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ON SOME PROPERTIES OF TRAVELING WATER WAVES WITH VORTICITY*

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Abstract. We prove that for a large class of vorticity functions the crests of any corresponding traveling gravity water wave of finite depth are necessarily points of maximal horizontal velocity. We also show that for waves with nonpositive vorticity the pressure everywhere in the fluid is larger than the atmospheric pressure. A related a priori estimate for waves with nonnegative vorticity is also given.

Key words. water waves, vorticity, maximum principle

AMS subject classifications. 76B15, 35R35, 35J65

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1. Introduction. In this article we consider the classical hydrodynamical problem concerning traveling two-dimensional gravity water waves with vorticity. This problem has attracted considerable interest in recent years, starting with the systematic study of Constantin and Strauss [8] on periodic waves of finite depth.

The problem arises from the following physical situation. A wave of permanent form moves with constant speed on the surface of an incompressible flow, the bottom of the flow domain being horizontal. With respect to a frame of reference moving with the speed of the wave, the flow is steady and occupies a fixed region Ω in the upper half of the (x, y)-plane, which lies between the real axis $\mathcal{B} := \{(x, 0) : x \in \mathbb{R}\}$ and some a priori unknown free surface $\mathcal{S} := \{(x, \eta(x)) : x \in \mathbb{R}\}$, where η is a periodic function. Since the fluid is incompressible, the flow can be described by a (relative) stream function ψ which is periodic in the horizontal direction and satisfies the following equations and boundary conditions:

$$\Delta\psi = -\gamma(\psi) \quad \text{in } \Omega, \tag{1.1a}$$

$$\psi = B \quad \text{on } \mathcal{B}, \tag{1.1b}$$

$$\psi = 0 \quad \text{on } \mathcal{S}, \tag{1.1c}$$

$$|\nabla\psi|^2 + 2gy = Q \quad \text{on } \mathcal{S}, \tag{1.1d}$$

$$\psi_y < 0 \quad \text{in } \Omega, \tag{1.1e}$$

where B, g, and Q are positive constants. Equation (1.1a) involves a vorticity function $\gamma : [0, B] \to \mathbb{R}$ and expresses the fact that the vorticity of the flow $\omega := -\Delta\psi$ and the stream function ψ are functionally dependent. Equations (1.1b) and (1.1c) mean that the bottom and the free surface are streamlines, while (1.1d) means that the pressure at the surface of the flow is a constant. The relative velocity of the fluid particles is given by $(\psi_y, -\psi_x)$. The requirement (1.1e) means that the horizontal velocity of each fluid particle is smaller than the speed of the wave and is motivated both by field observations and by laboratory experiments; see [8] for references. It is customary [8]

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to assume that the constants g, B and the vorticity function γ are given. The problem consists in determining the curves $\mathcal{S}$ for which there exists a function ψ in Ω which satisfies (1.1a)–(1.1e) for some value of the parameter Q. For a full justification of the equivalence between the problem of seeking solution triples $(\mathcal{S}, \psi, Q)$ of (1.1) and that of seeking traveling-wave solutions of the two-dimensional Euler equations, the reader is referred to [8].

When $\gamma \equiv 0$, the corresponding flow is called irrotational. Nowadays the mathematical theory dealing with this situation contains a wealth of results, mostly obtained during the last three decades, concerning the existence of large-amplitude solutions and their properties. Global bifurcation theories were given for various types of waves (periodic or solitary of finite depth; periodic of infinite depth) by Keady and Norbury [14] and by Amick and Toland [1, 2]. Moreover, it was shown by Toland [20] and McLeod [16] that in the closure of these continua of solutions there exist waves with stagnation points at their crests, a stagnation point being one at which the relative fluid velocity is zero, i.e., $|\nabla\psi| = 0$. The existence of such waves, called "extreme waves," was predicted by Stokes [18], who also conjectured that their profiles necessarily have corners with included angle of 120° at the crests. This conjecture was proved independently by Amick, Fraenkel, and Toland [3] and by Plotnikov [17]. Recently, the method of [3] was simplified and extended in [22].

On the other hand, when $\gamma \not\equiv 0$, the flow is called rotational or with vorticity, and significant advances in the corresponding mathematical theory have been made only in the last few years. The existence of global continua of smooth solutions was proved by Constantin and Strauss [8] for the periodic finite depth problem, and by Hur [13] for the related problem of periodic waves of infinite depth. For the solutions found in [8, 13] the wave profiles have exactly one crest and one trough per minimal period, are monotone between crests and troughs, and have a vertical axis of symmetry. (The symmetry assumption is in fact not a restriction since, for any vorticity function, any wave profile with the above monotonicity properties is necessarily symmetric [5, 11].) Of particular significance is the fact that the continuum of solutions found in [8] contains waves for which the values of $\max_{\overline{\Omega}} \psi_y$ are arbitrarily close to 0. Thus it is natural to expect that, as in the irrotational case, waves with stagnation points, referred to above as "extreme waves," exist for many vorticity functions, and that they can be obtained as limits, in a suitable sense, of certain sequences of regular waves found in [8]. In the case of constant vorticity, numerical evidence [15, 19] strongly points to the existence of extreme waves for any negative vorticity and for small positive vorticity, and also indicates that, for large positive vorticity, continua of solutions bifurcating from a line of trivial solutions develop into overhanging profiles (a situation which is not possible in the irrotational case; see [23] for references) and do not approach extreme waves. Further references to numerical investigations of waves with vorticity can be found in [15].

One of the questions addressed in this article concerns the location of the points at which the maximum over $\overline{\Omega}$ of the relative horizontal velocity ψ_y is attained for smooth waves with vorticity. In the irrotational case, the crests of the wave are the only such points; see Toland [21]. Very recently, Constantin and Strauss [9, Theorem 4.1] showed that this is also the case for the waves in the continuum in [8] under the assumption that γ is a nonpositive constant which satisfies a smallness condition involving B and g. Here we prove, with a novel approach, a slightly weaker result under substantially more general assumptions. Namely, for wave profiles with finitely many local extrema on a period, if the vorticity function γ satisfies $\gamma \leq 0$ and $\gamma' \geq 0$ everywhere on $[0, B]$, then any point of maximal relative horizontal velocity must lie on the free surface and the crests are necessarily such points. An immediate consequence of this result is that, whenever $\gamma \leq 0$ and $\gamma' \geq 0$, the continuum of solutions in [8] contains waves for which the values of $|\nabla\psi|$ at their crests are arbitrarily close to 0. Thus in this case the existence of waves with stagnation points at their crests is to be particularly expected.

Another contribution of this article is that we establish some new a priori bounds for waves corresponding to vorticity functions γ which do not change sign, without any assumptions on γ'. When $\gamma \leq 0$, the estimate in question means that the pressure everywhere in the fluid is larger than the atmospheric pressure. This estimate is the main ingredient in the proof of the previously mentioned result concerning the location of the points where $\max_{\overline{\Omega}} \psi_y$ is attained. When $\gamma \geq 0$, a slightly different, but related, estimate is given. Both these estimates play an essential role in the investigation in [24] concerning the existence of extreme waves with vorticity and the Stokes conjecture.

The proofs here are based on simple applications of the maximum principle [12, Chapters 2 and 3]. Analogous results to those of this article hold in the case of periodic rotational waves of infinite depth. They will be presented, together with some applications, in a subsequent article.

Of the many other directions in which the theory of traveling gravity water waves, with or without vorticity, has seen recent progress and is currently being further developed, we mention here only a few: variational formulations [7], stability [10], and properties of the fluid particle trajectories [4, 6].

2. The main results. We always deal with classical solutions of (1.1), in the sense that $\gamma \in C^1([0,B])$, $\eta \in C^3(\mathbb{R})$, $\psi \in C^3(\overline{\Omega})$. We assume that η is a periodic function of minimal period $2L$, and that ψ is $2L$-periodic in the horizontal direction. However, we do not assume that η has exactly one local maximum and one local minimum per minimal period.

Let $\hat{\Gamma} : [0,B] \to \mathbb{R}$ be given by

$$\hat{\Gamma}(s) = \int_0^s \gamma(t)\, dt \quad \text{for all } s \in [0,B]. \tag{2.1}$$

(Note that in [8] a function Γ is considered which is related to $\hat{\Gamma}$ by $\hat{\Gamma}(s) = -\Gamma(-s)$. The quantity of interest both here and there is $\hat{\Gamma}(\psi)$, which is denoted there by $-\Gamma(-\psi)$; we find our notation more convenient.) Let us also consider the function $R : \overline{\Omega} \to \mathbb{R}$ given by

$$R = \frac{1}{2}|\nabla\psi|^2 + gy - \frac{1}{2}Q + \hat{\Gamma}(\psi). \tag{2.2}$$

The function R is (up to a constant) the negative of the pressure in the fluid domain; see [8].

Our next result shows that when γ is everywhere nonpositive the pressure in the fluid domain is larger than the atmospheric pressure.

THEOREM 2.1. *Suppose that $\gamma(s) \leq 0$ for all $s \in [0,B]$. Then $R \leq 0$ in $\overline{\Omega}$.*

Remark 2.2. Under the much more restrictive assumptions that

$$\gamma \leq 0, \quad \gamma' \leq 0 \quad \text{and} \quad -\psi_y(x,0)\gamma(B) \geq -g \text{ for all } x \in \mathbb{R},$$

the conclusion of Theorem 2.1 was previously obtained in [9, Example 3.1].

The importance of the inequality $R \leq 0$ in $\overline{\Omega}$ in relation to the monotonicity of ψ_y along the free surface $\mathcal{S}$ was first recognized for waves with vorticity by Constantin and Strauss [9, Proposition 3.4]. We give here a slightly more general statement of their result and a somewhat more direct proof.

THEOREM 2.3. *Let $\eta : \mathbb{R} \to \mathbb{R}$ be such that there exists $N \in \mathbb{N}$ and points $x_0 < x_1 < \cdots < x_{2N} = x_0 + 2L$ with the property that $\eta'(x_j) = 0$ for all $j \in \{0, \ldots, 2N\}$, η is strictly increasing on $[x_{2j}, x_{2j+1}]$ for all $j \in \{0, \ldots, N-1\}$, and η is strictly decreasing on $[x_{2j-1}, x_{2j}]$ for all $j \in \{1, \ldots, N\}$. Suppose that $R \leq 0$ in $\overline{\Omega}$. Then the function $x \mapsto \psi_y(x, \eta(x))$ is increasing on $[x_{2j}, x_{2j+1}]$ for all $j \in \{0, \ldots, N-1\}$ and decreasing on $[x_{2j-1}, x_{2j}]$ for all $j \in \{1, \ldots, N\}$. Therefore, $\max_{\mathcal{S}} \psi_y$ is attained at the points of maximal height on $\mathcal{S}$.*

The preceding result leads with little effort to one concerning the location of the points where $\max_{\overline{\Omega}} \psi_y$ is attained.

THEOREM 2.4. *Let $\eta : \mathbb{R} \to \mathbb{R}$ be as in Theorem 2.3. Suppose that $\gamma(s) \leq 0$ and $\gamma'(s) \geq 0$ for all $s \in [0, B]$. Then any point at which $\max_{\overline{\Omega}} \psi_y$ is attained lies on $\mathcal{S}$, and the crests of the wave are necessarily such points.*

Remark 2.5. For a more restrictive class of wave profiles and under the assumption that γ is a nonpositive constant which satisfies

$$g^2 \geq 2g(-2B\gamma^3)^{1/2} - 2B\gamma^3,$$

Constantin and Strauss [9, Theorem 4.1] proved that the crests of the wave are the only points at which $\max_{\overline{\Omega}} \psi_y$ is attained. This slightly stronger conclusion does not seem to be readily obtainable by the methods used in the proof of Theorem 2.4.

The next result gives a new estimate in the case when γ is everywhere nonnegative, which is in the same spirit as that of Theorem 2.1. Let us consider the function $T : \overline{\Omega} \to \mathbb{R}$ given by

$$T := R - \varpi\psi, \tag{2.3}$$

where R is given by (2.2) and

$$\varpi := \frac{1}{2} \max_{s \in [0,B]} \gamma(s). \tag{2.4}$$

THEOREM 2.6. *Suppose that $\gamma(s) \geq 0$ for all $s \in [0, B]$. Then $T \leq 0$ in $\overline{\Omega}$.*

3. Proofs of the main results. A simple calculation shows that, everywhere in $\overline{\Omega}$,

$$R_x = \psi_y\psi_{xy} - \psi_x\psi_{yy}, \tag{3.1a}$$

$$R_y = \psi_x\psi_{xy} - \psi_y\psi_{xx} + g, \tag{3.1b}$$

and

$$\Delta R = 2\psi_{xy}^2 - 2\psi_{xx}\psi_{yy}. \tag{3.2}$$

Proof of Theorem 2.1. Note that $R = 0$ everywhere on the free surface $\mathcal{S}$. We claim that the maximum of R over $\overline{\Omega}$ must be attained on $\mathcal{S}$.

Observe first that, since $R_y = g > 0$ everywhere on the bottom $\mathcal{B}$, $\max_{\overline{\Omega}} R$ cannot be attained anywhere on $\mathcal{B}$.

Suppose now that $\max_{\overline{\Omega}} R$ is attained at some point A in Ω. Then necessarily

$$R_x(A) = 0, \quad R_y(A) = 0, \quad \Delta R(A) \le 0.$$

It follows from this, (3.1), and (3.2) that

$$\psi_y(A)\psi_{xy}(A) = \psi_x(A)\psi_{yy}(A), \tag{3.3a}$$

$$\psi_x(A)\psi_{xy}(A) < \psi_y(A)\psi_{xx}(A), \tag{3.3b}$$

$$\psi_{xy}^2(A) \le \psi_{xx}(A)\psi_{yy}(A). \tag{3.3c}$$

Since (1.1e) holds, it follows that $\psi_y(A) < 0$. We now distinguish two cases, depending on whether or not $\psi_{yy}(A) = 0$.

If $\psi_{yy}(A) = 0$, then (3.3a) implies that $\psi_{xy}(A) = 0$. It then follows from (3.3b) that $\psi_{xx}(A) < 0$, and hence $\gamma(\psi(A)) = -\Delta\psi(A) > 0$, which contradicts the assumption that $\gamma(s) \le 0$ for all $s \in [0, B]$.

If $\psi_{yy}(A) \ne 0$, then it follows from (3.3a) and (3.3b) that

$$\frac{\psi_y(A)\psi_{xy}^2(A)}{\psi_{yy}(A)} < \psi_y(A)\psi_{xx}(A).$$

It then follows from this and (3.3c) that $\psi_{yy}(A) < 0$. We now deduce from (3.3c) that $\psi_{xx}(A) \le 0$, and therefore $\gamma(\psi(A)) = -\Delta\psi(A) > 0$, which again contradicts the assumption that $\gamma(s) \le 0$ for all $s \in [0, B]$.

We conclude that the maximum of R over $\overline{\Omega}$ must be attained on $\mathcal{S}$, which implies that $R \le 0$ in $\overline{\Omega}$. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.3. The proof is based on a remarkable, though straightforward to verify, identity observed by Toland [21] in the irrotational case and by Constantin and Strauss [9] in the general case:

$$\frac{d}{dx}\left[\frac{1}{2}\psi_y^2(x, \eta(x))\right] = R_x(x, \eta(x)) \quad \text{for all } x \in \mathbb{R}. \tag{3.4}$$

Since $R \le 0$ in $\overline{\Omega}$ and $R = 0$ on $\mathcal{S}$, the required result concerning the monotonicity of $x \mapsto \psi_y(x, \eta(x))$ is immediate from (3.4). It follows that

$$\max_{\mathcal{S}} \psi_y = \max_{j \in \{0,\dots,N-1\}} \psi_y(x_{2j+1}, \eta(x_{2j+1})). \tag{3.5}$$

But for every $j \in \{0, \dots, 2N\}$, $\psi_x(x_j, \eta(x_j)) = 0$ and therefore

$$\psi_y(x_j, \eta(x_j)) = -(Q - 2g\eta(x_j))^{1/2}.$$

Hence $\max_{\mathcal{S}} \psi_y$ is attained at the points of maximal height on $\mathcal{S}$. This completes the proof of Theorem 2.3. □

Proof of Theorem 2.4. It follows from (1.1a) that

$$\Delta\psi_y = -\gamma'(\psi)\psi_y \quad \text{in } \Omega.$$

Since $\psi_y < 0$ in Ω and $\gamma'(s) \ge 0$ for all $s \in [0, B]$, it follows immediately from the maximum principle that $\max_{\overline{\Omega}} \psi_y$ cannot be attained anywhere in Ω.

We now show that $\max_{\overline{\Omega}} \psi_y$ cannot be attained anywhere on $\mathcal{B}$ either. This is trivial when $\gamma(B) < 0$, since then $\psi_{yy} = -\gamma(B) > 0$ everywhere on $\mathcal{B}$. When

$\gamma(B) = 0$, we use a reflection argument. Let $\tilde{\gamma} : [0, 2B] \to \mathbb{R}$ be an extension of γ such that $\tilde{\gamma}(s) = -\gamma(2B - s)$ for all $s \in (B, 2B]$. Let Ω^R be the reflection of Ω into $\mathcal{B}$,

$$\widetilde{\Omega} := \Omega \cup \mathcal{B} \cup \Omega^R,$$

and $\tilde{\psi} : \widetilde{\Omega} \to \mathbb{R}$ be an extension of ψ such that $\tilde{\psi}(x, y) = 2B - \psi(x, -y)$ for all $(x, y) \in \Omega^R$. Then it is easily checked that $\tilde{\gamma} \in C^1([0, 2B])$, $\tilde{\psi} \in C^3(\widetilde{\Omega})$ and

$$\Delta\tilde{\psi} = -\tilde{\gamma}(\tilde{\psi}) \quad \text{in } \widetilde{\Omega}.$$

Since $\tilde{\psi}_y < 0$ in $\widetilde{\Omega}$ and $\tilde{\gamma}'(s) \geq 0$ for all $s \in [0, 2B]$, the maximum principle yields the required result.

We conclude that $\max_{\overline{\Omega}} \psi_y$ can only be attained on $\mathcal{S}$. Next note that, since $\gamma(s) \leq 0$ for all $s \in [0, B]$, Theorem 2.1 shows that $R \leq 0$ in $\overline{\Omega}$. An application of Theorem 2.3 now yields that $\max_{\overline{\Omega}} \psi_y$ is attained at the crests of the wave. This completes the proof of Theorem 2.4. $\square$

Proof of Theorem 2.6. Note first that $T_y = g - \varpi\psi_y > 0$ everywhere on $\mathcal{B}$, so that the maximum of T over $\overline{\Omega}$ cannot be attained anywhere on $\mathcal{B}$.

Next note from (3.2) that

$$\Delta R \geq -\frac{1}{2}\gamma^2(\psi) \quad \text{in } \Omega.$$

Since

$$\Delta T = \Delta R + \varpi\gamma(\psi),$$

it is immediate, upon using (2.4) and the assumption that $\gamma(s) \geq 0$ for all $s \in [0, B]$, that T is a subharmonic function in Ω. Therefore, the maximum of T over $\overline{\Omega}$ cannot be attained anywhere in Ω.

We conclude that $\max_{\overline{\Omega}} T$ must be attained somewhere on $\mathcal{S}$. Since $T = 0$ everywhere on $\mathcal{S}$, it follows that $T \leq 0$ in $\overline{\Omega}$. This completes the proof of Theorem 2.6. $\square$

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