \frac{2}{27}\left(25\pi^2 - 246\right) \approx 0.0548 > 0.$$

Case $j = 2$. We choose $\boldsymbol{\tau}^+ = \left(\frac{3}{8}, \frac{5}{8}\right)$, cf. Figure 5 (left), yielding

$$S_2(r_2; \boldsymbol{\tau}^+) = \frac{23}{750}\left(243\pi^2 - 2320\right) \approx 2.4016 > 0.$$

Case $j = 3$. We choose $\boldsymbol{\tau}^+ = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$, cf. Figure 5 (right), as

$$S_3(r_3; \boldsymbol{\tau}^+) = \frac{535\,279\pi^2 - 5\,226\,560}{32\,000} \approx 1.7635 > 0.$$

Case $j = 4$. We choose, on the basis of some numerical experiments, $\boldsymbol{\tau}^+ = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{5}, \frac{7}{15}\right)$. We get

$$S_4(r_4; \boldsymbol{\tau}^+) = \frac{17\,179\,393\pi^2 - 169\,474\,000}{546\,875} \approx 0.1459 > 0.$$

The combination of all the cases completes the proof of (2.8). It remains to remark that (2.9) is just an explicit representation of $\left\{(\lambda, \mu) : \left(\frac{\mu}{\lambda}, \lambda\right) \in \mathrm{R}\Lambda_{\mathrm{II}}\right\}$, where we have taken extra care of the jumps of $\eta_{\mathrm{II}}(r)$ at $r = r_j$, $j = 1, 2, 3, 4$. $\square$

3 | TRAPEZOIDAL FLOOR SUMS FOR LIPSCHITZ FUNCTIONS AND THEIR BOUNDS

We want to compare, under various conditions on a function g, its trapezoidal floor sum with the integrals $\int_a^b g(x)\,\mathrm{d}x$. Some estimates of this type have been already established in [1] and [2]; however, proving Theorem 1.1 requires more delicate upper bounds for trapezoidal floor sums which we collect here.

We recall that a function $g : [a,b] \to \mathbb{R}$ is called *c-Lipschitz* ($c > 0$) if $|g(z) - g(w)| \leqslant c|z - w|$ for all $z, w \in [a,b]$. We will write this as $g \in \mathrm{Lip}_c$.

The following introductory result is obvious.

Proposition 3.1. *If $g(z)$ is concave, then we always have*

$$\mathbf{T}(g,a,b) \leqslant \int_a^b g(z)\,\mathrm{d}z.$$

Proof. In this case, $\mathbf{T}(g,a,b)$ is trivially less than or equal to the trapezoid rule approximation for the integral of g, which is a lower bound for the true integral. □

A sharper bound for a concave function g can be obtained subject to some additional constraints.

Theorem 3.2. *Let $a, b \in \mathbb{Z}$, $a < b$. Let $c \in (0,1)$. Assume that g is a decreasing concave* Lip_c *function on $[a,b]$ such that*

$$\lfloor g(a) \rfloor > \lfloor g(a+1) \rfloor .$$

Then

$$\mathbf{T}(g,a,b) \leqslant \int_a^b g(z)\,\mathrm{d}z - \frac{1-c}{2}(b-a). \tag{3.1}$$

Proof. By the conditions of the theorem, the function g is strictly monotone on some interval to the left of $a+1$, and therefore, by concavity, also on $[a+1,b]$, and thus is invertible there. In addition,

$$N := \lfloor g(a) \rfloor > \lfloor g(b) \rfloor =: M.$$

Denote

$$a_n = \max\{m \in \mathbb{Z} : g(m) \geqslant n\} = \left\lfloor g^{-1}(n) \right\rfloor, \qquad n = N-1, \dots, M+1.$$

Set also $a_N := a$ and $a_M := b$.

By additivity of the trapezoidal floor sums, we have

$$\mathbf{T}(g,a,b) = \mathbf{T}(g,a_N,a_{N-1}) + \cdots + T(g,a_{M+1},a_M),$$

and it is therefore sufficient to prove the statement of the theorem with (a,b) replaced by (a_n, a_{n-1}), $n = N, \dots, M-1$.

Consider any such interval, and set $A := a_n$, $B := a_{n-1}$, and $k := B - A \in \mathbb{N}$. Without loss of generality, we can assume $n = 1$, otherwise we just need to add the same constant $(n-1)k$ to both sides of (3.1) which does not affect the validity of the inequality. Then $g(A) \geqslant 1 > g(A+1) > \ldots > g(B) \geqslant 0$ and $\mathbf{T}(g, A, B) = \frac{1}{2}$. We therefore need to prove that

$$\int_A^B g(z)\,\mathrm{d}z \geqslant \frac{1}{2} + \frac{k(1-c)}{2} = \frac{k(1-c)+1}{2}. \tag{3.2}$$

By concavity of g and using $g(A) \geqslant 1$, $g(B) \geqslant 0$, we have

$$\int_A^B g(z)\,\mathrm{d}z - \frac{k(1-c)+1}{2} \geqslant \frac{(g(A)+g(B))k}{2} - \frac{k(1-c)+1}{2} \geqslant \frac{k}{2} - \frac{k(1-c)+1}{2} = \frac{ck-1}{2},$$

and therefore (3.2) holds if $k \geqslant \frac{1}{c}$.

Suppose now $k \leqslant \frac{1}{c}$. We have, by the Lipschitz condition on g, the bound $g(B) \geqslant 1 - ck$, and therefore,

$$\int_A^B g(z)\,\mathrm{d}z - \frac{k(1-c)+1}{2} \geqslant \frac{(2-ck)k}{2} - \frac{k(1-c)+1}{2} = \frac{2k - ck^2 + ck - k - 1}{2}$$
$$= \frac{(k-1)(1-ck)}{2} \geqslant 0.$$

Thus, (3.2) holds again. □

We can now easily give the following proof.

Proof of Theorem 1.4. By additivity,

$$\mathbf{T}(g, \alpha, \beta) = \mathbf{T}(g, \alpha, p) + \mathbf{T}(g, p, \beta),$$

and we estimate the first term by Proposition 3.1 with $[a,b] = [\alpha, p]$, and the second term by Theorem 3.2 with $[a,b] = [p, \beta]$. □

We now switch to the bounds for convex decreasing functions g. The next result is essentially a version of [1, Theorem 5.1].

Theorem 3.3. *Let $a, b \in \mathbb{Z}$, and let $g : [a,b] \to \mathbb{R}$ be non-negative, decreasing, convex and of class* $\mathrm{Lip}_{\frac{1}{2}}$. *Suppose also that $g(b)$ is an integer. Then*

$$\mathbf{T}\Big(g + \frac{1}{4}, a, b\Big) \leqslant \int_a^b g(z)\,\mathrm{d}z.$$

Moreover, equality implies that $g(z)$ is a constant on $[a,b]$.

Proof. If $g(b) = 0$, this is an immediate consequence of [1, Theorem 5.1], noting that the last term in the sum is zero and so it does not matter whether it has a factor of $1/2$. Adding an integer n to $g(z)$ increases both sides by $(b-a)n$, and so, equality is maintained. □

The bound in Theorem 3.3 can be improved given more information on the function g.

Theorem 3.4. *Let $a, b \in \mathbb{Z}$, and suppose $g(z)$ is decreasing, convex,* $\mathrm{Lip}_{\frac{1}{2}}$ *on $[a, b]$ and has $g(b) = 0$. Suppose further that there exists $t \in [a, b]$ for which $g(z)$ is* $\mathrm{Lip}_{\frac{1}{3}}$ *on $[t, b]$. Then*

$$\mathbf{T}\left(g + \frac{1}{4}, a, b\right) \leqslant \int_a^b g(z)\,\mathrm{d}z - \frac{1}{4}\lfloor g(t) \rfloor .$$

In order to prove Theorem 3.4, we require the following

Lemma 3.5. *Let $A, B \in \mathbb{Z}$. Suppose that $g(z)$ is decreasing and convex on $[A, B+1]$ with Lipschitz constant $\frac{1}{2}$, and is Lipschitz with constant $\frac{1}{3}$ for $z \geqslant A + 1$. Suppose further that for some $n \in \mathbb{Z}$,*

$$g(A) \geqslant n + 1 \geqslant g(A+1) \geqslant g(B) \geqslant n \geqslant g(B+1).$$

Then

$$\mathbf{T}\left(g + \frac{1}{4}, A, B\right) \leqslant \int_A^B g(t)\,\mathrm{d}t - \frac{1}{4}. \tag{3.3}$$

Proof of Lemma 3.5. Without loss of generality, we assume $A = 0$ and $n = 0$. Let

$$k = \#\{m \in [0, B] \,:\, g(m) \geqslant 3/4\},$$

and consider three cases.

Case 1: $k = 1$. The left-hand side of (3.3) is $\frac{1}{2}$. Since $g(0) \geqslant 1$, we must have $g(z) \geqslant 1 - \frac{z}{2}$ for $z \geqslant 0$. In particular, $g(1) \geqslant \frac{1}{2}$, and therefore, $g(2) \geqslant \frac{1}{6}$, and so $B \geqslant 2$. Thus,

$$\int_0^B g(z)\,\mathrm{d}z \geqslant \int_0^2 \left(1 - \frac{z}{2}\right)\mathrm{d}z = 1,$$

and so the right-hand side of (3.3) is at least $\frac{3}{4} > \frac{1}{2}$ as desired.

Case 2: $k = 2$. The left-hand side of (3.3) is $\frac{3}{2}$. Since $g(1) \geqslant \frac{3}{4}$, we have $g(z) \geqslant \frac{3}{4} - \frac{z-1}{3}$ whenever $z \geqslant 1$. Thus, $g(3) \geqslant \frac{1}{12}$ and $B \geqslant 3$. Now the right-hand side of (3.3) is at least

$$\int_0^3 g(z)\,\mathrm{d}z = \int_0^2 g(z)\,\mathrm{d}z + \int_2^3 g(z)\,\mathrm{d}z.$$

By convexity, the first integral is at least $2g(1) \geqslant \frac{3}{2}$. The second integral, using the bound $g(z) \geqslant \frac{3}{4} - \frac{z-1}{3}$, is at least $\frac{1}{4}$. So, the right-hand side of (3.3) is at least $\frac{3}{2}$, as desired.

Case 3: $k \geqslant 3$. The left-hand side of (3.3) is $k - \frac{1}{2}$. Now we know that $g(k-1) \geqslant \frac{3}{4}$. By convexity, $g(1) + g(2k-3) \geqslant 2g(k-1) \geqslant \frac{3}{2}$. Since $g(1) < 1$, we have $g(2k-3) \geqslant \frac{1}{2}$, and therefore, of course, $B \geqslant 2k - 2$. So,

$$\int_0^B g(z)\,\mathrm{d}z \geqslant \int_0^{2k-2} g(z)\,\mathrm{d}z,$$

which by convexity is at least $(2k-2)g(k-1) \geqslant \frac{3}{2}(k-1) \geqslant k$. Thus, the right-hand side of (3.3) is at least $k - \frac{1}{4} > k - \frac{1}{2}$, completing the proof. ∎

Proof of Theorem 3.4. Let $N = \lfloor g(a) \rfloor$, and for each $n \in [0, N]$, let $q_n = \max\{m \in \mathbb{Z} : g(m) \geqslant n\}$. Then

$$\mathbf{T}\Big(g + \frac{1}{4}, a, b\Big) = \mathbf{T}\Big(g + \frac{1}{4}, a, q_N\Big) + \sum_{n=0}^{N-1} \mathbf{T}\Big(g + \frac{1}{4}, q_{n+1}, q_n\Big).$$

Apply Lemma 3.5 for each n for which $q_{n+1} + 1 \geqslant t$, with $A = q_{n+1}$ and $B = q_n$, and use Theorem 3.3 to estimate each of the other sums. We get the integral we want plus an additional $-\frac{1}{4}$ for every n for which $q_{n+1} + 1 \geqslant t$. Since $g(t) \leqslant g(q_{n+1} + 1) < n + 1$ for any such n, there are at least $\lfloor g(t) \rfloor$ of these values of n, which completes the proof. □

4 | THE EIGENVALUE COUNTING FUNCTION AND THE BESSEL PHASE FUNCTION

Standard separation of variables implies that the eigenfunctions of $-\Delta_{A_r}^{\mathrm{Dir}}$, written in polar coordinates $\rho = \sqrt{x_1^2 + x_2^2}$ and ψ, are of the form

$$\Big(c_1 J_m\Big(\sqrt{\lambda}\rho\Big) + c_2 Y_m\Big(\sqrt{\lambda}\rho\Big)\Big)\mathrm{e}^{\pm \mathrm{i} m\psi}, \qquad m \in \mathbb{N}_0 := \{0\} \cup \mathbb{N},$$

where J_m and Y_m are the Bessel functions of the first and second kind, respectively, and $|c_1|^2 + |c_2|^2 > 0$. If we define the functions

$$L_{r,m}(x) := J_m(x)Y_m(rx) - Y_m(x)J_m(rx), \qquad m \in \mathbb{N}_0, \tag{4.1}$$

and denote the kth positive root of $L_{r,m}(x)$ by $\ell_{r,m,k}$, then the corresponding eigenvalues equal $\ell_{r,m,k}^2$, taken with multiplicity

$$\varkappa_m := \begin{cases} 1 & \text{if } m = 0. \\ 2 & \text{if } m > 0. \end{cases}$$

We recall that the Bessel *modulus* function $M_\nu(z)$ and *phase* function $\theta_\nu(x)$ are defined, for $\nu \geqslant 0$, via

$$J_\nu(x) + \mathrm{i}Y_\nu(x) = M_\nu(x)\mathrm{e}^{\mathrm{i}\theta_\nu(x)}, \qquad \lim_{x \to 0^+} \theta_\nu(x) = -\frac{\pi}{2}. \tag{4.2}$$

We refer to [13, Section 10.18], [9] and [3] for more information on the modulus and phase functions; for the moment, we only need to recall that

$$M_\nu(x) > 0 \quad \text{and} \quad \theta_\nu'(x) > 0 \qquad \text{for all } x > 0,$$

and that

$$\theta_\nu(x) = x - \Big(\frac{\nu}{2} + \frac{1}{4}\Big)\pi + O\Big(\frac{1}{x}\Big) \qquad \text{as } x \to +\infty \tag{4.3}$$

[13, Section 10.18.18]. Using notation (4.2) and definition (4.1), we get

$$L_{r,m}(x) = M_m(x)M_m(rx)(\cos\theta_m(x)\sin\theta_m(rx) - \cos\theta_m(rx)\sin\theta_m(x))$$
$$= -M_m(x)M_m(rx)\sin(\theta_m(x) - \theta_m(rx)).$$

It is easy to see that the difference

$$\Theta_{r,m}(x) := \theta_m(x) - \theta_m(rx) \tag{4.4}$$

is monotonically increasing in x for every $m \geqslant 0$ and every $r \in (0,1)$, with

$$\Theta_{r,m}(0) = 0 \qquad \text{and} \qquad \lim_{x\to\infty}\Theta_{r,m}(x) = \infty,$$

see Lemma 4.1, and therefore, the function $\Theta_{r,m}$ is invertible. Thus, the kth positive root $\ell_{r,m,k}$ of $L_{r,m}(x)$ is given by

$$\ell_{r,m,k} = \Theta_{r,m}^{-1}(\pi k),$$

and the eigenvalue counting function of an annulus can be rewritten as

$$\mathcal{N}_r(\lambda) = \sum_{m=0}^{\infty}\varkappa_m \#\{k\in\mathbb{N} : \ell_{r,m,k}\leqslant\lambda\} = \sum_{m=0}^{\infty}\varkappa_m\left\lfloor\frac{1}{\pi}\Theta_{r,m}(\lambda)\right\rfloor = \sum_{m=0}^{\infty}\varkappa_m\left\lfloor\frac{1}{\pi}(\theta_m(\lambda)-\theta_m(r\lambda))\right\rfloor. \tag{4.5}$$

We will require the following technical results.

Lemma 4.1. *Let $m \geqslant 0$, $0 < r < 1$. Then the function $\Theta_{r,m}(x)$ defined by (4.4) is strictly monotone increasing at any point $x > 0$, and $\Theta_{r,m}(\lambda)\to\infty$ as $\lambda\to\infty$. Moreover,*

$$\Theta_{r,m}(\lambda) < \pi \qquad \textit{for any } \lambda\in(0,m]. \tag{4.6}$$

Proof. We have

$$\Theta'_{r,m}(x) = \theta'_m(x) - r\theta'_m(rx) = \frac{2}{\pi x}\left(\frac{1}{(M_m(x))^2} - \frac{1}{(M_m(rx))^2}\right), \tag{4.7}$$

where the last equality is by [13, §10.18.8]. Nicholson's formula [13, §10.9.30],

$$(M_m(x))^2 = \frac{8}{\pi^2}\int_0^\infty \cosh(2mt)K_0(2x\sinh t)\,\mathrm{d}t,$$

in which $K_0(\cdot)$ is the modified Bessel function of the second kind (which is strictly decreasing on $(0,+\infty)$), ensures that $M_m(x)$ is strictly decreasing in x and therefore the right-hand side of (4.7) is positive.

Also,

$$\Theta_{r,m}(\lambda) = (1-r)\lambda + O\left(\frac{1}{\lambda}\right) \qquad \text{as } \lambda\to\infty$$

by (4.3).

We also have, for $0 < \lambda \leqslant m$,

$$\Theta_{r,m}(\lambda) < \theta_m(\lambda) + \frac{\pi}{2} \leqslant \theta_m(m) + \frac{\pi}{2} < \pi,$$

since

$$-\frac{\pi}{2} < \theta_m(rm) < \theta_m(m) < \theta_m\left(j_{m,1}\right) = \frac{\pi}{2}$$

by monotonicity of θ_m and the classical bound $j_{m,1} > m$. □

Lemma 4.1 immediately implies that $\ell_{r,m,1} = \Theta_{r,m}^{-1}(\pi) > m$, and thus,

Corollary 4.2. *For an integer $m > \lfloor\lambda\rfloor \geqslant 0$,*

$$\#\{k \in \mathbb{N} : \ell_{r,m,k} \leqslant \lambda\} = 0.$$

Therefore, (4.5) can be rewritten as

$$\mathcal{N}_r(\lambda) = \sum_{m=0}^{\lfloor\lambda\rfloor} \varkappa_m \left\lfloor \frac{1}{\pi}\Theta_{r,m}(\lambda)\right\rfloor, \tag{4.8}$$

and the statement of Theorem 1.1 is equivalent to the bound

$$\mathcal{N}_r(\lambda) = \sum_{m=0}^{\lfloor\lambda\rfloor} \varkappa_m \left\lfloor \frac{1}{\pi}\Theta_{r,m}(\lambda)\right\rfloor = \sum_{m=0}^{\lfloor\lambda\rfloor} \varkappa_m \left\lfloor \frac{1}{\pi}(\theta_m(\lambda) - \theta_m(r\lambda))\right\rfloor < \frac{\lambda^2(1-r^2)}{4} \tag{4.9}$$

being valid for all $0 < r < 1$ and all λ.

In the sequel, it will be convenient to interchange the independent variable and the parameter, and to switch to μ instead of r by setting

$$\gamma_{\lambda,\mu}(z) := \frac{1}{\pi}\Theta_{\frac{\mu}{\lambda},z}(\lambda) = \frac{1}{\pi}(\theta_z(\lambda) - \theta_z(\mu)), \tag{4.10}$$

noting that (4.6) in this notation reads

$$\gamma_{\lambda,\mu}(m) < 1 \qquad \text{for all } m \geqslant \lambda, \tag{4.11}$$

and thus rewriting (4.8) as

$$\mathcal{N}_r(\lambda) = \sum_{m=0}^{\lfloor\lambda\rfloor} \varkappa_m \left\lfloor \gamma_{\lambda,\mu}(m)\right\rfloor. \tag{4.12}$$

Remark 4.3. We note that since $\left\lfloor\gamma_{\lambda,\mu}\left(\lfloor\lambda\rfloor + 1\right)\right\rfloor = 0$ by (4.11), the right-hand side of (4.12) can be interpreted as a multiple of the trapezoidal floor sum:

$$\mathcal{N}_r(\lambda) = 2\mathbf{T}\left(\gamma_{\lambda,\mu}, 0, \lfloor\lambda\rfloor + 1\right). \tag{4.13}$$

5 | BOUNDS ON THE PHASE FUNCTIONS DIFFERENCE AND REDUCTION TO A LATTICE COUNTING PROBLEM

We proceed to estimating the values $\Theta_{r,m}(\lambda)$ appearing in the left-hand side of (4.9). Introduce, for $\lambda > 0$ and $\mu \geqslant 0$, the functions

$$G_\lambda(z) := \begin{cases} \frac{1}{\pi}\left(\sqrt{\lambda^2 - z^2} - z \arccos\frac{z}{\lambda}\right), & z \in [0, \lambda], \\ 0, & z > \lambda, \end{cases} \tag{5.1}$$

$$H_\mu(z) := \frac{3\mu^2 + 2z^2}{24\pi(\mu^2 - z^2)^{3/2}}, \qquad z \in [0, \mu), \tag{5.2}$$

$$F_\mu(z) := \begin{cases} \max\left\{G_\mu(z) - H_\mu(z), -\frac{1}{4}\right\}, & z \in [0, \mu), \\ -\frac{1}{4}, & z \geqslant \mu. \end{cases} \tag{5.3}$$

Also, for $0 \leqslant z \leqslant \mu < \lambda$, define

$$\Phi_{\lambda,\mu}(z) := G_\lambda(z) - G_\mu(z). \tag{5.4}$$

Theorem 5.1. *Let $z \geqslant 0$, $\lambda > 0$. Then*

$$F_\lambda(z) - \frac{1}{4} < \frac{1}{\pi}\theta_z(\lambda) < G_\lambda(z) - \frac{1}{4}.$$

Proof. For $\lambda > z$, the result is the re-statement of [3, Theorem 1.4] using notation (5.1)–(5.3). For $0 < \lambda \leqslant z$, the lower bound is the re-statement of the elementary bound $\theta_z(\lambda) > -\frac{\pi}{2}$, and the upper bound follows from monotonicity and continuous differentiability of both $\theta_z(\lambda)$ and $G_\lambda(z)$ in λ since

$$\frac{1}{\pi}\theta_z(\lambda) \leqslant \frac{1}{\pi}\theta_z(z) \leqslant \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\theta_z(z+\varepsilon) \leqslant G_z(z) - \frac{1}{4} = -\frac{1}{4},$$

and $\theta_z'(\lambda) > 0 = G_\lambda'(z)$, see also [1, formula (3.8)]. □

Theorem 5.1 plays the key role in the proof of the following bounds, which are illustrated in Figure 6.

Lemma 5.2. *Let $0 < r < 1$, $\lambda > 0$, $0 \leqslant z \leqslant \lambda$, $\mu = r\lambda$, $\gamma_{\lambda,\mu}(z) = \frac{1}{\pi}\Theta_{r,z}(\lambda)$ as before. Then*

$$\gamma_{\lambda,\mu}(z) < G_\lambda(z) - F_\mu(z). \tag{5.5}$$

In particular,

$$\gamma_{\lambda,\mu}(z) < G_\lambda(z) + \frac{1}{4}, \tag{5.6}$$

and if $z < \mu$, then also

$$\gamma_{\lambda,\mu}(z) < \Phi_{\lambda,\mu}(z) + H_\mu(z). \tag{5.7}$$

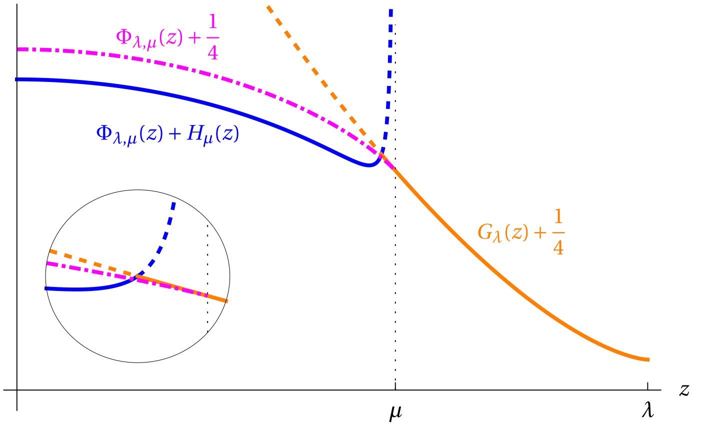

FIGURE 6 A typical behaviour of bounds (5.6) and (5.7), and of the upper bound (5.8), with the solid line showing bound (5.5). The inset zooms near intersections of the curves.

Additionally, if $z \leqslant \mu$, then also

$$\Phi_{\lambda,\mu}(z) < \gamma_{\lambda,\mu}(z) < \Phi_{\lambda,\mu}(z) + \frac{1}{4}. \tag{5.8}$$

Proof of Lemma 5.2. As $\gamma_{\lambda,\mu}(z) = \frac{1}{\pi}(\theta_z(\lambda) - \theta_z(\mu))$ by (4.10), bound (5.5) follows immediately from Theorem 5.1 by applying its upper bound to $\theta_z(\lambda)$ and its lower bound to $\theta_z(\mu)$. By definitions (5.3) and (5.4), the right-hand side of (5.5) becomes

$$G_\lambda(z) - F_\mu(z) = \begin{cases} \min\left\{\Phi_{\lambda,\mu}(z) + H_\mu(z), G_\lambda(z) + \frac{1}{4}\right\}, & z \in [0,\mu), \\ G_\lambda(z) + \frac{1}{4}, & z \geqslant \mu, \end{cases}$$

thus providing (5.6) and (5.7).

In order to prove (5.8), fix $z \geqslant 0$ and consider the function

$$\delta_z(\mu) := G_\mu(z) - \frac{1}{\pi}\theta_z(\mu), \qquad \mu \in [z, +\infty).$$

We have

$$\delta_z(z) = -\frac{1}{\pi}\theta_z(z) \leqslant -\frac{1}{\pi}\theta_z(0) = \frac{1}{2}$$

by the equality $G_z(z) = 0$ and monotonicity of θ,

$$\lim_{\mu\to+\infty} \delta_z(\mu) = \frac{1}{4}$$

by (4.3) and the asymptotics

$$G_\mu(z) = \frac{\mu}{\pi} - \frac{z}{2} + O\left(\frac{1}{\mu}\right) \qquad \text{as } \mu \to +\infty,$$

[1, formula (3.7)], and also

$$\delta_z'(\mu) = \frac{1}{\pi}\left(\frac{\sqrt{\mu^2 - z^2}}{\mu} - \theta_z'(\mu)\right) < 0 \qquad \text{for all } \mu \geqslant z$$

by [9]. Therefore, $\frac{1}{4} < \delta_z(\mu) \leqslant \frac{1}{2}$ for all $\mu \in [z, +\infty)$, and thus,

$$0 < \gamma_{\lambda,\mu}(z) - \Phi_{\lambda,\mu}(z) = \delta_z(\mu) - \delta_z(\lambda) < \frac{1}{4}$$

because $\lambda > \mu$. □

In the next section, we summarise some additional properties of the functions (5.1)–(5.4), which we will use afterwards.

6 | SOME PROPERTIES OF FUNCTIONS G_λ AND $\Phi_{\lambda,\mu}$

Lemma 6.1. *Let* $0 < z < \lambda$. *Then* G_λ *is decreasing and convex on* $[0, \lfloor\lambda\rfloor + 1]$,

$$\int_0^{\lfloor\lambda\rfloor+1} G_\lambda(z)\,\mathrm{d}z = \int_0^{\lambda} G_\lambda(z)\,\mathrm{d}z = \frac{\lambda^2}{8},$$

and also G_λ *is* $\mathrm{Lip}_{\frac{1}{2}}$ *on* $[0, \lfloor\lambda\rfloor + 1]$, *and* $\mathrm{Lip}_{\frac{1}{3}}$ *on* $\left[\frac{\lambda}{2}, \lfloor\lambda\rfloor + 1\right]$, *with*

$$G_\lambda\left(\frac{\lambda}{2}\right) = \omega_0\lambda, \qquad \omega_0 := \frac{\sqrt{3}}{2\pi} - \frac{1}{6} \approx 0.108998 > \frac{1}{10}. \tag{6.1}$$

Proof. The result follows by direct computation and is mostly contained in [1, Lemmas 4.5 and 4.6]. □

Lemma 6.2. *Let* $0 < z < \lambda$. *Then*[†]

$$\frac{3\lambda}{10}\left(1 - \frac{z}{\lambda}\right)^{3/2} < G_\lambda(z) < \frac{\lambda}{3}\left(1 - \frac{z}{\lambda}\right)^{3/2}.$$

Proof. We first prove that for $0 < w < \lambda$,

$$\frac{9}{20}\sqrt{1 - \frac{w}{\lambda}} < -G_\lambda'(w) = \frac{1}{\pi}\arccos\frac{w}{\lambda} < \frac{1}{2}\sqrt{1 - \frac{w}{\lambda}}, \tag{6.2}$$

or, equivalently,

$$\cos\left(\frac{\pi}{2}\sqrt{1 - v}\right) < v < \cos\left(\frac{9\pi}{20}\sqrt{1 - v}\right), \tag{6.3}$$

[†] We do not actually use the lower bound in this paper.

where $v := \frac{w}{\lambda} \in (0,1)$. The left inequality in (6.3) holds since $\cos\left(\frac{\pi}{2}\sqrt{1-v}\right) - v\Big|_{v\in\{0,1\}} = 0$ and

$$\frac{\mathrm{d}^2}{\mathrm{d}v^2}\left(\cos\left(\frac{\pi}{2}\sqrt{1-v}\right) - v\right) = \frac{\pi\cos\left(\frac{\pi}{2}\sqrt{1-v}\right)}{16(1-v)}\left(\frac{2}{\sqrt{1-v}}\tan\left(\frac{\pi}{2}\sqrt{1-v}\right) - \pi\right) > 0$$

for all $v \in (0,1)$. The right inequality holds since $v - \cos\left(\frac{9\pi}{20}\sqrt{1-v}\right)\Big|_{v=1} = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(v - \cos\left(\frac{9\pi}{20}\sqrt{1-v}\right)\right) = 1 - \frac{9\pi\sin\left(\frac{9\pi}{20}\pi\sqrt{1-v}\right)}{40\sqrt{1-v}} > 1 - \frac{81\pi^2}{800} > 0$$

for all $v \in (0,1)$.

Integrating (6.2) in w from z to λ gives the result. □

Lemma 6.2 implies

Corollary 6.3. *Let* $1 \leqslant \mu \leqslant \lambda$. *Then*

$$\int_{\lfloor\mu\rfloor}^{\mu} G_\mu(z)\,\mathrm{d}z = \int_{\lfloor\mu\rfloor}^{\mu} G_\lambda(z)\,\mathrm{d}z - \int_{\lfloor\mu\rfloor}^{\mu} \Phi_{\lambda,\mu}(z)\,\mathrm{d}z < \frac{2}{15\sqrt{\mu}}. \tag{6.4}$$

Proof. The first equality is obvious by the definitions (5.1) and (5.4). In order to estimate the left-hand side, we just integrate the upper bound in Lemma 6.2 with $\lambda = \mu$ from $\mu - 1 < \lfloor\mu\rfloor$ to μ, arriving at the right-hand side of (6.4). □

Lemma 6.4. *Let* $\lambda > \mu > 0$. *Then* $\Phi_{\lambda,\mu}(z)$ *is decreasing, concave and* $\mathrm{Lip}_{c_{\lambda,\mu}}$ *on* $[0,\mu]$, *with*

$$c_{\lambda,\mu} := \frac{1}{\pi}\arccos\frac{\mu}{\lambda} < \frac{1}{2}\sqrt{1-\frac{\mu}{\lambda}} < \frac{1}{2}. \tag{6.5}$$

Moreover, the first three derivatives of $\Phi_{\lambda,\mu}(z)$ *are all negative for* $z \in (0,\mu)$.

Proof. We have

$$\Phi'_{\lambda,\mu}(z) = \frac{1}{\pi}\left(\arcsin\frac{z}{\lambda} - \arcsin\frac{z}{\mu}\right) < 0,$$

$$\Phi''_{\lambda,\mu}(z) = \frac{1}{\pi}\left(\frac{1}{\sqrt{\lambda^2 - z^2}} - \frac{1}{\sqrt{\mu^2 - z^2}}\right) < 0,$$

$$\Phi'''_{\lambda,\mu}(z) = \frac{z}{\pi}\left(\left(\lambda^2 - z^2\right)^{-3/2} - \left(\mu^2 - z^2\right)^{-3/2}\right) < 0.$$

We also have

$$c_{\lambda,\mu} = -\Phi'_{\lambda,\mu}(\mu) = \frac{1}{2} - \frac{1}{\pi}\arcsin\frac{\mu}{\lambda} = \frac{1}{\pi}\arccos\frac{\mu}{\lambda},$$

and the upper bound in (6.5) follows from the upper bound in (6.2). □

We will later use the following.

Corollary 6.5. *Let $\lambda > \mu > 0$. Then*

$$\Phi_{\lambda,\mu}(0) - \Phi_{\lambda,\mu}(z) > \frac{(\lambda-\mu)z^2}{2\pi\lambda\mu}$$

for all $z \in (0,\mu)$.

Proof. We have

$$\Phi'_{\lambda,\mu}(0) = 0, \qquad \Phi''_{\lambda,\mu}(0) = -\frac{\lambda-\mu}{\pi\lambda\mu},$$

and, by Taylor's theorem with the remainder and by Lemma 6.4,

$$\Phi_{\lambda,\mu}(0) - \Phi_{\lambda,\mu}(z) > \frac{z^2}{2}\left(-\Phi''_{\lambda,\mu}(0)\right) = \frac{(\lambda-\mu)z^2}{2\pi\lambda\mu}$$

for all $z \in (0,\mu)$. □

7 | LATTICE POINT COUNT AND THE TRAPEZOIDAL FLOOR SUMS

7.1 | The outline of the scheme

Our approach will be as follows. Recall that by Remark 4.3, we need to prove that

$$\mathcal{N}_r(\lambda) = 2\mathbf{T}\big(\gamma_{\lambda,\mu}, 0, \lfloor\lambda\rfloor + 1\big) < \frac{\lambda^2-\mu^2}{4}.$$

Depending on the values of λ and μ, we will use the upper bounds on $\gamma_{\lambda,\mu}$ obtained in §5 and some further bounds, sometimes applied piecewise, to replace the function $\gamma_{\lambda,\mu}(z)$ by some more explicit function $\widetilde{\gamma}_{\lambda,\mu}(z) \geqslant \gamma_{\lambda,\mu}(z)$. Then we will use the estimates of trapezoidal floor sums from §3 to select the values of λ, μ for which we can guarantee that

$$\frac{\lambda^2-\mu^2}{4} - 2\mathbf{T}\big(\widetilde{\gamma}_{\lambda,\mu}, 0, \lfloor\lambda\rfloor + 1\big) > 0, \tag{7.1}$$

which, in turn, immediately implies (4.9).

We also recall, for further use, that

$$2\left(\int_0^\mu \Phi_{\lambda,\mu}(z)\,\mathrm{d}z + \int_\mu^\lambda G_\lambda(z)\,\mathrm{d}z\right) = 2\left(\int_0^\lambda G_\lambda(z)\,\mathrm{d}z - \int_0^\mu G_\mu(z)\,\mathrm{d}z\right) = \frac{\lambda^2-\mu^2}{4},$$

therefore (7.1) holds if

$$\boldsymbol{\Delta}(\lambda,\mu) := \int_0^\mu \Phi_{\lambda,\mu}(z)\,\mathrm{d}z + \int_\mu^\lambda G_\lambda(z)\,\mathrm{d}z - \mathbf{T}\big(\widetilde{\gamma}_{\lambda,\mu}, 0, \lfloor\lambda\rfloor + 1\big) \tag{7.2}$$

is positive. We will write $\boldsymbol{\Delta}_{\aleph}(\lambda,\mu)$ when estimating $\boldsymbol{\Delta}(\lambda,\mu)$ in region $\Lambda\mathrm{M}_{\aleph}$.

7.2 Region III

We use (5.6) to set

$$\widetilde{\gamma}_{\lambda,\mu}(z) := G_\lambda(z) + \frac{1}{4}, \qquad z \in [0, \lfloor\lambda\rfloor + 1],$$

then

$$2\mathbf{T}\big(\gamma_{\lambda,\mu}, 0, \lfloor\lambda\rfloor + 1\big) \leqslant 2\mathbf{T}\Big(G_\lambda + \frac{1}{4}, 0, \lfloor\lambda\rfloor + 1\Big).$$

Recalling Lemma 6.1, we apply Theorem 3.4 with $g(z) = G_\lambda(z)$, $[a, b] = [0, \lfloor\lambda\rfloor + 1]$, to deduce that

$$2\mathbf{T}\Big(G_\lambda + \frac{1}{4}, 0, \lfloor\lambda\rfloor + 1\Big) < \frac{\lambda^2}{4} - \frac{2\lfloor\omega_0\lambda\rfloor}{4}, \tag{7.3}$$

and (4.9) follows if

$$2\mathbf{\Delta}_{\mathrm{III}}(\lambda,\mu) = \frac{\lambda^2 - \mu^2}{4} - 2\mathbf{T}\Big(G_\lambda + \frac{1}{4}, 0, \lfloor\lambda\rfloor + 1\Big) > \frac{1}{4}\big(2\lfloor\omega_0\lambda\rfloor - \mu^2\big) \geqslant 0.$$

As $\lfloor\omega_0\lambda\rfloor > \omega_0\lambda - 1$, we have, using additionally (6.1),

$$\mathbf{\Delta}_{\mathrm{III}}(\lambda,\mu) > \frac{1}{8}\big(2\omega_0\lambda - 2 - \mu^2\big) > \frac{1}{8}\Big(\frac{\lambda}{5} - 2 - \mu^2\Big),$$

and keeping the right-hand side non-negative, we obtain the following

Theorem 7.1. *Set*

$$\eta_{\mathrm{III},\pm}(r) := \frac{1 \pm \sqrt{1 - 200r^2}}{10r^2}, \qquad \zeta_{\mathrm{III}}(\lambda) := \sqrt{\frac{\lambda}{5} - 2}.$$

Inequality (1.4) holds for all $(\lambda,\mu) \in \Lambda\mathrm{M}_{\mathrm{III}}$, where

$$\Lambda\mathrm{M}_{\mathrm{III}} := \{(\lambda,\mu) : \lambda > 10,\ 0 < \mu \leqslant \zeta_{\mathrm{III}}(\lambda)\} \subset \Lambda\mathrm{M}.$$

Equivalently, inequality (1.3) holds for all $(r,\lambda) \in \mathrm{R}\Lambda_{\mathrm{III}}$, where

$$\mathrm{R}\Lambda_{\mathrm{III}} := \left\{(r,\lambda) : 0 < r < \frac{1}{10\sqrt{2}},\ \eta_{\mathrm{III},-}(r) \leqslant \lambda \leqslant \eta_{\mathrm{III},+}(r)\right\} \subset \mathrm{R}\Lambda. \tag{7.4}$$

Proof. If $\lambda > 10$ and $\mu \leqslant \sqrt{\frac{\lambda}{5} - 2}$, then, obviously, $\mathbf{\Delta}_{\mathrm{III}}(\lambda,\mu) > 0$. Rewriting this condition in terms of r and λ yields (7.4). ☐

We also immediately obtain the improved version of Pólya's conjecture for the disk.

Proof of Theorem 1.5. By [1, Theorem 2.3], re-written using the notation of the present paper,

$$\mathcal{N}_{\mathbb{D}}^{\mathrm{Dir}}(\lambda) \leqslant 2\mathbf{T}\Big(G_\lambda + \frac{1}{4}, 0, \lfloor\lambda\rfloor + 1\Big),$$

and hence inequality (7.3) directly implies (1.5). ☐

7.3 | Sharper bounds: Region IV

Let $N_2 := \lfloor \mu \rfloor$ and $N_3 := \lfloor \lambda \rfloor + 1$. We have, by (5.8) and (5.6),

$$\gamma_{\lambda,\mu}(z) < \widetilde{\gamma}_{\lambda,\mu}(z) := \begin{cases} \Phi_{\lambda,\mu}(z) + \frac{1}{4} & \text{if } z \in [0, N_2], \\ G_\lambda(z) + \frac{1}{4} & \text{if } z \in [N_2, N_3]. \end{cases}$$

Applying Proposition 3.1 with $g = \Phi_{\lambda,\mu}(z) + \frac{1}{4}$, $[a,b] = [0, N_2]$, we obtain

$$\int_0^{N_2} \Phi_{\lambda,\mu}(z)\,\mathrm{d}z - \mathbf{T}\big(\widetilde{\gamma}_{\lambda,\mu}, 0, N_2\big) > -\frac{1}{4}N_2 = -\frac{1}{4}\lfloor \mu \rfloor. \tag{7.5}$$

Assume that

$$\mu \leqslant \frac{\lambda}{2}. \tag{7.6}$$

Applying Theorem 3.4 with $g(z) = G_\lambda(z)$, $[a,b] = [N_2, N_3]$, together with Lemma 6.1, we obtain

$$\int_{N_2}^{N_3} G_\lambda(z)\,\mathrm{d}z - \mathbf{T}\big(\widetilde{\gamma}_{\lambda,\mu}, N_2, N_3\big) > \frac{1}{4}\lfloor \omega_0 \lambda \rfloor. \tag{7.7}$$

Adding (7.5) and (7.7) and using Corollary 6.3, bound (6.1) and definition (7.2), we get

$$\mathbf{\Delta}_{\mathrm{IV}}(\lambda,\mu) > \frac{1}{4}\left(\left\lfloor \frac{\lambda}{10} \right\rfloor - \lfloor \mu \rfloor - \frac{8}{15\sqrt{\mu}}\right), \tag{7.8}$$

and we choose the conditions for the right-hand side to be non-negative.

Theorem 7.2. *Set*

$$\eta_{\mathrm{IV}}(r) := \max\left\{\frac{64}{225r}, \frac{10}{1-10r}\right\} = \begin{cases} \frac{64}{225r} & \textit{if } 0 < r \leqslant \frac{32}{1445}, \\ \frac{10}{1-10r} & \textit{if } \frac{32}{1445} < r < \frac{1}{10}, \end{cases} \tag{7.9}$$

$$\zeta_{\mathrm{IV},-}(\lambda) := \frac{64}{225}, \qquad \zeta_{\mathrm{IV},+}(\lambda) := \frac{\lambda}{10} - 1.$$

Inequality (1.4) holds for all $(\lambda,\mu) \in \Lambda\mathrm{M}_{\mathrm{IV}}$, *where*

$$\Lambda\mathrm{M}_{\mathrm{IV}} := \left\{(\lambda,\mu) : \lambda \geqslant \frac{578}{45}, \zeta_{\mathrm{IV},-}(\lambda) \leqslant \mu \leqslant \zeta_{\mathrm{IV},+}(\lambda)\right\} \subset \Lambda\mathrm{M}.$$

Equivalently, inequality (1.3) holds for all $(r,\lambda) \in \mathrm{R}\Lambda_{\mathrm{IV}}$, *where*

$$\mathrm{R}\Lambda_{\mathrm{IV}} := \left\{(r,\lambda) : 0 < r < \frac{1}{10}, \lambda \geqslant \eta_{\mathrm{IV}}(r)\right\} \subset \mathrm{R}\Lambda.$$

Proof. For $(\lambda,\mu) \in \Lambda\mathrm{M}_{\mathrm{IV}}$, the condition (7.6) holds, $\frac{\lambda}{10} \geqslant \mu + 1$, and $\frac{8}{15\sqrt{\mu}} \leqslant 1$, hence $\mathbf{\Delta}_{\mathrm{IV}}(\lambda,\mu) > 0$. The equivalent statement in terms of r and λ easily follows from the change of variables. □

7.4 | Even sharper bounds: Region V

In what follows, given $\lambda > \mu > 0$, we intend to choose an $s \in (0,1)$ and an integer number N_1 such that

$$0 \leqslant N_1 < N_2 = \lfloor \mu \rfloor < N_3 = \lfloor \lambda \rfloor + 1, \tag{7.10}$$

$$\gamma_{\lambda,\mu}(z) \leqslant \Phi_{\lambda,\mu}(z) + s, \qquad z \in [0, N_1] \tag{7.11}$$

and

$$\left\lfloor \Phi_{\lambda,\mu}(N_1+1) + \frac{1}{4} \right\rfloor = \left\lfloor \Phi_{\lambda,\mu}(N_1) + \frac{1}{4} \right\rfloor - 1. \tag{7.12}$$

Given such a choice, we have a bound

$$\gamma_{\lambda,\mu}(z) < \widetilde{\gamma}_{\lambda,\mu}(z) := \begin{cases} \Phi_{\lambda,\mu}(z) + s & \text{if } z \in [0, N_1], \\ \Phi_{\lambda,\mu}(z) + \frac{1}{4} & \text{if } z \in [N_1, N_2], \\ G_\lambda(z) + \frac{1}{4} & \text{if } z \in [N_2, N_3], \end{cases}$$

where the estimates from above on intervals $[N_1, N_2]$ and $[N_2, N_3]$ follow from (5.8) and (5.6), respectively.

The choice of N_1 *and s, and bounds on the interval* $[0, N_1]$.
Set

$$z_* := \sqrt{\frac{2\pi\lambda\mu}{\lambda - \mu}}.$$

By Corollary 6.5, $\Phi_{\lambda,\mu}(z_*) < \Phi_{\lambda,\mu}(0) - 1$, and therefore

$$\left\lfloor \Phi_{\lambda,\mu}(z_*) + \frac{1}{4} \right\rfloor < \left\lfloor \Phi_{\lambda,\mu}(0) + \frac{1}{4} \right\rfloor.$$

Let

$$N_1 := \max\left\{ m \in [0, z_*) \cap \mathbb{Z} : \left\lfloor \Phi_{\lambda,\mu}(m) + \frac{1}{4} \right\rfloor = \left\lfloor \Phi_{\lambda,\mu}(0) + \frac{1}{4} \right\rfloor \right\},$$

then (7.12) holds. It may happen that $N_1 = 0$, and we will treat this case separately.

Assume for the moment that $z_* < \mu$ and set

$$s := H_\mu(z_*).$$

Lemma 7.3. *Suppose that*

$$\mu > 4\pi \qquad \textit{and} \qquad \lambda > \frac{\mu^2}{\mu - 4\pi}, \tag{7.13}$$

and that N_1 and s are defined as above. Then both bounds (7.10) and (7.11) are true. Additionally, if $N_1 > 0$, then

$$\int_0^{N_1} \Phi_{\lambda,\mu}(z)\,\mathrm{d}z - \mathbf{T}\big(\widetilde{\gamma}_{\lambda,\mu}, 0, N_1\big) > -\frac{1}{3\pi}. \tag{7.14}$$

Proof. By the construction of N_1, we have $N_1 < z_*$. Moreover,

$$\mu^2 - 2z_*^2 = \mu^2 - \frac{4\pi\lambda\mu}{\lambda-\mu} = \frac{\mu}{\lambda-\mu}\big(\lambda(\mu-4\pi)-\mu^2\big) > 0$$

by (7.13), and therefore, $N_1 < z_* < \frac{\mu}{\sqrt{2}}$.

Also, $0 < \mu - 1 < \lfloor\mu\rfloor$, and thus, we will have $N_1 < \lfloor\mu\rfloor$ if $\frac{\mu}{\sqrt{2}} \leqslant \mu - 1$, which is satisfied due to the first inequality (7.13).

Finally, assuming $N_1 > 0$, using bound (5.7) for γ and the monotonicity of the function $H_\mu(z)$, we deduce, for $z \in [0, N_1]$,

$$\begin{aligned}\gamma_{\lambda,\mu}(z) &< \Phi_{\lambda,\mu}(z) + H_\mu(z) < \Phi_{\lambda,\mu}(z) + H_\mu(z_*) = \Phi_{\lambda,\mu}(z) + s \\ &= \Phi_{\lambda,\mu}(z) + \frac{3\mu^2 + 2z_*^2}{24\pi(\mu^2 - z_*^2)^{3/2}} \leqslant \Phi_{\lambda,\mu}(z) + \frac{4\mu^2}{24\pi(\mu^2/2)^{3/2}} = \Phi_{\lambda,\mu}(z) + \frac{\sqrt{2}}{3\pi\mu}\end{aligned}$$

(proving along the way (7.11) and $s \leqslant \frac{\sqrt{2}}{3\pi\mu}$). The function $\Phi_{\lambda,\mu}(z) + s$ is concave for $z \in [0, N_1]$. Applying Proposition 3.1 with $g(z) = \Phi_{\lambda,\mu}(z) + s$, $[a,b] = [0, N_1]$, yields

$$\mathbf{T}\big(\widetilde{\gamma}_{\lambda,\mu}, 0, N_1\big) \leqslant \int_0^{N_1} (\Phi_{\lambda,\mu}(z) + s)\,\mathrm{d}z \leqslant \int_0^{N_1} \Phi_{\lambda,\mu}(z)\,\mathrm{d}z + \frac{\sqrt{2}}{3\pi\mu}N_1 < \int_0^{N_1} \Phi_{\lambda,\mu}(z)\,\mathrm{d}z + \frac{1}{3\pi},$$

and the result follows. □

Remark 7.4. If it happens that $N_1 = 0$, we formally set $\mathbf{T}\left(\widetilde{\gamma}_{\lambda,\mu}, 0, N_1\right) := 0$, and then bound (7.14) remains valid.

Bound on the interval $[N_1, N_2]$.

The function $\Phi_{\lambda,\mu}(z) + \frac{1}{4}$ is decreasing, concave, and $\mathrm{Lip}_{c_{\lambda,\mu}}$ with $c_{\lambda,\mu} < \frac{1}{2}\sqrt{1 - \frac{\mu}{\lambda}}$ on the interval $[0, \mu]$, see Lemma 6.4. As $[0, \mu] \supset [N_1, N_2]$ under the conditions of Lemma 7.5 and $\widetilde{\gamma}_{\lambda,\mu}(z) = \Phi_{\lambda,\mu}(z) + \frac{1}{4}$ for $z \in [N_1, N_2]$, we can apply Theorem 3.2 to obtain

$$\begin{aligned}\int_{N_1}^{N_2} \Phi_{\lambda,\mu}(z)\,\mathrm{d}z - \mathbf{T}\big(\widetilde{\gamma}_{\lambda,\mu}, N_1, N_2\big) &> (N_2 - N_1)\left(\frac{1 - c_{\lambda,\mu}}{2} - \frac{1}{4}\right) \\ &> \frac{N_2 - N_1}{4}\left(1 - \sqrt{1 - \frac{\mu}{\lambda}}\right) > 0.\end{aligned} \tag{7.15}$$

We therefore have the following lemma.

Lemma 7.5. *Assume conditions (7.13). Then bound (7.15) holds. If, additionally, $\mu \geqslant \frac{\lambda}{2}$, then*

$$\int_{N_1}^{N_2} \Phi_{\lambda,\mu}(z)\,\mathrm{d}z - \mathbf{T}\big(\widetilde{\gamma}_{\lambda,\mu}, N_1, N_2\big) > \frac{3-2\sqrt{2}}{8}\mu - \frac{2-\sqrt{2}}{8}. \tag{7.16}$$

Proof. We have, as shown in the proof of Lemma 7.3, $N_2 - N_1 > \mu - 1 - \frac{\mu}{\sqrt{2}}$. Also, if $\mu \geqslant \frac{\lambda}{2}$, then $1 - \sqrt{1 - \frac{\mu}{\lambda}} \geqslant 1 - \frac{1}{\sqrt{2}}$, and (7.16) now follows from (7.15) after minor simplifications. □

Bounds on the interval $[N_2, N_3]$.
We consider the two cases $\mu < \frac{\lambda}{2}$ and $\mu \geqslant \frac{\lambda}{2}$ separately.

In the former case, we act as in §7.3 using Theorem 3.4 to get

$$\int_{N_2}^{N_3} G_\lambda(z)\,\mathrm{d}z - \mathbf{T}\big(\widetilde{\gamma}_{\lambda,\mu}, N_2, N_3\big) > \frac{1}{4}\lfloor \omega_0\lambda \rfloor,$$

which together with Corollary 6.3 and bound (6.1) yields

$$\int_{N_2}^{\mu} \Phi_{\lambda,\mu}(z)\,\mathrm{d}z + \int_{\mu}^{N_3} G_\lambda(z)\,\mathrm{d}z - \mathbf{T}\big(\widetilde{\gamma}_{\lambda,\mu}, N_2, N_3\big) > \frac{1}{4}\left(\left\lfloor \frac{\lambda}{10} \right\rfloor - \frac{8}{15\sqrt{\mu}}\right) \tag{7.17}$$

In the latter case, we use instead Theorem 3.3 to obtain

$$\int_{N_2}^{\mu} \Phi_{\lambda,\mu}(z)\,\mathrm{d}z + \int_{\mu}^{N_3} G_\lambda(z)\,\mathrm{d}z - \mathbf{T}\big(\widetilde{\gamma}_{\lambda,\mu}, N_2, N_3\big) > -\frac{2}{15\sqrt{\mu}}. \tag{7.18}$$

Putting everything together
We now combine the bounds we deduced, subject to appropriate conditions, on intervals $[0, N_1]$, $[N_1, N_2]$ and $[N_2, N_3]$. First of all, we remark that in terms of parameters (r, λ) conditions (7.13) translate as

$$\lambda > \frac{4\pi}{r(1-r)}, \tag{7.19}$$

and this, in turn, implies $\lambda > 16\pi$. Also, the first condition (7.13) implies $\frac{2}{15\sqrt{\mu}} < \frac{1}{15\sqrt{\pi}}$.

Theorem 7.6. *Let*

$$\eta_{\mathrm{V}}(r) := \frac{4\pi}{r(1-r)}, \qquad \zeta_{\mathrm{V},\pm}(\lambda) := \frac{1}{2}\Big(\lambda \pm \sqrt{\lambda(\lambda - 16\pi)}\Big). \tag{7.20}$$

Inequality (1.3) holds for all $(r, \lambda) \in \mathrm{R\Lambda_V}$, *where*

$$\mathrm{R\Lambda_V} := \{(r, \lambda) : 0 < r < 1,\ \lambda > \eta_{\mathrm{V}}(r)\} \subset \mathrm{R\Lambda}.$$

Equivalently, inequality (1.4) holds for all $(\lambda, \mu) \in \mathrm{\Lambda M_V}$, *where*

$$\mathrm{\Lambda M_V} := \big\{(\lambda, \mu) : \lambda > 16\pi,\ \zeta_{\mathrm{V},-}(\lambda) < \mu < \zeta_{\mathrm{V},+}(\lambda)\big\} \subset \mathrm{\Lambda M}.$$

Proof. Assume first, in addition to conditions (7.13), that $r = \frac{\mu}{\lambda} < \frac{1}{2}$. Adding together bounds (7.14), (7.15) and (7.17), and recalling (7.2), we obtain

$$\boldsymbol{\Delta}_{\mathrm{V}}(\lambda,\mu) > -\frac{1}{3\pi} + \frac{1}{4}\left\lfloor\frac{\lambda}{10}\right\rfloor - \frac{2}{15\sqrt{\mu}} > \frac{\lambda}{40} - \left(\frac{1}{3\pi} + \frac{1}{4} + \frac{1}{15\sqrt{\pi}}\right),$$

which is positive whenever $\lambda > 40\left(\frac{1}{3\pi} + \frac{1}{4} + \frac{1}{15\sqrt{\pi}}\right) \approx 15.7486$. As mentioned, this is automatically true by (7.19).

Assume now that $r = \frac{\mu}{\lambda} \geqslant \frac{1}{2}$. Adding together bounds (7.14), (7.16) and (7.18), and recalling once more (7.2), we obtain

$$\boldsymbol{\Delta}_{\mathrm{V}}(\lambda,\mu) > -\frac{1}{3\pi} + \frac{3-2\sqrt{2}}{8}\mu - \frac{2-\sqrt{2}}{8} - \frac{2}{15\sqrt{\mu}} > \frac{3-2\sqrt{2}}{8}\mu - \left(\frac{1}{3\pi} + \frac{2-\sqrt{2}}{8} + \frac{1}{15\sqrt{\pi}}\right),$$

which is positive whenever $\mu > \frac{8}{3-2\sqrt{2}}\left(\frac{1}{3\pi} + \frac{2-\sqrt{2}}{8} + \frac{1}{15\sqrt{\pi}}\right) \approx 10.1153$. As we already imposed the restriction $\mu > 4\pi$, this is automatically satisfied. □

8 | FILLING THE VOID: A COMPUTER-ASSISTED ALGORITHM

It remains to prove (1.4) in the region

$$\Lambda\mathrm{M} \setminus \Lambda\mathrm{M}_{\mathrm{theory}}. \tag{8.1}$$

Theorem 8.1. *The region (8.1) is bounded and is contained in*

$$\Lambda\mathrm{M}_{\mathrm{comp}} := \left\{(\lambda,\mu) : \frac{5}{2} \leqslant \lambda \leqslant 150, 0 \leqslant \mu \leqslant \zeta_{\mathrm{comp}}(\lambda)\right\}, \tag{8.2}$$

where

$$\zeta_{\mathrm{comp}}(\lambda) := \frac{22}{25}\lambda, \tag{8.3}$$

see Figure 7.

Proof. We will show that

$$(\lambda,\mu) \notin \Lambda\mathrm{M}_{\mathrm{comp}} \quad \Rightarrow \quad (\lambda,\mu) \in \Lambda\mathrm{M}_{\mathrm{theory}}$$

by considering several cases.

Case 1: $\lambda < \frac{5}{2}$ We immediately get $(\lambda,\mu) \in \Lambda\mathrm{M}_{\mathrm{I}}$ by Theorem 2.1.

Case 2: $\mu > \zeta_{\mathrm{comp}}(\lambda) = \frac{22}{25}\lambda$ This is the same as $r > \frac{22}{25}$. By (7.20) and (2.6) (with $j = 4$), we have in this case

$$\frac{\eta_{\mathrm{II}}(r)}{\eta_{\mathrm{V}}(r)} = \frac{5r^{3/2}}{4} > \frac{5\left(\frac{22}{25}\right)^{3/2}}{4} = \sqrt{\frac{1331}{1250}} > 1,$$

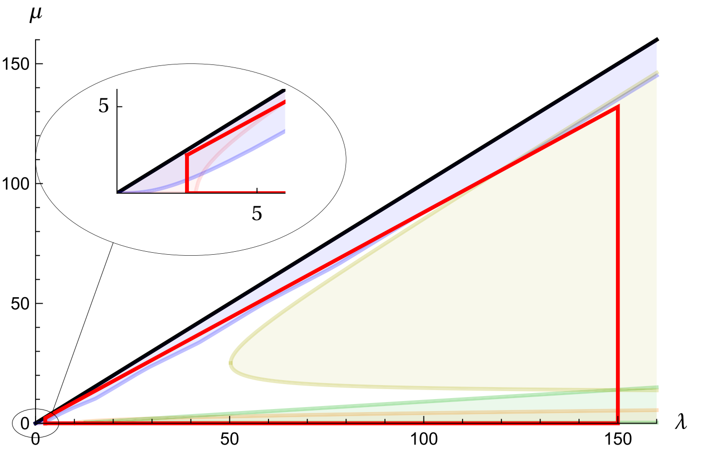

FIGURE 7 The region ΛM_{comp}.

and Theorems 2.3 and 7.6 imply that either $(r, \lambda) \in R\Lambda_{II}$ or $(r, \lambda) \in R\Lambda_V$, or, equivalently, $(\lambda, \mu) \in \Lambda M_{II} \cup \Lambda M_V$.

Case 3: $\lambda > 150,\ 0 < \mu \leqslant \zeta_{comp}(\lambda) = \frac{22}{25}\lambda$ We analyse this case by vertical ordering of the boundaries of individual regions in ΛM_{theory}. We will proceed from bottom to top. First, clearly,

$$\frac{\zeta_{III}(\lambda)}{\zeta_{IV,-}(\lambda)} = \frac{225}{64\sqrt{5}}\sqrt{\lambda - 10} > \frac{225\sqrt{7}}{32} > 1, \qquad \frac{\zeta_{IV,+}(\lambda)}{\zeta_{III}(\lambda)} = \frac{1}{2\sqrt{5}}\sqrt{\lambda - 10} > \sqrt{7} > 1.$$

Second, by (7.9) and (7.20),

$$\frac{\zeta_{IV,+}(\lambda)}{\zeta_{V,-}(\lambda)} = \frac{\left(1 - \frac{10}{\lambda}\right)\left(5\lambda + 5\sqrt{\lambda(\lambda - 16\pi)}\right)}{400\pi},$$

and since the right-hand side is monotone increasing in λ, we have

$$\frac{\zeta_{IV,+}(\lambda)}{\zeta_{V,-}(\lambda)} > \frac{\zeta_{IV,+}(150)}{\zeta_{V,-}(150)} = \frac{7\left(\sqrt{3(75 - 8\pi)} + 15\right)}{60\pi} \approx 1.01126 > 1.$$

Thus, by Theorems 7.1, 7.2 and 7.6, $(\lambda, \mu) \in \Lambda M_{theory}$ whenever $\lambda > 150$ and $0 < \mu < \zeta_{V,+}(\lambda)$, see the left image in Figure 8.

Finally, for $\lambda > 150$,

$$\frac{\zeta_{V,+}(\lambda)}{\zeta_{comp}(\lambda)} = \frac{25}{44}\left(1 + \sqrt{1 - \frac{16}{\pi\lambda}}\right) > \frac{25}{44}\left(1 + \sqrt{1 - \frac{8}{75\pi}}\right) \approx 1.03148 > 1,$$

see the right image in Figure 8, and therefore $0 < \mu \leqslant \zeta_{comp}(\lambda)$ implies $0 < \mu < \zeta_{V,+}(\lambda)$, completing the proof in Case 3. □

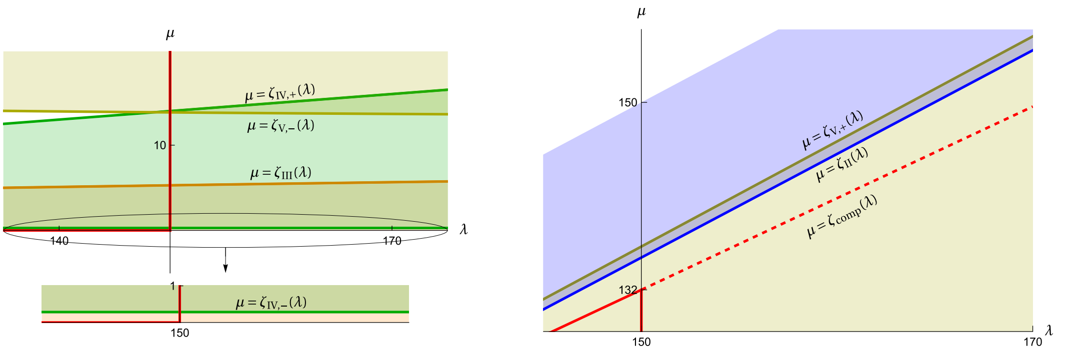

FIGURE 8 The order of the borders of subregions for $\lambda > 150$, bottom part on the left (with an extra zoom inset near $\mu = 0$), and top part on the right.

Let now

$$\mathcal{P}(\lambda, \mu) := 2\mathbf{T}\big(G_\lambda - F_\mu, 0, \lfloor\lambda\rfloor + 1\big),$$

see (5.1)–(5.3). By (5.5) and (4.13),

$$\mathcal{N}_{\mu/\lambda}(\lambda) \leqslant \mathcal{P}(\lambda, \mu),$$

and in order to prove (1.4) for a given $(\lambda, \mu) \in \Lambda\mathrm{M}$, it is enough to show that the inequality

$$\mathcal{P}(\lambda, \mu) < \frac{\lambda^2 - \mu^2}{4} \tag{8.4}$$

holds.

Computer-assisted Theorem 8.2. *Let $(\lambda, \mu) \in \Lambda\mathrm{M}_{\mathrm{comp}}$. Then (8.4) holds.*

Remark 8.3. We note that the region $\Lambda\mathrm{M}_{\mathrm{comp}}$ is much larger than the region (8.1) which we actually need to deal with; however, its shape is determined by the methods we employ. There is, as a result, an element of redundancy in our computer-assisted algorithm, which covers some parts of the regions already covered by analytic proofs.

To start describing the algorithm of proving Theorem 8.2, we note that $\mathcal{P}(\lambda, \mu)$ is non-decreasing in λ and non-increasing in μ: this follows from definitions (5.1)–(5.3) together with

$$\frac{\partial G_\lambda(z)}{\partial \lambda} = \frac{\sqrt{\lambda^2 - z^2}}{\pi\lambda} > 0 \qquad \text{for } 0 < z < \lambda,$$

$$\frac{\partial H_\mu(z)}{\partial \mu} = -\frac{\mu\big(\mu^2 + 4z^2\big)}{8\pi\big(\mu^2 - z^2\big)^{5/2}} < 0 \qquad \text{for } 0 < z < \mu.$$

To make the algorithm work in a finite number of exact calculations, we use the following two simple but important observations.

Lemma 8.4. *Let $0 < \mu_0 < \lambda_0$, and let $p_0 := \mathcal{P}(\lambda_0, \mu_0)$.*

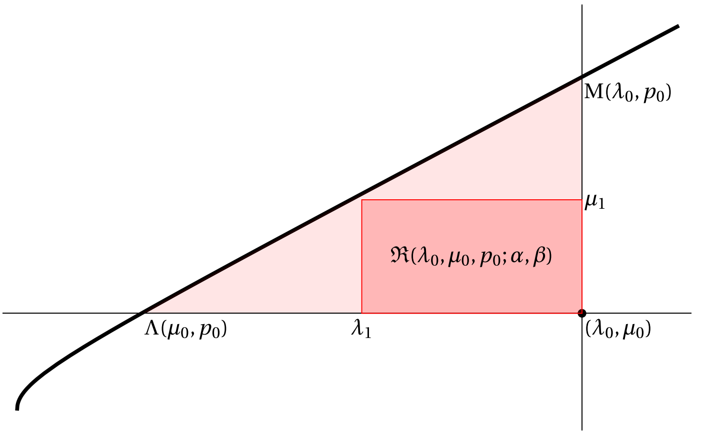

FIGURE 9 The regions (8.5) and (8.6).

(i) *If* $p_0 = 0$*, then (8.4) holds in*

$$\mathfrak{T}(\lambda_0, \mu_0) := \{(\lambda, \mu) : 0 < \lambda \leqslant \lambda_0, \mu_0 \leqslant \mu < \lambda\}.$$

(ii) *If (8.4) holds for* $\lambda = \lambda_0$*,* $\mu = \mu_0$ *with a positive* margin*, that is,*

$$\frac{\lambda_0^2 - \mu_0^2}{4} - p_0 > 0,$$

then it also holds in the following part of the interior of the hyperbola,

$$\left\{(\lambda, \mu) : \Lambda(\mu_0, p_0) := \sqrt{\mu_0^2 + 4p_0} < \lambda \leqslant \lambda_0, \mu_0 \leqslant \mu < \sqrt{\lambda^2 - 4p_0} =: \mathrm{M}(\lambda, p_0)\right\}. \tag{8.5}$$

In particular, it holds inside any rectangle

$$\mathfrak{R}(\lambda_0, \mu_0, p_0; \alpha, \beta) := [\lambda_1, \lambda_0] \times [\mu_0, \mu_1], \tag{8.6}$$

where $\alpha, \beta \in (0, 1)$ *and*

$$\lambda_1 = \lambda_1(\lambda_0, \mu_0, p_0; \alpha) := \alpha\Lambda(\mu_0, p_0) + (1 - \alpha)\lambda_0,$$

$$\mu_1 = \mu_1(\lambda_0, \mu_0, p_0; \alpha, \beta) := \beta\mathrm{M}(\lambda_1, p_0) + (1 - \beta)\mu_0,$$

see Figure 9.

Proof. Statement (i) of the Lemma follows immediately from the monotonicity properties of $\mathcal{P}(\lambda, \mu)$. Again, by monotonicity, we have, for $\lambda \leqslant \lambda_0$ and $\mu \geqslant \mu_0$,

$$\frac{\lambda^2 - \mu^2}{4} - \mathcal{P}(\lambda, \mu) \geqslant \frac{\lambda^2 - \mu^2}{4} - p_0,$$

and this is positive if $\lambda^2 - \mu^2 > 4p_0$, that is, inside (8.5). It is then elementary to check that any rectangle $R(\lambda_0, \mu_0, p_0; \alpha, \beta)$ with the restrictions as stated lies inside (8.5). □

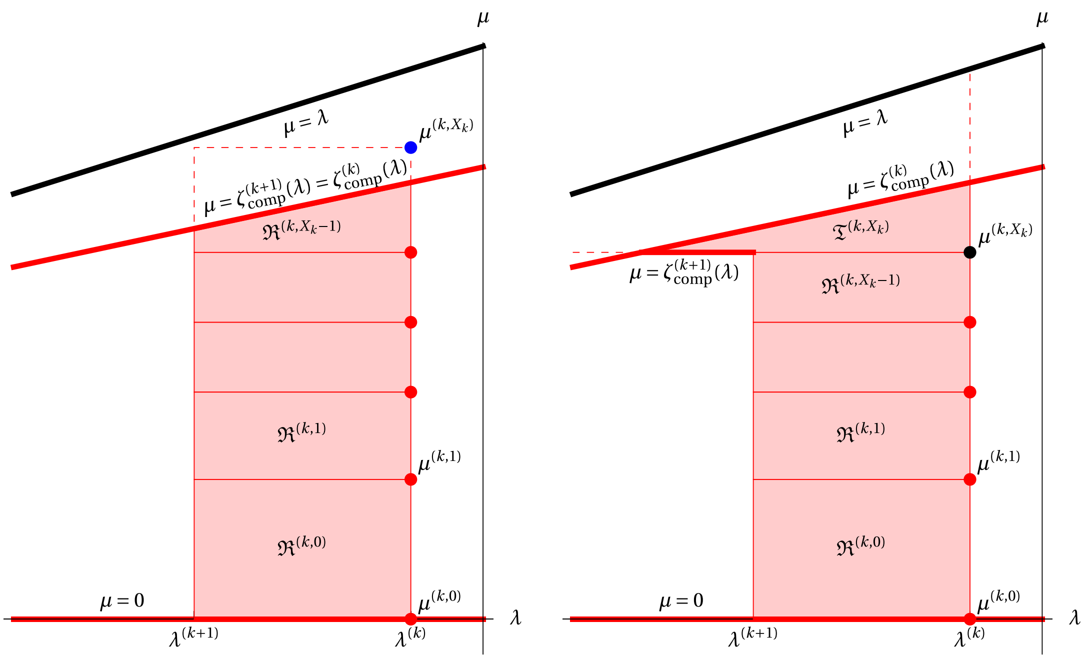

FIGURE 10 The two ways the kth vertical strip may stop: on the left, by reaching a point (blue) outside $\Lambda\mathrm{M}^{(k)}_{\mathrm{comp}}$, and on the right, by reaching a point (black) with the zero trapezoidal floor sum. The image is not to scale.

Lemma 8.4 opens a way of covering the region $\Lambda\mathrm{M}_{\mathrm{comp}}$ by a finite number of polygonal regions, always moving up and to the left: these directions explain the reasons for choosing the larger region for the computer-assisted proof than strictly necessary, see Remark 8.3.

Before describing the algorithm in detail, we also need to address the issue of avoiding floating-point calculations. To do so, we use the method of verified rational approximations we have introduced in [1]. For $x \in \mathbb{R}$, we denote by $\underline{x} \leqslant x \leqslant \overline{x}$, $\underline{x}, \overline{x} \in \mathbb{Q}$, some rational lower and upper bounds for x; see [1, §8] for description of algorithms of finding upper and lower rational approximations of some relevant functions including G_λ. Assuming that $\lambda, \mu \in \mathbb{Q}$, we replace the function $\mathcal{P}(\lambda, \mu)$ by the rational-valued function of rational arguments

$$\overline{\mathcal{P}}(\lambda, \mu) = 2\mathbf{T}\left(\overline{G_\lambda} - \underline{F_\mu}, 0, \lfloor \lambda \rfloor + 1\right) \geqslant \mathcal{P}(\lambda, \mu).$$

We will construct the finite sequence of rational points, $\left(\lambda^{(k)}, \mu^{(k,j)}\right)$, $k = 0, \dots, K$, $j = 0, \dots, X_k$, where the number of steps K and X_k are determined by the algorithm, and the finite sequence of rectangles

$$\mathfrak{R}^{(k,j)} := \left[\lambda^{(k+1)}, \lambda^{(k)}\right] \times \left[\mu^{(k,j)}, \mu^{(k,j+1)}\right] \tag{8.7}$$

(or, occasionally, triangles), each with its rightmost bottom point at $\left(\lambda^{(k)}, \mu^{(k,j)}\right)$, inside which (8.4) holds, and whose union will cover the region $\Lambda\mathrm{M}_{\mathrm{comp}}$.

To start with, set $\beta = \frac{99}{100}$ and $\alpha = \frac{2}{3}$ (these values are chosen empirically following some experiments; varying them does not have a radical effect on the required time of computations). Then set

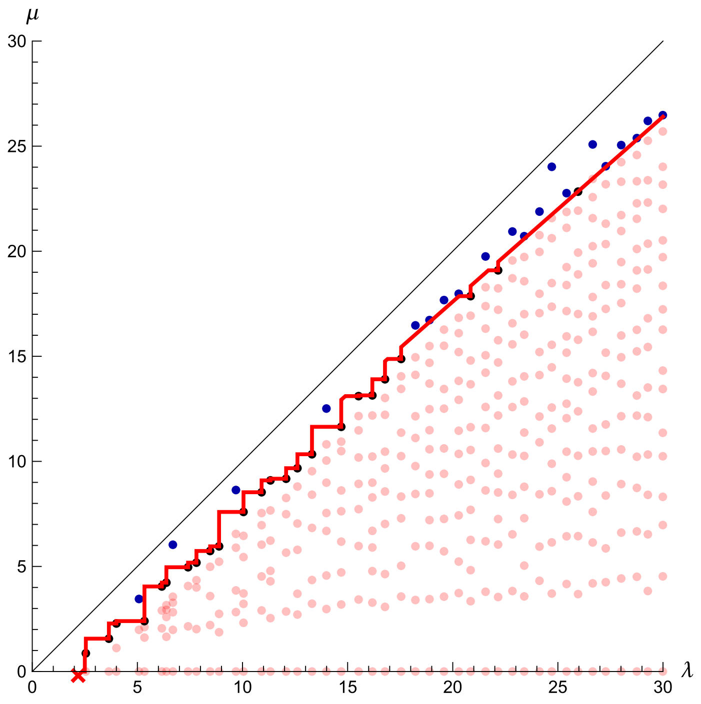

FIGURE 11 The final steps of the calculations. The red line shows the dynamically updated upper boundary of $\Lambda\mathrm{M}^{(k)}_{\mathrm{comp}}$, and the dots are the points where $\overline{\mathcal{P}}$ is evaluated (with blue and black ones as in Figure 10). The cross is at the point $\left(\lambda^{(K)}, 0\right) = \left(\frac{179}{82}, 0\right)$, $K = 227$, where the calculation stops.

$$\Lambda\mathrm{M}^{(0)}_{\mathrm{comp}} := \Lambda\mathrm{M}_{\mathrm{comp}}, \quad \zeta^{(0)}_{\mathrm{comp}}(\lambda) := \zeta_{\mathrm{comp}}(\lambda), \quad \lambda^{(0)} := \max_{(\lambda,\mu)\in\Lambda\mathrm{M}^{(0)}_{\mathrm{comp}}} \lambda = 150, \quad \mu^{(0,0)} := 0,$$

and compute

$$p^{(0,0)} := \overline{\mathcal{P}}\left(\lambda^{(0)}, \mu^{(0,0)}\right) \in \mathbb{Q},$$

$$\lambda^{(1)} := \alpha\sqrt{\mu^{(0,0)2} + 4p^{(0,0)}} + (1-\alpha)\lambda^{(0)} \in \mathbb{Q}, \quad \mu^{(0,1)} := \beta\sqrt{\lambda^{(1)2} - 4p^{(0,0)}} + (1-\beta)\mu^{(0,0)} \in \mathbb{Q},$$

and (temporarily) define the rectangle $\mathfrak{R}^{(0,0)}$ by (8.7) with $k = j = 0$. Lemma 8.4, adjusted to rational approximations, then guarantees that (8.4) holds for $(\lambda, \mu) \in \mathfrak{R}^{(0,0)}$.

We now compute a sequence of rectangles $\mathfrak{R}^{(0,j)}$ on top of the first one, by setting on the jth vertical step,

$$p^{(0,j)} := \overline{\mathcal{P}}\left(\lambda^{(0)}, \mu^{(0,j)}\right), \qquad j = 1, 2, \dots,$$

$$\lambda^{(1)}_{\mathrm{old}} := \lambda^{(1)}, \qquad \lambda^{(1)}_{\mathrm{temp}} := \alpha\sqrt{\mu^{(0,j)2} + 4p^{(0,j)}} + (1-\alpha)\lambda^{(0)},$$

$$\lambda^{(1)} = \lambda^{(1)}_{\mathrm{new}} := \max\left\{\lambda^{(1)}_{\mathrm{temp}}, \lambda^{(1)}_{\mathrm{old}}\right\}$$

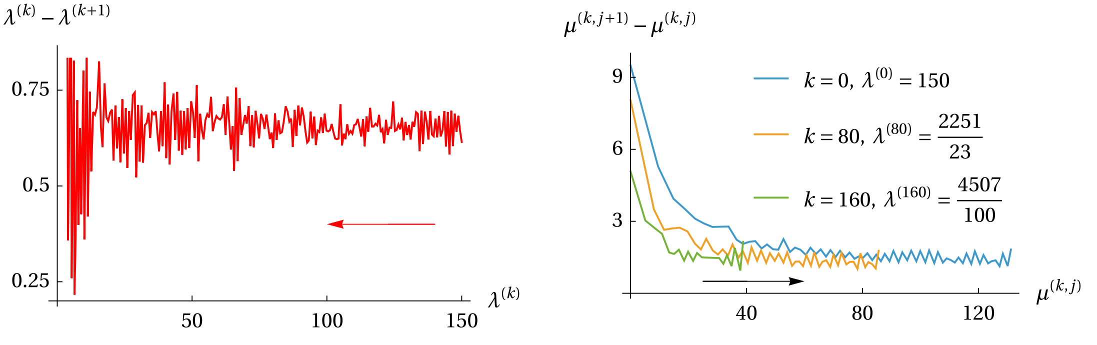

FIGURE 12 On the left, the width of the kth vertical strip plotted as a function of its starting abscissa $\lambda^{(k)}$. On the right, the heights of the rectangles $\mathfrak{R}^{(k,j)}$ plotted as functions of $\mu^{(k,j)}$ for some selected ks. The arrows indicate the directions of computations.

(i.e. we always choose the minimal width of all previously and currently computed rectangles in this vertical strip: if $\lambda^{(1)}_{\text{new}} > \lambda^{(1)}_{\text{old}}$, we also redefine the previously defined rectangles $\mathfrak{R}^{(0,j')}$, $j' = 0, \dots, j-1$, using (8.7)), and

$$\mu^{(0,j+1)} := \beta\sqrt{\lambda^{(1)2} - 4p^{(0,j)}} + (1-\beta)\mu^{(0,j)}.$$

The same argument using Lemma 8.4 guarantees that (8.4) holds for $(\lambda, \mu) \in \mathfrak{R}^{(0,j)}$.

The strip is finished in one of the two ways: either for some $j = X_0$ we get outside the region, $\left(\lambda^{(0)}, \mu^{(0,X_0)}\right) \notin \Lambda\mathrm{M}^{(0)}_{\text{comp}}$, or we get the zero count, $p^{(0,X_0)} = 0$. In both cases, we can now remove the vertical strip (or rectangle)

$$\left\{(\lambda,\mu) : \lambda^{(1)} \leqslant \lambda \leqslant \lambda^{(0)}\right\} = \bigcup_{j=0}^{X_0-1} \mathfrak{R}^{(0,j)}$$

from $\Lambda\mathrm{M}_{\text{comp}}$, and in the latter case, we can additionally remove the triangle $\mathfrak{T}^{(0,X_0)} = \left\{(\lambda,\mu) : \mu \geqslant \mu^{(0,X_0)}\right\}$, see Figure 10. In essence we redefine (8.2), (8.3) as

$$\Lambda\mathrm{M}^{(1)}_{\text{comp}} := \left\{(\lambda,\mu) : \frac{5}{2} \leqslant \lambda \leqslant \lambda^{(1)}, 0 \leqslant \mu \leqslant \zeta^{(1)}_{\text{comp}}(\lambda)\right\},$$

with

$$\zeta^{(1)}_{\text{comp}}(\lambda) := \min\left\{\zeta^{(0)}_{\text{comp}}(\lambda), \mu^{(0,X_0)}\right\}.$$

We can now proceed to construct another vertical strip, starting from the point $\left(\lambda^{(1)}, \mu^{(1,0)}\right)$, with $\mu^{(1,0)} := 0$, and so on, with

$$\Lambda\mathrm{M}^{(k+1)}_{\text{comp}} := \left\{(\lambda,\mu) : \frac{5}{2} \leqslant \lambda \leqslant \lambda^{(k+1)}, 0 \leqslant \mu \leqslant \zeta^{(k+1)}_{\text{comp}}(\lambda)\right\}, \quad \zeta^{(k+1)}_{\text{comp}}(\lambda) := \min\left\{\zeta^{(k)}_{\text{comp}}(\lambda), \mu^{(0,X_k)}\right\},$$

until on some step we reach $\lambda^{(K)} < \frac{5}{2}$ and have therefore covered all of $\Lambda\mathrm{M}_{\text{comp}}$.

The final stages of the calculations are illustrated in Figure 11. Altogether, we complete $K = 227$ columns, evaluating the trapezoidal floor sums 8473 times; the `Mathematica` script running process required slightly more than 2 min of CPU time on a standard Mac laptop.

Some further details on the progress of the computer-assisted algorithm are shown in Figure 12. Full data are available online, see Data Availability Statement.

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DATA AVAILABILITY STATEMENT

The accompanying `Mathematica` script, its printout, and data files are available for download at https://www.michaellevitin.net/polya.html#annuli or at https://github.com/michaellevitin/Polya.

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