Plane wave approximation in linear elasticity

Article

Accepted Version

Moiola, A. (2013) Plane wave approximation in linear elasticity. Applicable Analysis, 92 (6). pp. 1299-1307. ISSN 0003-6811 doi: https://doi.org/10.1080/00036811.2012.671300 Available at https://centaur.reading.ac.uk/28025/

Applicable Analysis
Vol. 00, No. 00, Month 200x, 1–8

Plane wave approximation in linear elasticity

Andrea Moiola

Department of Mathematics and Statistics – University of Reading, Whiteknights, PO Box 220, Reading RG6 6AX, UK;

(*Received XXX*)

We consider the approximation of solutions of the time-harmonic linear elastic wave equation by linear combinations of plane waves. We prove algebraic orders of convergence both with respect to the dimension of the approximating space and to the diameter of the domain. The error is measured in Sobolev norms and the constants in the estimates explicitly depend on the problem wavenumber. The obtained estimates can be used in the h- and p-convergence analysis of wave-based finite element schemes.

Keywords: Linear elasticity, approximation by plane waves, potential representation, hp-estimates.

AMS Subject Classification: 74B05; 41A30

1. Introduction

In order to efficiently discretize the time-harmonic elastic wave equation (Navier equation), some non-polynomial finite element methods with plane wave basis functions have been designed; see for example the schemes described in [1–5]. A rigorous convergence analysis of these methods requires the proof of a best-approximation estimate: a bound on the minimal error $\inf_{\boldsymbol{v}_N \in V_N} \|\boldsymbol{u} - \boldsymbol{v}_N\|$ where $\boldsymbol{u}$ is a given solution of the considered PDE, V_N is the discrete trial space, and $\|\cdot\|$ is a suitable norm.

For acoustic wave propagation, governed by the Helmholtz equation, approximation estimates have been proven in [6] using Vekua's theory, harmonic polynomial approximation results, and a careful residual estimate of Jacobi-Anger's expansion. Here we use the results of [6] to prove similar bounds for solutions of the time-harmonic Navier equation.

Using a balanced choice of pressure and shear waves, we obtain algebraic orders of convergence both in the diameter of the considered domain and in the dimension of the approximating space; these parameters are relevant for the h- and p-convergence of the corresponding finite element methods. The error is measured in weighted Sobolev norms on a bounded, star-shaped, Lipschitz domain. The dependence of the constants on the wavenumbers of pressure and shear waves is made explicit.

The proof follows the corresponding one for the Maxwell problem described in [7, Sec. 4]. It is based on a potential representation of time-harmonic elastic solutions

Email: A.Moiola@reading.ac.uk

ISSN: 0003-6811 print/ISSN 1563-504X online

DOI: 10.1080/0003681YYxxxxxxxx
http://www.informaworld.com

(see Section 2 below), in particular it relies on the approximation of the scalar and vector potentials using Helmholtz- and Maxwell-type plane waves, respectively.

The final convergence estimate is not expected to be sharp: the displacement is approximated by first-order derivatives of a (plane wave) approximation of the potentials. Thus, its approximation error measured in H^{j-1}-norm is related to the approximation of the potentials in H^j-norm, that is one algebraic order in h and p less than what was proved in the acoustic case (compare the indices of the Sobolev norms in equations (5) and (7)). Sharp bounds might be obtained by adapting Vekua's theory to the linear elasticity setting, but it has not been accomplished yet (see the comments at the end of Section 3).

The purpose of this short paper is not to provide a recipe for finding a plane wave approximation of the solution of a given a boundary value problem for the elastic wave equation. Several sophisticated methods capable of this have already been introduced in the literature, for example the Ultra Weak Variational Formulation (UWVF, see [1, 3]), the Discontinuous Enrichment Method (DEM, see [4, 5]) and the Variational Theory of Complex Rays (VTCR, see [2] and subsequent papers). As demonstrated by extensive numerical experiment in the cited articles, they perform very well compared to standard polynomial finite element schemes. More generally, wave-based methods constitute one of the most promising recent developments in the area of numerical algorithms for time-harmonic wave problems at medium frequencies. In particular, their main strength is that the use of oscillatory basis functions permits great accuracy with few degrees of freedom. However, to the author's knowledge, for no one of these schemes a complete convergence analysis is available in the case of elasticity problems. By looking at the similar analysis for the Helmholtz and Maxwell equations (for the UWVF, see [8, 9] and [7], respectively), it is clear that one of the most important and technical steps is the proof of best approximation estimates. This paper aims at filling this gap: the error bounds proved in Theorem 3.2 can be directly used in the h- and p-convergence analysis of *any* of the methods mentioned above.

2. Potential representation

In this section we define Navier's equation and we briefly study a well-known special Helmholtz decomposition of the displacement field, sometimes called *Lamé's solution*. For a more comprehensive treatment of potential representations in (time-dependent) elasticity problems we refer to Sections 1 and 2 of [10]. A different representation through a single vector potential that is solution of the iterated Helmholtz equation can be found in [11].

Time-harmonic elastic wave propagation in a homogeneous medium and in absence of body forces is described in frequency domain by Navier's equation (*cf.* [12, Sec. 5.1.1]):

$$(\lambda + 2\mu)\nabla \operatorname{div} \boldsymbol{u} - \mu \operatorname{curl}\operatorname{curl} \boldsymbol{u} + \omega^2 \rho\, \boldsymbol{u} = \boldsymbol{0} \qquad \text{in } D\,, \tag{1}$$

supplemented by appropriate boundary conditions (see for example [1, eq. (2.4)]); here

$$\begin{aligned} D \subset \mathbb{R}^3 &\quad \text{is an open domain,}\\ \boldsymbol{u} : D \to \mathbb{R}^3 &\quad \text{is the displacement vector field,}\\ \omega > 0 &\quad \text{is the angular frequency,} \end{aligned}$$

$$\begin{aligned} \lambda, \mu > 0 &\quad \text{are the Lamé constants, and} \\ \rho > 0 &\quad \text{is the density of the medium.} \end{aligned}$$

We assume λ, μ, ρ and ω to be constant in D, and define the wavenumber of pressure (longitudinal) and shear (transverse) waves, respectively, as:

$$\omega_P := \omega \left(\frac{\rho}{\lambda + 2\mu} \right)^{\frac{1}{2}}, \qquad \omega_S := \omega \left(\frac{\rho}{\mu} \right)^{\frac{1}{2}}.$$

Remark 1 : Thanks to the identity $\nabla \operatorname{div} = \mathbf{\Delta} + \operatorname{curl}\operatorname{curl}$, $\mathbf{\Delta}$ being the vector Laplacian, equation (1) can be written as

$$(\lambda + \mu)\nabla \operatorname{div} \boldsymbol{u} + \mu \mathbf{\Delta} \boldsymbol{u} + \omega^2 \rho \boldsymbol{u} = \mathbf{0} \qquad \text{in } D\,.$$

We denote by $\boldsymbol{D}\boldsymbol{v}$ the Jacobian of the vector field $\boldsymbol{v}$, by $\boldsymbol{D}^S \boldsymbol{v} = \frac{1}{2}(\boldsymbol{D}\boldsymbol{v} + \boldsymbol{D}^\top \boldsymbol{v})$ the symmetric gradient (or Cauchy's strain tensor), by $\mathbf{div}$ the (row-wise) vector divergence of matrix fields, and by Id the 3×3 identity matrix. Using the identity $2\,\mathbf{div}\,\boldsymbol{D}^S = \nabla \operatorname{div} + \mathbf{\Delta} = 2\nabla \operatorname{div} - \operatorname{curl}\operatorname{curl}$, equation (1) can be written in the form

$$\mathbf{div}\,\boldsymbol{\sigma} + \omega^2 \rho \boldsymbol{u} = \mathbf{0}\,,$$

where $\boldsymbol{\sigma} = 2\mu \boldsymbol{D}^S \boldsymbol{u} + \lambda(\operatorname{div} \boldsymbol{u})\,\mathrm{Id}$ is the Cauchy stress tensor.

In this section we assume $\boldsymbol{u}$ to be a solution of (1) in the sense of distributions; we define the scalar and vector potential, respectively, as

$$\chi := -\frac{\lambda + 2\mu}{\omega^2 \rho} \operatorname{div} \boldsymbol{u} = -\frac{\operatorname{div} \boldsymbol{u}}{\omega_P^2}, \qquad \boldsymbol{\psi} := \frac{\mu}{\omega^2 \rho} \operatorname{curl} \boldsymbol{u} = \frac{\operatorname{curl} \boldsymbol{u}}{\omega_S^2}. \tag{2}$$

From (1), we can use these potentials to represent $\boldsymbol{u}$:

$$\boldsymbol{u} = -\frac{\lambda + 2\mu}{\omega^2 \rho} \nabla \operatorname{div} \boldsymbol{u} + \frac{\mu}{\omega^2 \rho} \operatorname{curl}\operatorname{curl} \boldsymbol{u} = \nabla \chi + \operatorname{curl} \boldsymbol{\psi}\,, \tag{3}$$

which is a Helmholtz decomposition of the displacement field. Moreover, the scalar and the vector potentials satisfy Helmholtz's and Maxwell's equations, respectively:

$$\begin{aligned} -\Delta\chi - \omega_P^2 \chi &\overset{(2),\Delta=\operatorname{div}\nabla}{=} \operatorname{div} \nabla \frac{\operatorname{div} \boldsymbol{u}}{\omega_P^2} + \operatorname{div} \boldsymbol{u} \\ &\overset{(1)}{=} \frac{1}{\omega_P^2} \operatorname{div} \left(\frac{\mu}{\lambda + 2\mu} \operatorname{curl}\operatorname{curl} \boldsymbol{u} - \omega_P^2 \boldsymbol{u} \right) + \operatorname{div} \boldsymbol{u} \overset{\operatorname{div}\operatorname{curl}=0}{=} 0\,, \\ \operatorname{curl}\operatorname{curl} \boldsymbol{\psi} - \omega_S^2 \boldsymbol{\psi} &\overset{(2)}{=} \operatorname{curl}\operatorname{curl} \frac{\operatorname{curl} \boldsymbol{u}}{\omega_S^2} - \operatorname{curl} \boldsymbol{u} \\ &\overset{(1)}{=} \frac{1}{\omega_S^2} \operatorname{curl} \left(\frac{\lambda + 2\mu}{\mu} \nabla \operatorname{div} \boldsymbol{u} + \omega_S^2 \boldsymbol{u} \right) - \operatorname{curl} \boldsymbol{u} \overset{\operatorname{curl}\nabla=\mathbf{0}}{=} \mathbf{0}\,. \end{aligned} \tag{4}$$

As a consequence, the vector potential $\boldsymbol{\psi}$ satisfies also $\operatorname{div} \boldsymbol{\psi} = 0$ and the vector Helmholtz equation $-\mathbf{\Delta}\boldsymbol{\psi} - \omega_S^2 \boldsymbol{\psi} = \mathbf{0}$.

Remark 2 : The potentials χ and $\boldsymbol{\psi}$ defined in (2) are the only couple of scalar and vector fields such that: *(i)* they are solution of Helmholtz's equation with wavenumber ω_P and Maxwell's equations with wavenumber ω_S, respectively; *(ii)* they constitute a Helmholtz decomposition (3) of $\boldsymbol{u}$. Indeed, if $\widetilde{\chi}$ and $\widetilde{\boldsymbol{\psi}}$ satisfy conditions *(i)* and *(ii)*, then

$$\widetilde{\chi} = -\omega_P^{-2}\Delta\widetilde{\chi} = -\omega_P^{-2}\operatorname{div}\nabla\widetilde{\chi} = -\omega_P^{-2}\operatorname{div}(\boldsymbol{u} - \operatorname{curl}\widetilde{\boldsymbol{\psi}}) = -\omega_P^{-2}\operatorname{div}\boldsymbol{u} = \chi \ ,$$
$$\widetilde{\boldsymbol{\psi}} = \omega_S^{-2}\operatorname{curl}\operatorname{curl}\widetilde{\boldsymbol{\psi}} = \omega_S^{-2}\operatorname{curl}(\boldsymbol{u} - \nabla\widetilde{\chi}) = \omega_S^{-2}\operatorname{curl}\boldsymbol{u} = \boldsymbol{\psi} \ .$$

3. Approximation by plane waves

From now on, we assume for the domain D:

(D1) $D \subset \mathbb{R}^3$ is open, Lipschitz and bounded,
(D2) there exists $\rho \in (0, 1/2]$ such that the ball with centre in a point $\boldsymbol{x}_0$ and radius ρh is included in D, where h is the diameter of D,
(D3) there exists $\rho_0 \in (0, \rho]$ such that D is star-shaped with respect to the ball with centre in the same point $\boldsymbol{x}_0$ and radius $\rho_0 h$.

For instance, every convex polyhedron satisfies these assumptions; this is not a severe restriction since D is meant to be an element of a finite element mesh.

Given $j \in \mathbb{N}$ and $\widetilde{\omega} \in \mathbb{R}$, $\widetilde{\omega} > 0$, we define the $\widetilde{\omega}$-weighted Sobolev norm

$$\|v\|^2_{j,\widetilde{\omega},D} := \sum_{j_0=0}^{j} \widetilde{\omega}^{2(j-j_0)} \ |v|^2_{j_0,D} \qquad \forall\, v \in H^j(D) \ ,$$

where $|\cdot|_{j_0,D}$ is the usual Sobolev seminorm in $H^{j_0}(D)$. We use the same notation for the analogous norm of vector fields in $H^j(D)^3$. We denote the unit sphere in $\mathbb{R}^3$ by $\mathbb{S}^2 = \{\boldsymbol{x} \in \mathbb{R}^3, \ |\boldsymbol{x}| = 1\}$.

We report in the following Lemma the result of Lemma 3.4.6 and Corollary 3.5.5 of [9] concerning the approximation of solutions of Helmholtz equation by linear combinations of plane waves. The same result with a slightly worse dependence of the bounding constant on the wavenumber was proved in Lemma 4.5 and Corollary 5.5 of [6].

Lemma 3.1: *Let D be a domain satisfying* (D1)–(D3)*, fix*

$$0 < \widetilde{\omega} \in \mathbb{R} \ , \qquad k \in \mathbb{N} \ , \qquad q \in \mathbb{N} \quad \text{such that} \quad q \geq 2k+1 \ , \quad q \geq 2(1 + 2^{1/\lambda_D}) \ ,$$

where λ_D is a positive parameter which depends only on the shape of D, as defined in [9, Th. 3.2.12].

Then, there exists a set $\{\boldsymbol{d}_\ell\}_{1\leq\ell\leq p} \subset \mathbb{S}^2$ of $p = (q+1)^2$ different plane wave propagation directions such that, for every $v \in H^{k+1}(D)$ that is solution of the homogeneous Helmholtz equation

$$-\Delta v - \widetilde{\omega}^2 v = 0 \qquad \text{in } D \ ,$$

there exist some coefficients $\alpha_1, \ldots, \alpha_p \in \mathbb{C}$ such that the following bound holds for every $0 \leq j \leq k+1$:

$$\left\| v - \sum_{\ell=1}^{p} \alpha_\ell e^{i\widetilde{\omega}\, \boldsymbol{x}\cdot\boldsymbol{d}_\ell} \right\|_{j,\widetilde{\omega},D} \leq C \ \left(1 + (\widetilde{\omega}h)^{j+6}\right) e^{\left(\frac{7}{4} - \frac{3}{4}\rho\right)\widetilde{\omega}h} \, h^{k+1-j}$$

$$\cdot \left[q^{-\lambda_D(k+1-j)} + \frac{1+(\widetilde{\omega}h)^{q-k+2}}{(\sqrt{2}\,\rho\, q)^{\frac{q-3}{2}}} M \right] \|v\|_{k+1,\widetilde{\omega},D} \cdot \tag{5}$$

Here, the constant $C > 0$ depends only on j, k and on the shape of D, and the constant M satisfies $M \le 2\sqrt{\pi}\,p$.

The bound on the constant M is given by an "optimal" choice of the directions which is not explicitly available. A good choice is provided by the system of directions introduced in [13] and available on the website [14]. In this case, the bound on M is only slightly weaker, namely, $M \le 4\sqrt{\pi}\,pq$ (*cf.* [6, Rem. 4.6]).

Our policy is to apply Lemma 3.1 to the potentials χ and $\boldsymbol{\psi}$. Thus we use two kinds of plane wave functions to approximate the solutions of Navier's equation (1): pressure (longitudinal) waves

$$\boldsymbol{w}^P_{\boldsymbol{d}} : \boldsymbol{x} \mapsto \boldsymbol{d}\, e^{i\omega_P \boldsymbol{x}\cdot\boldsymbol{d}} \qquad \boldsymbol{d} \in \mathbb{S}^2 ,$$

and shear (transverse) waves

$$\boldsymbol{w}^S_{\boldsymbol{d},\boldsymbol{A}} : \boldsymbol{x} \mapsto \boldsymbol{A}\, e^{i\omega_S \boldsymbol{x}\cdot\boldsymbol{d}} \qquad \boldsymbol{d}, \boldsymbol{A} \in \mathbb{S}^2, \quad \boldsymbol{A}\cdot\boldsymbol{d} = 0 .$$

Given $\boldsymbol{d} \in \mathbb{S}^2$, there exist two linearly independent shear waves propagating along $\boldsymbol{d}$ ($\boldsymbol{w}^S_{\boldsymbol{d},\boldsymbol{A}}$ and $\boldsymbol{w}^S_{\boldsymbol{d},\boldsymbol{d}\times\boldsymbol{A}}$) and only one pressure wave ($\boldsymbol{w}^P_{\boldsymbol{d}}$). They satisfy the relations

$$\begin{aligned}
\operatorname{div} \boldsymbol{w}^P_{\boldsymbol{d}} &= i\omega_P\, e^{i\omega_P \boldsymbol{x}\cdot\boldsymbol{d}} , & \operatorname{div} \boldsymbol{w}^S_{\boldsymbol{d},\boldsymbol{A}} &= 0 ,\\
\operatorname{curl} \boldsymbol{w}^P_{\boldsymbol{d}} &= 0 , & \operatorname{curl} \boldsymbol{w}^S_{\boldsymbol{d},\boldsymbol{A}} &= i\omega_S \boldsymbol{d}\times\boldsymbol{A}\, e^{i\omega_S \boldsymbol{x}\cdot\boldsymbol{d}} = i\omega_S \boldsymbol{w}^S_{\boldsymbol{d},\boldsymbol{d}\times\boldsymbol{A}} ,\\
\nabla \operatorname{div} \boldsymbol{w}^P_{\boldsymbol{d}} &= -\omega_P^2 \boldsymbol{w}^P_{\boldsymbol{d}} , & \operatorname{curl}\operatorname{curl} \boldsymbol{w}^S_{\boldsymbol{d},\boldsymbol{A}} &= -\omega_S^2 \boldsymbol{w}^S_{\boldsymbol{d},\boldsymbol{A}} ,\\
i\omega_P\, \boldsymbol{w}^P_{\boldsymbol{d}} &= \nabla\big(e^{i\omega_P \boldsymbol{x}\cdot\boldsymbol{d}}\big) .
\end{aligned} \tag{6}$$

It is intuitive to guess that the two components of $\boldsymbol{u}$, namely, $\nabla\chi$ and $\operatorname{curl}\boldsymbol{\psi}$, can be approximated separately by pressure and shear waves, respectively. This is the basic idea we will exploit in the proof of Theorem 3.2.

Given $p \in \mathbb{N}$ distinct unit propagation directions $\{\boldsymbol{d}_\ell\}_{1\le\ell\le p} \subset \mathbb{S}^2$, we associate p unit amplitude vectors $\{\boldsymbol{A}_\ell\}_{1\le\ell\le p} \subset \mathbb{S}^2$ such that $\boldsymbol{d}_\ell \cdot \boldsymbol{A}_\ell = 0$ for $1 \le \ell \le p$. We use them to define the linear space

$$\begin{aligned}
V_{3p} = \Bigg\{ \sum_{\ell=1}^{p} \alpha^P_\ell \boldsymbol{d}_\ell\, e^{i\omega_P \boldsymbol{x}\cdot\boldsymbol{d}_\ell} + \alpha^{S,1}_\ell \boldsymbol{A}_\ell\, e^{i\omega_S \boldsymbol{x}\cdot\boldsymbol{d}_\ell} + \alpha^{S,2}_\ell (\boldsymbol{d}_\ell \times \boldsymbol{A}_\ell)\, e^{i\omega_S \boldsymbol{x}\cdot\boldsymbol{d}_\ell},\\
\alpha^P_\ell,\ \alpha^{S,1}_\ell,\ \alpha^{S,2}_\ell \in \mathbb{C} \Bigg\}\\
= \operatorname{span}\Big\{ \boldsymbol{w}^P_{\boldsymbol{d}_\ell},\ \boldsymbol{w}^S_{\boldsymbol{d}_\ell,\boldsymbol{A}_\ell},\ \boldsymbol{w}^S_{\boldsymbol{d}_\ell,\boldsymbol{d}_\ell\times\boldsymbol{A}_\ell} \Big\}_{\ell=1,\dots,p} .
\end{aligned}$$

Notice that V_{3p} depends on the choice of $\boldsymbol{d}_\ell$'s but not on $\boldsymbol{A}_\ell$'s, and that $\dim(V_{3p}) = 3p$.

Now we can state our main result.

Theorem 3.2: *Let $D \subset \mathbb{R}^3$ be a domain satisfying* (D1)–(D3)*, k and $q \in \mathbb{N}$, $q \ge 2k+1$, $q \ge 2(1+2^{1/\lambda_D})$, where λ_D is the positive parameter that depends only* *on the shape of D defined in [9, Th. 3.2.12]. Then, there exists a set of $p = (q+1)^2$ propagation directions $\{\boldsymbol{d}_\ell\}_{1\le\ell\le p} \subset \mathbb{S}^2$, such that, for every solution $\boldsymbol{u}$ of Navier's equation* (1) *that belongs to*

$$H^{k+1}(\mathrm{div}; D) \cap H^{k+1}(\mathrm{curl}; D) \\ = \left\{ \boldsymbol{v} \in H^{k+1}(D)^3 :\ \mathrm{div}\,\boldsymbol{v} \in H^{k+1}(D),\ \mathrm{curl}\,\boldsymbol{v} \in H^{k+1}(D)^3 \right\},$$

there exists $\boldsymbol{\xi} \in V_{3p}$, namely, a linear combination of p pressure and $2p$ shear plane waves, such that, for $1 \le j \le k+1$,

$$\|\boldsymbol{u} - \boldsymbol{\xi}\|_{j-1,\omega_S,D} \le\ C\ \left(1 + (\omega_S h)^{j+6}\right) e^{(\frac{7}{4}-\frac{3}{4}\rho)\omega_S h}\ h^{k+1-j} \\ \cdot \left[q^{-\lambda_D(k+1-j)} + \frac{1+(\omega_S h)^{q-k+2}}{(\sqrt{2}\,\rho\, q)^{\frac{q-3}{2}}} M \right] \\ \cdot \left(\omega_P^{-2}\, \|\mathrm{div}\,\boldsymbol{u}\|_{k+1,\omega_P,D} + \omega_S^{-2}\, \|\mathrm{curl}\,\boldsymbol{u}\|_{k+1,\omega_S,D} \right). \tag{7}$$

Here, the constant $C > 0$ depends only on j, k and on the shape of D, the constant M is bounded by $2\sqrt{\pi}\,p$.

Proof: This proof follows the lines of the one of Theorem 5.4 in [7].

We fix the directions $\{\boldsymbol{d}_\ell\}_{1\le\ell\le p}$ to be the ones provided by Lemma 3.1, and separately approximate the two potentials χ and $\boldsymbol{\psi}$.

In (4) we have seen that the scalar potential χ is solution of the Helmholtz equation with wavenumber ω_P; Lemma 3.1 provides a combination of scalar plane waves $\xi_\chi = \sum_{\ell=1}^p \alpha_\ell^\chi e^{i\omega_P \boldsymbol{x}\cdot\boldsymbol{d}_\ell}$ such that, for $0 \le j \le k+1$,

$$|\chi - \xi_\chi|_{j,D} \le C\left(1 + (\omega_P h)^{j+6}\right) e^{(\frac{7}{4}-\frac{3}{4}\rho)\omega_P h} h^{k+1-j} \\ \cdot \left[q^{-\lambda_D(k+1-j)} + \frac{1+(\omega_P h)^{q-k+2}}{(\sqrt{2}\,\rho\, q)^{\frac{q-3}{2}}} M \right] \|\chi\|_{k+1,\omega_P,D}\ . \tag{8}$$

The three Cartesian components of the vector potential $\boldsymbol{\psi}$ are solutions of the Helmholtz equation with wavenumber ω_S. For every $\ell \in \{1,\dots,p\}$, the three vectors $\boldsymbol{d}_\ell$, $\boldsymbol{A}_\ell$ and $\boldsymbol{d}_\ell \times \boldsymbol{A}_\ell$ constitute an orthonormal basis of $\mathbb{R}^3$. Thus, according to Lemma 3.1, $\boldsymbol{\psi}$ can be approximated by a linear combination of $3p$ vector Helmholtz plane waves

$$\boldsymbol{\xi}_{\boldsymbol{\psi}} = \sum_{l=1}^{p} \alpha_\ell^{\boldsymbol{\psi},1} \boldsymbol{d}_\ell e^{i\omega_S \boldsymbol{x}\cdot\boldsymbol{d}_\ell} + \alpha_\ell^{\boldsymbol{\psi},2} \boldsymbol{A}_\ell e^{i\omega_S \boldsymbol{x}\cdot\boldsymbol{d}_\ell} + \alpha_\ell^{\boldsymbol{\psi},3} \boldsymbol{d}_\ell \times \boldsymbol{A}_\ell e^{i\omega_S \boldsymbol{x}\cdot\boldsymbol{d}_\ell}$$

with the error bound, for $0 \le j \le k+1$,

$$\left|\boldsymbol{\psi} - \boldsymbol{\xi}_{\boldsymbol{\psi}}\right|_{j,D} \le C\left(1 + (\omega_S h)^{j+6}\right) e^{(\frac{7}{4}-\frac{3}{4}\rho)\omega_S h}\ h^{k+1-j} \\ \cdot \left[q^{-\lambda_D(k+1-j)} + \frac{1+(\omega_S h)^{q-k+2}}{(\sqrt{2}\,\rho\, q)^{\frac{q-3}{2}}} M \right] \|\boldsymbol{\psi}\|_{k+1,\omega_S,D}\ . \tag{9}$$

Now we define

$$\boldsymbol{\xi} = \nabla \xi_\chi + \mathrm{curl}\,\boldsymbol{\xi}_{\boldsymbol{\psi}}$$

$$\overset{(6)}{=} i \sum_{l=1}^{p} \Big(\omega_P \boldsymbol{d}_\ell \alpha_\ell^{\chi} e^{i\omega_P \boldsymbol{x}\cdot\boldsymbol{d}_\ell} + \omega_S \alpha_\ell^{\boldsymbol{\psi},2} \boldsymbol{d}_\ell \times \boldsymbol{A}_\ell e^{i\omega_S \boldsymbol{x}\cdot\boldsymbol{d}_\ell} - \omega_S \alpha_\ell^{\boldsymbol{\psi},3} \boldsymbol{A}_\ell e^{i\omega_S \boldsymbol{x}\cdot\boldsymbol{d}_\ell} \Big)$$

which clearly belongs to V_{3p}. This vector field provides the desired approximation of the displacement $\boldsymbol{u}$:

$$\begin{aligned}
&\|\boldsymbol{u}-\boldsymbol{\xi}\|_{j-1,\omega_S,D} = \big\|\nabla\chi + \operatorname{curl}\boldsymbol{\psi} - \nabla\xi_\chi - \operatorname{curl}\boldsymbol{\xi}_{\boldsymbol{\psi}}\big\|_{j-1,\omega_S,D} \\
&\le \sum_{j_0=0}^{j-1} \omega_S^{j-1-j_0} \left|\nabla(\chi-\xi_\chi) + \operatorname{curl}(\boldsymbol{\psi}-\boldsymbol{\xi}_{\boldsymbol{\psi}})\right|_{j_0,D} \\
&\le \sum_{j_1=1}^{j} \omega_S^{j-j_1} \Big(|\chi-\xi_\chi|_{j_1,D} + |\boldsymbol{\psi}-\boldsymbol{\xi}_{\boldsymbol{\psi}}|_{j_1,D} \Big) \\
&\overset{\substack{(8),(9)\\ \omega_P<\omega_S}}{\le} C \left(\sum_{j_1=1}^{j} \omega_S^{j-j_1} \left(1+(\omega_S h)^{j_1+6}\right) h^{k+1-j_1} \right) e^{(\frac{7}{4}-\frac{3}{4}\rho)\omega_S h} \\
&\qquad\qquad \left[q^{-\lambda_D(k+1-j)} + \frac{1+(\omega_S h)^{q-k+2}}{(\sqrt{2}\,\rho\, q)^{\frac{q-3}{2}}} M \right] \Big(\|\chi\|_{k+1,\omega_P,D} + \|\boldsymbol{\psi}\|_{k+1,\omega_S,D} \Big) \\
&\le C \left(1+(\omega_S h)^{j+6}\right) e^{(\frac{7}{4}-\frac{3}{4}\rho)\omega_S h}\, h^{k+1-j} \left[q^{-\lambda_D(k+1-j)} + \frac{1+(\omega_S h)^{q-k+2}}{(\sqrt{2}\,\rho\, q)^{\frac{q-3}{2}}} M \right] \\
&\qquad\qquad \Big(\|\chi\|_{k+1,\omega_P,D} + \|\boldsymbol{\psi}\|_{k+1,\omega_S,D} \Big) \\
&\overset{(2)}{=} C \left(1+(\omega_S h)^{j+6}\right) e^{(\frac{7}{4}-\frac{3}{4}\rho)\omega_S h}\, h^{k+1-j} \left[q^{-\lambda_D(k+1-j)} + \frac{1+(\omega_S h)^{q-k+2}}{(\sqrt{2}\,\rho\, q)^{\frac{q-3}{2}}} M \right] \\
&\qquad\qquad \Big(\omega_P^{-2} \|\operatorname{div}\boldsymbol{u}\|_{k+1,\omega_P,D} + \omega_S^{-2} \|\operatorname{curl}\boldsymbol{u}\|_{k+1,\omega_S,D} \Big) .
\end{aligned}$$

□

Notice that, in order to have convergence in the bound (7), either in h or p, the potentials $\operatorname{div}\boldsymbol{u}$ and $\operatorname{curl}\boldsymbol{u}$ have to belong to $H^2(D)$.

Since $\omega_P < \omega_S$, the bound (7) holds true also in the case where the norm on the left-hand side is substituted by $\|\boldsymbol{u}-\boldsymbol{\xi}\|_{j-1,\omega_P,D}$; on the contrary we can not substitute the algebraic and exponential terms in $\omega_S h$ on the right-hand side with the analogous ones containing $\omega_P h$.

The bound proven in Theorem 3.2 shows algebraic orders of convergence both with respect to the size h of the domain and to the dimension p of the approximating space. If the solution $\boldsymbol{u}$ can be smoothly extended outside D, the order in p is exponential, see [8, Rem. 3.14], and [9, Rem. 3.5.8]. The bounding constant depends on the problem parameters ω, λ, μ and ρ only through ω_P and ω_S, with the dependence shown in the estimate.

The approximation results for the Maxwell equations suggest some extensions of Theorem 3.2. For example, error bounds for elastic spherical waves (i.e., solutions of (1) defined via vector spherical harmonics and spherical Bessel functions) could be proved by following the technique used in Section 6.2.2 of [9]. The order of convergence with respect to the diameter h might be improved, as it was done in the electromagnetic case in Section 6.3 of [9]: a special Taylor approximation bound for harmonic functions can be used in the time-harmonic setting thanks to a careful use of Vekua's theory; some lengthy manipulations of vector spherical harmonics could provide approximating fields which are exact solution of the elastic wave equation; a special vector Jacobi–Anger formula (see equation 6.34 in [9]) is then needed to find approximants in the form of vector plane waves. The proof of improved orders of convergence in p seems to be much harder: it requires better understanding of the Vekua transform in a vector setting and is not available even in the electromagnetic case.

In the almost incompressible case, i.e., for very large values of λ, both ω_P and $\operatorname{div} \boldsymbol{u}$ go to zero. Therefore, estimate (7) is useful only if $\omega_P^{-2} \|\operatorname{div} \boldsymbol{u}\|_{k+1,\omega_P,D}$ remains bounded in dependence of λ. In the limit case we recover Maxwell's equations and Theorem 3.2 reduces to Theorem 5.4 of [7].

Acknowledgement

The author is grateful to R. Hiptmair, I. Perugia and C. Schwab for the support in the preparation of this paper.

References

[1] T. Huttunen, P. Monk, F. Collino, and J.P. Kaipio, *The ultra-weak variational formulation for elastic wave problems*, SIAM J. Sci. Comput. 25 (2004), pp. 1717–1742.

[2] P. Ladevèze, L. Arnaud, P. Rouch, and C. Blanzé, *The variational theory of complex rays for the calculation of medium-frequency vibrations.*, Eng. Comput. (Bradf.) 18 (2001), pp. 193–214.

[3] T. Luostari, T. Huttunen, and P. Monk, *The ultra weak variational formulation for 3D elastic wave problems*, in *Proceedings of 20th International Congress on Acoustics, ICA 2010, 23-27 August 2010, Sydney, Australia*, 2010.

[4] P. Massimi, R. Tezaur, and C. Farhat, *A discontinuous enrichment method for three-dimensional multiscale harmonic wave propagation problems in multi-fluid and fluid-solid media*, Internat. J. Numer. Methods Engrg. 76 (2008), pp. 400–425.

[5] L. Zhang, R. Tezaur, and C. Farhat, *The discontinuous enrichment method for elastic wave propagation in the medium-frequency regime*, Internat. J. Numer. Methods Engrg. 66 (2006), pp. 2086–2114.

[6] A. Moiola, R. Hiptmair, and I. Perugia, *Plane wave approximation of homogeneous Helmholtz solutions*, Z. Angew. Math. Phys. 62 (2011), pp. 809–837.

[7] R. Hiptmair, A. Moiola, and I. Perugia, *Error analysis of Trefftz-discontinuous Galerkin methods for the time-harmonic Maxwell equations*, Technical report 2011-09, SAM-ETH Zürich, Zürich, Switzerland, 2011 To appear in Math. Comput.

[8] ———, *Plane wave discontinuous Galerkin methods for the 2D Helmholtz equation: analysis of the p-version*, SIAM J. Numer. Anal. 49 (2011), pp. 264–284.

[9] A. Moiola, *Trefftz-discontinuous Galerkin methods for time-harmonic wave problems*, PhD Thesis, Seminar for applied mathematics, ETH Zürich, 2011.

[10] E. Sternberg, *On the integration of the equations of motion in the classical theory of elasticity*, Arch. Rational Mech. Anal. 6 (1960), pp. 34–50.

[11] W. Noll, *Verschiebungsfunktionen für elastische Schwingungsprobleme*, Z. Angew. Math. Mech. 37 (1957), pp. 81–87.

[12] K.F. Graff *Wave motion in elastic solids.*, Oxford Engineering Science Series. Oxford: Clarendon Press; Oxford University Press. XVII, 649 p., 1975, reprint 1991.

[13] I.H. Sloan and R.S. Womersley, *Extremal systems of points and numerical integration on the sphere*, Adv. Comput. Math. 21 (2004), pp. 107–125.

[14] R.S. Womersley and I.H. Sloan, Interpolation and Cubature on the Sphere; http://web.maths.unsw.edu.au/˜rsw/Sphere/.