$\varrho \rightarrow 4\pi$ in chirally symmetric models

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The decays $\rho^0 \to 2\pi^+2\pi^-$ and $\rho^0 \to 2\pi^0\pi^+\pi^-$ are studied using various effective Lagrangians for π and ρ (and in some cases a_1) mesons, all of which respect the approximate chiral symmetry of the strong interaction. Partial widths of the order of 1 keV or less are found in all cases. These are an order of magnitude smaller than recent predictions based on non-chiral models.

There have been recent hopes [1,2] that the rare decays $\rho^0 \to 2\pi^+2\pi^-$ and $\rho^0 \to 2\pi^0\pi^+\pi^-$ might be observable in the near future in experiments at high-luminosity e^+e^- machines, such as VEPP-2M [3] or DAΦNE [4]. These decays might test various aspects of the low-energy effective Lagrangians that have been proposed for the interactions of ρ-mesons and pions [5–7]. For example, the $2\pi^0\pi^+\pi^-$ mode is sensitive to the 3ρ coupling that appears in theories with gauge-type couplings of the ρ. Current experimental limits on the partial widths for the decays are 30 keV for the $2\pi^+2\pi^-$ mode [8] and 6 keV for the $2\pi^0\pi^+\pi^-$ mode [9]. These are already sufficient to rule out some earlier estimates which did not incorporate chiral symmetry, for example one dominated by πa_1 and πa_2 intermediate states [10].

Bramon, Grau and Pancheri [1] studied the $2\pi^+2\pi^-$ decay with two commonly-used formalisms for including the ρ meson in low-energy effective Lagrangians. For the simplest "hidden-gauge" theory of Bando *et al.* [11,12] they obtained a partial width of 7.5 ± 0.8 keV. In contrast, for a simplified "massive Yang-Mills" theory, where the ρ is coupled to a sigma model as an SU(2)$_V$ gauge boson [13,5], they obtained 60 ± 7 keV. This suggested that the process could distinguish between these approaches, and indeed that the massive Yang-Mills one is already ruled out by the data.

More recently, Eidelman, Silagadze and Kuraev [2] have pointed out that the massive Yang-Mills model of Ref. [1] does not respect chiral symmetry. One possibility is to use gauge couplings corresponding to the full $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L$ symmetry by introducing the a_1 as the chiral partner of the ρ [13,5]. An alternative, and the one followed in Ref. [2], is that proposed by Brihaye, Pak and Rossi [14] where additional terms are added to the naive π-ρ Lagrangian in order to preserve low-energy theorems. Including 4π and $\rho 4\pi$ correction terms, the authors of Ref. [2] obtained partial widths of 16 ± 1 keV for the $2\pi^+ 2\pi^-$ mode and 6.0 ± 0.2 keV for the $2\pi^0 \pi^+ \pi^-$ mode, consistent with the present experimental limits and within reach of future experiments.

Although the Lagrangian of Ref. [2] is constructed to satisfy vector-meson dominance and a range of low-energy theorems for $\pi\pi$ scattering and the couplings of photons, it is still not chirally symmetric. A Yang-Mills Lagrangian for pions and ρ mesons only contains a momentum-independent $2\pi 2\rho$ coupling of the form

$$\frac{1}{2} g_{\rho\pi\pi}^2 (\underline{\pi} \times \underline{\rho}_\mu)^2. \tag{1}$$

This leads to a π-ρ scattering amplitude that does not vanish at threshold in the chiral limit, and so is in conflict with low-energy theorems of chiral symmetry [15]. One can add further terms to such a Lagrangian to cancel out this unwanted behaviour, as described in Ref. [7]. It is, however, straightforward to work with a manifestly chiral approach from the start. We have therefore calculated the $\rho^0 \to 4\pi$ decay rates using several such Lagrangians.

The low-energy interactions among pions can be described by a chirally symmetric sigma model. There is, however, considerable freedom in extending such a description to incorporate spin-1 mesons [5–7]. We have examined three such approaches: i) the hidden-gauge theories of Bando *et al.* [11,12] in which a local SU(2) symmetry is introduced into the non-linear sigma model and the ρ is treated as the gauge boson of this symmetry; ii) the massive Yang-Mills theories in which ρ and a_1 fields transform linearly under the chiral symmetry;

iii) the approach suggested by Weinberg [16] and developed by Callan, Coleman, Wess and Zumino [17] (CCWZ) in which the spin-1 mesons transform homogeneously under a non-linear realisation of chiral symmetry. One should remember that, in general, all of these are equivalent and any Lagrangian of one approach can be converted into a corresponding Lagrangian of another [5–7].

If we work at tree level, the relevant diagrams are shown in Figure 1. Including anomalous terms in the effective Lagrangian leads to additional contributions, similar to those of 1(f) and 1(g) but involving an intermediate ω meson. In all cases we include a symmetry-breaking term of the form

$$\frac{f_\pi^2}{4} m_\pi^2 Tr(U + U^\dagger), \tag{2}$$

where $U = \exp(i\underline{\tau}.\underline{\pi}/f_\pi)$. This term produces the non-zero pion mass. It also has the effect of modifying the 4π vertex in diagram 1(b). This is quantitatively important for our results, since they are much smaller that those of Refs. [1,2], but it does not affect their qualitative features.

First, we consider the simplest example of a hidden-gauge theory, defined by the Lagrangian of Bando *et al.* [11]. To allow for contributions from the anomalous part of the Lagrangian, we introduce the ω meson by enlarging the gauge group to $U(2)_V$. There are then three possible gauge-covariant anomalous terms, with undetermined coefficients. We adopt the suggestion of Ref. [18] with regard to these coefficients, including an $\omega\rho\pi$ vertex but no $\omega 3\pi$ contact term. The parameters of this model satisfy the relations

$$m_\rho^2 = a g^2 f_\pi^2, \qquad g_{\rho\pi\pi} = \frac{a}{2} g. \tag{3}$$

The choice $a = 2$ yields the KSRF relation [19], universal coupling of the ρ [20] and allows vector dominance to be realised in the $\gamma\pi\pi$ coupling. In our calculations we take the following values for the parameters:

$$f_\pi = 92.4\text{MeV}, \quad m_\pi = 139.6\text{MeV}, \quad m_\rho = m_\omega = 770\text{MeV}, \quad a = 2, \quad g = 5.89. \tag{4}$$

The amplitudes for the $\rho^0 \to 4\pi$ decays are straightforward to derive, but the final expressions are rather long and so we do not present them here. Having found them, we must integrate over phase space to obtain the corresponding partial widths. We express the integrals in terms of the variables of Ref. [21] and evaluate them numerically using the NAG routine D01FDF, which maps the region of integration onto an n-dimensional sphere and uses the method of Sag and Szekeres [22] to perform the integration. With a suitable choice of the parameter controlling the mapping onto the sphere, we find that about 50,000 integration points are sufficient to give the integrals to an accuracy of one part in a thousand.

The results obtained in this model are shown in the first line of Table 1, labelled HG. They are roughly an order of magnitude smaller than any of the results given in Refs. [1,2]. Bramon *et al.* [1] have calculated the process $\rho^0 \to 2\pi^+2\pi^-$ in the same model and they quote a value of 7.5 ± 0.8keV, in sharp contrast with our result. The crucial difference between the calculations lies in the strength of the direct $\rho 4\pi$ coupling. Bramon *et al.* assumed that the expression for this vertex was the same as in a massive Yang-Mills model, specifically the $\rho 4\pi$ term of

$$-ig_{\rho\pi\pi}\frac{f_\pi^2}{2}\mathrm{Tr}[\rho^\mu(U^\dagger\partial_\mu U + U\partial_\mu U^\dagger)] = g_{\rho\pi\pi}\left(1 - \frac{1}{3f_\pi^2}\underline{\pi}^2 + \cdots\right)\underline{\rho}_\mu.\underline{\pi}\times\partial^\mu\underline{\pi}. \tag{5}$$

In fact, the corresponding term in the hidden-gauge model should be written, in the unitary gauge, as

$$-2iagf_\pi^2\mathrm{Tr}[\rho^\mu(u^\dagger\partial_\mu u + u\partial_\mu u^\dagger)] = \frac{a}{2}g\left(1 - \frac{1}{12f_\pi^2}\underline{\pi}^2 + \cdots\right)\underline{\rho}_\mu.\underline{\pi}\times\partial^\mu\underline{\pi}. \tag{6}$$

where u is the square root of U. Although both expressions yield the same $\rho\pi\pi$ coupling, the $\rho 4\pi$ terms differ by a factor of four. Hence one cannot take the latter coupling to be the same as in a massive Yang-Mills model. Reducing the contribution of diagram 1(a) by a factor of four has a large effect on the total amplitude, explaining the difference between the result of Ref. [1] and ours.[1]

[1]Comparison of the amplitudes given by Refs. [1,2] also indicates a discrepancy in the evaluation of diagram 1(b). Our evaluation concurs with that of Ref. [2]. However, the impact of this error is fairly small.

We have examined the importance of the anomalous processes for our results. Keeping only the non-anomalous contributions leads to the result for $\rho^0 \to 2\pi^0\pi^+\pi^-$ labelled HGNA, showing that the two types of term are of similar importance. We have also looked at the effect of omitting the symmetry-breaking 4π interaction of Eq. (2) (but still keeping the physical pion mass in the propagator etc.). The results, labelled HGCS, indicate that this term is a smaller, but still significant, contribution. As a simple estimate of contributions beyond tree level, we have examined the effect of including the finite width of the ρ in its propagator (as in Ref. [2]). We find that this does not alter the results significantly.

We now turn to the massive Yang-Mills type of theory, so named because the spin-1 mesons are introduced as if they were gauge bosons of chiral symmetry. Local chiral symmetry is broken by the mass terms for those mesons. As mentioned above, chiral symmetry requires that the a_1 be included as the chiral partner of the ρ in these models. We start with the simplest example of such a Lagrangian, which is just a version of that in Refs. [23,24] but using a non-linear rather than a linear sigma model. In this minimal model the ρ and a_1 masses satisfy Weinberg's relation, $m_{a_1} = \sqrt{2}m_\rho$ [25]. Note that, in this approach, chiral symmetry requires strong cancellations among the contributions to any amplitude. One cannot replace by hand, say, the a_1 mass by its physical value in such a calculation without violating chiral symmetry. The constraints imposed by the symmetry mean also that one must not omit the a_1 field from this approach without introducing compensating terms that maintain chiral low-energy theorems [14,7].

A feature of this approach is the appearance of a π-a_1 mixing term in the Lagrangian. To remove this and diagonalise the free-field part of the Lagrangian, one can subtract a term proportional to $\partial_\mu \underline{\pi}$ from the axial field and renormalise the pion field. This constitutes a minimal diagonalization procedure and leads to physical fields with complicated chiral-transformation properties, but it is sufficient for our present purposes. The procedure generates various additional three- and four-point interactions arising from the gauge-invariant kinetic energies of the spin-1 fields. These include a $\rho\pi\pi$ interaction of order $\mathcal{O}(p^3)$ which reduces the $\rho\pi\pi$ coupling strength at m_ρ by a factor of $\frac{3}{4}$ compared to its value at zero momentum. The minimal model is thus unable to give a good description of the ρ and a_1 masses and widths. To remove this deficiency, the model must be supplemented with additional terms [13,5], a point we return to below.

For the moment though, we consider the minimal model with the physical value of m_ρ and $g = g_{\rho\pi\pi} = 5.89$. We extend the model to include the ω meson, taking the relevant anomalous $\omega\rho\pi$ and $\omega 3\pi$ vertices from Ref. [13]. The results for $\rho^0 \to 4\pi$ are shown in Table 1, labelled MMYM. They are similar in magnitude to those of the hidden-gauge model. As already mentioned, the calculations of Refs. [1,2] used Yang-Mills models that do not respect chiral symmetry. Without the ensuing cancellations, they lead to partial widths that are an order of magnitude larger than ours.

Finally we consider several Lagrangians of the CCWZ type [17,6]. These are expressed in terms of spin-1 fields that transform homogeneously under a non-linear realisation of chiral symmetry [16,17]. Our reasons for this are twofold. First, the models just described can be converted by a change of variables into equivalent CCWZ Lagrangians [6,7], which should yield the same predictions for any observable as the original models. These therefore provide a useful check on our results above. Second, the CCWZ formalism provides a convenient framework to examine the sensitivity of our results to assumptions about the a_1. In contrast to the massive Yang-Mills approach, the parameters describing the a_1 mass and couplings may be independently changed without the need to introduce further terms into the Lagrangian.

Apart from labelling of the coefficients, we use the notation of Ref. [7] to express the relevant terms in the non-anomalous Lagrangian as:

$$\begin{aligned}\mathcal{L} = {} & \frac{f_\pi^2}{4} Tr(u_\mu u^\mu) + m_\rho^2 Tr(V_\mu V^\mu) + m_{a_1}^2 Tr(A_\mu A^\mu) - \frac{1}{2} Tr(V_{\mu\nu} V^{\mu\nu}) - \frac{1}{2} Tr(A_{\mu\nu} A^{\mu\nu}) \\ & - \frac{i}{2} g_1 Tr(V_{\mu\nu}[u^\mu, u^\nu]) + \frac{i}{2} g_2 Tr(V_{\mu\nu}[V^\mu, V^\nu]) + \frac{i}{2} g_3 Tr\left(V_{\mu\nu}([u^\mu, A^\nu] - [u^\nu, A^\mu])\right) \\ & + \frac{i}{2} g_4 Tr\left(A_{\mu\nu}([u^\mu, V^\nu] - [u^\nu, V^\mu])\right) + \frac{1}{8} c_1 Tr\left([u_\mu, u_\nu]^2\right) - \frac{1}{4} c_2 Tr\left([u_\mu, u_\nu][V^\mu, V^\nu]\right)\end{aligned}$$

$$+\frac{1}{8}c_3 Tr\left(([u_\mu,V_\nu]-[u_\mu,V_\nu])^2\right)-\frac{1}{4}c_4 Tr\left([u_\mu,u_\nu]([u^\mu,A^\nu]-[u^\nu,A^\mu])\right). \tag{7}$$

The minimal Yang-Mills model discussed above corresponds to the following set of coefficients:

$$g_1=\frac{1}{2g}(1-Z^4),\quad g_2=2g,\quad g_3=g_4=Z^2,$$
$$c_1=g_1^2,\quad c_2=1-Z^4,\quad c_3=Z^4,\quad c_4=g_1g_3, \tag{8}$$

where

$$Z^2=1-\frac{g^2f_\pi^2}{m_\rho^2}. \tag{9}$$

The appropriate coefficients for the simplest hidden-gauge model [11] can be obtained by setting $Z=0$ in Eq. (8) and dropping all terms involving the a_1. We have calculated the amplitudes for $\rho^0\to 4\pi$ within this framework and verified the results for the non-anomalous parts of the hidden-gauge and massive Yang-Mills models described above.

The anomalous sectors of these models can also be converted into CCWZ form, although this is somewhat involved for the massive Yang–Mills approach. However the sum of amplitudes for the anomalous diagrams must remain unaltered by the change of variables. For simplicity, therefore, we take the anomalous part of the amplitude directly from the original version of the massive Yang-Mills model. For the couplings given in Eqs. (8,9), $m_\omega=783$ MeV and $m_{a_1}=1230$ MeV, we get the results labelled MYM+1 in Table 1. These correspond to a massive Yang-Mills Lagrangian with non-minimal terms of the type suggested in Refs. [13,5].

The parameter choice of Eqs. (8,9) still suffers from the fact that it gives too small a width for $\rho\to 2\pi$. It is straightforward to modify the CCWZ parameters to remove this deficiency. This is equivalent to adding a further non-minimal term to the massive Yang-Mills Lagrangian [13,5] in order to cancel the diagonalization-induced $\mathcal{O}(p^3)$ $\rho\pi\pi$ coupling. In the CCWZ representation, the corresponding parameters are

$$g_1=\frac{1}{2g},\qquad g_2=2g,\qquad g_3=0,\qquad g_4=Z^2,$$
$$c_1=\frac{1}{4g^2}(1-Z^8),\qquad c_2=1,\qquad c_3=Z^4,\qquad c_4=0. \tag{10}$$

The results for this set, with the empirical meson masses, are labelled MYM+2 in Table 1.

We have examined the sensitivity of our results to the 3ρ coupling present in these Lagrangians. In the hidden-gauge and Yang-Mills models described above this coupling is equal to the $\rho\pi\pi$ one because of the assumed universal coupling of the ρ. Using the CCWZ equivalents of these models, we have varied g_2 by $\pm 30\%$ and found that the decay rate for $\rho^0 \to 2\pi^0\pi^+\pi^-$ changes by about $\pm 1\%$.

These results show that, for all of the chirally symmetric models considered, the partial widths for the decays $\rho^0 \to 2\pi^+2\pi^-$ and $\rho^0 \to 2\pi^0\pi^+\pi^-$ are of the order of 1 keV, corresponding to cross sections of the order of 5 pb. These are an order of magnitude smaller than the predictions of Refs. [1,2]. Although the processes may be hard to observe in future experiments, they should not be beyond the reach of DAΦNE which is designed to have a luminosity of 5×10^8 $\mathrm{b}^{-1}\mathrm{s}^{-1}$ [4]. The precise results are sensitive to the choice of Lagrangian and parameters, receiving significant contributions from anomalous processes (about which there is still some debate [26]) and symmetry-breaking interactions. However, any uncertainties in the determination of the relevant parameters are not sufficient to affect our general conclusion on the magnitudes of the decay rates for for $\rho^0 \to 4\pi$.

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[1] A. Bramon, A. Grau and G. Pancheri, Phys. Lett. **B317** (1993) 190.

[2] S. I. Eidelman, Z. K. Silagadze and E. A. Kuraev, Phys. Lett. **B346** (1995) 186.

[3] S. I. Dolinsky *et al.*, Phys. Rep. **202** (1991) 99.

[4] G. Pancheri, Nuovo Cim. **A107** (1994) 2149.

[5] U.-G. Meissner, Phys. Rep. **161** (1988) 213.

[6] G. Ecker, H. Leutwyler, J. Gasser, A. Pich and E. de Rafael, Phys. Lett. **B223** (1989) 425.

[7] M. C. Birse, University of Manchester preprint MC/TH 95/08, hep-ph/9507202.

[8] L. M. Kurdadze *et al.*, JETP Lett. **47** (1988) 512.

[9] V. M. Aulchenko *et al.*, Novosibirsk preprint YAF 87-90 (1987).

[10] D. Parashar, Phys. Rev. **D26** (1982) 1183.

[11] M. Bando, T. Kugo S. Uehara, K. Yamawaki and T. Yanagida, Phys. Rev. Lett. **54** (1985) 1215.

[12] M. Bando, T. Kugo and K. Yamawaki, Phys. Rep. **164** (1988) 217.

[13] Ö. Kaymakcalan, S. Rajeev and J. Schechter, Phys. Rev. **D30** (1984) 594; H. Gomm, Ö. Kaymakcalan and J. Schechter, *ibid.* 2345.

[14] Y. Brihaye, N. K. Pak and P. Rossi, Nucl. Phys. **B254** (1985) 71.

[15] J.F. Donoghue, E. Golowich and B.R. Holstein, *Dynamics of the standard model*, (Cambridge University Press, Cambridge 1992).

[16] S. Weinberg, Phys. Rev. **166** (1968) 1568.

[17] S. Coleman, J. Wess and B. Zumino, Phys. Rev. **177** (1969) 2239; C. G. Callan, S. Coleman, J. Wess and B. Zumino, *ibid.* 2247.

[18] T. Fujiwara, T. Kugo, H. Terao, S. Uehara and K.Yamawaki, Prog. Theor. Phys. **73** (1985) 926.

[19] K. Kawarabayashi and M. Suzuki, Phys. Rev. Lett. **16** (1966) 255; Riazuddin and Fayyazuddin, Phys. Rev. **147** (1966) 1071.

[20] V. de Alfaro, S. Fubini, G. Furlan and C. Rossetti, *Currents in hadron physics*, (North Holland,

Amsterdam, 1973).

[21] R. Kumar, Phys. Rev. **185** (1969) 1865.

[22] T. W. Sag and G. Szekeres, Math. Comput. **18** (1964) 245.

[23] B. W. Lee and H. T. Nieh, Phys. Rev. **166** (1968) 1507.

[24] S. Gasiorowicz and D. A. Geffen, Rev. Mod. Phys. **41** (1969) 531.

[25] S. Weinberg, Phys. Rev. Lett. **18** (1967) 507.

[26] E. A. Kuraev and Z. K. Silagadze, hep-ph/9502406.

TABLE

Model	$\rho^0 \to 2\pi^+2\pi^-$	$\rho^0 \to 2\pi^0\pi^+\pi^-$
HG	0.89	0.44
HGNA	0.89	0.24
HGCS	0.59	0.37
MMYM	0.68	0.37
MYM+1	0.63	0.34
MYM+2	1.03	0.39

Table 1. Decay widths (in keV) for $\rho^0 \to 4\pi$ in various versions of the chiral models. See the text for the definitions of these.

FIGURE

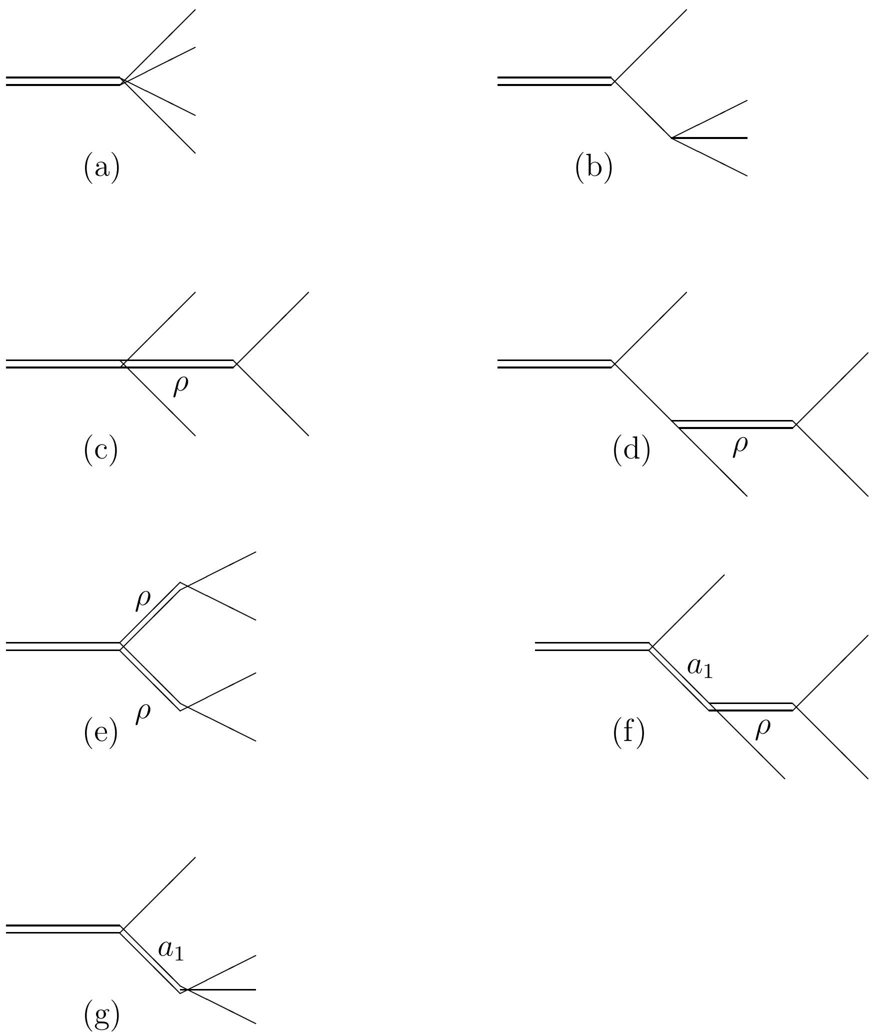

Figure 1. Diagrams contributing to the $\rho^0 \to 4\pi$ decays in chiral effective Lagrangians of π, ρ and a_1 mesons. Single lines denote pions, double lines spin-1 mesons. Anomalous terms in the effective Lagrangian introduce diagrams similar to (f) and (g) but with an ω meson replacing the a_1.