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OPTIMAL CONVERGENCE ESTIMATES FOR THE TRACE OF THE POLYNOMIAL L^2-PROJECTION OPERATOR ON A SIMPLEX

ALEXEY CHERNOV

ABSTRACT. In this paper we study convergence of the L^2-projection onto the space of polynomials up to degree p on a simplex in $\mathbb{R}^d$, $d \geq 2$. Optimal error estimates are established in the case of Sobolev regularity and illustrated on several numerical examples. The proof is based on the collapsed coordinate transform and the expansion into various polynomial bases involving Jacobi polynomials and their antiderivatives. The results of the present paper generalize corresponding estimates for cubes in $\mathbb{R}^d$ from [P.Houston, C.Schwab, E.Süli, *Discontinuous hp-finite element methods for advection-diffusion-reaction problems.* SIAM J. Numer. Anal. 39 (2002), no. 6, 2133–2163].

1. Introduction

Polynomial approximation in $\mathbb{R}^d$ plays an important role in spectral, hp- and discontinuous hp-finite elements [3, 22, 16, 14, 7, 15, 19, 13]. Compared to approximation in the tensor product domains, approximation on simplices has several favorable properties, such as greater flexibility for complex geometries and local hp-adaptivity. However, approximation analysis on simplices for $d \geq 2$ is more involved, since the tensor product arguments are not applicable anymore.

Recently, numerical analysis of spectral and hp-methods on simplices has received an increasing attention. A number of publications has been devoted to studies of approximation by polynomials [28, 23] and rational functions [24] in weighted Sobolev spaces on triangles in $\mathbb{R}^2$. Much less theory is developed on simplices in $d \geq 3$, see however [18, 6].

In this paper we study approximation property of the L^2-projection onto the space of polynomials up to degree p on a simplex $T \subset \mathbb{R}^d$, $d \geq 2$, measured in $L^2(\partial T)$. Our analysis generalizes results for the tensor product domains from [15, Section 3.3] onto the case of simplices. Results of the present paper in the case $d = 2, 3$ play an important role in the convergence analysis of Discontinuous Galerkin Methods on the meshes of simplices (cf. e.g. [15, Theorem 4.7] and [19, Proposition 8.1] for corresponding results for d-dimensional parallelepipeds). Another application is in the convergence analysis of interface problems on non-matching grids, discretized by Nitsche's method [8]. At the present time we have no particular application of our approximation estimates in the case $d \geq 4$, which is, however, an interesting and challenging result from approximation theory.

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Let $\mathcal{P}_p(T)$ be the space of d-variate polynomials on T of degree up to p and Π_p be the $L^2(T)$-orthogonal projection operator onto $\mathcal{P}_p(T)$, i.e.

$$\int_T (u - \Pi_p u)v = 0 \qquad \forall v \in \mathcal{P}_p(T).$$

We denote by $\|u\|_\Omega = \left(\int_\Omega |u|^2\right)^{\frac{1}{2}}$ the L^2-norm on a subset (of a manifold) $\Omega \subset \mathbb{R}^d$. The aim of this paper is to prove the following approximation result:

Theorem 1.1. *Suppose $u \in H^s(T)$ with $s \geq 1$. Then there exists $C > 0$ independent of p, such that*

$$\|u - \Pi_p u\|_{\partial T} \leq C p^{-s+\frac{1}{2}} \|u\|_{H^s(T)}. \tag{1.1}$$

The basic idea of the proof is the following. We prove first the estimate

$$\|u - \Pi_p u\|_F \leq C(p+1)^{-\frac{1}{2}} \|\nabla u\|_T \qquad \forall u \in H^1(T) \tag{1.2}$$

where F is one of the faces of T and the constant $C > 0$ depends only on the spatial dimension d. Having this, we proceed as in the one-dimensional case in [15, Lemma 3.5]. The projection Π_p clearly preserves polynomials $v \in \mathcal{P}_p(T)$, hence by (1.2)

$$\|u - \Pi_p u\|_{\partial T} = \|u - v - \Pi_p(u - v)\|_{\partial T} \leq C(p+1)^{-\frac{1}{2}} \inf_{v \in \mathcal{P}_p(T)} \|\nabla(u - v)\|_T.$$

Then we estimate the best approximation error as in [2, 18] or [22, Remark 4.74]:

$$\inf_{v \in \mathcal{P}_p(T)} \|\nabla(u - v)\|_T \leq C p^{-s+1} \|u\|_{H^s(T)}. \tag{1.3}$$

Here we utilize a regularity preserving extension into a larger cube, [25, Chapter 6] and approximation properties of tensor product polynomials of degree p in each direction on this cube. Collecting the terms we arrive at (1.1).

Our main concern now is the estimate (1.2), which is the result of Theorem 3.1 below if T is a triangle and of Theorem 4.1 if T is a general simplex in $\mathbb{R}^d$. Although the structure of the proofs is similar for $d = 2$ and $d \geq 3$ (moreover, Theorem 3.1 is a special case of Theorem 4.1 with $d = 2$) we give the proof in $\mathbb{R}^2$ explicitly, since it is more transparent and helps to understand the general case which requires more involved notations.

The structure of the proofs is as follows. We shall find a suitable auxiliary function $g \in H^1(T)$ and a vector field $\vec{a}$ on T, $\|\vec{a}\| \sim 1$, such that

$$\|u - \Pi_p u\|_F \leq C\left(p + \tfrac{d}{2}\right)^{-\frac{1}{2}} \|\partial_{\vec{a}} u - g\|_T \quad \text{and} \quad \|g\|_T \leq C\|\nabla u\|_T. \tag{1.4}$$

Then we arrive at (1.2) using the triangle inequality. The proof of the left inequality in (1.4) relies on expansions of u in certain suitable polynomial bases. We shall see that the auxiliary function g naturally appears in an expansion form with respect to a set of *rational functions*, see (3.14) and (4.20) below. The proof of the right inequality in (1.4) involves the directional derivatives of u in certain remaining linear independent directions and is realized by induction in the case of an arbitrary dimension d.

The remainder of the paper is organized as follows. Section 2 includes some basic definitions and properties of Jacobi polynomials in one dimension. Section 3 contains approximation analysis in $\mathbb{R}^2$. Section 4 is devoted to the generalization to $\mathbb{R}^d$. In Section 5 we give several numerical experiments in $\mathbb{R}^2$ and $\mathbb{R}^3$ supporting our convergence estimates. Several important transformations between families of Jacobi polynomials are collected in the Appendix.

2. Basic facts on Jacobi polynomials in one dimension

In this section we collect some basic facts on Jacobi polynomials in one dimension which will be important in what follows. We refer to [1, 26] for more details.

For any $n \in \mathbb{N}_0$ and $\alpha, \beta > -1$ we denote by

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n n!(x-1)^\alpha (x+1)^\beta} \frac{d^n}{dx^n} \left((x-1)^{\alpha+n}(x+1)^{\beta+n}\right) \tag{2.1}$$

the classical Jacobi polynomials, orthogonal on $[-1,1]$ with respect to the weighting function $(1-x)^\alpha(1+x)^\beta$. In the special case $\beta = 0$ we shall skip the second index and write

$$P^\alpha(x) := P^{(\alpha,\beta)}(x).$$

Let us recall the orthogonality relation for this family

$$\int_{-1}^{1} P_m^\alpha(x) P_n^\alpha(x) \left(\frac{1-x}{2}\right)^\alpha dx = \frac{2\delta_{mn}}{2n+\alpha+1} \tag{2.2}$$

and the normalization

$$P_n^\alpha(-1) = (-1)^n. \tag{2.3}$$

We shall write Df for the derivative with respect to the argument of $f = f(x)$: $Df \equiv \frac{d}{dx} f$ for any differentiable f. It is known that the derivative DP_n^α is proportional to $P_{n-1}^{(\alpha+1,1)}$ [26, (4.21.7)]. In particular, we have the orthogonality relation

$$\int_{-1}^{1} DP_n^\alpha(x) DP_m^\alpha(x) \left(\frac{1-x}{2}\right)^\alpha (1-x^2)\, dx = \frac{2\delta_{mn}}{2n+\alpha+1} n(n+\alpha+1). \tag{2.4}$$

We define

$$\hat{P}_n^\alpha(x) = \int_{-1}^{x} P_{n-1}^\alpha(t)\, dt, \quad n \geq 1, \qquad \hat{P}_0^\alpha(x) = 1. \tag{2.5}$$

the antiderivatives of the Jacobi polynomials. Clearly, P_n^α and $\hat{P}_n^\alpha$ are polynomials of degree n and hence form two different polynomial bases on $[-1,1]$. In what follows we shall extensively use expansions involving these polynomial bases and employ the basis transformations listed in the Appendix. It will be convenient to write such basis expansions in vector notations. Suppose $\{P_i\}_{i=0}^\infty$ is a basis. Then we abbreviate the infinite sum

$$\underline{U}^\top \underline{P}(x) := \sum_{i=0}^{\infty} U_i P_i(x)$$

for a vector of coefficients $\underline{U} = (U_0, U_1, \dots)^\top$ and a vector of basis functions $\underline{P} = (P_0, P_1, \dots)^\top$.

Before we continue, let us prove the following technical lemma which is a partial generalization of [15, Lemma 3.5].

Lemma 2.1. *Consider a function $U : [-1,1] \to \mathbb{R}$ such that*

$$\int_{-1}^{1} (1-x)^\alpha |U'(x)|^2\, dx < \infty$$

given by the infinite series

$$U(x) = \sum_{j=r}^{\infty} U_j P_j^\alpha(x), \qquad r \geq 0,\ \alpha \geq 0. \tag{2.6}$$

Then

$$U(-1) = \begin{cases} (-1)^r (a_r^\alpha V_r + c_{r+1}^\alpha V_{r+1}), & r \geq 1, \\ V_0, & r = 0, \end{cases}$$

where V_j is the j-th coefficient of U in the basis $\{\hat{P}_j^\alpha(x)\}$

$$U(x) = \sum_{j=0}^{\infty} V_j \hat{P}_j^\alpha(x), \qquad r \geq 0, \ \alpha \geq 0, \tag{2.7}$$

and a_r^α and c_{r+1}^α are transformation coefficients from (A.2).

Proof. According to (2.3) we have $U(-1) = \sum_{i=r}^{\infty} (-1)^j U_j$ and it remains to establish the relation between coefficients $\{U_j\}$ and $\{V_j\}$. To this end we set $U_0 = \cdots = U_{r-1} = 0$ and rewrite (2.6), (2.7) in the vector form:

$$\underline{U}^\top \underline{P}^\alpha(x) = U(x) = \underline{V}^\top \underline{\hat{P}}^\alpha(x) = \underline{V}^\top \mathcal{A}^\alpha \underline{P}^\alpha(x),$$

where $\mathcal{A}^\alpha$ is the transformation matrix from (A.2). Comparing the coefficients we obtain

$$U_j = V_j a_j^\alpha + V_{j+1} b_{j+1}^\alpha + V_{j+2} c_{j+2}^\alpha, \qquad j \geq 0$$

where $a_0^\alpha = 1$, and hence

$$\begin{aligned} U(-1) &= \sum_{i=r}^{\infty} (-1)^j U_j = \sum_{j=r}^{\infty} (-1)^j a_j^\alpha V_j + \sum_{j=r+1}^{\infty} (-1)^{j+1} b_j^\alpha V_j + \sum_{j=r+2}^{\infty} (-1)^j c_j^\alpha V_j \\ &= (-1)^r \left(a_r^\alpha V_r + (b_{r+1}^\alpha - a_{r+1}^\alpha) V_{r+1} \right) + \sum_{j=r+2}^{\infty} (-1)^j \left(a_j^\alpha - b_j^\alpha + c_j^\alpha \right) V_j \end{aligned}$$

A direct substitution of (A.2) gives

$$b_j^\alpha - a_j^\alpha = \begin{cases} c_j^\alpha, & j \geq 2, \\ 0, & j = 1, \end{cases}$$

and hence the assertion follows. $\square$

3. Convergence estimates on a triangle in $\mathbb{R}^2$

3.1. **Notations and formulation of the convergence theorem in $\mathbb{R}^2$.** Let T be a reference triangle with the vertices $(-1,-1)$, $(1,-1)$ and $(0,1)$ and $F = \partial T \cap \{y = -1\}$ be its lower face. Although this choice of the reference element is rather nonstandard (see however [5]), it suites the best for our purpose due to its symmetry with respect to the vertical axis. T can be parametrized as

$$T = \left\{ \left(z \frac{1-y}{2}, y \right) \in \mathbb{R}^2 : (z, y) \in [-1,1]^2 \right\} \tag{3.1}$$

with the Jacobian determinant

$$\det \left(\frac{\partial(x,y)}{\partial(z,y)} \right) = \frac{1-y}{2}. \tag{3.2}$$

The main result of this section is the following theorem:

Theorem 3.1. *For an arbitrary $u \in H^1(T)$ there holds*

$$\|u - \Pi_p u\|_F \leq \frac{\sqrt{5}+1}{\sqrt{2}} (p+1)^{-\frac{1}{2}} \|\nabla u\|_T. \tag{3.3}$$

We prove (3.3) using various convenient basis expansions for u. Clearly, an orthogonal basis suites the best for representation of $u - \Pi_p u$. We define

$$\phi_{ij}(x,y) = P_i^0\left(\frac{2x}{1-y}\right) P_j^{2i+1}(y)\left(\frac{1-y}{2}\right)^i, \qquad i,j \geq 0, \tag{3.4}$$

which is a variant of Dubiner or Koornwinder polynomial basis on a triangle [10, 16, 17]. ϕ_{ij} is indeed a polynomial of degree $i+j$, since for $P_i^0(z) = \sum_{n=0}^i B_n z^n$

$$\phi_{ij}(x,y) = P_j^{2i+1}(y) \sum_{n=0}^{i} B_n x^n \left(\frac{1-y}{2}\right)^{i-n}. \tag{3.5}$$

Using parametrization (3.1), orthogonality relation (2.2) and (3.2) we find

$$\begin{aligned}\int_T \phi_{ij}\phi_{kl} = \int_{-1}^1 P_i^0(z)P_k^0(z)\,dz \int_{-1}^1 P_j^{2i+1}(y)P_l^{2k+1}(y)\left(\frac{1-y}{2}\right)^{i+k+1} dy \\ = \frac{2\delta_{ik}}{2i+1}\frac{\delta_{jl}}{i+j+1}.\end{aligned} \tag{3.6}$$

Hence $\{\phi_{ij}\}_{i+j\leq p}$ forms an orthogonal basis in $\mathcal{P}_p(T)$. On the other hand, as we shall see, $\|\nabla u\|_T$ has a lower bound, which is convenient to represent in another polynomial basis $\{\psi_{ij}\}_{i+j\leq p}$ involving antiderivatives in the y-direction

$$\psi_{ij}(x,y) = P_i^0\left(\frac{2x}{1-y}\right) \hat{P}_j^{2i+1}(y)\left(\frac{1-y}{2}\right)^i, \qquad i,j \geq 0,\ i+j \leq p, \tag{3.7}$$

see the proof of Theorem 3.1 below. Any function $u \in L^2(T)$ admits two equivalent expansions

$$u(x,y) = \sum_{i,j=0}^{\infty} u_{ij}\phi_{ij}(x,y) = \sum_{i,j=0}^{\infty} v_{ij}\psi_{ij}(x,y) \tag{3.8}$$

converging in $L^2(T)$. Here, due to (3.6), the Jacobi-Fourier coefficients are

$$u_{ij} = \frac{2i+1}{2}(i+j+1)\int_T u\phi_{ij}$$

and $\{v_{ij}\}$ can be found using the three-term polynomial relationship (A.1).

3.2. **An auxiliary upper bound for $\|u-\Pi_p u\|_F$.** First, we prove a simple bound for $\|u - \Pi_p u\|_F$, which involves expansion coefficients v_{ij} of u in the basis with antiderivatives in the y-direction.

Lemma 3.2. *For a sufficiently smooth function u we have the bound*

$$\|u - \Pi_p u\|_F^2 \leq \sum_{i=0}^{p} \sum_{j=p+1-i}^{p+2-i} \frac{2}{2i+1}\frac{2v_{ij}^2}{(i+j+\frac{1}{2})^2} + \sum_{i=p+1}^{\infty} \frac{2}{2i+1} v_{i0}^2 \tag{3.9}$$

with expansion coefficients v_{ij} from (3.8).

Proof. Firstly, we express the approximation error on the face F via Jacobi-Fourier coefficients u_{ij}. Orthogonality (3.6) of $\{\phi_{ij}\}$ provides

$$(u - \Pi_p u)(x,y) = \sum_{i+j\geq p+1} u_{ij}\phi_{ij}(x,y), \tag{3.10}$$

hence

$$(u-\Pi_p u)|_F = \sum_{i+j\geq p+1} u_{ij}\phi_{ij}(x,-1) = \sum_{i=0}^{\infty} P_i^0(x) \sum_{j=\max(p+1-i,0)}^{\infty} (-1)^j u_{ij}.$$

and thus by orthogonality (2.2) we have the identity

$$\|u-\Pi_p u\|_F^2 = \sum_{i=0}^{\infty} \frac{2}{2i+1}\left(\sum_{j=\max(p+1-i,0)}^{\infty} (-1)^j u_{ij}\right)^2.$$

Let us consider relation (3.8) connecting expansion coefficients $\{u_{ij}\}$ and $\{v_{ij}\}$. Lemma 2.1 with $U_j \to u_{ij}$, $V_j \to v_{ij}$ for every fixed i gives

$$\|u-\Pi_p u\|_F^2 = \sum_{i=0}^{p} \frac{2}{2i+1}\left(a_{p+1-i}^{2i+1} v_{i,p+1-i} + c_{p+2-i}^{2i+1} v_{i,p+2-i}\right)^2 + \sum_{i=p+1}^{\infty} \frac{2}{2i+1} v_{i0}^2.$$

Using explicit expressions for the coefficients in (A.2) we get directly the bounds $|c_n^\alpha| \leq a_n^\alpha$ for $n \geq 2$ and

$$a_{n-i}^{2i+1} = \frac{2(n+i+1)}{2n(2n+1)} \leq \frac{2}{2n+1}, \quad i=0,\ldots,n-1,\ n\geq 1.$$

Applying the above estimate with $n = p+1$ and $n = p+2$ we get

$$\|u-\Pi_p u\|_F^2 \leq \sum_{i=0}^{p} \frac{2}{2i+1}\left(\frac{2v_{i,p+1-i}^2}{(p+\frac{3}{2})^2} + \frac{2v_{i,p+2-i}^2}{(p+\frac{5}{2})^2}\right) + \sum_{i=p+1}^{\infty} \frac{2}{2i+1} v_{i0}^2$$

and the assertion (3.9) follows. □

3.3. **Directional derivatives and the proof of Theorem 3.1.** We start with explicit computation of gradients of basis functions ψ_{ij} defined in (3.7). To simplify the calculations, we set $P_k^\alpha(x) = 0$ if $k=-1$. Recall that $P_0^0(z) \equiv 1$. Then

$$\nabla\psi_{0j} = \left(0, P_{j-1}^1(y)\right) \quad \text{and} \quad \nabla\psi_{ij} = (\partial_x\psi_{ij}, \partial_y\psi_{ij}) \quad i\geq 1,$$

where

$$\partial_x\psi_{ij} = DP_i^0\left(\frac{2x}{1-y}\right)\left(\frac{1-y}{2}\right)^{i-1} \hat{P}_j^{2i+1}(y),$$

$$\begin{aligned}\partial_y\psi_{ij} = {} & DP_i^0\left(\frac{2x}{1-y}\right)\frac{x}{1-y}\left(\frac{1-y}{2}\right)^{i-1}\hat{P}_j^{2i+1}(y) \\ & + P_i^0\left(\frac{2x}{1-y}\right)\left\{P_{j-1}^{2i+1}(y)\left(\frac{1-y}{2}\right)^{i} - \hat{P}_j^{2i+1}(y)\frac{i}{2}\left(\frac{1-y}{2}\right)^{i-1}\right\}.\end{aligned} \tag{3.11}$$

Consider a vector field pointing towards the vertex $(0,1)$ of T:

$$\vec{a} = \left(-\frac{x}{1-y}, 1\right), \qquad 1 \leq \|\vec{a}\| \leq \tfrac{\sqrt{5}}{2} \quad \text{on } T. \tag{3.12}$$

Then the directional derivative along $\vec{a}$ has a simple form

(3.13)
$$\partial_{\vec{a}}\psi_{ij} = \begin{cases} P^1_{j-1}(y) & i=0, \\ P^0_i\left(\frac{2x}{1-y}\right)\left\{P^{2i+1}_{j-1}(y)\left(\frac{1-y}{2}\right)^i - \hat{P}^{2i+1}_j(y)\frac{i}{2}\left(\frac{1-y}{2}\right)^{i-1}\right\} & i\geq 1. \end{cases}$$

Recalling expansion (3.8) with coefficients v_{ij} we introduce a formal sum

$$g(x,y) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{i}{2}v_{ij}P^0_i\left(\frac{2x}{1-y}\right)\hat{P}^{2i+1}_j(y)\left(\frac{1-y}{2}\right)^{i-1} \tag{3.14}$$

which plays an important role in the following

Proof of Theorem 3.1. We make the proof in three steps:

Step 1. Firstly, we prove the bound

$$\frac{p+1}{2}\|u-\Pi_p u\|^2_F \leq \|\partial_{\vec{a}}u - g\|^2_T. \tag{3.15}$$

We start with an explicit computation of $\partial_{\vec{a}}u - g$. According to (3.8), (3.13) and (3.14) we find

$$\begin{aligned} \partial_{\vec{a}}u - g &= \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}v_{ij}P^0_i\left(\frac{2x}{1-y}\right)P^{2i+1}_{j-1}(y)\left(\frac{1-y}{2}\right)^i \\ &\quad - \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}iv_{ij}P^0_i\left(\frac{2x}{1-y}\right)\hat{P}^{2i+1}_j(y)\left(\frac{1-y}{2}\right)^{i-1} \\ &= \sum_{j=1}^{\infty}v_{0j}P^1_{j-1}(y) - \sum_{i=1}^{\infty}iv_{i0}P^0_i\left(\frac{2x}{1-y}\right)\left(\frac{1-y}{2}\right)^{i-1} \\ &\quad - \sum_{i,j=1}^{\infty}v_{ij}P^0_i\left(\frac{2x}{1-y}\right)\left(\frac{1-y}{2}\right)^{i-1}\left\{\frac{y-1}{2}P^{2i+1}_{j-1}(y) + i\hat{P}^{2i+1}_j(y)\right\}. \end{aligned} \tag{3.16}$$

Now, we utilize Lemma A.3 providing $\frac{y-1}{2}P^{2i+1}_{j-1}(y) + i\hat{P}^{2i+1}_j(y) = P^{2i-1}_j(y)$. Hence (3.16) is an expansion in an orthogonal on $L^2(T)$ basis and by orthogonality (2.2)

$$\begin{aligned} \|\partial_{\vec{a}}u - g\|^2_T &= \sum_{j=1}^{\infty}2\frac{v^2_{0j}}{j} + \sum_{i=1}^{\infty}\frac{2}{2i+1}\frac{(iv_{i0})^2}{i} + \sum_{i,j=1}^{\infty}\frac{2}{2i+1}\frac{v^2_{ij}}{i+j} \\ &= \sum_{i=1}^{\infty}\frac{2}{2i+1}iv^2_{i0} + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\frac{2}{2i+1}\frac{v^2_{ij}}{i+j} \\ &\geq \sum_{i=p+1}^{\infty}\frac{2}{2i+1}iv^2_{i0} + \sum_{i=0}^{p}\sum_{j=p+1-i}^{p+2-i}\frac{2}{2i+1}\frac{v^2_{ij}}{i+j} \\ &\geq (p+1)\left\{\sum_{i=p+1}^{\infty}\frac{2v^2_{i0}}{2i+1} + \sum_{i=0}^{p}\sum_{j=p+1-i}^{p+2-i}\frac{2}{2i+1}\frac{v^2_{ij}}{(i+j)^2}\right\} \\ &\geq \frac{p+1}{2}\|u-\Pi_p u\|^2_F, \end{aligned}$$

where we used Lemma 3.2 in the last step.

Step 2. Secondly, we prove the bound

$$\|g\|_T \le \frac{1}{2}\Big\|\sqrt{1-\big(\tfrac{2x}{1-y}\big)^2}\,\partial_x u\Big\|_T. \tag{3.17}$$

Let us consider the inner sum over j in (3.14) for every fixed $i \ge 1$. Using vector notations $\underline{v_i} = (v_{i0}, v_{i1}, \dots)^\top$ we introduce auxiliary expansion coefficients $\{w_{ij}\}$ satisfying

$$\underline{w_i}^\top \underline{P}^{2i-1}(y) = \underline{u_i}^\top \underline{P}^{2i+1}(y) = \underline{v_i}^\top \underline{\hat{P}}^{2i+1}(y) \tag{3.18}$$

hence by Lemmas A.1 and A.2 we find an explicit relation

$$\underline{v_i}^\top \mathcal{A}^{2i+1} = \underline{u_i}^\top = \underline{w_i}^\top \mathcal{W}^{2i}\mathcal{W}^{2i+1}.$$

In the new basis

$$2g(x,y) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} i w_{ij} P_i^0\left(\frac{2x}{1-y}\right) P_j^{2i-1}(y)\left(\frac{1-y}{2}\right)^{i-1}.$$

This is *not a polynomial basis* expansion anymore but consists of *rational functions*, cf. also [24]. However, according to (2.2), the new basis functions are L^2-orthogonal on T and hence

$$\|2g\|_T^2 = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \frac{2}{2i+1}\frac{(iw_{ij})^2}{i+j} \tag{3.19}$$

On the other hand, (3.4) and (3.18) yield the representation

$$\begin{aligned}\partial_x u(x,y) &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} u_{ij} DP_i^0\left(\frac{2x}{1-y}\right)\left(\frac{1-y}{2}\right)^{i-1} P_j^{2i+1}(y)\\ &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} w_{ij} DP_i^0\left(\frac{2x}{1-y}\right)\left(\frac{1-y}{2}\right)^{i-1} P_j^{2i-1}(y).\end{aligned}$$

Orthogonality (2.2), orthogonality of the derivatives (2.4) and (3.19) give

$$\Big\|\sqrt{1-\big(\tfrac{2x}{1-y}\big)^2}\,\partial_x u\Big\|_T^2 = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{2i(i+1)}{2i+1}\frac{w_{ij}^2}{i+j} \ge \|2g\|_T^2.$$

Step 3. The weight $1-\big(\frac{2x}{1-y}\big)^2$ is always positive and bounded by 1 from above on T. Furthermore, $\|\vec{a}\| \le \frac{\sqrt{5}}{2}$, yielding

$$\|g\|_T \le \frac{1}{2}\Big\|\sqrt{1-\big(\tfrac{2x}{1-y}\big)^2}\,\partial_x u\Big\|_T \le \frac{1}{2}\|\nabla u\|_T, \qquad \|\partial_{\vec{a}} u\|_T \le \frac{\sqrt{5}}{2}\|\nabla u\|_T,$$

hence (3.3) follows from (3.15) by the triangle inequality. □

4. Convergence estimates on a simplex in $\mathbb{R}^d$

In this section we generalize the approach in Section 3 onto the case of a simplex in $\mathbb{R}^d$. We pursue the same strategy. Firstly, we use a suitable L^2-orthogonal polynomial basis and a basis involving antiderivatives to represent $\|u - \Pi_p u\|_F$. Then, we bound this expression by a sum of L^2-norms of directional derivatives of u, which in turn is bounded by the L^2-norm of the gradient ∇u.

4.1. **Notations and formulation of the convergence theorem in $\mathbb{R}^d$.** In this section we denote by T the simplex in $\mathbb{R}^d$ with $d+1$ vertices $\vec{V}^k$, $k=0,\ldots,d$

$$(\vec{V}^k)_j = \begin{cases} 0, & 1 \le j \le k-1, \\ 1, & 1 \le j = k \le d, \\ -1, & k+1 \le j \le d, \end{cases}$$

or explicitly

$$\begin{array}{rlrrrcrrl} \vec{V}^0 = (& -1, & -1, & -1, & \ldots & -1 & -1 &) \\ \vec{V}^1 = (& 1, & -1, & -1, & \ldots & -1 & -1 &) \\ \vec{V}^2 = (& 0, & 1, & -1, & \ldots & -1 & -1 &) \\ \vdots & & & & & & & \\ \vec{V}^d = (& 0, & 0, & 0, & \ldots & 0 & 1 &) \end{array} \tag{4.1}$$

Let $F = \partial T \cap \{x_d = -1\}$ be the lower face of T. The simplex T possesses convenient symmetries and allows for a simple representation of directional derivatives.

For simplicity of presentation we agree that a product over an empty index set equals to 1 and a sum over an empty index set equals to zero. This agreement allows to shorten many expressions in what follows, i.e. the definition of the following parametrization of T.

Let us consider the following collapsed coordinate transformation

$$\Phi : \left\{ \begin{array}{ccc} [-1,1]^d & \to & T \\ \vec{z} & \mapsto & \vec{x} \end{array} \right. , \qquad x_k = z_k \prod_{m=k+1}^{d} \frac{1-z_m}{2}, \qquad k = 1\ldots,d, \tag{4.2}$$

which parametrizes $\vec{x} = (x_1,\ldots,x_d) \in T$ by $\vec{z} = (z_1,\ldots,z_d) \in [-1,1]^d$ and is a variant of a d-dimensional generalization of the Duffy transformation [11]. Note that $x_d = z_d$ according to the above agreement. The Jacobian matrix of this parametrization is triangular and we obtain for the Jacobian determinant

$$\det\left(\frac{\partial(x_1,\ldots,x_d)}{\partial(z_1,\ldots,z_d)}\right) = \prod_{k=2}^{d} \left(\frac{1-z_k}{2}\right)^{k-1}. \tag{4.3}$$

Using (4.2) we obtain

$$\vec{x} = \Phi(\vec{z}) \quad \Leftrightarrow \quad 1 - \sum_{m=k+1}^{d} 2^{d-m} x_m = \prod_{m=k+1}^{d} (1-z_m), \qquad k = 1,\ldots,d$$

which holds also for $k=d$ according to the convention on the sums and products over empty index sets. This gives an explicit form of the inverse parametrization

$$\Phi^{-1} : \left\{ \begin{array}{ccc} T & \to & [-1,1]^d \\ \vec{x} & \mapsto & \vec{z} \end{array} \right. , \qquad z_k = \frac{2^{d-k} x_k}{1 - \sum\limits_{m=k+1}^{d} 2^{d-m} x_m}, \qquad k = 1\ldots,d. \tag{4.4}$$

In this section we shall use multivariate basis expansions as sums over multiindices $\vec{i} = (i_1,\ldots,i_d) \in \mathbb{N}_0^d$. For convenience we introduce the notations

$$[\vec{i}]_n := \sum_{k=1}^{n} i_k, \quad n = 1,\ldots,d, \qquad [\vec{i}]_0 = 0, \qquad [\vec{i}]_d = |\vec{i}|.$$

The main result of this section is given by the following theorem.

Theorem 4.1. *For an arbitrary $u \in H^1(T)$ there holds*

$$\|u - \Pi_p u\|_F \leq C \left(p + \tfrac{d}{2}\right)^{-\frac{1}{2}} \|\nabla u\|_T, \tag{4.5}$$

where the constant $C > 0$ depends only on the spatial dimension d.

We follow the approach in Section 3 and prove first several auxiliary results. We start with definition of polynomial bases. Generalizing (3.4), let us consider a family of rational functions on T

$$\pi_{i_1,\dots,i_{d-1}}(\vec{x}) = \prod_{k=1}^{d-1} \left\{ P_{i_k}^{2[\vec{i}]_{k-1}+k-1}(z_k) \left(\frac{1-z_k}{2}\right)^{[\vec{i}]_{k-1}} \right\} \Bigg|_{\vec{z}=\Phi^{-1}(\vec{x})} \tag{4.6}$$

and a family of polynomials on T

$$\phi_{\vec{i}}(\vec{x}) = \pi_{i_1,\dots,i_{d-1}}(\vec{x}) P_{i_d}^{2[\vec{i}]_{d-1}+d-1}(x_d) \left(\frac{1-x_d}{2}\right)^{[\vec{i}]_{d-1}}, \tag{4.7}$$

see [27] for a similar construction of an orthogonal basis in $\mathbb{R}^d$ and e.g. [16] for $\mathbb{R}^3$. Analogously to (3.5) we check that $\phi_{\vec{i}}$ is a polynomial of degree $|\vec{i}|$. Using orthogonality (2.2), parametrization (4.2) and (4.3) we find

$$\int_T \phi_{\vec{i}} \phi_{\vec{l}} = \prod_{k=1}^{d} \frac{2\delta_{i_k,l_k}}{2[\vec{i}]_k + k}, \tag{4.8}$$

hence $\{\phi_{\vec{i}}\}_{|\vec{i}|\leq p}$ forms an L^2-orthogonal basis in $\mathcal{P}_p(T)$. Next, we define

$$\psi_{\vec{i}}(\vec{x}) = \pi_{i_1,\dots,i_{d-1}}(\vec{x}) \hat{P}_{i_d}^{2[\vec{i}]_{d-1}+d-1}(x_d) \left(\frac{1-x_d}{2}\right)^{[\vec{i}]_{d-1}}, \tag{4.9}$$

involving an antiderivative along x_d. Let us consider two expansions

$$u(\vec{x}) = \sum_{\vec{i}\in\mathbb{N}_0^d} u_{\vec{i}} \phi_{\vec{i}}(\vec{x}) = \sum_{\vec{i}\in\mathbb{N}_0^d} v_{\vec{i}} \psi_{\vec{i}}(\vec{x}) \tag{4.10}$$

which are convergent in $L^2(T)$ for every $u \in L^2(T)$.

4.2. **An auxiliary upper bound for $\|u - \Pi_p u\|_F$.** A direct generalization of Lemma 3.2 to the case of $T \subset \mathbb{R}^d$ is given by the following result.

Lemma 4.2. *For a sufficiently smooth function u there holds the bound*

$$\|u - \Pi_p u\|_F^2 \leq \sum_{\vec{i}\in\mathcal{I}_p^d} \gamma_{i_1,\dots,i_{d-1}} \frac{2v_{\vec{i}}^2}{\left(|\vec{i}| + \frac{d-1}{2}\right)^2} + \sum_{\vec{i}\in\mathcal{J}_p^d} \gamma_{i_1,\dots,i_{d-1}} v_{\vec{i}}^2 \tag{4.11}$$

where $\gamma_{i_1,\dots,i_{d-1}} = \prod_{k=1}^{d-1} \frac{2}{2(i_1+\dots+i_k)+k}$ and

$$\begin{aligned} \mathcal{I}_p^d &= \left\{ \vec{i} \in \mathbb{N}_0^d : [\vec{i}]_{d-1} \leq p \wedge p+1 \leq |\vec{i}| \leq p+2 \right\}, \\ \mathcal{J}_p^d &= \left\{ \vec{i} \in \mathbb{N}_0^d : [\vec{i}]_{d-1} \geq p+1 \wedge i_d = 0 \right\}. \end{aligned} \tag{4.12}$$

Proof. Using orthogonality (4.8) we find

$$(u - \Pi_p u)(\vec{x}) = \sum_{|\vec{i}|\geq p+1} u_{\vec{i}} \phi_{\vec{i}}(\vec{x}).$$

Next, we utilize normalization (2.3) and rearranging the terms obtain

$$(u - \Pi_p u)|_F = \sum_{i_1,\dots,i_{d-1}=0}^{\infty} A_{i_1,\dots,i_{d-1}} \pi_{i_1,\dots,i_{d-1}}(\vec{x})\big|_{x_d=-1},$$

with $\pi_{i_1,\dots,i_{d-1}}$ from (4.6) and

$$A_{i_1,\dots,i_{d-1}} = \sum_{i_d=\max(p+1-[\vec{i}]_{d-1},0)}^{\infty} (-1)^{i_d} u_{\vec{i}}.$$

Restricted to F, $\pi_{i_1,\dots,i_{d-1}}$ becomes a polynomial of the form $\phi_{\vec{i}}$ in space dimension $d-1$. Analogously to (4.8) we find that $\pi_{i_1,\dots,i_{d-1}}$ are orthogonal in $L^2(F)$ with the norm $\|\pi_{i_1,\dots,i_{d-1}}\|_F^2 = \gamma_{i_1,\dots,i_{d-1}}$ and therefore

$$\|u - \Pi_p u\|_F^2 = \sum_{i_1,\dots,i_{d-1}=0}^{\infty} \gamma_{i_1,\dots,i_{d-1}} A^2_{i_1,\dots,i_{d-1}}.$$

For every fixed index combination $i_1,\dots,i_{d-1}$ we use Lemma 2.1 with $U_j \to u_{i_1,\dots,i_{d-1},j}$ and $V_j \to v_{i_1,\dots,i_{d-1},j}$ and obtain

$$A^2_{i_1,\dots,i_{d-1}} = \begin{cases} (a_r^{2[\vec{i}]_{d-1}+d-1} v_{i_1,\dots,i_{d-1},r} + c_{r+1}^{2[\vec{i}]_{d-1}+d-1} v_{i_1,\dots,i_{d-1},r+1})^2, & [\vec{i}]_{d-1} \le p, \\ v^2_{i_1,\dots,i_{d-1},0}, & [\vec{i}]_{d-1} \ge p+1. \end{cases}$$

where $r = p+1-[\vec{i}]_{d-1}$. According to (A.2) there holds $|c_n^\alpha| \le a_n^\alpha$ for $n \ge 2$ and

$$a_{n-q}^{2q+d-1} = \frac{2(n+q+d-1)}{(2n+d-2)(2n+d-1)} \le \frac{2}{2n+d-1}, \qquad q = 0,\dots,n-1.$$

We use this bound with $n = p+1$, $n = p+2$ and $q = [\vec{i}]_{d-1}$ and obtain the assertion. □

4.3. **Directional derivatives and the proof of Theorem 4.1.** Suppose we are given a scalar function $f = f(\vec{x}) : T \to \mathbb{R}$. Using parametrization $\vec{x} = \Phi(\vec{z})$ (4.2) we pull this function back to $[-1,1]^d$ and denote $\tilde{f}(\vec{z}) = f \circ \Phi(\vec{z})$, $\tilde{f} : [-1,1]^d \to \mathbb{R}$

$$f(\vec{x}) = \tilde{f} \circ \Phi^{-1}(\vec{x}) = \tilde{f}\left(\frac{2^{d-1}x_1}{1-\sum_{m=2}^{d} 2^{d-m} x_m}, \dots, \frac{2x_{d-1}}{1-x_d}, x_d \right). \tag{4.13}$$

We are going to find certain directions $\vec{b}$ in T, such that the derivative along $\vec{b}$ has a simple form (4.14) below. This result will be important for finding a suitable lower bound for $\|\nabla u\|_T$ in the proof of Theorem 4.1 below.

Lemma 4.3. *Suppose $f : T \to \mathbb{R}$ and $\tilde{f} : [-1,1]^d \to \mathbb{R}$ are related via (4.13). For every $k = 1,\dots,d$ there exists a vector field $\vec{b} : T \to \mathbb{R}^d$ such that*

$$\partial_{\vec{b}} f \circ \Phi(\vec{z}) = \frac{\partial \tilde{f}}{\partial z_k}(\vec{z}) \frac{\partial z_k}{\partial x_k}\bigg|_{\vec{x}=\Phi(\vec{z})} \quad \textit{where} \quad \frac{\partial z_k}{\partial x_k}\bigg|_{\vec{x}=\Phi(\vec{z})} = \prod_{m=k+1}^{d} \left(\frac{1-z_m}{2}\right)^{-1} \tag{4.14}$$

and

$$1 \le \|\vec{b}\| \le \frac{\sqrt{k+3}}{2}. \tag{4.15}$$

Proof. Transformation (4.4) yields

$$\nabla_{\vec{x}} f = \nabla_{\vec{z}} \tilde{f} \, \frac{\partial(z_1, \ldots, z_d)}{\partial(x_1, \ldots, x_d)} \tag{4.16}$$

where the Jacobian matrix is an upper triangular matrix with entries

$$\frac{\partial z_i}{\partial x_j} = \begin{cases} 0 & i > j, \\ \frac{2^{d-i}}{1-\sum_{m=i+1}^{d} 2^{d-m} x_m} & i = j, \\ \frac{2^{d-i} x_i 2^{d-j}}{(1-\sum_{m=i+1}^{d} 2^{d-m} x_m)^2} & i < j. \end{cases} \tag{4.17}$$

Multiplying (4.16) by $\vec{b}$ we find that in order to satisfy (4.14) the vector $\frac{\partial(z_1,\ldots,z_d)}{\partial(x_1,\ldots,x_d)} \vec{b}^\top$ should vanish except its k-th component which should be equal to $\frac{\partial z_k}{\partial x_k}$. It suffices to choose $b_{k+1} = \cdots = b_d = 0$, $b_k = 1$ and $b_1, \ldots, b_{k-1}$ satisfying

$$\frac{\partial(z_1, \ldots, z_k)}{\partial(x_1, \ldots, x_k)} \begin{pmatrix} b_1 \\ \vdots \\ b_{k-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial z_k}{\partial x_k} \end{pmatrix} \tag{4.18}$$

Solving this triangular system explicitly we find for $i = 1, \ldots, k-1$

$$b_i + \sum_{j=i+1}^{k} \frac{2^{d-j} x_i}{1 - \sum_{m=i+1}^{d} 2^{d-m} x_m} b_j = 0, \qquad b_k = 1.$$

According to (4.4), the multipliers in the sum are equal to $2^{i-j} z_i$ respectively and $|z_i| \leq 1$. This yields the bound

$$|b_i| \leq \sum_{j=i+1}^{k} 2^{i-j} |b_j|.$$

This inequality gives $|b_i| \leq \frac{1}{2}$, $i = 1, \ldots, k-1$ by induction, hence (4.15). Equation (4.14) follows from (4.16), (4.18) and the relation

$$\frac{\partial z_k}{\partial x_k} = \frac{2^{d-k}}{1 - \sum_{m=k+1}^{d} 2^{d-m} x_m} = \frac{z_k}{x_k} = \prod_{i=k+1}^{d} \frac{2}{1 - z_i}, \quad k = 1, \ldots, d-1,$$

for $\vec{x} = \Phi(\vec{z})$, cf. (4.2) and (4.4). $\square$

According to the previous Lemma, there exists a vector field $\vec{a}$ on T such that $\partial_{\vec{a}} f \circ \Phi(\vec{z}) = \frac{\partial \tilde{f}}{\partial z_d}$. Suppose u is developed in the basis $\psi_{\vec{i}}$

$$u(\vec{x}) = \sum_{\vec{i} \in \mathbb{N}_0^d} v_{\vec{i}} \psi_{\vec{i}}(\vec{x})$$

Recalling (4.9) and (4.6) we find that $\tilde{\pi}_{i_1,\ldots,i_{d-1}}(\vec{z}) = \pi_{i_1,\ldots,i_{d-1}} \circ \Phi(\vec{z})$ does not depend on z_d, hence

$$\begin{aligned} \partial_{\vec{a}} u(\vec{x}) = & \sum_{\vec{i} \in \mathbb{N}_0^d, i_d \geq 1} v_{\vec{i}} \pi_{i_1,\ldots,i_{d-1}}(\vec{x}) P_{i_d - 1}^{2[\vec{i}]_{d-1}+d-1}(x_d) \left(\frac{1 - x_d}{2} \right)^{[\vec{i}]_{d-1}} \\ & - \sum_{\vec{i} \in \mathbb{N}_0^d, [\vec{i}]_{d-1} \geq 1} v_{\vec{i}} \pi_{i_1,\ldots,i_{d-1}}(\vec{x}) \frac{[\vec{i}]_{d-1}}{2} \hat{P}_{i_d}^{2[\vec{i}]_{d-1}+d-1}(x_d) \left(\frac{1 - x_d}{2} \right)^{[\vec{i}]_{d-1} - 1} . \end{aligned} \tag{4.19}$$

Let us introduce an auxiliary function

$$g(\vec{x}) = \sum_{\vec{i}\in\mathbb{N}_0^d, [\vec{i}]_{d-1}\geq 1} v_{\vec{i}}\pi_{i_1,\ldots,i_{d-1}}(\vec{x}) \left(\tfrac{[\vec{i}]_{d-1}+d-2}{2}\right) \hat{P}_{i_d}^{2[\vec{i}]_{d-1}+d-1}(x_d) \left(\frac{1-x_d}{2}\right)^{[\vec{i}]_{d-1}-1}. \tag{4.20}$$

Proof of Theorem 4.1.

Step 1. We start by proving the estimate

$$\frac{1}{2}\left(p+\frac{d}{2}\right)\|u-\Pi_p u\|_F^2 \leq \|\partial_{\vec{a}}u - g\|_T^2. \tag{4.21}$$

Analogously to (3.16) we obtain from (4.19), (4.20)

$$\begin{aligned}\partial_{\vec{a}}u - g &= \sum_{i_d=1}^{\infty} v_{0,\ldots,0,i_d}\pi_{0,\ldots,0}(\vec{x})P_{i_d-1}^{d-1}(x_d)\\ &- \sum_{[\vec{i}]_{d-1}\geq 1, i_d=0} v_{\vec{i}}\pi_{i_1,\ldots,i_{d-1}}(\vec{x})\left(\tfrac{2[\vec{i}]_{d-1}+d-2}{2}\right)\left(\frac{1-x_d}{2}\right)^{[\vec{i}]_{d-1}-1}\\ &- \sum_{[\vec{i}]_{d-1}\geq 1, i_d\geq 1} v_{\vec{i}}\pi_{i_1,\ldots,i_{d-1}}(\vec{x})P_{i_d}^{2[\vec{i}]_{d-1}+d-3}(x_d)\left(\frac{1-x_d}{2}\right)^{[\vec{i}]_{d-1}-1}\end{aligned} \tag{4.22}$$

where we have utilized identity (A.5):

$$P_{i_d-1}^{2[\vec{i}]_{d-1}+d-1}(x_d)\frac{1-x_d}{2} - \left(\tfrac{2[\vec{i}]_{d-1}+d-2}{2}\right)\hat{P}_{i_d}^{2[\vec{i}]_{d-1}+d-1}(x_d) = -P_{i_d}^{2[\vec{i}]_{d-1}+d-3}(x_d).$$

Orthogonality (2.2) yields orthogonality of the functions

$$\pi_{i_1,\ldots,i_{d-1}}P_{i_d}^{2[\vec{i}]_{d-1}+d-3}(z_d)\left(\frac{1-z_d}{2}\right)^{[\vec{i}]_{d-1}-1}$$

in $L^2(T)$. Thus (4.22) is an orthogonal expansion in $L^2(T)$ yielding

$$\|\partial_{\vec{a}}u - g\|_T^2 = \sum_{\vec{i}\in\mathbb{N}_0^d, i_d\geq 1}\gamma_{i_1,\ldots,i_{d-1}}\frac{v_{\vec{i}}^2}{|\vec{i}|+\frac{d-2}{2}} + \sum_{[\vec{i}]_{d-1}\geq 1, i_d=0}\gamma_{i_1,\ldots,i_{d-1}}\frac{\left(\frac{2[\vec{i}]_{d-1}+d-2}{2}v_{\vec{i}}\right)^2}{|\vec{i}|+\frac{d-2}{2}}.$$

This is a sum of positive numbers, hence restricting the summation to $\vec{i}\in\mathcal{I}_p^d$ and $\vec{i}\in\mathcal{J}_p^d$ from (4.12) we get using (4.11) a lower bound

$$\begin{aligned}\|\partial_{\vec{a}}u - g\|_T^2 &\geq \sum_{\vec{i}\in\mathcal{I}_p^d}\gamma_{i_1,\ldots,i_{d-1}}\frac{v_{\vec{i}}^2}{|\vec{i}|+\frac{d-2}{2}} + \sum_{\vec{i}\in\mathcal{J}_p^d}\gamma_{i_1,\ldots,i_{d-1}}\left([\vec{i}]_{d-1}+\tfrac{d-2}{2}\right)v_{\vec{i}}^2\\ &\geq (p+\tfrac{d}{2})\left\{\sum_{\vec{i}\in\mathcal{I}_p^d}\gamma_{i_1,\ldots,i_{d-1}}\frac{v_{\vec{i}}^2}{(|\vec{i}|+\frac{d-2}{2})^2} + \sum_{\vec{i}\in\mathcal{J}_p^d}\gamma_{i_1,\ldots,i_{d-1}}v_{\vec{i}}^2\right\}\\ &\geq \frac{1}{2}(p+\tfrac{d}{2})\|u-\Pi_p u\|_F^2.\end{aligned}$$

Step 2. We prove the bound $\|g\|_T \leq C\|\nabla u\|_T$ where $C>0$ depends only on the spatial dimension d. This estimate relies on Lemma 4.4 which is proven below separately. Recalling (4.10) we rewrite g as an expansion with coefficients $\{u_{\vec{i}}\}$:

$$g(\vec{x}) = \sum_{\vec{i}\in\mathbb{N}_0^d, [\vec{i}]_{d-1}\geq 1} u_{\vec{i}} \pi_{i_1,\dots,i_{d-1}}(\vec{x}) \left(\tfrac{[\vec{i}]_{d-1}+d-2}{2}\right) P_{i_d}^{2[\vec{i}]_{d-1}+d-1}(x_d) \left(\frac{1-x_d}{2}\right)^{[\vec{i}]_{d-1}-1}.$$

Suppose f_d satisfies the assumptions of Lemma 4.4 for $m = d$ and $\vec{\sigma} = (0, \dots, 0, 1)$. For this parameters g has a form similar to (4.26) differing by the coefficient $\frac{[\vec{i}]_{d-1}+d-2}{2} \leq \frac{d-1}{2}[\vec{i}]_{d-1}$. Then by (4.27) with $m = d$ and using (4.28) we obtain

$$\|g\|_T \leq \frac{d-1}{2}\|f_d\|_T \leq C\|\nabla u\|_T. \tag{4.23}$$

Step 3. Combining (4.21), the triangle inequality, (4.15) and (4.23) we get

$$\|u - \Pi_p u\|_F \leq \sqrt{2}(p + \tfrac{d}{2})^{-\frac{1}{2}} (\|\partial_{\vec{a}} u\|_T + \|g\|_T) \leq C(p + \tfrac{d}{2})^{-\frac{1}{2}} \|\nabla u\|_T \tag{4.24}$$

where C depends only on the spatial dimension d. The proof is complete. $\square$

The proof of the right inequality in (4.23) is done by induction and is a subject of the following lemma.

Lemma 4.4. *Suppose $u \in H^1(T)$ and let $\{u_{\vec{i}}\}$ be its expansion coefficients in the $L^2(T)$-orthogonal basis $\phi_{\vec{i}}$, see (4.7), (4.10). Suppose $2 \leq m \leq d$ is an integer and $\vec{\sigma} = (\sigma_1, \dots, \sigma_d)$ with the components σ_k satisfying*

$$\sigma_k = \begin{cases} 0, & 1 \leq k \leq m-1, \\ \frac{1}{2}, & k = m, \qquad (\textit{if } m \leq d-1), \\ 0 \textit{ or } \frac{1}{2}, & m+1 \leq k \leq d-1, \qquad (\textit{if } m \leq d-2), \\ 1, & k = d, \end{cases} \tag{4.25}$$

in particular $\vec{\sigma} = (0, \dots, 0, 1)$ if $m = d$. Consider the formal sum

$$\tilde{f}_m(\vec{z}) := \sum_{\vec{i}\in\mathbb{N}_0^d, [\vec{i}]_{m-1}\geq 1} u_{\vec{i}} [\vec{i}]_{m-1} \prod_{k=1}^d P_{i_k}^{2[\vec{i}]_{k-1}+k-1}(z_k) \left(\frac{1-z_k}{2}\right)^{[\vec{i}]_{k-1}-\sigma_k} \tag{4.26}$$

and its push forward $f_m := \tilde{f}_m \circ \Phi^{-1} : T \to \mathbb{R}$ as in (4.13). Then $f_m \in L^2(T)$ for any $2 \leq m \leq d$ and there holds

$$\|f_m\|_T \leq C\|\nabla u\|_T \tag{4.27}$$

with a constant $C > 0$ depending only on the spatial dimension d.

Proof. For any fixed $\vec{\sigma}$ satisfying (4.25) we have an equivalent representation

$$\tilde{f}_m(\vec{z}) = \sum_{\vec{i}\in\mathbb{N}_0^d, [i]_{m-1}\geq 1} w_{\vec{i}} [\vec{i}]_{m-1} \prod_{k=1}^d P_{i_k}^{2([\vec{i}]_{k-1}-\sigma_k)+k-1}(z_k) \left(\frac{1-z_k}{2}\right)^{[\vec{i}]_{k-1}-\sigma_k} \tag{4.28}$$

where the relation between $\{w_{\vec{i}}\}$ and $\{u_{\vec{i}}\}$ follows from (A.3). Note that $[\vec{i}]_{k-1} \geq \sigma_k$. Expansion (4.28) is orthogonal in $L^2(T)$ and we obtain by (2.2)

$$\|f_m\|_T^2 = \sum_{\vec{i}\in\mathbb{N}_0^d, [\vec{i}]_{m-1}\geq 1} (w_{\vec{i}} [\vec{i}]_{m-1})^2 \hat{\gamma}_{\vec{i}}, \qquad \hat{\gamma}_{\vec{i}} = \prod_{k=1}^d \frac{2}{2([\vec{i}]_k - \sigma_k) + k} > 0. \tag{4.29}$$

Although both $\{w_{\vec{i}}\}$ and $\{\hat{\gamma}_{\vec{i}}\}$ depend on a particular $\vec{\sigma}$, we do not trace this dependence explicitly to simplify the notations.

We prove (4.27) by induction on m. Suppose $m = 2$ and $\vec{\sigma}$ is fixed. Then

$$\|f_2\|_T^2 = \sum_{\vec{i}\in\mathbb{N}_0^d, i_1\geq 1} (w_{\vec{i}} i_1)^2 \hat{\gamma}_{\vec{i}}. \tag{4.30}$$

by (4.29). Using (4.14) for $k = 1$ and changing $\{u_{\vec{i}}\}$ to $\{w_{\vec{i}}\}$, cf. (4.28), we find (4.31)

$$\frac{\partial u}{\partial x_1} \circ \Phi(\vec{z}) = \sum_{\vec{i}\in\mathbb{N}_0^d, i_1\geq 1} w_{\vec{i}} D P_{i_1}^0(z_1) \prod_{k=2}^{d} P_{i_k}^{2([\vec{i}]_{k-1}-\sigma_k)+k-1}(z_k) \left(\frac{1-z_k}{2}\right)^{[\vec{i}]_{k-1}-1}.$$

Consider a weighting function W_1 on T with $W_1 := \tilde{W}_1 \circ \Phi^{-1}$ and

$$\tilde{W}_1(\vec{z}) := \sqrt{1-z_1^2} \prod_{k=2}^{d} \left(\frac{1-z_k}{2}\right)^{1-\sigma_k}.$$

Using orthogonality relations (2.2), (2.4), representation (4.30) and (4.31) we find

$$\|W_1 \frac{\partial u}{\partial x_1}\|_T^2 = \sum_{\vec{i}\in\mathbb{N}_0^d, i_1\geq 1} w_{\vec{i}}^2 i_1(i_1+1)\hat{\gamma}_{\vec{i}} \geq \|f_2\|_T^2. \tag{4.32}$$

Clearly $W_1 \leq 1$ on T and (4.32) implies (4.27) for $m = 2$ with $C = 1$.

Assume now that (4.27) holds for $m = 2, \ldots, n$ with $2 \leq n \leq d-1$, and prove (4.27) for $m = n+1$. Suppose $\vec{\sigma}$ is fixed for $m = n+1$. Denote $\tilde{u} = u \circ \Phi$ the pull back of u. By Lemma 4.3 there exists $\vec{b}^m$ such that

$$\partial_{\vec{b}^m} u \circ \Phi(\vec{z}) = \frac{\partial \tilde{u}}{\partial z_m} \prod_{k=m+1}^{d} \left(\frac{1-z_k}{2}\right)^{-1} =: \tilde{h}_m^1(\vec{z}) - \tilde{h}_m^2(\vec{z}), \tag{4.33}$$

where

$$\begin{aligned}\tilde{h}_m^1(\vec{z}) = \sum_{\vec{i}\in\mathbb{N}_0^d, i_m\geq 1} u_{\vec{i}} \Bigg\{ &\prod_{k=1}^{m-1} P_{i_k}^{2[\vec{i}]_{k-1}+k-1}(z_k) \left(\frac{1-z_k}{2}\right)^{[\vec{i}]_{k-1}} \times \\ &\times D P_{i_m}^{2[\vec{i}]_{m-1}+m-1}(z_m) \left(\frac{1-z_m}{2}\right)^{[\vec{i}]_{m-1}} \times \\ &\times \prod_{k=m+1}^{d} P_{i_k}^{2[\vec{i}]_{k-1}+k-1}(z_k) \left(\frac{1-z_k}{2}\right)^{[\vec{i}]_{k-1}-1} \Bigg\},\end{aligned} \tag{4.34}$$

$$\begin{aligned}\tilde{h}_m^2(\vec{z}) = \sum_{\vec{i}\in\mathbb{N}_0^d, [\vec{i}]_{m-1}\geq 1} u_{\vec{i}} \Bigg\{ &\prod_{k=1}^{m-1} P_{i_k}^{2[\vec{i}]_{k-1}+k-1}(z_k) \left(\frac{1-z_k}{2}\right)^{[\vec{i}]_{k-1}} \times \\ &\times \frac{[\vec{i}]_{m-1}}{2} P_{i_m}^{2[\vec{i}]_{m-1}+m-1}(z_m) \left(\frac{1-z_m}{2}\right)^{[\vec{i}]_{m-1}-1} \times \\ &\times \prod_{k=m+1}^{d} P_{i_k}^{2[\vec{i}]_{k-1}+k-1}(z_k) \left(\frac{1-z_k}{2}\right)^{[\vec{i}]_{k-1}-1} \Bigg\}.\end{aligned} \tag{4.35}$$

Denote $h_m^i := \tilde{h}_m^i \circ \Phi^{-1}$. Then for the product of h_m^1 with the function

$$W_m := \tilde{W}_m \circ \Phi^{-1}, \qquad \tilde{W}_m(\vec{z}) = \sqrt{1-z_m^2} \prod_{k=m+1}^{d} \left(\frac{1-z_k}{2}\right)^{1-\sigma_k} \tag{4.36}$$

we get the representation

$$\|W_m h_m^1\|_T^2 = \sum_{\vec{i}\in\mathbb{N}_0^d, i_m\geq 1} w_{\vec{i}}^2 \hat{\gamma}_{\vec{i}} i_m (i_m + 2[\vec{i}]_{m-1} + m). \tag{4.37}$$

Recall (4.29), then (4.37) and (4.33) yield by simple calculations

$$\|f_{n+1}\|_T^2 \leq \sum_{m=1}^{n} \|W_m h_m^1\|_T^2 \leq \sum_{m=1}^{n} 2\|\partial_{\vec{b}^m} u\|_T^2 + \sum_{m=1}^{n} 2\|W_m h_m^2\|_T^2 \tag{4.38}$$

Every term in the first sum is bounded by $\frac{m+3}{2}\|\nabla u\|_T^2$, cf. (4.15). Definitions (4.35), (4.36) yield the identity $\tilde{W}_m \tilde{h}_m^2 = \sqrt{\frac{1+z_m}{2}}\tilde{\varphi}_m$ where

$$\tilde{\varphi}_m(\vec{z}) = \sum_{\vec{i}\in\mathbb{N}_0^d, [\vec{i}]_{m-1}\geq 1} u_{\vec{i}} [\vec{i}]_{m-1} \prod_{k=1}^{d} P_{i_k}^{2[\vec{i}]_{k-1}+k-1}(z_k) \left(\frac{1-z_k}{2}\right)^{[\vec{i}]_{k-1}-\sigma_k^*} \tag{4.39}$$

with $\sigma_m^* = 1/2$ and $\sigma_k^* = \sigma_k$ for $k \neq m$. Then $\tilde{\varphi}_m$ has the form (4.26) with $m \leq n$ and hence satisfies the inductive hypothesis yielding $\|\varphi_m\|_T \leq C\|\nabla u\|_T$ for $\varphi_m := \tilde{\varphi}_m \circ \Phi^{-1}$. Inequality (4.38) yields

$$\|f_{n+1}\|_T^2 \leq \frac{n(n+7)}{4}\|\nabla u\|_T^2 + 2\sum_{m=1}^{n} \|\varphi_m\|_T^2 \leq \tilde{C}\|\nabla u\|_T^2 \tag{4.40}$$

where $\tilde{C}$ depends only on the spatial dimension d. The proof is complete. $\square$

5. Numerical experiments

In this section we demonstrate the convergence behavior of the $L^2(T)$-projection on the boundary of a simplex $T \subset \mathbb{R}^d$ for $d = 2, 3$. In the numerical tests below we show the convergence behavior of the L^2-error on the lower face F of T and compare it to the well studied convergence of the L^2-error in the entire simplex T:

$$\|u - \Pi_p u\|_F \qquad \text{and} \qquad \|u - \Pi_p u\|_T$$

for a class of functions $u \in H^s(T)$. We also provide convergence curves for the best approximation error in the $H^1(T)$-seminorm used in (1.3):

$$\|\nabla(u - u_p)\|_T = \inf_{v\in\mathcal{P}_p} \|\nabla(u - v)\|_T. \tag{5.1}$$

As a prototype of a function in $H^s(T)$ we take a function with an isotropic radial singularity located in $\vec{X} \in \mathbb{R}^d$ of order γ

$$u_{\vec{X},\gamma}(\vec{x}) = \left(\sum_{i=1}^{d}(x_i - X_i)^2\right)^{\frac{\gamma}{2}}.$$

It is known (cf. e.g. [4, 12]) that $u_{\vec{X},\gamma} \in H^{\frac{d}{2}+\gamma-\epsilon}(T)$ for arbitrary $\epsilon > 0$ if the singularity is located inside the simplex or on its boundary. Hence there holds

$$\inf_{v\in\mathcal{P}_p(T)} \|u_{\vec{X},\gamma} - v\|_{H^m(T)} \leq Cp^{\frac{d}{2}+\gamma-m-\epsilon}. \tag{5.2}$$

Moreover, if $\vec{X}$ coincides with one of the vertices of T a sharper analysis involving Jacobi-weighted spaces yields at least the doubled convergence rate:

$$\inf_{v\in\mathcal{P}_p(T)} \|u_{\vec{X},\gamma} - v\|_{H^m(T)} \leq Cp^{d+2\gamma-2m-\epsilon}. \tag{5.3}$$

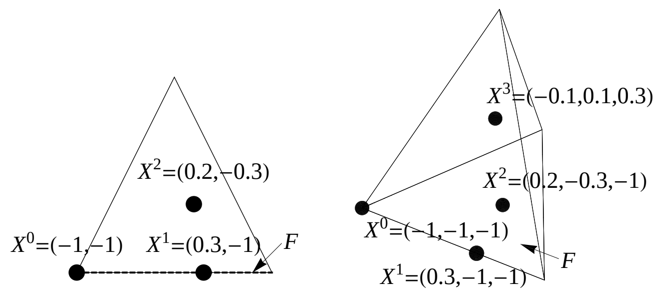

FIGURE 1. Location of an isotropic radial singularity in $\mathbb{R}^2$ and $\mathbb{R}^3$

We perform our numerical tests on the reference simplex T with vertices listed in (4.1). As before, let $F = \partial T \cap \{x_d = -1\}$ be the lower face of T. In our tests we examine functions $u_{\vec{X},\gamma}$ with singularity located at $\vec{X} = \vec{X}^0, \ldots, \vec{X}^d$ as shown on Fig. 1 (not on the lines of symmetry of T).

The computation is organized as follows. First, we compute the expansion coefficients $\{u_{\vec{i}}\}_{|\vec{i}|\le p}$ of the $L^2(T)$ projection

$$\Pi_p u_{\vec{X}^k,\gamma}(\vec{x}) = \sum_{|\vec{i}|\le p} u_{\vec{i}}\phi_{\vec{i}}(\vec{x}) \qquad u_{\vec{i}} = \|\phi_{\vec{i}}\|_T^{-2} \int_T u_{\vec{X}^k,\gamma}(\vec{x})\phi_{\vec{i}}(\vec{x})\, d\vec{x}.$$

For this, we decompose T into $k+1$ smaller simplices T^j in such a way that the restriction of the function $u_{\vec{X}^k,\gamma}|_{T^j}$ has a singularity in $\vec{X}^k$ being a vertex of T^j. E.g. in the case $d = 2$, $k = 1$ we have $\vec{X}^1 = (0.3, -1)$ and $T = T^0 \cup T^1$ with $T^0 = \mathrm{convhull}\{(-1,-1), (0,1), (0.3,-1)\}$, $T^1 = \mathrm{convhull}\{(1,-1), (0,1), (0.3,-1)\}$. Next, we parametrize T^j by $[-1,1]^d$ using a d-dimensional Duffy transformation collapsing at the vertex $\vec{X}^k$ [11]. The integrand transformed to $[-1,1]^d$ has a one-dimensional singularity in one direction and is analytic in the other direction(s). An efficient integration is then performed on $[-1,1]^d$ by the product-type quadrature rule consisting of the one-dimensional Gauss-Legendre quadrature rules in the regular directions and a one-dimensional composite variable order Gauss-Legendre quadrature rule in the singular direction, see e.g. [20, 21, 9] for more details. Having this, we evaluate the $L^2(T)$-error as

$$\|u_{\vec{X}^k,\gamma} - \Pi_p u_{\vec{X}^k,\gamma}\|_T^2 = \|u_{\vec{X}^k,\gamma}\|_T^2 - \sum_{|\vec{i}|\le p} u_{\vec{i}}^2 \|\phi_{\vec{i}}\|_T^2.$$

The order of the quadrature is chosen in such a way that the quadrature error is smaller than the approximation error and does not affect the convergence behavior. The $L^2(F)$-error is evaluated directly in the form

$$\|u_{\vec{X}^k,\gamma} - \sum_{|\vec{i}|\le p} u_{\vec{i}}\phi_{\vec{i}}\|_F^2$$

using an analogous decomposition of F into smaller simplices with a singularity in one of their vertices.

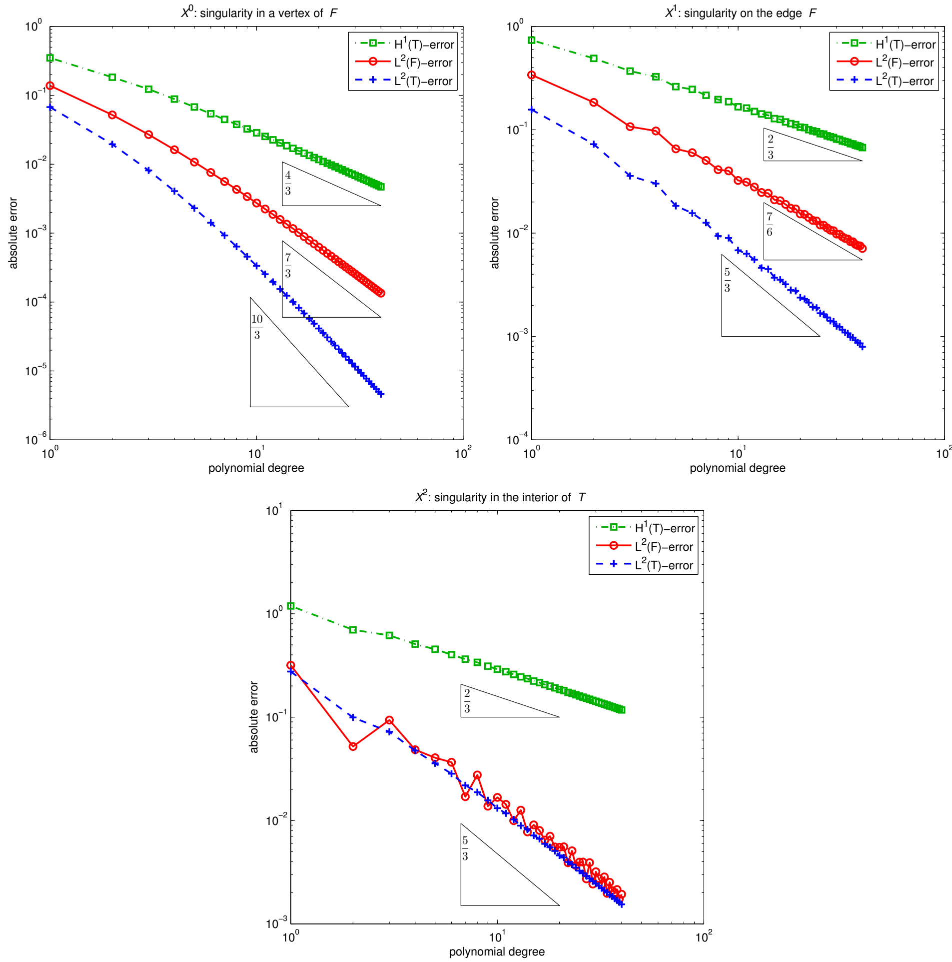

FIGURE 2. Approximation of an isotropic radial singularity in $\mathbb{R}^2$

The computation of the best approximation error (5.1) is equivalent to the computation of the unique up to a constant Galerkin projection $u_p \in \mathcal{P}_p$:

$$\int_T \nabla u_p \cdot \nabla v = \int_T \nabla u_{\vec{X},\gamma} \cdot \nabla v \qquad \forall v \in \mathcal{P}_p.$$

The best approximation error (5.1) is evaluated via $\|\nabla(u_{\vec{X},\gamma} - u_p)\|_T^2 = \|\nabla u_{\vec{X},\gamma}\|_T^2 - \|\nabla u_p\|_T^2$ which holds by the Galerkin orthogonality.

In Fig. 2 and Fig. 3 we give the convergence results for the radial singularity of the order $\gamma = \frac{2}{3}$. Theorem 1.1 predicts the convergence rate at least of the order $\frac{7}{6} - \epsilon$ for $d = 2$ and of the order $\frac{5}{3} - \epsilon$ for $d = 3$ for the $L^2(F)$-error, which is achieved in all tests. Furthermore, we observe superconvergence in the case when $\vec{X} = \vec{X}^0$ is a vertex of T or when $\vec{X} = \vec{X}^d$ is located in the interior of T. In the first case, the doubled order of convergence, $\sim \frac{7}{3}$ for $d = 2$ and $\sim \frac{10}{3}$ for $d = 3$, is achieved, which corresponds to the doubling of the convergence order in (5.3).

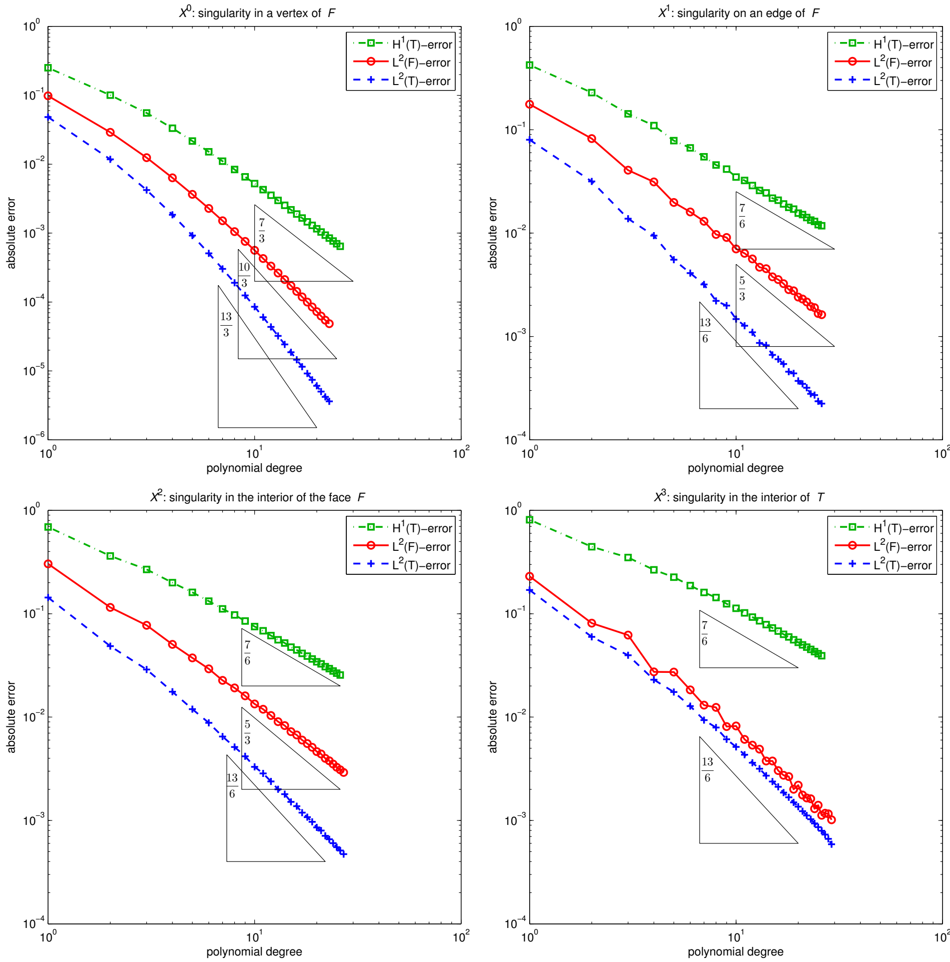

FIGURE 3. Approximation of an isotropic radial singularity in $\mathbb{R}^3$

Superconvergence in the second case is connected, presumably, with the location of the singularity, $\vec{X}^d$, outside the face F.

APPENDIX A. RELATIONS BETWEEN JACOBI POLYNOMIALS AND THEIR ANTIDERIVATIVES

Lemma A.1. *There holds*

$$\underline{\hat{P}}^{\alpha} = \mathcal{A}^{\alpha}\underline{P}^{\alpha}, \qquad \alpha \geq 0 \tag{A.1}$$

where

$$\mathcal{A}^{\alpha} = \begin{pmatrix} 1 & & & & \\ b_1^{\alpha} & a_1^{\alpha} & & & 0 \\ c_2^{\alpha} & b_2^{\alpha} & a_2^{\alpha} & & \\ 0 & c_3^{\alpha} & b_3^{\alpha} & a_3^{\alpha} & \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \begin{aligned} a_n^{\alpha} &= \tfrac{2(n+\alpha)}{(2n+\alpha-1)(2n+\alpha)}, \\ b_n^{\alpha} &= \tfrac{2\alpha}{(2n+\alpha-2)(2n+\alpha)}, \\ c_n^{\alpha} &= \tfrac{-2(n-1)}{(2n+\alpha-2)(2n+\alpha-1)}. \end{aligned} \tag{A.2}$$

Proof. The relation

$$\hat P_n^\alpha(x) = a_n^\alpha P_n^\alpha(x) + b_n^\alpha P_{n-1}^\alpha(x) + c_n^\alpha P_{n-2}^\alpha(x), \quad n \ge 2$$

has been shown in [5, Lemma 2.2]. For $n = 0, 1$ the assertion follows from (2.5) by the direct computation. $\square$

Lemma A.2. *There holds*

$$\underline{P}^{\alpha-1} = \mathcal{W}^\alpha \underline{P}^\alpha, \qquad \alpha \ge 1 \tag{A.3}$$

where

$$\mathcal{W}^\alpha = \begin{pmatrix} 1 & & & \\ \hat b_1^\alpha & \hat a_1^\alpha & & 0 \\ 0 & \hat b_2^\alpha & \hat a_2^\alpha & \\ 0 & & \ddots & \ddots \end{pmatrix}, \qquad \begin{aligned} \hat a_n^\alpha &= \tfrac{n+\alpha}{2n+\alpha}, \\ \hat b_n^\alpha &= \tfrac{-n}{2n+\alpha}. \end{aligned} \tag{A.4}$$

Proof. The required transformation follows from [1, 22.7.18] for $n \ge 1$ and the identity $P_0^{\alpha-1}(x) \equiv 1 \equiv P_0^\alpha(x)$. $\square$

Lemma A.3. *For all $\alpha > 0$ there holds*

$$2P_n^{\alpha-1}(x) = (x-1)P_{n-1}^{\alpha+1}(x) + \alpha \hat P_n^{\alpha+1}(x), \qquad n \ge 1. \tag{A.5}$$

Proof. Let us denote

$$f(x) = (x-1)P_{n-1}^{\alpha+1}(x) + \alpha \hat P_n^{\alpha+1}(x)$$

and show that $f = 2P_n^{\alpha-1}$. We have

$$\begin{aligned} (x-1)^{\alpha-1} f &= (x-1)^\alpha P_{n-1}^{\alpha+1} + \alpha (x-1)^{\alpha-1} \hat P_n^{\alpha+1} \\ &= \frac{d}{dx}\left\{(x-1)^\alpha \hat P_n^{\alpha+1}\right\} \end{aligned} \tag{A.6}$$

On the other hand, using Rodrigues formula (2.1)

$$\begin{aligned} (x-1)^{\alpha-1} P_n^{\alpha-1}\, dx &= \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x-1)^{\alpha-1}(x^2-1)^n\right) \\ &= \frac{1}{2^n n!}\frac{d^n}{dx^n}\left\{(x-1)^\alpha (x+1)(x^2-1)^{n-1}\right\} \\ &= \frac{1}{2n}\frac{d}{dx}\left\{(x-1)^\alpha (x+1) P_{n-1}^{(\alpha,1)}\right\} \end{aligned} \tag{A.7}$$

Comparing (A.6) and (A.7) we find that (A.5) is equivalent to

$$\hat P_n^{\alpha+1}(x) = \frac{x+1}{n} P_{n-1}^{(\alpha,1)}(x). \tag{A.8}$$

It remains to prove (A.8). To this end, we use classical relations between Jacobi polynomials. Using [26, (4.5.4)] and [26, (4.21.7)] and [1, (22.7.19)] in sequence we obtain

$$\begin{aligned} \frac{d}{dx}\left\{\frac{x+1}{n} P_{n-1}^{(\alpha,1)}(x)\right\} &= \frac{2}{2n+\alpha}\frac{d}{dx}\left\{P_{n-1}^\alpha(x) + P_n^\alpha(x)\right\} \\ &= \frac{1}{2n+\alpha}\left\{(n+\alpha)P_{n-2}^{(\alpha+1,1)}(x) + (n+\alpha+1)P_{n-1}^{(\alpha+1,1)}(x)\right\} \\ &= P_{n-1}^{\alpha+1}(x). \end{aligned}$$

This finishes the proof. $\square$

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