Vekua theory for the Helmholtz operator

Article

Accepted Version

Moiola, A., Hiptmair, R. and Perugia, I. (2011) Vekua theory for the Helmholtz operator. Zeitschrift für Angewandte Mathematik und Physik, 62 (5). pp. 779-807. ISSN 0044-2275 doi: 10.1007/s00033-011-0142-3 Available at https://centaur.reading.ac.uk/28022/

To link to this article DOI: http://dx.doi.org/10.1007/s00033-011-0142-3

Publisher: Springer

Vekua theory for the Helmholtz operator

A. Moiola, R. Hiptmair and I. Perugia

Abstract. Vekua[1] operators map harmonic functions defined on domain in $\mathbb{R}^2$ to solutions of elliptic partial differential equations on the same domain and vice versa. In this paper, following the original work of I. Vekua, we define Vekua operators in the case of the Helmholtz equation in a completely explicit fashion, in any space dimension $N \geq 2$. We prove *i)* that they actually transform harmonic functions and Helmholtz solutions into each other; *ii)* that they are inverse to each other; *iii)* that they are continuous in any Sobolev norm in star-shaped Lipschitz domains.

Finally, we define and compute the generalized harmonic polynomials as the Vekua transforms of harmonic polynomials. These results are instrumental in proving approximation estimates for solutions of the Helmholtz equation in spaces of circular, spherical and plane waves.

Mathematics Subject Classification (2010). 35C15, 35J05.

Keywords. Vekua transform, Helmholtz equation, generalized harmonic polynomials, Sobolev continuity.

1. Introduction and Motivation

Vekua's theory (see [20, 36]) is a tool for linking properties of harmonic functions (solutions of the Laplace equation $\Delta u = 0$) to solutions of general second-order elliptic PDEs $\mathcal{L}u = 0$: the so-called Vekua operators (inverses of each other) map harmonic functions to solutions of $\mathcal{L}u = 0$ and vice versa.

The original formulation targets elliptic PDEs with analytic coefficients in two space dimensions. Some generalizations to higher space dimensions have been attempted, see [10–12, 18, 23, 24] and the references therein, but the Vekua operators in these general cases are not completely explicit.

Here, the PDE we are interested in is the homogeneous Helmholtz equation $\mathcal{L}u := \Delta u + \omega^2 u = 0$. In this particular case, simple explicit integral operators have been defined in the original work of Vekua in any space dimension $N \geq 2$ (see [34, 35], [36, p. 59], and Fig. 1), but no proofs of their properties are provided and, to the best of our knowledge, these results have been used later on only in very few cases [9, 25].

Vekua's theory has surprising relevance for numerical analysis. Several finite element methods used in the numerical discretization of the Helmholtz equation $\Delta u + \omega^2 u = 0$ are based on incorporating a priori knowledge about the differential equation into the local approximation spaces by using Trefftz-type basis functions, namely functions which belong to the kernel of the Helmholtz operator.

Examples of methods using local approximating spaces spanned by plane wave functions $x \mapsto e^{i\omega x\cdot d}$, $d \in S^{N-1}$, are the *Plane Wave Partition of Unit Method* (see, [4]), the *Ultra Weak Variational Formulation* (see [8]), the *Plane Wave Least Squares Method* (see [32]), the *Discontinuous Enrichment Method* (see [16]), and the *Plane Wave Discontinuous Galerkin Method* (see [19, 22]). Other methods are based on generalized harmonic polynomials (Fourier-Bessel functions), like the *Partition of Unit*

[1]Ilja Vekua (1907-1977), Soviet-Georgian mathematician

> 3. Notice that formula (13.9) may be generalized to the case of a space of any number of dimensions. In fact, the formula
>
> $$u(x_1, \dots, x_n) = u_0(x_1, \dots, x_n) - \int_0^1 u_0(x_1 t, \dots, x_n t)\, t^{\frac{1}{2}(n-2)} \frac{\partial}{\partial t} J_0(\lambda r \sqrt{1-t})\, dt\,, \qquad (13.14)$$
>
> ...
>
> The formula (13.14) also can be inverted, in fact,
>
> $$u_0(x_1, \dots, x_n) = u(x_1, \dots, x_n) + \tfrac{1}{2}\lambda r \int_0^1 u(x_1 t, \dots, x_n t)\, t^{\frac{1}{2}(n-2)}\, I_1(\lambda r \sqrt{t(1-t)}) \frac{dt}{\sqrt{t(1-t)}}\,. \qquad (13.17)$$

FIGURE 1. Two paragraphs of Vekua's book [36] addressing the theory for the Helmholtz equation

Method of [29], the version of the *Least Squares Method* presented again in [32] and the method of [6], or on Hankel functions, like the *Method of Fundamental Solutions* of [5].

The convergence analysis of each of these techniques requires *best approximation estimates*: the finite element space must contain a function which approximates the analytic solution of the problem with an error that tends to zero when the mesh size h is reduced (h-convergence), or when the dimension p of the local approximating space is raised (p-convergence). This error is usually measured in Sobolev norms and an explicit estimation of the convergence rate with respect to the parameters h and p is very desirable.

In the case of plane waves, only few approximation estimates are available in the literature. A first one is contained in Theorem 3.7 of [8]: the proof was based on Taylor expansion and only h–convergence for two-dimensional domains was proved; moreover, the obtained order of convergence is not sharp. A more sophisticated result is Proposition 8.4.14 of [27]: in this case, p–estimates were obtained in the two-dimensional case by using complex analysis techniques and Vekua's theory. A similar approach was used in [30] to prove sharp estimates in h for the PWDG method in 2D; there, the dependence on the wave number was made explicit. In order to generalize and make precise the results of [27] and [30], it is necessary to study in more details the basic tool used: Vekua's theory. This paper is devoted to this purpose: the results developed here will be the main ingredients in the proof of best approximation estimates by circular, spherical and plane waves. This has been done in [21] and greately improved in [31].

We proceed as follows: in Section 2, we will start by defining the Vekua operators for the Helmholtz equation with $N \geq 2$ and prove their basic properties, namely, that they are inverse to each other and map harmonic functions to solutions of the homogeneous Helmholtz equation and vice versa (see Theorem 2.5). Next, in Section 3, we establish their continuity properties in (weighted) Sobolev norms, like in [27], but with continuity constants explicit in the domain shape parameter, in the Sobolev regularity exponent and in the product of the wavenumber times the diameter of the domain (see Theorem 3.1). The main difficulty in proving these continuity estimates consists in establishing precise interior estimates. Finally, in Section 4, we introduce the generalized harmonic polynomials, which are the mapping through the direct Vekua operator of the harmonic polynomials, and derive their explicit expression. They correspond to circular and spherical waves in two and three dimensions, respectively.

All the proofs are self-contained and do not need the use of other results connected with Vekua's theory. Theorem 2.5 was already stated in [36], but the proof given in this paper is new; all the other results presented in this paper are new, although many ideas come from the work of M. Melenk (see [27, 28]).

We conclude this introduction by fixing some notation used throughout this paper.

1.1. Notation

In order to prove inequalities with constants that are explicit and sharp with respect to the indices, we need precise definitions of Sobolev norms and seminorms, because equivalent norms give different bounds.

We denote by $\mathbb{N}$ the set of natural numbers, including 0. We set

$$B_r(x_0) = \{x \in \mathbb{R}^N,\ |x - x_0| < r\}\ , \qquad B_r = B_r(0)\ , \qquad S^{N-1} = \partial B_1 \subset \mathbb{R}^N\ .$$

We introduce the standard multi-index notation

$$D^\alpha \phi = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}, \qquad |\alpha| = \sum_{j=1}^N \alpha_j \qquad \forall\, \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N\ , \tag{1}$$

and define the Sobolev seminorms and norms

$$\begin{aligned}
|u|_{W^{k,p}(\Omega)} &= \left(\sum_{\alpha \in \mathbb{N}^N, |\alpha| = k} \int_\Omega |D^\alpha u(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}}, \\
\|u\|_{W^{k,p}(\Omega)} &= \left(\sum_{j=1}^k |u|^p_{W^{j,p}(\Omega)} \right)^{\frac{1}{p}} = \left(\sum_{\alpha \in \mathbb{N}^N, |\alpha| \le k} \int_\Omega |D^\alpha u(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}}, \\
|u|_{k,\Omega} &= |u|_{W^{k,2}(\Omega)}\ , \qquad\qquad \|u\|_{k,\Omega} = \|u\|_{W^{k,2}(\Omega)}\ , \\
|u|_{W^{k,\infty}(\Omega)} &= \sup_{\alpha \in \mathbb{N}^N, |\alpha| = k}\ \operatorname*{ess\,sup}_{x \in \Omega} |D^\alpha u(x)|, \\
\|u\|_{W^{k,\infty}(\Omega)} &= \sup_{j=0,\ldots,k} |u|_{W^{j,\infty}(\Omega)}\ ,
\end{aligned}$$

and the ω–weighted Sobolev norms

$$\|u\|_{k,\omega,\Omega} = \left(\sum_{j=0}^k \omega^{2(k-j)}\, |u|^2_{j,\Omega} \right)^{\frac{1}{2}} \qquad \forall\, \omega > 0\ . \tag{2}$$

We denote the space of harmonic functions and of solutions to the homogeneous Helmholtz equation, respectively, by

$$\begin{aligned}
\mathcal{H}^j(D) &:= \left\{ \phi \in H^j(D) :\ \Delta \phi = 0 \right\} && \forall\, j \in \mathbb{N}\ , \\
\mathcal{H}^j_\omega(D) &:= \left\{ u \in H^j(D) :\ \Delta u + \omega^2 u = 0 \right\} && \forall\, j \in \mathbb{N},\ \omega \in \mathbb{C}\ .
\end{aligned}$$

Finally, we denote the number of the independent spherical harmonics of degree l in $\mathbb{R}^N$ (see [33, eq. (11)] and [3, Prop. 5.8]) by

$$n(N,l) := \begin{cases} 1 & \text{if } l = 0, \\ \dfrac{(2l + N - 2)(l + N - 3)!}{l!\,(N-2)!} & \text{if } l \ge 1\ . \end{cases} \tag{3}$$

2. N-Dimensional Vekua Theory for the Helmholtz Operator

Throughout domains satisfy the following assumption.

Assumption 2.1. *The domain $D \subset \mathbb{R}^N$, $N \ge 2$, is a bounded open set such that*

- *D is star-shaped with respect to the origin,*
- *and there exists $\rho \in (0, 1/2]$ such that $B_{\rho h} \subseteq D$, where $h := \operatorname{diam} D$.*

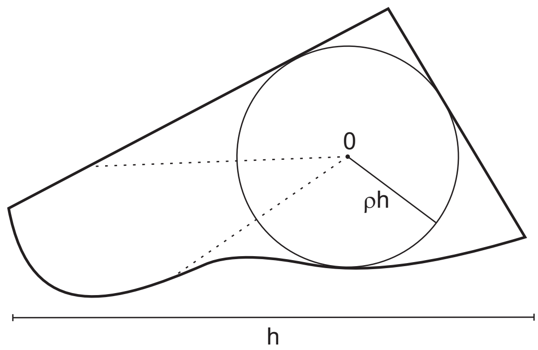
Not all these assumptions are necessary in order to establish the results of this section (see Remark 2.7 below).

Remark 2.2. *If D is a domain as in Assumption 2.1, then*

$$B_{\rho h} \subseteq D \subseteq B_{(1-\rho)h}.$$

The maximum $1/2$ for the parameter ρ is achieved when the domain is a sphere: $D = B_{\frac{h}{2}}$.

FIGURE 2. A domain D that satisfies Assumption 2.1



Definition 2.3. Given a positive number ω, we define two continuous functions $M_1, M_2 : D \times [0,1) \to \mathbb{R}$ as follows

$$\begin{aligned} M_1(x,t) &:= -\frac{\omega|x|}{2} \frac{\sqrt{t}^{N-2}}{\sqrt{1-t}} J_1(\omega|x|\sqrt{1-t}), \\ M_2(x,t) &:= -\frac{i\omega|x|}{2} \frac{\sqrt{t}^{N-3}}{\sqrt{1-t}} J_1(i\omega|x|\sqrt{t(1-t)}), \end{aligned} \tag{4}$$

where J_1 is the 1-st order Bessel function of the first kind, see Appendix A.

Using the expression (60), we can write

$$M_1(x,t) = -t^{\frac{N}{2}-1} \sum_{k\geq 0} \frac{(-1)^k \left(\frac{\omega|x|}{2}\right)^{2k+2} (1-t)^k}{k!\,(k+1)!},$$

$$M_2(x,t) = \sum_{k\geq 0} \frac{\left(\frac{\omega|x|}{2}\right)^{2k+2} (1-t)^k\, t^{k+\frac{N}{2}-1}}{k!\,(k+1)!}.$$

Note that M_1 and M_2 are radially symmetric in x and belong to $C^\infty(D \times (0,1])$; if N is even, they have a C^∞-extension to $\mathbb{R}^N \times \mathbb{R}$.

Definition 2.4. We define the *Vekua operator* $V_1 : C(D) \to C(D)$ and the *inverse Vekua operator* $V_2 : C(D) \to C(D)$ for the Helmholtz equation according to

$$V_j[\phi](x) = \phi(x) + \int_0^1 M_j(x,t)\phi(tx)\,\mathrm{d}t \qquad \forall\, x \in D,\ j = 1,2, \tag{5}$$

where $C(D)$ is the space of the complex-valued continuous functions on D. $V_1[\phi]$ is called the Vekua transform of ϕ.

Notice that $t \mapsto M_j(x,t)\phi(tx)$, $j = 1,2$, belong to $L^1([0,1])$ for every $x \in D$; consequently, V_1 and V_2 are well defined. The operators V_1 and V_2 can also be defined with the same formulas from the space of the essentially bounded functions $L^\infty(D)$ to itself, or from $L^p(D)$ to $L^2(D)$, with p sufficiently large, depending on the spatial dimension N. In the following theorem, we summarize general results about the Vekua operators, while their continuity will be proved in Theorem 3.1 below.

Theorem 2.5. *Let D be a domain as in Assumption 2.1; the Vekua operators satisfy:*

(i) *V_2 is the inverse of V_1:*

$$V_1\big[V_2[\phi]\big] = V_2\big[V_1[\phi]\big] = \phi \qquad \forall\, \phi \in C(D)\ . \tag{6}$$

(ii) *If ϕ is harmonic in D, i.e.,*

$$\Delta\phi = 0 \qquad \text{in } D\ , \tag{7}$$

then

$$\Delta V_1[\phi] + \omega^2 V_1[\phi] = 0 \qquad \text{in } D\ .$$

(iii) *If u is a solution of the homogeneous Helmholtz equation with wavenumber $\omega > 0$ in D, i.e.,*

$$\Delta u + \omega^2 u = 0 \qquad \text{in } D\ , \tag{8}$$

then

$$\Delta V_2[u] = 0 \qquad \text{in } D\ .$$

Theorem 2.5 states that the operators V_1 and V_2 are inverse to each other and map harmonic functions to solutions of the homogeneous Helmholtz equation and vice versa.

The results of this theorem were stated in [36, Chapter 1, § 13.2-3]. In two space dimensions, the operator V_1 followed from the general Vekua theory for elliptic PDEs; this implies that V_1 is a bijection between the space of complex harmonic function and the space of solutions of the homogeneous Helmholtz equation [1]. The fact that the inverse of V_1 can be written as the operator V_2 (part *(i)* of Theorem 2.5) was stated in [35], and the proof was skipped as an "easy calculation", after reducing the problem to a one-dimensional Volterra integral equation. Here, we give a completely self-contained and general proof of Theorem 2.5 merely using elementary calculus.

As in Theorem 2.5, in the following we will usually denote the solutions of the homogeneous Helmholtz equation with the letter u, and harmonic functions, as well as generic functions defined on D, with the letter ϕ.

Remark 2.6. *Theorem 2.5 holds with the same proof also for every $\omega \in \mathbb{C}$, i.e., for the Helmholtz equation in lossy materials.*

Remark 2.7. *Theorem 2.5 holds also for an unbounded or irregular domain: the only necessary hypotheses are that D has to be open and star-shaped with respect to the origin. In fact the proof only relies on the local properties of the functions on the segment $[0, x]$. For the same reason, the singularities of ϕ and u on the boundary of D do not affect the results of the theorem.*

Theorem 2.5 can be proved by using elementary mathematical analysis results. We proceed by proving the parts *(i)* and *(ii)* separately.

Proof of Theorem 2.5, part (i). We define a function

$$g : [0,\infty) \times [0,\infty) \to \mathbb{R},$$
$$g(r,t) = \frac{\omega\sqrt{r}\, t}{2\sqrt{r-t}}\, J_1(\omega\sqrt{r}\sqrt{r-t}).$$

Note that if $r < t$ the argument of the Bessel function J_1 is imaginary on the standard branch cut but the function g is always real-valued.

Using the change of variable $s = t|x|$, for every $\phi \in C(D)$ and for every $x \in D$, we can compute

$$\begin{aligned} V_1[\phi](x) &= \phi(x) + \int_0^{|x|} M_1\Big(x, \frac{s}{|x|}\Big)\phi\Big(s\frac{x}{|x|}\Big)\frac{1}{|x|}\,\mathrm{d}s \\ &= \phi(x) - \int_0^{|x|} \frac{\omega|x|}{2}\sqrt{\frac{s}{|x|}}^{N-2} \frac{\sqrt{|x|}}{\sqrt{|x|-s}}\,\frac{1}{|x|}\, J_1\Big(\omega\sqrt{|x|}\sqrt{|x|-s}\Big)\,\phi\Big(s\frac{x}{|x|}\Big)\,\mathrm{d}s \end{aligned}$$

[1]The proof in higher space dimensions might be contained in the Georgian language article [34] that is hard to obtain.

$$= \phi(x) - \int_0^{|x|} \frac{s^{\frac{N-4}{2}}}{|x|^{\frac{N-2}{2}}}\; g(|x|,s)\; \phi\Big(s\frac{x}{|x|}\Big)\,\mathrm{d}s,$$

$$\begin{aligned}
V_2[\phi](x) &= \phi(x) + \int_0^{|x|} M_2\Big(x,\frac{s}{|x|}\Big)\phi\Big(s\frac{x}{|x|}\Big)\frac{1}{|x|}\,\mathrm{d}s\\
&= \phi(x) - \int_0^{|x|} \frac{i\omega|x|}{2}\sqrt{\frac{s}{|x|}}^{N-3}\frac{\sqrt{|x|}}{\sqrt{|x|-s}}\,\frac{1}{|x|}\;J_1\Big(i\omega\sqrt{s}\sqrt{|x|-s}\Big)\;\phi\Big(s\frac{x}{|x|}\Big)\,\mathrm{d}s\\
&= \phi(x) + \int_0^{|x|} \frac{s^{\frac{N-4}{2}}}{|x|^{\frac{N-2}{2}}}\; g(s,|x|)\; \phi\Big(s\frac{x}{|x|}\Big)\,\mathrm{d}s
\end{aligned}$$

because $s \le |x|$ and we have fixed the sign $\sqrt{s-|x|} = i\sqrt{|x|-s}$. Note that in the expressions for the two operators the arguments of the functions g are swapped. Now we apply the first operator after the second one, switch the order of the integration in the resulting double integral and get

$$\begin{aligned}
V_1\big[V_2[\phi]\big](x) = &\left[\phi(x) + \int_0^{|x|} \frac{s^{\frac{N-4}{2}}}{|x|^{\frac{N-2}{2}}}\; g(s,|x|)\; \phi\Big(s\frac{x}{|x|}\Big)\,\mathrm{d}s\right]\\
&- \int_0^{|x|} \frac{s^{\frac{N-4}{2}}}{|x|^{\frac{N-2}{2}}}\; g(|x|,s)\left[\phi\Big(s\frac{x}{|x|}\Big) + \int_0^{s} \frac{z^{\frac{N-4}{2}}}{s^{\frac{N-2}{2}}} g(z,s)\phi\Big(z\frac{x}{|x|}\Big)\,\mathrm{d}z\right]\mathrm{d}s\\
=&\phi(x) + \int_0^{|x|} \frac{s^{\frac{N-4}{2}}}{|x|^{\frac{N-2}{2}}}\;\big(g(s,|x|) - g(|x|,s)\big)\phi\Big(s\frac{x}{|x|}\Big)\,\mathrm{d}s\\
&- \int_0^{|x|} \frac{z^{\frac{N-4}{2}}}{|x|^{\frac{N-2}{2}}}\;\phi\Big(z\frac{x}{|x|}\Big)\int_z^{|x|}\frac{1}{s}\; g(z,s)\; g(|x|,s)\;\,\mathrm{d}s\,\mathrm{d}z.
\end{aligned}$$

The exchange of the order of integration is possible because ϕ is continuous and in the domain of integration $|s^{-1}z^{-1}g(|x|,s)g(z,s)| \le \frac{\omega^4}{16}\; s\; |x|\; e^{\omega|x|}$ thanks to (63), so Fubini theorem can be applied.

Notice that $V_1\big[V_2[\phi]\big] = V_2\big[V_1[\phi]\big]$, so we only have to show that V_2 is right inverse of V_1. In order to prove that $V_1\big[V_2[\phi]\big] = \phi$ it is enough to show that

$$g(t,r) - g(r,t) = \int_t^r \frac{g(t,s)\; g(r,s)}{s}\;\mathrm{d}s \qquad \forall\, r \ge t \ge 0, \tag{9}$$

so that all the integrals in the previous expression vanish, and we are done. Using (60), we expand g in power series (recall that, for $k \ge 0$ integer, $\Gamma(k+1) = k!$):

$$g(r,t) = \frac{\omega^2\; r\; t}{4}\sum_{l\ge 0}\frac{(-1)^l\;\omega^{2l}\; r^l\;(r-t)^l}{2^{2l}\; l!\;(l+1)!}\;, \tag{10}$$

from which we get

$$g(t,r) - g(r,t) = \frac{\omega^2\; r\; t}{4}\sum_{l\ge 0}\frac{(-1)^l\;\omega^{2l}\;(r-t)^l\;\big((-t)^l - r^l\big)}{2^{2l}\; l!\;(l+1)!}. \tag{11}$$

We compute the following integral using the change of variables $z = \frac{s-t}{r-t}$ and the expression of the beta integral $\int_0^1 (1-z)^p z^q\,\mathrm{d}z = B(p+1,q+1) = \dfrac{p!\;q!}{(p+q+1)!}$:

$$\begin{aligned}
\int_t^r s(r-s)^j(t-s)^k\,\mathrm{d}s &= (-1)^k(r-t)^{j+k+1}\int_0^1 (1-z)^j z^k\big(zr + (1-z)t\big)\,\mathrm{d}z\\
&= (-1)^k(r-t)^{j+k+1}\frac{j!\;k!}{(j+k+2)!}\big(r(k+1) + t(j+1)\big).
\end{aligned} \tag{12}$$

Thus, expanding the product of $g(t,s)\,g(r,s)$ in a double power series, integrating term by term and using the previous identity give

$$\begin{aligned}
&\int_t^r \frac{g(t,s)\,g(r,s)}{s}\,\mathrm{d}s\\
&\overset{(10)}{=} \frac{\omega^2\,r\,t}{4}\sum_{j,k\geq 0}\frac{(-1)^{j+k}\,\omega^{2(j+k+1)}\,r^j\,t^k}{2^{2(j+k+1)}\,j!\,(j+1)!\,k!\,(k+1)!}\int_t^r \frac{s^2(r-s)^j(t-s)^k}{s}\,\mathrm{d}s\\
&\overset{(12)}{=} \frac{\omega^2\,r\,t}{4}\sum_{j,k\geq 0}\frac{(-1)^{j}\,\omega^{2(j+k+1)}\,r^j\,t^k\,(r-t)^{j+k+1}}{2^{2(j+k+1)}\,(j+1)!\,(k+1)!\,(j+k+2)!}\big(r(k+1)+t(j+1)\big)\\
&\overset{(l=j+k+1)}{=} \frac{\omega^2\,r\,t}{4}\sum_{l\geq 1}\frac{\omega^{2l}\,(r-t)^l}{2^{2l}\,(l+1)!}\,\frac{1}{l!}\sum_{j=0}^{l-1} l!\,\frac{(-1)^j\,r^j\,t^{l-j-1}}{(j+1)!\,(l-j)!}\big(r(l-j)+t(j+1)\big)\\
&= \frac{\omega^2\,r\,t}{4}\sum_{l\geq 1}\frac{\omega^{2l}\,(r-t)^l}{2^{2l}\,(l+1)!\,l!}\sum_{j=0}^{l-1}\left[-\binom{l}{j+1}(-r)^{j+1}\,t^{l-j-1}+\binom{l}{j}(-r)^j\,t^{l-j}\right]\\
&= \frac{\omega^2\,r\,t}{4}\sum_{l\geq 1}\frac{\omega^{2l}\,(r-t)^l}{2^{2l}\,(l+1)!\,l!}\left[-(t-r)^l+t^l+(t-r)^l-(-r)^l\right]\\
&\overset{(11)}{=} g(t,r)-g(r,t),
\end{aligned}$$

thanks to the binomial theorem and (11), where the term corresponding to $l=0$ is zero. This proves (9), and the proof is complete. □

Proof of Theorem 2.5, part (ii). Let ϕ be a harmonic function, then $\phi\in C^\infty(D)$, thanks to the regularity theorem for harmonic functions (see, e.g., [17, Corollary 8.11]). We prove that $(\Delta+\omega^2)V_1[\phi](x)=0$. In order to do that, we establish some useful identities.

We set $r:=|x|$ and compute

$$\begin{aligned}
\frac{\partial}{\partial|x|}M_1(x,t) &= \omega\sqrt{1-t}\,\frac{\partial}{\partial(\omega r\sqrt{1-t})}\left[-\frac{\sqrt{t}^{N-2}}{2(1-t)}\;\omega r\sqrt{1-t}\;\;J_1(\omega r\sqrt{1-t})\right]\\
&\overset{(65)}{=} -\frac{\omega^2 r\sqrt{t}^{N-2}}{2}\;J_0(\omega r\sqrt{1-t}),\\
\Delta M_1(x,t) &= \frac{N-1}{r}\,\frac{\partial}{\partial|x|}M_1(x,t)+\frac{\partial^2}{\partial|x|^2}M_1(x,t)\\
&= -\frac{\omega^2\sqrt{t}^{N-2}}{2}\left(N\,J_0(\omega r\sqrt{1-t})-\omega r\sqrt{1-t}\,J_1(\omega r\sqrt{1-t})\right),
\end{aligned}\tag{13}$$

where the Laplacian acts on the x variable.

Since M_1 depends on x only through r, we can compute

$$\begin{aligned}
\Delta\Big(M_1(x,t)\phi(tx)\Big) &= \Delta M_1(x,t)\;\phi(tx)+2\nabla M_1(x,t)\cdot\nabla\phi(tx)+M_1(x,t)\Delta\phi(tx)\\
&= \Delta M_1(x,t)\;\phi(tx)+2\frac{\partial}{\partial|x|}M_1(x,t)\frac{x}{r}\cdot t\nabla\phi\Big|_{tx}+0\\
&= \Delta M_1(x,t)\;\phi(tx)+2\frac{t}{r}\frac{\partial}{\partial|x|}M_1(x,t)\frac{\partial}{\partial t}\phi(tx),
\end{aligned}$$

because $\frac{\partial}{\partial t}\phi(tx)=x\cdot\nabla\phi\Big|_{tx}$.

Finally, we define an auxiliary function $f_1:[0,h]\times[0,1]\to\mathbb{R}$ by

$$f_1(r,t)=\sqrt{t}^N\;J_0(\omega r\sqrt{1-t}).$$

This function verifies

$$\frac{\partial}{\partial t} f_1(r,t) = \frac{N\sqrt{t}^{N-2}}{2}\; J_0(\omega r\sqrt{1-t}) + \frac{\sqrt{t}^N \omega r}{2\sqrt{1-t}} J_1(\omega r\sqrt{1-t}),$$
$$f_1(r,0)=0, \qquad\qquad f_1(r,1)=1.$$

At this point, we can use all these identities to prove that $V_1[\phi]$ is a solution of the homogeneous Helmholtz equation:

$$\begin{aligned}
(\Delta+\omega^2)V_1[\phi](x) &= \Delta\phi(x) + \omega^2\phi(x) + \int_0^1 \Delta\Big(M_1(x,t)\phi(tx)\Big)\,\mathrm{d}t + \int_0^1 \omega^2 M_1(x,t)\phi(tx)\,\mathrm{d}t \\
&= \omega^2\phi(x) - \omega^2 \int_0^1 \sqrt{t}^N\; J_0(\omega r\sqrt{1-t})\frac{\partial}{\partial t}\phi(tx)\,\mathrm{d}t \\
&\quad - \omega^2 \int_0^1 \Bigg(\frac{N\sqrt{t}^{N-2}}{2}\; J_0(\omega r\sqrt{1-t}) - \frac{\omega r\sqrt{t}^{N-2}}{2}\frac{1-t}{\sqrt{1-t}}\; J_1(\omega r\sqrt{1-t}) \\
&\quad + \frac{\omega r\sqrt{t}^{N-2}}{2\sqrt{1-t}} J_1(\omega r\sqrt{1-t})\Bigg)\;\phi(tx)\,\mathrm{d}t \\
&= \omega^2\phi(x) - \omega^2 \int_0^1 \left(f_1(r,t)\frac{\partial}{\partial t}\phi(tx) + \frac{\partial}{\partial t} f_1(r,t)\phi(tx)\right)\,\mathrm{d}t \\
&= \omega^2\left(\phi(x) - \Big[f_1(r,t)\phi(tx)\Big]_{t=0}^{t=1}\right) = 0.
\end{aligned}$$

We have used the values assumed by ϕ only in the segment $[0,x]$ that lies inside D, because D is star-shaped with respect to 0. Thus, the values of the function ϕ and of its derivative are well defined and the fundamental theorem of calculus applies, thanks to the regularity theorem for harmonic functions.

Now, let u be a solution of the homogeneous Helmholtz equation. Since interior regularity results also hold for solutions of the homogeneous Helmholtz equation, we infer $u \in C^\infty(D)$. In order to prove that $\Delta V_2[u] = 0$, we proceed as before and compute

$$\begin{aligned}
\frac{\partial}{\partial |x|} M_2(x,t) &= \frac{\omega^2 r\sqrt{t}^{N-2}}{2}\; J_0(i\omega r\sqrt{t(1-t)}), \\
\Delta M_2(x,t) &= \frac{\omega^2\sqrt{t}^{N-2}}{2}\left(N\; J_0(i\omega r\sqrt{t(1-t)}) - i\omega r\sqrt{t(1-t)}\; J_1(i\omega r\sqrt{t(1-t)})\right), \\
\Delta\left(M_2(x,t)u(tx)\right) &= \Delta M_2(x,t)u(tx) + 2\frac{t}{r}\frac{\partial}{\partial r}M_2(x,t)\frac{\partial}{\partial t}u(tx) - \omega^2 t^2 M_2(x,t)u(tx),
\end{aligned}$$

and we define the function

$$f_2(r,t) = \sqrt{t}^N\; J_0(i\omega r\sqrt{t(1-t)}),$$

which verifies

$$\frac{\partial}{\partial t} f_2(r,t) = \frac{N\sqrt{t}^{N-2}}{2}\; J_0(i\omega r\sqrt{t(1-t)}) - \frac{\sqrt{t}^N i\omega r(1-2t)}{2\sqrt{t(1-t)}} J_1(i\omega r\sqrt{t(1-t)}),$$
$$f_2(r,0)=0, \qquad\qquad f_2(r,1)=1.$$

We conclude by computing the Laplacian of $V_2[u]$:

$$\begin{aligned}
\Delta V_2[u](x) &= \Delta u(x) + \int_0^1 \Delta\Big(M_2(x,t)u(tx)\Big)\,\mathrm{d}t \\
&= -\omega^2 u(x) + \omega^2 \int_0^1 \sqrt{t}^N\; J_0(i\omega r\sqrt{t(1-t)})\frac{\partial}{\partial t}u(tx)\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+\omega^2 \int_0^1 \frac{\sqrt{t}^{N-2}}{2} \Big(N\ J_0(i\omega r\sqrt{t(1-t)}) \\
&-i\omega r\sqrt{t}\frac{1-t}{\sqrt{1-t}} J_1(i\omega r\sqrt{t(1-t)}) + \frac{i\omega r t\sqrt{t}}{\sqrt{1-t}}\ J_1(i\omega r\sqrt{t(1-t)})\Big)\, u(tx)\,\mathrm{d}t \\
&= -\omega^2 u(x) + \omega^2 \int_0^1 \left(f_2(r,t)\frac{\partial}{\partial t}u(tx) + \frac{\partial}{\partial t} f_2(r,t) u(tx)\right)\,\mathrm{d}t = 0. \qquad \square
\end{aligned}
$$

Remark 2.8. *With a slight modification in the proof, it is possible to show that V_1 transforms the solutions of the homogeneous Helmholtz equation*

$$\Delta\phi + \omega_0^2\phi = 0$$

into solutions of

$$\Delta\phi + (\omega_0^2 + \omega^2)\phi = 0$$

for every ω and $\omega_0 \in \mathbb{C}$, and V_2 does the converse.

3. Continuity of the Vekua Operators

In the following theorem, we establish the continuity of V_1 and V_2 in Sobolev norms with continuity constants as explicit as possible.

Theorem 3.1. *Let D be a domain as in the Assumption 2.1; the Vekua operators*

$$V_1 : \mathcal{H}^j(D) \to \mathcal{H}^j_\omega(D) \quad , \quad V_2 : \mathcal{H}^j_\omega(D) \to \mathcal{H}^j(D)\ ,$$

with $\mathcal{H}^j(D)$ and $\mathcal{H}^j_\omega(D)$ both endowed with the norm $\|\cdot\|_{j,\omega,D}$ defined in (2), are continuous. More precisely, for all space dimensions $N \geq 2$, for all ϕ and u in $H^j(D)$, $j \geq 0$, solutions to (7) and (8), respectively, the following continuity estimates hold:

$$\|V_1[\phi]\|_{j,\omega,D} \leq C_1(N)\ \rho^{\frac{1-N}{2}}\ (1+j)^{\frac{3}{2}N+\frac{1}{2}}\ e^j\ \left(1+(\omega h)^2\right)\ \|\phi\|_{j,\omega,D}\ , \tag{14}$$

$$\|V_2[u]\|_{j,\omega,D} \leq C_2(N,\omega h,\rho)\ (1+j)^{\frac{3}{2}N-\frac{1}{2}}\ e^j\ \|u\|_{j,\omega,D}\ , \tag{15}$$

where the constant $C_1 > 0$ depends only on the space dimension N, and $C_2 > 0$ also depends on the product ωh and the shape parameter ρ. Moreover, we can establish the following continuity estimates for V_2 with constants depending only on N:

$$\|V_2[u]\|_{0,D} \leq C_N\ \rho^{\frac{1-N}{2}} \left(1+(\omega h)^4\right) e^{\frac{1}{2}(1-\rho)\omega h} \left(\|u\|_{0,D} + h\,|u|_{1,D}\right) \quad \text{if } N = 2,\dots,5,\ u \in H^1(D), \tag{16}$$

$$\|V_2[u]\|_{j,\omega,D} \leq C_N\ \rho^{\frac{1-N}{2}}\ (1+j)^{2N-1}\ e^j\ \left(1+(\omega h)^4\right)\ e^{\frac{3}{4}(1-\rho)\omega h}\ \|u\|_{j,\omega,D} \quad \text{if } N = 2,3,\ j \geq 1,\ u \in H^j(D), \tag{17}$$

and the following continuity estimates in L^∞–norm:

$$\|V_1[\phi]\|_{L^\infty(D)} \leq \left(1 + \frac{\left((1-\rho)\omega h\right)^2}{4}\right) \|\phi\|_{L^\infty(D)} \tag{18}$$

$$\|V_2[u]\|_{L^\infty(D)} \leq \left(1 + \frac{\left((1-\rho)\omega h\right)^2}{4}\ e^{\frac{1}{2}(1-\rho)\omega h}\right) \|u\|_{L^\infty(D)} \quad \text{if } N \geq 2,\ \phi,\ u \in L^\infty(D). \tag{19}$$

Theorem 3.1 states that the operators V_1 and V_2 preserve the Sobolev regularity when applied to harmonic functions and solutions of the homogeneous Helmholtz equation (see Theorem 2.5). For such functions, these operators are continuous from $H^j(D)$ to itself with continuity constants that depend on the wavenumber ω only through the product ωh. In two and three space dimensions, we can make explicit the dependence of the bounds on ωh. The only exception is the L^2-continuity of V_2 (see (16)), where a weighted H^1-norm appears on the right-hand side; this is due to the poor explicit interior estimates available for the solutions of the homogeneous Helmholtz equation.

All the continuity constants are explicit with respect to the order of the Sobolev norm and depend on D only through its shape parameter ρ and its diameter h, the latter only appearing within the product ωh.

In the literature, there exist many proofs of the continuity of V_1 and V_2 in L^∞-norm (in two space dimensions); see, for example, [7, 15]. To our knowledge, the only continuity result in Sobolev norms is the one given in [27, Section 4.2]: this holds for general PDEs and for norms with non-integer indices, but is restricted to the two-dimensional case, and the constants in the bounds are not explicit in the various parameters.

Since the proof of Theorem 3.1 is quite lengthy and requires several preliminary results, we give here a short outline. In Lemma 3.2, a direct attempt to compute the Sobolev norms of $V_\xi[\phi]$ shows that two types of intermediate estimates are required. The first ones consist in bounds of the kernel functions M_1 and M_2 in $W^{j,\infty}$–norms; these are proved in Lemma 3.3. The second ones are interior estimates for harmonic functions and for Helmholtz solutions: the former are well-known and recalled in Lemma 3.8, while the latter are proved in Lemma 3.11. Since we want explicit dependence of the bounding constants on the wave number, this step turns out to be the hardest one. Finally, we combine all these ingredients and prove Theorem 3.1.

From here on, if β is a multi-index in $\mathbb{N}^N$, we will denote by D^β the corresponding differential operator with respect to the space variable $x \in \mathbb{R}^N$; see (1).

Lemma 3.2. *For $\xi = 1, 2$, $j \geq 0$ and $\phi \in H^j(D)$, we have*

$$|V_\xi[\phi]|^2_{j,D} \leq 2\,|\phi|^2_{j,D} + 2(j+1)^{3N-2} e^{2j} \sum_{k=0}^{j} \sup_{t\in[0,1]} |M_\xi(\cdot,t)|^2_{W^{j-k,\infty}(D)} \cdot \sum_{|\beta|=k} \int_0^1 \int_D \left|D^\beta \phi(tx)\right|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{20}$$

Proof. From Definition 2.4, we have

$$\begin{aligned}
\left|V_\xi[\phi]\right|^2_{j,D} &\leq 2\,|\phi|^2_{j,D} + 2\sum_{|\alpha|=j} \int_D \left| \int_0^1 D^\alpha \left(M_\xi(x,t)\phi(tx)\right)\,\mathrm{d}t\right|^2 \mathrm{d}x \\
&\leq 2\,|\phi|^2_{j,D} + 2\sum_{|\alpha|=j} \int_D \int_0^1 \left| \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} M_\xi(x,t) D^\beta\phi(tx)\right|^2 \mathrm{d}t\,\mathrm{d}x \\
&\leq 2\,|\phi|^2_{j,D} + 2\int_D \int_0^1 \left| \sum_{k=0}^{j} \sum_{|\beta|=k} \left|D^\beta\phi(tx)\right| \sum_{\substack{|\alpha|=j\\ \alpha\geq\beta}} \binom{\alpha}{\beta} \left|D^{\alpha-\beta} M_\xi(x,t)\right| \right|^2 \mathrm{d}t\,\mathrm{d}x,
\end{aligned}$$

where in the second inequality we have applied the Jensen inequality and the product (Leibniz) rule for multi-indices (see [1, Sec. 1.1]); here, the binomial coefficient for multi-indices is $\binom{\alpha}{\beta} = \prod_{i=1}^N \binom{\alpha_i}{\beta_i}$. We multiply by the number $\binom{N+k-1}{N-1}$ of the multi-indices β of length k in $\mathbb{N}^N$, in order to move the square inside the sum, and we obtain

$$\left|V_\xi[\phi]\right|^2_{j,D} \leq 2\,|\phi|^2_{j,D} + 2\int_D \int_0^1 (j+1) \sum_{k=0}^{j} \binom{N+k-1}{N-1}$$

$$\cdot \sum_{|\beta|=k} \left|D^\beta \phi(tx)\right|^2 \left| \sum_{\substack{|\alpha|=j\\ \alpha\geq\beta}} \binom{\alpha}{\beta} \left|D^{\alpha-\beta} M_\xi(x,t)\right| \right|^2 \mathrm{d}t\,\mathrm{d}x$$

$$\leq 2\,|\phi|^2_{j,D} + 2(j+1)\tbinom{N+j-1}{N-1} \sum_{k=0}^{j} \sum_{|\beta|=k} \int_D \int_0^1 \left|D^\beta \phi(tx)\right|^2 \mathrm{d}t\,\mathrm{d}x$$

$$\cdot \sup_{t\in[0,1]} |M_\xi(\cdot,t)|^2_{W^{j-k,\infty}(D)} \sup_{|\beta|=k} \left[\sum_{\substack{|\alpha|=j\\ \alpha\geq\beta}} \binom{\alpha}{\beta} \right]^2 ;$$

the last factor can be bounded as

$$\sup_{|\beta|=k} \sum_{\substack{|\alpha|=j\\ \alpha\geq\beta}} \prod_{i=1}^{N} \binom{\alpha_i}{\beta_i} \leq \sup_{|\beta|=k} \sum_{\substack{|\alpha|=j\\ \alpha\geq\beta}} \prod_{i=1}^{N} \frac{\alpha_i^{\beta_i}}{\beta_i!} \leq \sum_{|\alpha|=j} e^{\sum_{i=1}^N \alpha_i}$$

$$\leq e^j \cdot \#\{\alpha\in\mathbb{N}^N,\ |\alpha|=j\} = e^j\ \tbinom{N+k-1}{N-1}.$$

Finally, we note that, for every $j\in\mathbb{N}$, $N\geq 2$, we have

$$\binom{N+j-1}{N-1} = \frac{N+j-1}{N-1}\ \frac{N+j-2}{N-2} \cdots \frac{1+j}{1} \leq (1+j)^{N-1}\ , \tag{21}$$

from which the assertion follows. □

Now we need to bound the terms present in (20). The next lemma provides $W^{j,\infty}(D)$–estimates for M_1 and M_2 uniformly in t. The proof relies on some properties of Bessel functions.

Lemma 3.3. *The functions M_1 and M_2 satisfy the following bounds:*

$$\|M_1\|_{L^\infty(D\times[0,1])} \leq \frac{\big((1-\rho)\ \omega\ h\big)^2}{4}, \tag{22}$$

$$\sup_{t\in[0,1]} |M_1(\cdot,t)|_{W^{1,\infty}(D)} \leq \frac{(1-\rho)\ \omega^2\ h}{2}, \tag{23}$$

$$\sup_{t\in[0,1]} |M_1(\cdot,t)|_{W^{j,\infty}(D)} \leq \frac{\omega^j}{2}\ (j+(1-\rho)\ \omega\ h) \qquad \forall j\geq 2, \tag{24}$$

$$\|M_2\|_{L^\infty(D\times[0,1])} \leq \frac{\big((1-\rho)\ \omega\ h\big)^2}{4}\ e^{\frac12(1-\rho)\omega h}\ , \tag{25}$$

$$\sup_{t\in[0,1]} |M_2(\cdot,t)|_{W^{1,\infty}(D)} \leq \frac{(1-\rho)\ \omega^2\ h}{2}\ e^{\frac12(1-\rho)\omega h}, \tag{26}$$

$$\sup_{t\in[0,1]} |M_2(\cdot,t)|_{W^{j,\infty}(D)} \leq \frac{\omega^j}{2^{j-1}}\ \left(j+\frac{(1-\rho)\ \omega\ h}{2}\right) e^{\frac34(1-\rho)\omega h} \qquad \forall j\geq 2. \tag{27}$$

Proof. Thanks to Remark 2.2, we have that $\sup_{x\in D}|x| \leq (1-\rho)\ h$. Now, the L^∞–inequalities (22) and (25) follow directly from (63).

Since M_1 and M_2 depend on x only through $|x|$, we obtain the $W^{1,\infty}$ bounds (23) and (26):

$$\sup_{t\in[0,1]} |M_1(\cdot,t)|_{W^{1,\infty}(D)} = \sup_{t\in[0,1],\ x\in D} \left| \frac{\partial}{\partial|x|} M_1(x,t) \right|$$

$$\overset{(65)}{\leq} \sup_{\substack{t\in[0,1],\\ |x|\in[0,(1-\rho)h]}} \left| \frac{\omega^2 |x| \sqrt{t}^{N-2}}{2}\ J_0(\omega|x|\sqrt{1-t}) \right| \overset{(62)}{\leq} \frac{(1-\rho)\ \omega^2\ h}{2},$$

$$\sup_{t\in[0,1]} |M_2(\cdot,t)|_{W^{1,\infty}(D)} \overset{(65)}{\leq} \sup_{\substack{t\in[0,1],\\ |x|\in[0,(1-\rho)h]}} \left| \frac{\omega^2|x|\sqrt{t}^{N-2}}{2} \; J_0(i\omega|x|\sqrt{t(1-t)}) \right|$$
$$\overset{(63)}{\leq} \frac{(1-\rho)\,\omega^2\,h}{2}\; e^{\frac{1}{2}(1-\rho)\omega h}.$$

In order to prove (24) and (27), we define the auxiliary complex-valued function $f(s) = s\,J_1(s)$. It is easy to verify by induction that its derivative of order k is

$$\frac{\partial^k}{\partial s^k} f(s) = k\,\frac{\partial^{k-1}}{\partial s^{k-1}} J_1(s) + s\,\frac{\partial^k}{\partial s^k} J_1(s).$$

We can bound this derivative using (66) and the binomial theorem:

$$\left|\frac{\partial^k}{\partial s^k} f(s)\right| = \left| k\,\frac{1}{2^{k-1}}\sum_{m=0}^{k-1}(-1)^m\binom{k-1}{m} J_{2m-k+2}(s) + s\,\frac{1}{2^k}\sum_{m=0}^{k}(-1)^m\binom{k}{m} J_{2m-k+1}(s)\right| \tag{28}$$
$$\leq (k+|s|) \max_{l=1-k,\dots,1+k} |J_l(s)|\;.$$

The functions M_1 and M_2 are related to f by

$$M_1(x,t) = -\frac{\sqrt{t}^{N-2}}{2(1-t)}\; f(\omega|x|\sqrt{1-t}),$$
$$M_2(x,t) = -\frac{\sqrt{t}^{N-4}}{2(1-t)}\; f(i\omega|x|\sqrt{t(1-t)}),$$

so we can bound their derivatives of order $j\geq 2$:

$$\sup_{t\in[0,1]} |M_1|_{W^{j,\infty}(D)} \leq \sup_{t\in[0,1],\, x\in D} \left| \frac{\partial^j}{\partial |x|^j} M_1(x,t)\right|$$
$$\leq \sup_{t\in[0,1],\, x\in D} \left| \frac{\sqrt{t}^{N-2}}{2(1-t)} \left(\omega\sqrt{1-t}\right)^j \frac{\partial^j}{\partial(\omega|x|\sqrt{1-t})^j} f(\omega|x|\sqrt{1-t}) \right|$$
$$\overset{(28),\,(62)}{\leq} \frac{\omega^j}{2}\;(j+(1-\rho)\omega h),$$

$$\sup_{t\in[0,1]} |M_2|_{W^{j,\infty}(D)} \leq \sup_{t\in[0,1],\, x\in D} \left| \frac{\sqrt{t}^{N-4}}{2(1-t)} \left(i\omega\sqrt{t(1-t)}\right)^j \frac{\partial^j}{\partial(i\omega|x|\sqrt{t(1-t)})^j} f(i\omega|x|\sqrt{t(1-t)}) \right|$$
$$\overset{(28),\,(63)}{\leq} \frac{\omega^j}{2^{j-1}}\; \left(j + \frac{(1-\rho)\omega h}{2}\right) e^{\frac{3}{4}(1-\rho)\omega h}.$$

□

Remark 3.4. *With less detail the bounds of Lemma 3.3 for every $j\geq 0$ can be summarized as:*

$$\sup_{t\in[0,1]} |M_1(\cdot,t)|_{W^{j,\infty}(D)} \leq \omega^j\,\left(j+(\omega h)^2\right), \tag{29}$$
$$\sup_{t\in[0,1]} |M_2(\cdot,t)|_{W^{j,\infty}(D)} \leq \omega^j\,(1+\omega h)\, e^{\frac{3}{4}(1-\rho)\omega h}. \tag{30}$$

We ignore the algebraic dependence on ρ because it will be absorbed in a generic bounding constant. In a shape regular domain, a precise lower bound for $\rho\in(0,\frac{1}{2}]$ can be used to reduce the exponential dependence on ωh.

Remark 3.5. *By performing some small changes in the proof of Lemma 3.3, we can extend Theorem 3.1 to every $\omega\in\mathbb{C}$, similarly to Theorem 2.5 (see Remark 2.6). In fact, the case $\omega=0$ is trivial, since V_1 and V_2 reduce to the identity, while in general, Theorem 3.1 holds by substituting ω with $|\omega|$ in the estimates and in the definition of the weighted norm (2), and multiplying the right-hand side of (14) by $e^{\frac{3}{2}|\omega|h}$ (see Remark 1.2.5 in [21]).*

Lemma 3.6. *Let $\phi \in H^k(D)$, $\beta \in \mathbb{N}^N$ be a multi-index of length $|\beta| = k$ and D^β be the corresponding differential operator in the variable x. Then*

$$\int_0^1 \int_D \left|D^\beta \phi(tx)\right|^2 \mathrm{d}x\,\mathrm{d}t \le \begin{cases} \dfrac{1}{2k-N+1} \left\|D^\beta \phi\right\|_{0,D}^2 & \text{if } 2k-N \ge 0, \\ K \left\|D^\beta \phi\right\|_{0,D}^2 + \left(\dfrac{\rho}{2}\right)^{2k+1} \dfrac{|D|}{2k+1} \left\|D^\beta \phi\right\|_{L^\infty(B_{\frac{\rho h}{2}})}^2 & \text{if } 2k-N < 0, \end{cases}$$

where $K = \log \frac{2}{\rho}$ if $2k - N = -1$, $K = \left(\frac{2}{\rho}\right)^{N-1}$ if $2k - N < -1$, $|D|$ denotes the measure of D and ρ is given in Assumption 2.1.

Proof. In the first case, we can simply compute the integral with respect to t with the change of variables $y = tx$:

$$\begin{aligned} \int_0^1 \int_D \left|D^\beta \phi(tx)\right|^2 \mathrm{d}x\,\mathrm{d}t &= \int_0^1 \int_{tD} t^{2|\beta|} \left|D^\beta \phi(y)\right|^2 \frac{\mathrm{d}y}{t^N}\,\mathrm{d}t \\ &\le \frac{1}{2k-N+1} \left\|D^\beta \phi\right\|_{0,D}^2 ; \end{aligned}$$

the set tD is included in D because D is star-shaped with respect to 0.

In the case $2k - N < 0$, the integral in t is not bounded so we need to split it in two parts, treating the second part as before:

$$\begin{aligned} \int_0^1 \int_D \left|D^\beta \phi(tx)\right|^2 \mathrm{d}x\,\mathrm{d}t &= \int_0^{\frac{\rho}{2}} \int_D \left|D^\beta \phi(tx)\right|^2 \mathrm{d}x\,\mathrm{d}t + \int_{\frac{\rho}{2}}^1 \int_D \left|D^\beta \phi(tx)\right|^2 \mathrm{d}x\,\mathrm{d}t \\ &\le \int_0^{\frac{\rho}{2}} t^{2|\beta|}\,\mathrm{d}t |D| \left\|D^\beta \phi\right\|_{L^\infty(B_{\frac{\rho h}{2}})}^2 + \int_{\frac{\rho}{2}}^1 t^{2k-N} \left\|D^\beta \phi\right\|_{0,tD}^2 \mathrm{d}t \\ &= \frac{1}{2k+1} \left(\frac{\rho}{2}\right)^{2k+1} |D| \left\|D^\beta \phi\right\|_{L^\infty(B_{\frac{\rho h}{2}})}^2 + \int_{\frac{\rho}{2}}^1 t^{2k-N} \left\|D^\beta \phi\right\|_{0,tD}^2 \mathrm{d}t, \end{aligned}$$

and the assertion comes from the expression

$$\int_{\frac{\rho}{2}}^{1} t^{2k-N}\,\mathrm{d}t = \begin{cases} \log \dfrac{2}{\rho} & \text{if } 2k-N = -1, \\ \dfrac{1 - \left(\frac{\rho}{2}\right)^{2k-N+1}}{2k-N+1} \le \left(\dfrac{2}{\rho}\right)^{N-1} & \text{if } 2k-N < -1. \end{cases} \qquad \square$$

Remark 3.7. *We can improve the bounds of Lemma 3.6 for every value of the multi-index length k with the estimate*

$$\int_0^1 \int_D \left|D^\beta \phi(tx)\right|^2 \mathrm{d}x\,\mathrm{d}t \le \left(\frac{2}{\rho}\right)^{N-1} \left\|D^\beta \phi\right\|_{0,D}^2 + \left(\frac{\rho}{2}\right)^{2k+1} \frac{|D|}{2k+1} \left\|D^\beta \phi\right\|_{L^\infty(B_{\frac{\rho h}{2}})}^2 . \tag{31}$$

From Lemma 3.6, it is clear that, in order to prove the continuity of V_1 and V_2 in the L^2-norm and in high-order Sobolev norms, we need interior estimates that bound the L^∞-norm of ϕ and its derivatives in a small ball contained in D with its L^2-norm and H^j-norms in D. It is easy to find such estimates for harmonic functions, thanks to the mean value theorem (see, e.g., Theorem 2.1 of [17]).

Notice that it is not possible to avoid the use of interior estimates for the continuity in $H^j(D)$ when $j \ge \frac{N}{2}$, as the assertion of Lemma 3.6 might suggest: in fact, Lemma 3.2 requires to estimate $\int_0^1 \int_D \left|D^\beta \phi(tx)\right|^2 \mathrm{d}x\,\mathrm{d}t$ for all the multi-index lengths $|\beta| = k \le j$, so we inevitably confront the cases $2k - N = -1$ and $2k - N < -1$.

Lemma 3.8 (Interior estimates for harmonic functions). *Let ϕ be a harmonic function in $B_R(x)$, $R > 0$, then*

$$|\phi(x)|^2 \le \frac{1}{R^N\,|B_1|} \|\phi\|_{0,B_R(x)}^2 , \tag{32}$$

where $|B_1| = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}$ *is the volume of the unit ball in* $\mathbb{R}^N$. *If* $\phi \in H^k(D)$ *and* $\beta \in \mathbb{N}^N$, $|\beta| \le k$, *then*

$$\left\|D^\beta \phi\right\|^2_{L^\infty(B_{\frac{\rho h}{2}})} \le \frac{1}{|B_1|}\left(\frac{2}{\rho h}\right)^N \left\|D^\beta \phi\right\|^2_{0,D}, \tag{33}$$

Proof. By the mean value property of harmonic functions (see Theorem 2.1 of [17]) and the Jensen inequality, we get the first estimate:

$$\begin{aligned}|\phi(x)|^2 &= \left|\frac{1}{|B_R(x)|}\int_{B_R(x)} \phi(y)\,\mathrm{d}y\right|^2 \\ &\le \frac{1}{|B_R|}\int_{B_R(x)} |\phi(y)|^2\;\mathrm{d}y = \frac{1}{R^N\;|B_1|}\left\|\phi\right\|^2_{0,B_R(x)}.\end{aligned}$$

The second bound follows by applying the first one to the derivatives of ϕ, which are harmonic in the ball $B_{\frac{\rho h}{2}}(x) \subset B_{\rho h} \subset D$. □

Remark 3.9. *The interior estimates for harmonic functions are related to Cauchy's estimates for their derivatives. Theorem 2.10 in [17] states that, given two domains* $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^N$ *such that* $d(\Omega_1, \partial\Omega_2) = d$, *if* ϕ *is harmonic in* Ω_2, *then for every multi-index* α *it holds*

$$\left\|D^\alpha \phi\right\|_{L^\infty(\Omega_1)} \le \left(\frac{N|\alpha|}{d}\right)^{|\alpha|} \left\|\phi\right\|_{L^\infty(\Omega_2)}. \tag{34}$$

In order to find analogous estimates for the Sobolev norms, we can combine (34) *and* (32) *using the intermediate domain* $\{x \in \mathbb{R}^N\;:\; d(x,\Omega_1) < \frac{d}{2}\}$ *and obtain*

$$\left\|D^\alpha \phi\right\|_{0,\Omega_1} \le C_{N,\alpha} |\Omega_1|^{N/2}\, d^{-|\alpha|-N/2} \left\|\phi\right\|^2_{0,\Omega_2},$$

but the order of the power of d is not satisfactory. In order to improve it, we represent the derivatives of a harmonic function ψ *in* $\overline{B}_1 \subset \mathbb{R}^N$ *using the Poisson kernel* P:

$$D^\alpha \psi(y) = \int_{S^{N-1}} \psi(z)\; D_1^\alpha P(y,z)\,\mathrm{d}\sigma(z) \qquad y \in B_1,\ \forall\, \alpha \in \mathbb{N}^N,$$

where the derivatives of P *are taken with respect to the first variable (see (1.22) in [3]). Rewriting this formula in* $y = 0$ *and then translating in a point* x, *if* ψ *is harmonic in* $\overline{B}_1(x)$, *we have*

$$D^\alpha \psi(x) = \int_{S^{N-1}} \psi(x+z)\; D_1^\alpha P(0,z)\,\mathrm{d}\sigma(z) \qquad \forall\, \alpha \in \mathbb{N}^N.$$

Given two domains $\hat\Omega_1 \subset \hat\Omega_2$ *such that* $d(\hat\Omega_1, \partial\hat\Omega_2) = 1$, *if* $\hat\phi$ *is harmonic in* $\hat\Omega_2$, *it holds*

$$\begin{aligned}\left\|D^\alpha \hat\phi\right\|_{0,\hat\Omega_1} &= \int_{\hat\Omega_1} |D^\alpha \hat\phi(x)|^2\,\mathrm{d}x = \int_{\hat\Omega_1}\left|\int_{S^{N-1}} \hat\phi(x+z)\; D_1^\alpha P(0,z)\,\mathrm{d}\sigma(z)\right|^2 \mathrm{d}x \\ &\overset{y=x+z}{\le} |S^{N-1}| \int_{S^{N-1}}\left(\int_{\hat\Omega_2} |\hat\phi(y)|^2\;\mathrm{d}y\right)|D_1^\alpha P(0,z)|^2\;\mathrm{d}\sigma(z) \le\; C_{N,\alpha}\;\left\|\hat\phi\right\|_{0,\hat\Omega_2},\end{aligned}$$

where we have used the Jensen inequality and the Fubini theorem. By summing over all the multi-indices of the same length and scaling the domains such that $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^N$ *and* $d(\Omega_1, \partial\Omega_2) = d$, *we finally obtain*

$$|\phi|_{j+k,\Omega_1} \le C_{N,j,k}\; d^{-k}\;|\phi|_{j,\Omega_2}, \qquad j,\,k \in \mathbb{N}. \tag{35}$$

We can use the bicontinuity of the Vekua operator to prove an analogous result for the solutions of the Helmholtz equations; see Lemma 3.2.1 of [21].

The main tool used to prove the interior estimates for harmonic functions is the mean value theorem. For the solutions of the homogeneous Helmholtz equation, we have an analogous mean value formula [14, page 289] but it does not provide good estimates.

Another way to prove interior estimates for the solutions of the homogeneous Helmholtz equation is to use the Green formula for the Laplacian in a ball, but this gives estimates that either involve the H^1-norm of u on the right-hand side of the bound or give bad order in the domain diameter R.

A third way is to use the technique presented in Lemma 4.2.7 of [27] for the two-dimensional case. This method can be generalized only to three space dimensions, and does not provide estimates with only the L^2-norm of u on the right-hand side. On the other hand, it is possible to make the dependence of the bounding constants on ωR explicit. We will prove these interior estimates in Lemma 3.11.

A more general way is to use Theorem 8.17 of [17]. This holds in every space dimension with the desired norms and the desired order in R. The only shortcoming of this result is that the bounding constant still depends on the product ωR but this dependence is not explicit. We report this result in Theorem 3.10.

Summarizing: we are able to prove interior estimates for homogeneous Helmholtz solutions with sharp order in R in two fashions. Theorem 3.10 works in any space dimension and with only the L^2–norm on the right-hand side. Lemma 3.11 works only in low space dimensions and with different norms but the constant in front of the estimates is explicit in ωR. Both techniques, however, allow to prove the final best approximation results we are looking for with the same order and in the same norms.

Theorem 3.10 (Interior estimates for Helmholtz solutions, version 1[2]). *For every $N \geq 2$, let $u \in H^1(B_R(x_0))$ be a solution of the homogeneous Helmholtz equation. Then there exists a constant $C > 0$ depending only on the product ωR and the dimension N, such that*

$$\|u\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq C(\omega R, N)\, R^{-\frac{N}{2}}\; \|u\|_{0,B_R(x_0)}\,. \tag{36}$$

Lemma 3.11 (Interior estimates for Helmholtz solutions, version 2). *Let $u \in H^1(B_R(x_0))$ be a solution of the inhomogeneous Helmholtz equation*

$$-\Delta u - \omega^2 u = f,$$

with $f \in H^1(B_R(x_0))$. Then there exists a constant $C > 0$ depending only on the space dimension N such that

$$\begin{aligned} \|u\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq{}& C\, R^{-1} \Big(\left(1+\omega^2R^2\right) \|u\|_{0,B_R(x_0)} + R\; \|\nabla u\|_{0,B_R(x_0)} \\ &+ R^2 \, \|f\|_{0,B_R(x_0)} \Big) \qquad \text{for } N=2, \end{aligned} \tag{37}$$

$$\begin{aligned} \|u\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq{}& C\, R^{-\frac{N}{2}} \Big((1+\omega^2R^2) \left(\|u\|_{0,B_R(x_0)} + R\; \|\nabla u\|_{0,B_R(x_0)}\right) \\ &+ R^2 \, \|f\|_{0,B_R(x_0)} + R^3 \, \|\nabla f\|_{0,B_R(x_0)} \Big) \qquad \text{for } N=3,4,5, \end{aligned} \tag{38}$$

$$\begin{aligned} \|\nabla u\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq{}& C\, R^{-\frac{N}{2}} \Big(\omega^2 R\, \|u\|_{0,B_R(x_0)} + (1+\omega^2R^2)\; \|\nabla u\|_{0,B_R(x_0)} \\ &+ R\, \|f\|_{0,B_R(x_0)} + R^2 \, \|\nabla f\|_{0,B_R(x_0)} \Big) \qquad \text{for } N=2,3. \end{aligned} \tag{39}$$

Remark 3.12. *In the homogeneous case, Lemma 3.11 reads as follows. Let $u \in H^1(B_R(x_0))$ be a solution of the homogeneous Helmholtz equation. Then there exists a constant $C > 0$ depending only on the space dimension N such that for*

$$\begin{aligned} & N = 2,3,4,5: \\ & \|u\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq C\, R^{-\frac{N}{2}} (1+\omega^2R^2)\, \left(\|u\|_{0,B_R(x_0)} + R\; \|\nabla u\|_{0,B_R(x_0)}\right), \\ & N = 2,3: \end{aligned} \tag{40}$$

[2] This is exactly Theorem 8.17 of [17]; with that notation, for the homogeneous Helmholtz equation we have $k(R) = 0$, $\lambda = 1$, $\Lambda = \sqrt{N}$ $\nu = \omega$ and $p = 2$ (q is not relevant for the homogeneous problem); see also [17], p. 178.

$$\|\nabla u\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq C\, R^{-\frac{N}{2}} \left(\omega^2 R\, \|u\|_{0,B_R(x_0)} + (1+\omega^2 R^2)\; \|\nabla u\|_{0,B_R(x_0)}\right). \tag{41}$$

Proof of Lemma 3.11. It is enough to bound $|u(x_0)|$ and $|\nabla u(x_0)|$, because for all $x \in B_{\frac{R}{2}}(x_0)$ we can repeat the proof using $B_{\frac{R}{2}}(x)$ instead of $B_R(x_0)$ with the same constants. We can also fix $x_0 = 0$.

Let $\varphi : \mathbb{R}^+ \to [0,1]$ be a smooth cut-off function such that

$$\varphi(r) = \begin{cases} 1 & |r| \leq \frac{1}{4}, \\ 0 & |r| \geq \frac{3}{4}, \end{cases}$$

and $\varphi_R : \mathbb{R}^N \to [0,1]$, $\varphi_R(x) := \varphi(\frac{|x|}{R})$. Then

$$\nabla\varphi_R(x) = \varphi'\left(\frac{|x|}{R}\right)\frac{x}{R|x|}, \qquad \Delta\varphi_R(x) = \frac{1}{R^2}\varphi''\left(\frac{|x|}{R}\right) + \frac{N-1}{R|x|}\varphi'\left(\frac{|x|}{R}\right).$$

We define the average of u and two auxiliary functions on B_R:

$$\overline{u} := \frac{1}{|B_R|}\int_{B_R} u(y)\,\mathrm{d}y,$$

$$g(x) := u(x)\;\varphi_R(x), \qquad \overline{g}(x) := (u(x) - \overline{u})\;\varphi_R(x);$$

their Laplacians are:

$$\begin{aligned}
\tilde{f}(x) :&= \tilde{f}_1(x) + \tilde{f}_2(x) + \tilde{f}_3(x) := -\Delta g(x) \\
&= -\left[\frac{1}{R^2}\varphi''\big(\tfrac{|x|}{R}\big) + \frac{N-1}{R|x|}\varphi'\big(\tfrac{|x|}{R}\big)\right] u(x) - 2\varphi'\big(\tfrac{|x|}{R}\big)\frac{x}{R|x|}\cdot\nabla u(x) + \varphi\big(\tfrac{|x|}{R}\big)\;(\omega^2 u(x) + f(x)), \\
\overline{f}(x) :&= \overline{f}_1(x) + \overline{f}_2(x) + \overline{f}_3(x) := -\Delta\overline{g}(x) \\
&= -\left[\frac{1}{R^2}\varphi''\big(\tfrac{|x|}{R}\big) + \frac{N-1}{R|x|}\varphi'\big(\tfrac{|x|}{R}\big)\right](u(x) - \overline{u}) - 2\varphi'\big(\tfrac{|x|}{R}\big)\frac{x}{R|x|}\cdot\nabla u(x) + \varphi\big(\tfrac{|x|}{R}\big)\;(\omega^2 u(x) + f(x)).
\end{aligned}$$

The fundamental solution formula for Poisson equation states that, if $-\Delta a = b$ in $\mathbb{R}^N$, then

$$a(x) = \int_{\mathbb{R}^N}\Phi(x-y)\;b(y)\,\mathrm{d}y, \quad \text{with} \quad \Phi(x) = \begin{cases} -\dfrac{1}{2\pi}\;\log|x| & N = 2, \\ \dfrac{|x|^{2-N}}{N(N-2)|B_1|} & N \geq 3. \end{cases} \tag{42}$$

The identity (42) holds for all $b \in L^2(B_R)$, thanks to Theorem 9.9 of [17]. We notice that

$$|\nabla\Phi(x)| = \left|-\frac{1}{N\,|B_1|}\;\frac{x}{|x|^N}\right| = \frac{1}{N\,|B_1|}\;|x|^{1-N} \qquad \forall\, N \geq 2.$$

We start by bounding $|u(0)|$ for $N = 2$. In this case, it is easy to see that, for all $R > 0$, we have

$$\int_{B_R}\big(\log|x| - \log R\big)^2\,\mathrm{d}x = \frac{\pi}{2}R^2. \tag{43}$$

We note that from the divergence theorem

$$\int_{B_R}\tilde{f}(y)\,\mathrm{d}y = -\int_{B_R}\Delta g(y)\,\mathrm{d}y = -\int_{\partial B_R}\nabla g(s)\cdot\mathbf{n}\,\mathrm{d}s = 0,$$

because $g \equiv 0$ in $\mathbb{R}^2 \setminus B_{\frac{3}{4}R}$ and, since $\tilde{f} = 0$ outside $B_{\frac{3}{4}R}$ then $\tilde{f}$ has zero mean value in the whole $\mathbb{R}^2$.

We apply (42) with $a = g$ and $b = \tilde{f}$; using the Cauchy–Schwarz inequality, the identity (43) and the fact that $\tilde{f}$ has zero mean value in $\mathbb{R}^2$, we obtain:

$$\begin{aligned}|u(0)| = |g(0)| &= \left|-\frac{1}{2\pi}\int_{\mathbb{R}^2}\big(\log|y| - \log R\big)\tilde{f}(y)\,\mathrm{d}y\right| \le \frac{1}{2\pi}\sqrt{\frac{\pi}{2}}\,R\,\|\tilde{f}\|_{0,B_{\frac{3}{4}R}}\\ &\le C_{N,\varphi}R\left(\frac{1}{R^2}\|u\|_{0,B_R} + \frac{1}{R}\|\nabla u\|_{0,B_R} + \omega^2\|u\|_{0,B_R} + \|f\|_{0,B_R}\right),\end{aligned}$$

where the constant $C_{N,\varphi}$ depends only on N and φ; in the last step we have used the definition of $\tilde{f}$ and the fact that $\varphi'(\frac{|x|}{R}) = 0$ in $B_{\frac{R}{4}}$. The estimate (37) easily follows.

Proving all the other bounds (on $|u(0)|$ for $N \ge 2$ and on $|\nabla u(0)|$ for $N \ge 2$) is more involved. We fix p, $p' > 1$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. For $\alpha > 0$, we calculate

$$\begin{aligned}\||y|^\alpha\|_{L^{p'}(B_R)} &= \left(\int_{S^{N-1}}\int_0^R r^{\alpha p'}r^{N-1}\,\mathrm{d}r\,\mathrm{d}S\right)^{\frac{1}{p'}}\\ &= \left(\frac{|S^{N-1}|}{\alpha p' + N}\right)^{\frac{1}{p'}} R^{\alpha+\frac{N}{p'}} = C_{N,p',\alpha}R^{\alpha+N-\frac{N}{p}},\end{aligned} \tag{44}$$

that holds if $\alpha p' + N \ne 0$, that is equivalent to $(\alpha + N)p \ne N$, for every $N \ge 2$. We compute also

$$\begin{aligned}\|\Phi\|_{L^p(B_{\frac{3}{4}R}\setminus B_{\frac{1}{4}R})} &= C_{N,p}\left(|S^{N-1}|\int_{\frac{1}{4}R}^{\frac{3}{4}R} r^{(2-N)p}\,r^{N-1}\,\mathrm{d}r\right)^{\frac{1}{p}}\\ &= C_{N,p}\,|S^{N-1}|^{\frac{1}{p}}\left(\left(\frac{3}{4}R\right)^{(2-N)p+N} - \left(\frac{1}{4}R\right)^{(2-N)p+N}\right)^{\frac{1}{p}}\\ &= C_{N,p}\,R^{2-N+\frac{N}{p}},\end{aligned} \tag{45}$$

for every $p \ne \frac{N}{N-2}$, $N \ge 3$, and the analogue

$$\begin{aligned}\|\nabla\Phi\|_{L^p(B_{\frac{3}{4}R}\setminus B_{\frac{1}{4}R})} &= C_{N,p}\left(|S^{N-1}|\int_{\frac{1}{4}R}^{\frac{3}{4}R} r^{(1-N)p}\,r^{N-1}\,\mathrm{d}r\right)^{\frac{1}{p}}\\ &= C_{N,p}\,R^{1-N+\frac{N}{p}},\end{aligned} \tag{46}$$

that holds for every $p \ne \frac{N}{N-1}$, $N \ge 2$.

Then, for all $\psi \in H_0^1(B_R)$, using scaling arguments, the continuity of the Sobolev embeddings $H_0^1(B_1) \hookrightarrow L^p(B_1)$ which hold provided that $2 \le p \le \frac{2N}{N-2}$, if $N \ge 3$, and $2 \le p < \infty$, if $N = 2$ (see [1, Th. 5.4,I,A-B]), and the Poincaré inequality, we obtain

$$\begin{aligned}\|\psi\|_{L^p(B_R)} &= R^{\frac{N}{p}}\,\|\hat{\psi}\|_{L^p(B_1)} \le C_{N,p}\,R^{\frac{N}{p}}\,\|\hat{\psi}\|_{1,B_1}\\ &\le C_{N,p}\,R^{\frac{N}{p}}\,\|\nabla\hat{\psi}\|_{0,B_1} \le C_{N,p}\,R^{\frac{N}{p}+1-\frac{N}{2}}\,\|\nabla\psi\|_{0,B_R}.\end{aligned} \tag{47}$$

Now we can estimate u in the case $N \ge 3$. From the Hölder inequality for the pair of spaces $L^{p'}$, L^p, $p > 2$ (thus, $p' < 2$), and the fact that $\tilde{f}_1 \equiv \tilde{f}_2 \equiv 0$ in $B_{\frac{1}{4}R}$ (see the definition of $\tilde{f}$), we can write

$$\begin{aligned}|u(0)| = |g(0)| &= \left|\int_{\mathbb{R}^N}\Phi(x)\tilde{f}(x)\,\mathrm{d}x\right|\\ &\le \|\Phi\|_{L^p(B_{\frac{3}{4}R}\setminus B_{\frac{1}{4}R})}\,\|\tilde{f}_1 + \tilde{f}_2\|_{L^{p'}(B_{\frac{3}{4}R}\setminus B_{\frac{1}{4}R})} + \|\Phi\|_{L^{p'}(B_R)}\,\|\tilde{f}_3\|_{L^p(B_R)}.\end{aligned}$$

Using (45) to bound the L^p-norm of Φ, the continuity of the embedding of $L^{p'}(B_{\frac34 R}\setminus B_{\frac14 R})$ into $L^2(B_{\frac34 R}\setminus B_{\frac14 R})$ (recall that $1<p'<2$) with constant $|B_{\frac34 R}\setminus B_{\frac14 R}|^{\frac{1}{p'}-\frac12}$ for the norm of $\tilde f_1+\tilde f_2$, the definition (42) of Φ and (44) with $\alpha = 2-N$, which requires $p>\frac N2$, to bound the $L^{p'}$-norm of Φ, and finally (47)which requires $2\le p\le \frac{2N}{N-2}$, to bound the norm of $\tilde f_3$ (recall that $\tilde f_3\in H^1_0(B_R)$), we have

$$|u(0)| \le C_{N,p} R^{2-N+\frac Np}\; |B_{\frac34 R}|^{\frac{1}{p'}-\frac12} \left\|\tilde f_1+\tilde f_2\right\|_{0,B_{\frac34 R}\setminus B_{\frac14 R}} + C_{N,p}R^{2-\frac Np}\; R^{\frac Np+1-\frac N2}\left\|\nabla\tilde f_3\right\|_{0,B_R}$$

Finally, using the definitions of the $\tilde f_i$'s, $|\nabla\varphi_R|\le \frac1R C_\varphi$ and $\frac1p+\frac1{p'}=1$ we obtain

$$\begin{aligned}
|u(0)| \le& C_{N,p,\varphi} R^{2-N+\frac Np}\; R^{\frac{N}{p'}-\frac N2}\left(\frac{1}{R^2}\|u\|_{0,B_R}+\frac1R\|\nabla u\|_{0,B_R}\right)\\
&+C_{N,p,\varphi}R^{3-\frac N2}\;\left(\omega^2\|\nabla u\|_{0,B_R}+\|\nabla f\|_{0,B_R}+\frac1R\omega^2\|u\|_{0,B_R}+\frac1R\|f\|_{0,B_R}\right)\\
\le& C_{N,p,\varphi}\;R^{-\frac N2}\left((1+\omega^2R^2)\|u\|_{0,B_R}+R\,(1+\omega^2R^2)\|\nabla u\|_{0,B_R}+R^2\|f\|_{0,B_R}+R^3\|\nabla f\|_{0,B_R}\right).
\end{aligned}$$

The previous argument for bounding $|u(0)|$ requires that there exists p such that $\frac N2<p\le\frac{2N}{N-2}$, which is possible only if $N<6$; this is the reason of the upper bound on the space dimension in the statement.

In order to conclude this proof, we have to estimate $|\nabla u(0)|$. We use the same technique as before, after differentiating the relation (42) with $a=\overline g$ and $b=\overline f$. For every $N\ge 2$, thanks to (46), the embedding of $L^{p'}(B_{\frac34 R}\setminus B_{\frac14 R})$ into $L^2(B_{\frac34 R}\setminus B_{\frac14 R})$, (44) with $\alpha=1-N$ and (47), that require $N<p\le\frac{2N}{N-2}$, we have

$$\begin{aligned}
|\nabla u(0)| = |\nabla\overline g(0)| &= \left|\int_{\mathbb R^N}\nabla\Phi(x)\overline f(x)\,\mathrm dx\right|\\
&\le \|\nabla\Phi\|_{L^p(B_{\frac34 R}\setminus B_{\frac14 R})}\|\overline f_1+\overline f_2\|_{L^{p'}(B_{\frac34 R}\setminus B_{\frac14 R})}+\|\nabla\Phi\|_{L^{p'}(B_R)}\|\overline f_3\|_{L^p(B_R)}\\
&\le C_{N,p}R^{1-N+\frac Np}\;|B_{\frac34 R}|^{\frac1{p'}-\frac12}\|\overline f_1+\overline f_2\|_{0,B_{\frac34 R}\setminus B_{\frac14 R}}+C_{N,p}R^{1-\frac Np}\;R^{\frac Np+1-\frac N2}\|\nabla\tilde f_3\|_{0,B_R}.
\end{aligned}$$

By using the Poincaré-Wirtinger inequality, whose constant scales with R, to bound $\|u-\overline u\|_{0,B_R}$, we obtain

$$\begin{aligned}
|\nabla u(0)| &\le C_{N,p,\varphi}\;R^{-1-\frac N2}\left(R^{-2}\|u-\overline u\|_{0,B_R}+R^{-1}\|\nabla u\|_{0,B_R}\right)\\
&\quad + C_{N,p,\varphi}\;R^{2-\frac N2}\left(R^{-1}\left\|\omega^2u+f\right\|_{0,B_R}+\left\|\nabla(\omega^2u+f)\right\|_{0,B_R}\right)\\
&\le C_{N,p,\varphi}\;R^{-\frac N2}\left(\omega^2R\|u\|_{0,B_R}+(1+\omega^2R^2)\|\nabla u\|_{0,B_R}+R\|f\|_{0,B_R}+R^2\|\nabla f\|_{0,B_R}\right),
\end{aligned}$$

The requirement that there exists p such that $N<p\le\frac{2N}{N-2}$ can be satisfied only if $N<4$. □

Lemma 3.11 is the only result in this section which we are not able to generalize to all the space dimensions $N\ge 2$. This is because in its proof we make use of a pair of conjugate exponents p and p' such that the fundamental solution Φ of the Laplace equation (together with its gradient) belongs to $L^{p'}(B_R)$ and, at the same time, $H^1(B_R)$ is continuously embedded in $L^p(B_R)$. This requirement yields the upper bounds on the space dimension we have required in the statement of Lemma 3.11.

Combining the results of the previous lemmas, we can now prove Theorem 3.1.

Proof of Theorem 3.1. We start by proving the continuity bound (14) for V_1. For every $j \in \mathbb{N}$, $N \geq 2$, $\phi \in \mathcal{H}^j(D)$, inserting (29) and (31) into (20) with $\xi = 1$, we have

$$|V_1[\phi]|_{j,D} \leq \Bigg[2\,|\phi|^2_{j,D} + 2(1+j)^{3N-2} e^{2j} \sum_{k=0}^{j} \omega^{2(j-k)} \big(j-k+(\omega h)^2\big)^2 \cdot \left(\left(\frac{2}{\rho}\right)^{N-1} |\phi|^2_{k,D} + \left(\frac{\rho}{2}\right)^{2k+1} \frac{|D|}{2k+1} \sum_{|\beta|=k} \left\| D^\beta \phi \right\|^2_{L^\infty(B_{\frac{\rho h}{2}})} \right) \Bigg]^{\frac{1}{2}}.$$

Then, using the interior estimates (33), we get

$$\begin{aligned}|V_1[\phi]|_{j,D} &\leq C_N\ (1+j)^{\frac{3}{2}N-1+1}\ e^j\ \big(1+(\omega h)^2\big) \left[\sum_{k=0}^{j} \omega^{2(j-k)} \left(\rho^{1-N} + \rho^{2k+1} \frac{|D|}{(\rho h)^N} \right) |\phi|^2_{k,D} \right]^{\frac{1}{2}} \\ &\leq C_N\ \rho^{\frac{1-N}{2}}\ (1+j)^{\frac{3}{2}N}\ e^j\ \big(1+(\omega h)^2\big)\ \|\phi\|_{j,\omega,D}\,,\end{aligned}$$

by the definition of weighted Sobolev norms (2), and because $|D| \leq h^N$ and $\rho < 1$. The constant C_N depends only on the dimension N of the space. Passing from the seminorms to the complete Sobolev norms gives an extra coefficient $(1+j)^{1/2}$ and the bound (14) follows.

In order to prove the continuity bound (15) for V_2, we proceed similarly. For every $j \in \mathbb{N}$, $N \geq 2$, $u \in \mathcal{H}^j_\omega(D)$, inserting (30) and (31) into (20) with $\xi = 2$, we have

$$\begin{aligned}|V_2[u]|_{j,D} &\leq \Bigg[2\,|u|^2_{j,D} + 2(1+j)^{3N-2} e^{2j} \sum_{k=0}^{j} \omega^{2(j-k)} (1+\omega h)^2 e^{\frac{3}{2}(1-\rho)\omega h} \cdot \left(\left(\frac{2}{\rho}\right)^{N-1} |u|^2_{k,D} + \left(\frac{\rho}{2}\right)^{2k+1} \frac{|D|}{2k+1} \sum_{|\beta|=k} \left\| D^\beta u \right\|^2_{L^\infty(B_{\frac{\rho h}{2}})} \right) \Bigg]^{\frac{1}{2}} \\ &\overset{(36)}{\leq} C(N,\omega h,\omega\rho h)\ (1+j)^{\frac{3}{2}N-1}\ e^j \left[\sum_{k=0}^{j} \omega^{2(j-k)} \left(\rho^{1-N} + \rho^{2k+1} \frac{|D|}{(\rho h)^N} \right) |u|^2_{k,D} \right]^{\frac{1}{2}} \\ &\leq C(N,\omega h,\rho)\ (1+j)^{\frac{3}{2}N-1}\ e^j\ \|u\|_{j,\omega,D}\,.\end{aligned}$$

Again, passing from the seminorms to the complete Sobolev norms gives an extra coefficient $(1+j)^{1/2}$ and the bound (15) follows.

Now we proceed by proving the bounds (16), (17) and (19) for V_2 with constants depending only on N.

For the continuity bound (16) for the V_2 operator from $H^1(D)$ to $L^2(D)$, we repeat the same reasoning as above. If $u \in \mathcal{H}^1_\omega(D)$, $N = 2, \ldots, 5$, using the definition of V_2, (25), (31) and (40), we have

$$\begin{aligned}\|V_2[u]\|_{0,D} &\leq \left[2\,\|u\|^2_{0,D} + 2\,\|M_2\|^2_{L^\infty(D\times[0,1])} \int_0^1 \int_D |u(tx)|^2\ \mathrm{d}x\,\mathrm{d}t \right]^{\frac{1}{2}} \\ &\leq \Bigg[2\,\|u\|^2_{0,D} + 2\left(\frac{(\omega h)^2}{4} e^{\frac{1}{2}(1-\rho)\omega h} \right)^2 \Bigg[\left(\frac{2}{\rho}\right)^{N-1} \|u\|^2_{0,D} \\ &\qquad + \frac{\rho}{2}|D| \Big(C_N (\rho h)^{-\frac{N}{2}} \big(1+(\omega\rho h)^2\big) \big(\,\|u\|_{0,D} + \rho h\,\|\nabla u\|_{0,D}\,\big) \Big)^2 \Bigg] \Bigg]^{\frac{1}{2}} \\ &\leq C_N\ \rho^{\frac{1-N}{2}}\ \big(1+(\omega h)^4\big)\ e^{\frac{1}{2}(1-\rho)\omega h} \big(\,\|u\|_{0,D} + \rho h\,\|\nabla u\|_{0,D}\,\big),\end{aligned}$$

which immediately gives (16).

Let us now prove (17). To this end, given a multi-index $\beta \in \mathbb{N}^N$, we need to bound $\left\|D^\beta u\right\|_{L^\infty(B_{\frac{\rho h}{2}})}$. If $|\beta| = 0$, for $N = 2, 3, 4, 5$, we simply use (40) and get

$$\begin{aligned} \left\|D^\beta u\right\|_{L^\infty(B_{\frac{\rho h}{2}})} &= \|u\|_{L^\infty(B_{\frac{\rho h}{2}})} \\ &\le C_N (\rho h)^{-\frac{N}{2}} (1 + \omega^2\rho^2 h^2) \left(\|u\|_{0,D} + \rho h \|\nabla u\|_{0,D}\right). \end{aligned} \tag{48}$$

If $|\beta| = j \ge 1$, we note that there exists another multi-index $\alpha \in \mathbb{N}^N$ of length $|\alpha| = j - 1$, such that for $N = 2, 3$ and $u \in \mathcal{H}^j_\omega(D)$ it holds

$$\begin{aligned} \left\|D^\beta u\right\|_{L^\infty(B_{\frac{\rho h}{2}})} &\le \|\nabla D^\alpha u\|_{L^\infty(B_{\frac{\rho h}{2}})} \\ &\le C_N (\rho h)^{-\frac{N}{2}} \left(\omega^2 \rho h \|D^\alpha u\|_{0,D} + \left(1 + (\omega\rho h)^2\right) \|\nabla D^\alpha u\|_{0,D}\right), \end{aligned} \tag{49}$$

thanks to (41). Notice that the restriction to $N = 2, 3$ in this proof is due to the use of (41). Again, inserting (30) and (31) into (20) with $\xi = 2$ gives

$$\begin{aligned} |V_2[u]|_{j,D} \le C_N \Bigg[&|u|^2_{j,D} + (1+j)^{3N-2}\, e^{2j} \sum_{k=0}^{j} \omega^{2(j-k)} (1+\omega h)^2 e^{\frac{3}{2}(1-\rho)\omega h} \\ &\cdot \left(\rho^{1-N} |u|^2_{k,D} + \rho^{2k+1} |D| \sum_{|\beta|=k} \left\|D^\beta u\right\|^2_{L^\infty(B_{\frac{\rho h}{2}})}\right)\Bigg]^{\frac{1}{2}} \\ \le C_N\, &(1+j)^{\frac{3}{2}N-1}\, e^j\, (1+\omega h)\, e^{\frac{3}{4}(1-\rho)\omega h} \\ &\cdot \left[\sum_{k=0}^{j} \omega^{2(j-k)} \left(\rho^{1-N} |u|^2_{k,D} + \rho^{2k+1} |D| \sum_{|\beta|=k} \left\|D^\beta u\right\|^2_{L^\infty(B_{\frac{\rho h}{2}})}\right)\right]^{\frac{1}{2}}, \end{aligned}$$

and thus, as a consequence of (48) and (49), we obtain

$$\begin{aligned} |V_2[u]|_{j,D} \le C_N\ &(1+j)^{\frac{3}{2}N-1}\, e^j\, (1+\omega h)\, e^{\frac{3}{4}(1-\rho)\omega h} \\ &\cdot \Bigg[\omega^{2j} \rho^{1-N} \left(\|u\|^2_{0,D} + \frac{|D|}{h^N} (1+\omega^2\rho^2h^2)^2 \left(\|u\|_{0,D} + \rho h \|\nabla u\|_{0,D}\right)^2\right) \\ &\quad + \sum_{k=1}^{j} \omega^{2(j-k)} \rho^{1-N} \ \Bigg(|u|^2_{k,D} + \rho^{2k} \binom{N+k-1}{N-1} \frac{|D|}{h^N} \\ &\qquad \cdot \left(\omega^2\rho h\, |u|_{k-1,D} + (1+\omega^2\rho^2h^2)\, |u|_{k,D}\right)^2\Bigg)\Bigg]^{\frac{1}{2}} \\ \le C_N\ &(1+j)^{\frac{3}{2}N-1}\, \rho^{\frac{1-N}{2}}\, e^j\, (1+\omega h)\, e^{\frac{3}{4}(1-\rho)\omega h} \\ &\cdot \Bigg[\omega^{2j} (1+\omega^2h^2)^2 \left(\|u\|_{0,D} + h \|\nabla u\|_{0,D}\right)^2 \\ &\quad + \sum_{k=1}^{j} \omega^{2(j-k)} (1+k)^{N-1} \left(\omega^2 h\, |u|_{k-1,D} + (1+\omega^2h^2)\, |u|_{k,D}\right)^2\Bigg]^{\frac{1}{2}} \\ \le C_N\ &(1+j)^{2N-\frac{3}{2}}\, \rho^{\frac{1-N}{2}}\, e^j\, (1+\omega h)\, e^{\frac{3}{4}(1-\rho)\omega h} \\ &\cdot \Bigg[\left(1+(\omega h)^2\right)^2 \omega^{2j}\, \|u\|^2_{0,D} + \left((\omega h)^2 + (\omega h)^6\right) \omega^{2(j-1)}\, |u|^2_{1,D} \end{aligned}$$

$$+ (\omega h)^2 \sum_{k=1}^{j} \omega^{2(j-k+1)} \, |u|^2_{k-1,D} + \big(1+(\omega h)^2\big)^2 \sum_{k=1}^{j} \omega^{2(j-k)} \, |u|^2_{k,D} \Bigg]^{\frac{1}{2}}$$
$$\leq C_N \;\; (1+j)^{2N-\frac{3}{2}} \, \rho^{\frac{1-N}{2}} \, e^j \left(1+(\omega h)^4\right) e^{\frac{3}{4}(1-\rho)\omega h} \, \|u\|_{j,\omega,D} \, ,$$

where the binomial coefficient comes from the number of the multi-indices $|\beta| = k$ and is bounded by (21). As before, passing from the seminorms to the complete Sobolev norms gives an extra coefficient $(1+j)^{1/2}$ and the bound (17) follows.

Finally, we prove the continuity of V_1 and V_2 in the L^∞-norm stated in (18), (19). Thanks to the definition of V_1 and V_2, the bounds (22) and (25), we have

$$\|V_1[\phi]\|_{L^\infty(D)} \leq \Big(1 + \|M_1\|_{L^\infty(D\times[0,1])}\Big) \, \|\phi\|_{L^\infty(D)}$$
$$\leq \left(1 + \frac{\big((1-\rho)\omega h\big)^2}{4}\right) \|\phi\|_{L^\infty(D)} \, ,$$
$$\|V_2[u]\|_{L^\infty(D)} \leq \left(1 + \frac{\big((1-\rho)\omega h\big)^2}{4} e^{\frac{1}{2}(1-\rho)\omega h}\right) \|u\|_{L^\infty(D)} \, ,$$

that holds for every $\phi,\; u \in L^\infty(D)$ and for every $N \geq 2$. This proves (18) and (19), the proof of Theorem 3.1 is complete. □

4. Generalized Harmonic Polynomials

Vekua's theory can be used to derive approximation estimates for the solutions of the homogeneous Helmholtz equation by using finite dimensional spaces of particular functions, the *generalized harmonic polynomials*, for instance.

Definition 4.1. A function $u \in C(D)$ is called a *generalized harmonic polynomial* of degree L if its inverse Vekua transform $V_2[u]$ is a harmonic polynomial of degree L.

Thanks to the results of the previous sections, the generalized harmonic polynomials are solutions of the homogeneous Helmholtz equation with wavenumber ω and belong to $H^k(D)$ for every $k \in \mathbb{N}$, so they are also in $C^\infty(D)$.

Let u be a solution to the homogeneous Helmholtz equation in D, and let P_L be an approximation of the harmonic function $V_2[u]$ in the space of harmonic polynomials of degree at most L in a suitable Sobolev norm, for which an estimate of the approximation error is available. Then, using the continuity of V_1 and V_2 given by (14) and (17), respectively, one can derive an approximation estimate for $u - V_1[P_L]$ ($V_1[P_L]$ is a generalized harmonic polynomial) in a suitable ω–weighted Sobolev norm (see Chapter 2 of [21]). This also implies that, if D is such that the harmonic polynomials are dense in $\mathcal{H}^k(D)$ for some k, then the generalized harmonic polynomials are dense in $\mathcal{H}^k_\omega(D)$.

In the next section, we show that the generalized harmonic polynomials in 2D and 3D are circular and spherical waves, respectively.

4.1. Generalized Harmonic Polynomials in 2D and 3D

In order to explicitly write the generalized harmonic polynomials, we prove the following lemma.

Lemma 4.2. *If $\phi \in C(D)$ is an l-homogeneous function with $l \in \mathbb{R}$, $l > -\frac{N}{2}$, i.e., there exists $g \in L^2(S^{N-1})$ such that*

$$\phi(x) = g\Big(\frac{x}{|x|}\Big) \, |x|^l \qquad \forall \, x \in D,$$

then its Vekua transform is

$$V_1[\phi](x) = \Gamma\Big(l + \frac{N}{2}\Big) \, \Big(\frac{2}{\omega}\Big)^{l+\frac{N}{2}-1} \, g\Big(\frac{x}{|x|}\Big) \, |x|^{1-\frac{N}{2}} \, J_{l+\frac{N}{2}-1}(\omega|x|) \quad \forall \, x \in D. \tag{50}$$

Proof. Using the Beta integral $\int_0^1 t^a(1-t)^b\,\mathrm{d}t = \frac{\Gamma(a+1)\,\Gamma(b+1)}{\Gamma(a+b+2)}$, $a,b>-1$, we can directly compute the Vekua transform from the definition of V_1:

$$\begin{aligned}
V_1[\phi](x) &= g\Big(\frac{x}{|x|}\Big)\,|x|^l + \int_0^1 g\Big(\frac{x}{|x|}\Big)\,(|x|t)^l\;M_1(x,t)\,\mathrm{d}t\\
&= g\Big(\frac{x}{|x|}\Big)\,|x|^l\left(1+\int_0^1 t^l M_1(x,t)\,\mathrm{d}t\right)\\
&= g\Big(\frac{x}{|x|}\Big)\,|x|^l\left(1-\int_0^1 t^{l+\frac{N}{2}-1}\sum_{j\geq 0}\frac{(-1)^j\left(\frac{\omega|x|}{2}\right)^{2j+2}(1-t)^j}{j!\,(j+1)!}\,\mathrm{d}t\right)\\
&= g\Big(\frac{x}{|x|}\Big)\,|x|^l\left(1-\sum_{j\geq 0}\frac{(-1)^j\left(\frac{\omega|x|}{2}\right)^{2j+2}}{j!\,(j+1)!}\;\frac{\Gamma\Big(l+\frac{N}{2}\Big)\,\Gamma(j+1)}{\Gamma\Big(l+\frac{N}{2}+j+1\Big)}\right)\\
&\overset{k=j+1}{=} g\Big(\frac{x}{|x|}\Big)\,|x|^l\left(1+\sum_{k\geq 1}\frac{(-1)^k\left(\frac{\omega|x|}{2}\right)^{2k}}{k!\,\Gamma\Big(l+\frac{N}{2}+k\Big)}\,\Gamma\Big(l+\frac{N}{2}\Big)\right)\\
&= g\Big(\frac{x}{|x|}\Big)\,|x|^l\sum_{k\geq 0}\frac{(-1)^k\left(\frac{\omega|x|}{2}\right)^{2k}}{k!\,\Gamma\Big(l+\frac{N}{2}+k\Big)}\,\Gamma\Big(l+\frac{N}{2}\Big)\\
&= \Gamma\Big(l+\frac{N}{2}\Big)\,g\Big(\frac{x}{|x|}\Big)|x|^{1-\frac{N}{2}}\,\left(\frac{2}{\omega}\right)^{l+\frac{N}{2}-1}\sum_{k\geq 0}\frac{(-1)^k\left(\frac{\omega|x|}{2}\right)^{2k+l+\frac{N}{2}-1}}{k!\,\Gamma\Big(l+\frac{N}{2}+k\Big)}\\
&= \Gamma\Big(l+\frac{N}{2}\Big)\,\left(\frac{2}{\omega}\right)^{l+\frac{N}{2}-1} g\Big(\frac{x}{|x|}\Big)\,|x|^{1-\frac{N}{2}}\,J_{l+\frac{N}{2}-1}(\omega|x|).
\end{aligned}$$

The condition $l>-\frac{N}{2}$ is necessary to ensure a finite value of the integral $\int_0^1 t^{l+\frac{N}{2}-1}(1-t)^j\,\mathrm{d}t$. □

As a consequence, the general (non homogeneous) harmonic polynomial of degree L and its Vekua transform can be written, in terms of spherical harmonics $Y_{l,m}$ (see [2,33]) and hyperspherical Bessel functions j_l^N (see the Appendix), by

$$P(x) = \sum_{l=0}^{L}\sum_{m=1}^{n(N,l)} a_{l,m}\,|x|^l\,Y_{l,m}\Big(\frac{x}{|x|}\Big), \tag{51}$$

$$\begin{aligned}
V_1[P](x) &= |x|^{1-\frac{N}{2}}\sum_{l=0}^{L}\sum_{m=1}^{n(N,l)} a_{l,m}\,\Gamma\big(l+\tfrac{N}{2}\big)\left(\frac{2}{\omega}\right)^{l+\frac{N}{2}-1} Y_{l,m}\Big(\frac{x}{|x|}\Big)J_{l+\frac{N}{2}-1}(\omega|x|)\\
&= \begin{cases} 2^{\frac{N}{2}-1}\sum_{l=0}^{L}\sum_{m=1}^{n(N,l)} a_{l,m}\,\Gamma\left(l+\frac{N}{2}\right)\,\left(\frac{2}{\omega}\right)^l\,Y_{l,m}\big(\frac{x}{|x|}\big)\,j_l^N(\omega|x|), & N \text{ even},\\ \frac{2^{\frac{N-1}{2}}}{\sqrt{\pi}}\sum_{l=0}^{L}\sum_{m=1}^{n(N,l)} a_{l,m}\,\Gamma\left(l+\frac{N}{2}\right)\,\left(\frac{2}{\omega}\right)^l\,Y_{l,m}\big(\frac{x}{|x|}\big)\,j_l^N(\omega|x|), & N \text{ odd}.\end{cases}
\end{aligned} \tag{52}$$

If $N=2$, identifying $\mathbb{R}^2=\mathbb{C}$ and using the complex variable $z=re^{i\psi}$, using directly (50), we have

$$P(z) = \sum_{l=-L}^{L} a_l\,r^{|l|}\,e^{il\psi}, \tag{53}$$

$$V_1[P](z) = \sum_{l=-L}^{L} a_l\,|l|!\,\left(\frac{2}{\omega}\right)^{|l|} e^{il\psi}\,J_{|l|}(\omega r). \tag{54}$$

If $N = 3$, we use the definition of spherical Bessel function (67) to get

$$P(x) = \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m} \, |x|^l \, Y_{l,m}\Big(\frac{x}{|x|}\Big), \tag{55}$$

$$\begin{aligned} V_1[P](x) &= \frac{2}{\sqrt{\pi}} \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m} \;\; \Gamma\Big(l+\frac{3}{2}\Big) \; \Big(\frac{2}{\omega}\Big)^l \; Y_{l,m}\Big(\frac{x}{|x|}\Big) \, j_l(\omega|x|) \\ &= \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m} \;\; \frac{(2l+1)!}{l!} \; \Big(\frac{1}{2\omega}\Big)^l \; Y_{l,m}\Big(\frac{x}{|x|}\Big) \, j_l(\omega|x|), \end{aligned} \tag{56}$$

where $\{Y_{l,m}\}_{m=-l,\dots,l}$ are a basis of spherical harmonics of order l, and we have used $\Gamma(l+\frac{3}{2}) = \frac{\sqrt{\pi}\,(2l+1)!}{2^{2l+1}\,l!}$, which follows from $\Gamma(s+1) = s\,\Gamma(s)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. This means that the generalized harmonic polynomials in 2D and 3D are the well-known circular and spherical waves, respectively.

4.2. Generalized Harmonic Polynomials as Herglotz Functions

In this section, we define an important family of solutions of the homogeneous Helmholtz equation: the Herglotz functions (see [13, Def. 3.14]), and prove that the generalized harmonic polynomials belong to this class. This result can be used to prove approximation properties of homogeneous Helmholtz solutions by plane waves, as in [27, Prop. 8.4.14].

Definition 4.3. Given a function $g \in L^2(S^{N-1})$ we define the *Herglotz function* w_g with Herglotz kernel g and wavenumber ω as the the function in $C^\infty(\mathbb{R}^N)$ defined by

$$w_g(x) = \int_{S^{N-1}} g(d)\, e^{i\omega x\cdot d}\, \mathrm{d}\sigma(d) \qquad\qquad x \in \mathbb{R}^N. \tag{57}$$

The Herglotz functions are entire solutions of the homogeneous Helmholtz equation; it is known that they are dense in $\mathcal{H}^k_\omega(\mathcal{D})$ with respect to the $H^k(\mathcal{D})$-norm or the $C^\infty(\mathcal{D})$ topology, whenever $\mathcal{D}$ is a $C^{k-1,1}$ domain; the proof is given in Theorem 2 of [38]. As already mentioned, if $\mathcal{D}$ is such that the harmonic polynomials are dense in $\mathcal{H}^k(\mathcal{D})$, then the generalized harmonic polynomials, which are Herglotz functions, are dense in $\mathcal{H}^k_\omega(\mathcal{D})$. This means that, for $k \geq 2$, we generalize the result of [38] to weaker assumptions on the domain $\mathcal{D}$ (see Section 1.3.1 and Theorem 2.2.1 of [21] for details).

Lemma 4.4. *Let P be a harmonic polynomial of degree $L \in \mathbb{N}$ in $\mathbb{R}^2$ or $\mathbb{R}^N$, $N \geq 3$, defined as in* (53) *or in* (51)*, respectively. Then the corresponding generalized harmonic polynomial $V_1[P]$ is a Herglotz function w_g with Herglotz kernel*

$$g(\theta) = \sum_{l=-L}^{L} a_l \, \frac{|l|!}{2\pi} \, \Big(\frac{2}{i\omega}\Big)^{|l|} e^{il\theta} \qquad\qquad N = 2,$$

$$g(d) = \sum_{l=0}^{L} \sum_{m=1}^{n(N,l)} a_{l,m} \, \frac{\Gamma\Big(l+\frac{N}{2}\Big)}{2\,\pi^{\frac{N}{2}}} \, \Big(\frac{2}{i\omega}\Big)^l Y_{l,m}(d) \qquad\qquad N \geq 3.$$

Proof. We write the Jacobi-Anger expansions combined with the addition theorem for (orthonormal) spherical harmonics, in two and N dimensions, see [2, 13, 33]:

$$e^{it\cos\theta} = \sum_{l\in\mathbb{Z}} i^l J_l(t)\, e^{il\theta} \qquad\qquad \forall\,\theta,\ t \in \mathbb{R}, \tag{58}$$

$$e^{ir\xi\cdot\eta} = (N-2)!!\,|S^{N-1}| \sum_{l\geq 0} i^l \, j_l^N(r) \sum_{m=1}^{n(N,l)} Y_{l,m}(\xi)\overline{Y_{l,m}(\eta)} \qquad \forall\,\xi,\ \eta \in S^{N-1},\ r \geq 0,\ N \geq 3. \tag{59}$$

These series converge absolutely and uniformly on compact subsets of $\mathbb{R}^N$. Now we only have to use these formulas to verify that the Herglotz functions with the kernels written above correspond to (54) and (52), respectively.

In two space dimensions with the polar coordinates $z = r\, e^{i\psi}$ we have

$$\begin{aligned}
w_g(z) &= \int_0^{2\pi} \sum_{l=-L}^{L} a_l \frac{|l|!}{2\pi} \left(\frac{2}{i\omega}\right)^{|l|} e^{il\theta}\, e^{i\omega r(\cos\psi,\sin\psi)\cdot(\cos\theta,\sin\theta)}\, \mathrm{d}\theta \\
&= \sum_{l=-L}^{L} a_l \frac{|l|!}{2\pi} \left(\frac{2}{i\omega}\right)^{|l|} \int_0^{2\pi} e^{il\theta}\, e^{i\omega r\cos(\psi-\theta)}\, \mathrm{d}\theta \\
&\overset{(58)}{=} \sum_{l=-L}^{L} a_l \frac{|l|!}{2\pi} \left(\frac{2}{i\omega}\right)^{|l|} \int_0^{2\pi} e^{il\theta} \sum_{l'\in\mathbb{Z}} i^{l'} J_{l'}(\omega r)\, e^{il'(\psi-\theta)}\, \mathrm{d}\theta \\
&= \sum_{l=-L}^{L} \sum_{l'\in\mathbb{Z}} a_l \frac{|l|!}{2\pi} \left(\frac{2}{i\omega}\right)^{|l|} i^{l'} J_{l'}(\omega r)\, e^{il\psi} \int_0^{2\pi} e^{i(l-l')\theta}\, \mathrm{d}\theta \\
&\overset{(61)}{=} \sum_{l=-L}^{L} a_l\, |l|! \left(\frac{2}{\omega}\right)^{|l|} J_{|l|}(\omega r)\, e^{il\psi} \overset{(54)}{=} V_1[P](z),
\end{aligned}$$

where in the second last step we have used the identity $\int_0^{2\pi} e^{i(l-l')\theta}\,\mathrm{d}\theta = 2\pi\,\delta_{l,l'}$. In the previous chain of equalities, we could exchange the order of summation and integration because the serie in l' converges uniformly and absolutely in $[0, 2\pi]$, thanks to (63).

In higher space dimensions, we use the orthonormality of the spherical harmonics $\int_{S^{N-1}} Y_{l,m}\overline{Y_{l',m'}} = \delta_{l,l'}\delta_{m,m'}$:

$$\begin{aligned}
w_g(x) &= \int_{S^{N-1}} \sum_{l=0}^{L} \sum_{m=1}^{n(N,l)} a_{l,m} \frac{\Gamma\left(l+\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \left(\frac{2}{i\omega}\right)^{l} Y_{l,m}(d)\, e^{i\omega x\cdot d}\, \mathrm{d}\sigma(d) \\
&\overset{(59)}{=} \int_{S^{N-1}} \sum_{l=0}^{L} \sum_{m=1}^{n(N,l)} a_{l,m} \frac{\Gamma\left(l+\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \left(\frac{2}{i\omega}\right)^{l} Y_{l,m}(d) \\
&\quad \cdot \sum_{l'\geq 0} \sum_{m'=1}^{n(N,l')} (N-2)!!\, |S^{N-1}|\, i^{l'}\, j_{l'}^{N}(\omega|x|)\, Y_{l',m'}\left(\frac{x}{|x|}\right) \overline{Y_{l',m'}(d)}\, \mathrm{d}\sigma(d) \\
&= \frac{(N-2)!!}{\Gamma\left(\frac{N}{2}\right)} \sum_{l=0}^{L} \sum_{m=1}^{n(N,l)} a_{l,m}\, \Gamma\left(l+\frac{N}{2}\right) \left(\frac{2}{\omega}\right)^{l} Y_{l,m}\left(\frac{x}{|x|}\right) j_l^N(\omega|x|) \\
&\overset{(52),\,(68)}{=} V_1[P](x),
\end{aligned}$$

where in the second last step we have used the formula $|S^{N-1}| = 2\pi^{\frac{N}{2}}/\Gamma\left(\frac{N}{2}\right)$. □

Lemma 4.4 also gives an easy formula to compute the Vekua transform of any Herglotz function w_g, given the expansion of its kernel g in harmonics.

Appendix A. Bessel Functions

We denote the usual Bessel functions of the first kind by $J_\nu(z)$ and the spherical Bessel functions of the first kind by $j_\nu(z)$. The first ones are defined, for every $\nu, z \in \mathbb{C}$, as

$$J_\nu(z) = \sum_{t=0}^{\infty} \frac{(-1)^t}{t!\,\Gamma(t+\nu+1)} \left(\frac{z}{2}\right)^{2t+\nu}, \tag{60}$$

where Γ is the gamma function. When $\nu \notin \mathbb{Z}$ and z belongs to the segment $[-\infty, 0]$, $J_\nu(z)$ is not single-valued. When $\nu \in \mathbb{Z}$, J_ν is an entire function.

We list some properties of these functions (references can be found in [26, 37]):

$$J_{-k}(z) = (-1)^k J_k(z) \qquad \forall\, k \in \mathbb{Z}, \tag{61}$$

$$\operatorname{Im}\big(J_k(t)\big) = 0, \qquad \operatorname{Re}\big(J_k(it)\big) = 0 \qquad \forall\, k \in \mathbb{Z},\ t \in \mathbb{R},$$

$$|J_k(t)| \le 1 \qquad \forall\, k \in \mathbb{Z},\ t \in \mathbb{R}, \tag{62}$$

$$|J_\nu(z)| \le \frac{e^{|\operatorname{Im} z|}}{\Gamma(\nu+1)} \left(\frac{|z|}{2}\right)^\nu \qquad \forall\, \nu > -\frac{1}{2},\ z \in \mathbb{C}, \tag{63}$$

$$J_0(0) = 1, \qquad J_k(0) = 0 \qquad \forall\, k \in \mathbb{Z} \setminus \{0\},$$

$$\frac{\partial}{\partial z} J_\nu(z) = \frac{1}{2}\left(J_{\nu-1}(z) - J_{\nu+1}(z)\right), \tag{64}$$

$$\frac{\partial}{\partial z}\big(z^k J_k(z)\big) = z^k J_{k-1}(z),$$

$$\frac{\partial}{\partial z} J_0(z) = -J_1(z), \qquad \frac{\partial}{\partial z}\left(z J_1(z)\right) = z J_0(z), \tag{65}$$

$$\frac{\partial^l}{\partial z^l} J_k(z) = \frac{1}{2^l} \sum_{m=0}^{l} (-1)^m \binom{l}{m} J_{2m-l+k}(z). \tag{66}$$

The last equality can be easily proved by induction from (64).

The spherical Bessel functions are defined as

$$j_\nu(z) = \sqrt{\frac{\pi}{2z}}\, J_{\nu+\frac{1}{2}}(z). \tag{67}$$

These functions are a particular case of the so-called hyperspherical Bessel functions (see [2] p. 52):

$$j_k^N(z) = \sum_{t=0}^{\infty} \frac{(-1)^t\, z^{2t+k}}{(2t)!!\,(N+2t+2k-2)!!} = \begin{cases} z^{1-\frac{N}{2}}\, J_{k+\frac{N}{2}-1}(z), & N \text{ even}, \\ \sqrt{\frac{\pi}{2}}\, z^{1-\frac{N}{2}}\, J_{k+\frac{N}{2}-1}(z), & N \text{ odd}, \end{cases}$$

$$J_k(z) = j_k^2(z), \qquad j_k(z) = j_k^3(z).$$

The above equality is proved using (60) and

$$(2m)!! = 2^m m!, \qquad (2m+1)!! = \frac{\Gamma\big(m+\frac{3}{2}\big)\, 2^{m+1}}{\sqrt{\pi}}, \quad m \in \mathbb{N}. \tag{68}$$

References

[1] R. Adams, *Sobolev spaces*, Pure and Applied Mathematics, Academic Press, 1975.

[2] J. Avery, *Hyperspherical Harmonics: applications in quantum theory*, Reidel texts in the mathematical sciences, Kluwer Academic Publishers, 1989.

[3] S. Axler, P. Bourdon, and W. Ramey, *Harmonic function theory*, Graduate Texts in Mathematics, Springer-Verlag, New York, 2001.

[4] I. Babuška and J. Melenk, *The partition of unity method*, Int. J. Numer. Methods Eng., 40 (1997), pp. 727–758.

[5] A. H. Barnett and T. Betcke, *Stability and convergence of the method of fundamental solutions for Helmholtz problems on analytic domains*, J. Comput. Phys., 227 (2008), pp. 7003–7026.

[6] ———, *An exponentially convergent nonpolynomial finite element method for time-harmonic scattering from polygons*, SIAM J. Sci. Comput., 32 (2010), pp. 1417–1441.

[7] T. Betcke, *Numerical computation of eigenfunctions of planar regions*, PhD thesis, University of Oxford, 2005.

[8] O. Cessenat and B. Després, *Application of an ultra weak variational formulation of elliptic PDEs to the two-dimensional Helmholtz equation*, SIAM J. Numer. Anal., 35 (1998), pp. 255–299.

[9] A. Charalambopoulos and G. Dassios, *On the Vekua pair in spheroidal geometry and its role in solving boundary value problems*, Appl. Anal., 81 (2002), pp. 85–113.

[10] D. Colton, *Bergman operators for elliptic equations in three independent variables*, Bull. Amer. Math. Soc., 77 (1971), pp. 752–756.

[11] ———, *Integral operators for elliptic equations in three independent variables. I*, Applicable Anal., 4 (1974/75), pp. 77–95.

[12] ———, *Integral operators for elliptic equations in three independent variables. II*, Applicable Anal., 4 (1974/75), pp. 283–295.

[13] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, vol. 93 of Applied Mathematical Sciences, Springer, Heidelberg, 2nd ed., 1998.

[14] R. Courant and D. Hilbert, *Methods of mathematical physics. Volume II: partial differential equations*, Interscience Publishers, 1962.

[15] S. C. Eisenstat, *On the rate of convergence of the Bergman-Vekua method for the numerical solution of elliptic boundary value problems*, SIAM J. Numer. Anal., 11 (1974), pp. 654–680.

[16] C. Farhat, I.Harari, and L. Franca, *The discontinuous enrichment method*, Comput. Methods Appl. Mech. Eng., 190 (2001), pp. 6455–6479.

[17] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, 2nd ed., 1983.

[18] R. P. Gilbert and C. Y. Lo, *On the approximation of solutions of elliptic partial differential equations in two and three dimensions*, SIAM J. Math. Anal., 2 (1971), pp. 17–30.

[19] C. Gittelson, R. Hiptmair, and I. Perugia, *Plane wave discontinuous Galerkin methods: analysis of the h-version*, M2AN Math. Model. Numer. Anal., 43 (2009), pp. 297–332.

[20] P. Henrici, *A survey of I. N. Vekua's theory of elliptic partial differential equations with analytic coefficients*, Z. Angew. Math. Phys., 8 (1957), pp. 169–202.

[21] R. Hiptmair, A. Moiola, and I. Perugia, *Approximation by plane waves*, Preprint 2009-27, SAM Report, ETH Zürich, Switzerland, 2009.

[22] ———, *Plane wave discontinuous Galerkin methods for the 2D Helmholtz equation: analysis of the p-version*, SIAM J. Numer. Anal., 49 (2011), pp. 264–284.

[23] N. T. Hop, *Some extensions of I. N. Vekua's method in higher dimensions. I. Integral representation of solutions of an elliptic system. Properties of solutions*, Math. Nachr., 130 (1987), pp. 17–34.

[24] ———, *Some extensions of I. N. Vekua's method in higher dimensions. II. Boundary value problems*, Math. Nachr., 130 (1987), pp. 35–46.

[25] M. Ikehata, *Probe method and a Carleman function*, Inverse Problems, 23 (2007), pp. 1871–1894.

[26] N. Lebedev, *Special functions and their applications*, Prentice-Hall, Englewood Cliffs, N.J., 1965.

[27] J. Melenk, *On Generalized Finite Element Methods*, PhD thesis, University of Maryland, 1995.

[28] ———, *Operator adapted spectral element methods I: harmonic and generalized harmonic polynomials*, Numer. Math., 84 (1999), pp. 35–69.

[29] J. Melenk and I. Babuška, *Approximation with harmonic and generalized harmonic polinomials in the partition of unit method*, Comput. Assist. Mech. Eng. Sci., 4 (1997), pp. 607–632.

[30] A. Moiola, *Approximation properties of plane wave spaces and application to the analysis of the plane wave discontinuous Galerkin method*, Tech. Rep. 2009-06, SAM Report, ETH Zürich, Switzerland, 2009.

[31] A. Moiola, R. Hiptmair, and I. Perugia, *Plane wave approximation of homogeneous Helmholtz solutions*, Z. Angew. Math. Phys., to appear (2011).

[32] P. Monk and D. Wang, *A least squares method for the Helmholtz equation*, Comput. Methods Appl. Mech. Eng., 175 (1999), pp. 121–136.

[33] C. Müller, *Spherical harmonics*, Lecture notes in Mathematics, Springer-Verlag, 1966.

[34] I. Vekua, *Solutions of the equation* $\Delta u + \lambda^2 u = 0$, Soobshch. Akad. Nauk Gruz. SSR, 3 (1942), pp. 307–314.

[35] ———, *Inversion of an integral transformation and some applications*, Soobshch. Akad. Nauk Gruz. SSR, 6 (1945), pp. 177–183.

[36] ———, *New methods for solving elliptic equations*, North Holland, 1967.

[37] G. Watson, *Theory of Bessel functions*, Cambridge University Press, 1966.
[38] N. Weck, *Approximation by Herglotz wave functions*, Math. Methods Appl. Sci., 27 (2004), pp. 155–162.

A. Moiola
SAM, ETH Zürich, CH-8092 Zürich, Switzerland
e-mail: andrea.moiola@sam.math.ethz.ch

R. Hiptmair
SAM, ETH Zürich, CH-8092 Zürich, Switzerland
e-mail: hiptmair@sam.math.ethz.ch

I. Perugia
Dipartimento di Matematica, Università di Pavia, Via Ferrata 1, 27100 Pavia, Italy
e-mail: ilaria.perugia@unipv.it